§5.3 (10) a) Let X and Y be the two incomes. Then X and Y are indep. normal (μ, σ^2) with $\mu = 60{,}000$ and $\sigma = 10{,}000$. Thus $X+Y \sim$ normal $(2\mu, 2\sigma^2)$ and $\frac{X+Y}{2} \sim$ normal $(\mu, \sigma^2/2)$. So

$$P\left(\frac{X+Y}{2} > 65{,}000\right) = 1 - \Phi\left(\frac{65{,}000-\mu}{\sqrt{\sigma^2/2}}\right) = 1 - \Phi\left(\frac{65{,}000-60{,}000}{10{,}000/\sqrt{2}}\right)$$
$$= 1 - \Phi(1/\sqrt{2}) = 1 - \Phi(.71) = 1 - .7611 = .2389.$$

b) Let X = older person's income, Y = younger person's income. Then X, Y indep. and $X \sim$ normal (μ, σ^2), $Y \sim$ normal (ν, σ^2) with $\mu = 60{,}000$, $\nu = 40{,}000$, $\sigma = 10{,}000$. Thus $X-Y \sim$ normal $(\mu-\nu, 2\sigma^2)$ and so

$$P(X<Y) = P(X-Y<0) = \Phi\left(