

§3.2 ② Let  $x_1, \dots, x_{100}$  be the first list,  $y_1, \dots, y_{100}$  the second list, and  $z_1, \dots, z_{100}$  the third.

$$\begin{aligned} \text{a) } \bar{z} &= (z_1 + \dots + z_{100})/100 = (x_1 + y_1 + \dots + x_{100} + y_{100})/100 = (x_1 + \dots + x_{100})/100 + (y_1 + \dots + y_{100})/100 \\ &= \bar{x} + \bar{y} = (0.2 \times 1 + 0.8 \times 2) + (0.5 \times 3 + 0.5 \times 5) = 1.8 + 4 = 5.8 \end{aligned}$$

(Note: no assumptions were made regarding the order of ones, twos, threes and fives in the lists!)

b) Similarly, if  $z_i = x_i - y_i$  for  $i=1, \dots, n$ , then  $\bar{z} = \bar{x} - \bar{y} = 1.8 - 4 = -2.2$

c) & d) If  $z_i = x_i y_i$  or  $z_i = x_i / y_i$  for  $i=1, \dots, n$ , then  $\bar{z}$  depends on the order within the lists! e.g. for  $z_i = x_i y_i$ ,

$$\begin{aligned} \text{Case 1. } x_i &: 1, 1, \dots, 1, 2, 2, \dots, 2 \\ y_i &: 3, 3, \dots, 3, 5, 5, \dots, 5 \\ z_i &: \underbrace{3, 3, \dots, 3}_{20}, \underbrace{6, \dots, 6}_{30}, \underbrace{10, \dots, 10}_{50} \Rightarrow \bar{z} = 7.4 \end{aligned}$$

$$\begin{aligned} \text{Case 2. } x_i &: 2, 2, \dots, 2, 1, 1, \dots, 1 \\ y_i &: 3, 3, \dots, 3, 5, 5, \dots, 5 \\ z_i &: \underbrace{6, 6, \dots, 6}_{50}, \underbrace{10, \dots, 10}_{30}, \underbrace{5, \dots, 5}_{20} \Rightarrow \bar{z} = 7 \end{aligned}$$

⑩ a)  $P(X_i = k) = P(X_1 = k) = P(\text{first } k \text{ non-aces, then an ace})$

$= P(\text{first } k \text{ are not aces}) P((k+1)\text{-st is ace} \mid \text{first } k \text{ are not aces})$

$$= \frac{\binom{48}{k}}{\binom{52}{k}} \cdot \frac{4}{52-k}$$

b) Note that  $X_1 + X_2 + X_3 + X_4 + X_5 = 48$  whatever the order of the cards.

Since each  $X_i$  has the same distribution, we get

$$48 = E(X_1 + \dots + X_5) = E(X_1) + E(X_2) + \dots + E(X_5) = 5E(X_1) \Rightarrow E(X_1) = \frac{48}{5} = 9.6$$

and then  $E(X_i) = \frac{48}{5}$  for  $i=1, \dots, 5$

c) No: e.g.  $P(X_1 = 48, X_2 = 1) = 0$  but  $P(X_1 = 48)P(X_2 = 1) \neq 0$ .

$$\S 3.2 \quad (10) \quad a) \quad I_A + I_B = \begin{cases} 0 & \text{if neither } A \text{ nor } B \text{ happens} \\ 1 & \text{if exactly one of } A \text{ and } B \text{ happens} \\ 2 & \text{if } A \text{ and } B \text{ both happen} \end{cases}$$

$$\text{Thus, } P(I_A + I_B = 0) = P(A^c B^c) = P(A^c) P(B^c) = (1 - P(A))(1 - P(B))$$

$$P(I_A + I_B = 1) = P(AB^c) + P(A^c B) = P(A)P(B^c) + P(A^c)P(B) \\ = P(A)(1 - P(B)) + P(B)(1 - P(A)) = P(A) + P(B) - 2P(AB)$$

$$P(I_A + I_B = 2) = P(AB) = P(A)P(B).$$

Then with  $Y = (I_A + I_B)^2$ ,  $\text{Range}(Y) = \{0, 1, 4\}$  and

$$P(Y=0) = 1 - P(A) - P(B) + P(A)P(B), \quad P(Y=1) = P(A) + P(B) - 2P(AB),$$

$$P(Y=4) = P(A)P(B)$$

b) Note that  $I_A^2 = I_A$ ,  $I_B^2 = I_B$  ~~and  $I_A^2 = I_A$~~ , so:

$$E[(I_A + I_B)^2] = E(I_A^2 + 2I_A I_B + I_B^2) = E(I_A + 2I_A I_B + I_B) \\ = E(I_A) + 2E(I_A I_B) + E(I_B) \stackrel{(\text{indep.})}{=} E(I_A) + 2E(I_A)E(I_B) + E(I_B) \\ = P(A) + 2P(A)P(B) + P(B).$$

(Alternative: use part a).

(18)  $\mu = E(X) = aP(X=a) + bP(X=b)$  and  $P(X=a) + P(X=b) = 1$ , so

$$\mu = aP(X=a) + b(1 - P(X=a)) = b - (b-a)P(X=a)$$

$$\therefore P(X=a) = \frac{b-\mu}{b-a}, \quad P(X=b) = \frac{\mu-a}{b-a}.$$