Answer Key, Hw. #4

§1.5 ① Let $O = \{\text{odd box}\}$, $E = \{\text{even box}\}$, $B = \{\text{marble is black}\}$.
Then $P(O) = P(E) = \frac{1}{2}$, $P(B|O) = \frac{1}{4}$, $P(B|E) = \frac{2}{6} = \frac{1}{3}$.

a) $P(B) = P(B|O)P(O) + P(B|E)P(E) = \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{8} + \frac{1}{6} = \frac{7}{24}$.

b) $P(B^c) = 1 - P(B) = \frac{17}{24}$, so $P(E|B^c) = \frac{P(EB^c)}{P(B^c)} = \frac{P(B^c|E)P(E)}{P(B^c)}$
$= \frac{\frac{2}{3}\cdot\frac{1}{2}}{17/24} = \frac{1}{3}\cdot\frac{24}{17} = \frac{8}{17}$.

§1.5 ⑤ Let $D = \{\text{person has disease}\}$, $A = \{\text{test is positive}\}$.

a) $P(A) = P(A|D)P(D) + P(A|D^c)P(D^c) = 0.8 \times 0.01 + 0.05 \times 0.99 = 0.0575$

b) Note that set-theoretically, "but" is equivalent to "and"! Thus,
$P(\text{person has disease but is diagnosed as healthy}) = P(D \cap A^c)$
$= P(A^c|D)P(D) = [1 - P(A|D)]P(D) = 0.2 \times 0.01 = 0.002$

c) $P(\text{correctly diagnosed and healthy}) = P(D^c \cap A^c) = P(A^c|D^c)P(D^c)$
$= 0.95 \times 0.99 = 0.9405$

d) $P(D|A) = \frac{P(A|D)P(D)}{P(A)} = \frac{0.8 \times 0.01}{0.0575} \approx 0.139$

e) All probabilities that "went in" [$P(D)$, $P(A|D)$, $P(A^c|D^c)$] are proportions of the population, and therefore have a long-run frequency interpretation. Then so do the probabilities that "come out".

§1.6 ② Assume that the at bats are independent, and in each at bat, the probability of a hit is $0.300 = \frac{3}{10}$.
Let $H_i = \{\text{hit in the } i^{th} \text{ at bat}\}$. Then from the above assumption, $H_1, H_2, \ldots$ are independent and $P(H_i) = \frac{3}{10}$ for $i = 1, 2, \ldots$.
So:

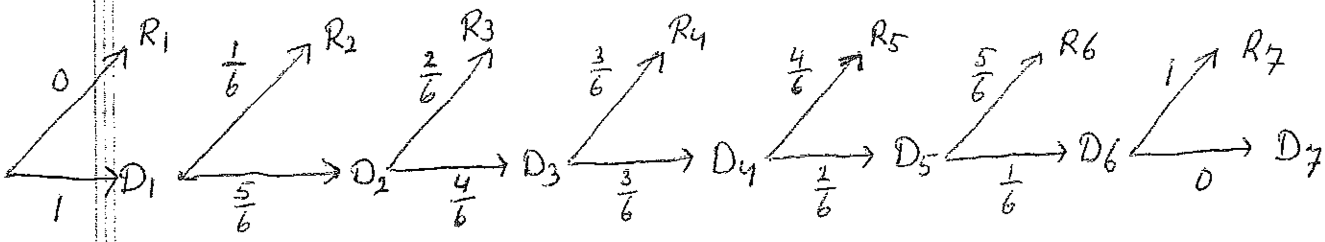
a) $P(\text{at least one hit in next two at bats}) = 1 - P(\text{no hit in next two at bats})$
$= 1 - P(H_1^c H_2^c) = 1 - P(H_1^c)P(H_2^c) = 1 - \left(\frac{7}{10}\right)^2 = 0.51$

b) $P(\text{at least one hit in next three at bats}) = 1 - P(H_1^c H_2^c H_3^c) = 1 - \left(\frac{7}{10}\right)^3 = 0.657$

c) $P(\text{at least one hit in next } n \text{ at bats}) = 1 - P(H_1^c H_2^c \cdots H_n^c) = 1 - \left(\frac{7}{10}\right)^n$.

§1.6 (6) Let $D_i = \{$the first i rolls are all different$\}$, and
$R_i = \{$the i^{th} roll is a number you have rolled before$\}$.

Tree diagram:



a) $p_1 = P(R_1) = 0$

$p_2 = P(D_1 R_2) = P(D_1)P(R_2|D_1) = 1 \cdot \frac{1}{6} = \frac{1}{6}$

$p_3 = P(D_1 D_2 R_3) = P(D_1)P(D_2|D_1)P(R_3|D_1 D_2) = 1 \cdot \frac{5}{6} \cdot \frac{2}{6} = \frac{5}{18}$

Continuing this way:

$p_4 = 1 \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} = \frac{5}{18}$

$p_5 = 1 \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{4}{6} = \frac{5}{27}$

$p_6 = 1 \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{5}{6} = \frac{25}{324}$

$p_7 = 1 \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6} \cdot 1 = \frac{5}{324}$

$p_8 = p_9 = p_{10} = \dots = 0$

b) It never takes more than 7 rolls to get a number you've rolled before.
The events $\{$it takes exactly i rolls$\}$, $i = 1, 2, \dots, 7$, form a partition of the outcome space Ω. Thus, $p_1 + p_2 + \dots + p_{10} = p_1 + p_2 + \dots + p_7 = 1$.

c) routine exercise in adding fractions.
[Note, however, that the answers from part (a) should <u>not</u> be written in decimal form, or else round-off errors would make it impossible to check whether the sum is <u>exactly</u> 1.]

~~Answer Key, with hints~~

App. 1 (ii) $\binom{n-1}{k-1}+\binom{n-1}{k} = \frac{(n-1)!}{(k-1)![n-1-(k-1)]!} + \frac{(n-1)!}{k!(n-1-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$

$$= \frac{(n-1)!\,k}{k(k-1)!(n-k)!} + \frac{(n-1)!\,(n-k)}{k!\,(n-k)(n-k-1)!} = \frac{(n-1)!\,k + (n-1)!\,(n-k)}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,n}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}.$$

(xi) $\binom{52}{5} = \frac{52!}{5!\,47!}$ different (unordered!) 5-card hands, ...

(xii) $\binom{48}{5} = \frac{48!}{5!\,43!}$ of which contain no aces.

§2.1: (2) The number of boys in a 4-child family has a binomial $(4, \frac{1}{2})$ distribution So

$$P(2 \text{ boys} + 2 \text{ girls}) = P(2 \text{ boys}) = \binom{4}{2}\left(\tfrac{1}{2}\right)^4 = 6 \cdot \tfrac{1}{16} = \tfrac{3}{8}$$

and

$$P(\text{different numbers of boys and girls}) = 1 - P(2 \text{ boys} + 2 \text{ girls}) = \tfrac{5}{8}.$$

Hence, a different number of boys and girls is more common.

(4)

$P(2 \text{ sixes in first 5 rolls} \mid 3 \text{ sixes in the 8 rolls}) =$

$$= \frac{P(2 \text{ sixes in first 5 rolls and 3 sixes in the 8 rolls})}{P(3 \text{ sixes in the 8 rolls})}$$

$$= \frac{P(2 \text{ sixes in first 5 rolls and 1 six in last 3 rolls})}{P(3 \text{ sixes in the 8 rolls})}$$

$$= \frac{P(2 \text{ sixes in 5 rolls})\, P(1 \text{ six in 3 rolls})}{P(3 \text{ sixes in 8 rolls})}$$

(since the first 5 rolls are independent of the last 3 rolls!)

$$= \frac{\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 \cdot \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2}{\binom{8}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^5} = \frac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}} = \frac{10 \times 3}{56} = \frac{15}{28}$$

$$\left[\binom{5}{3} = \frac{5!}{2!\,3!} = \frac{4\cdot 5}{2} = 10, \quad \binom{3}{1} = \frac{3!}{1!\,2!} = 3, \quad \binom{8}{3} = \frac{8!}{3!\,5!} = \frac{6\cdot 7\cdot 8}{6} = 56 \right].$$

(7) The number of times you win has a binomial $(5, p)$ distribution, where $p = \frac{15}{36} = \frac{5}{12}$. [15 out of 36 ordered pairs are favorable for you.] So

$$P(\text{you win at least 4 times}) = P(4) + P(5)$$

$$= \binom{5}{4} p^4 q + \binom{5}{5} p^5 = 5p^4(1-p) + p^5$$

$$= p^4(5-4p) = \left(\frac{5}{12}\right)^4 \cdot \frac{40}{12} = 8\left(\frac{5}{12}\right)^5 = .1005$$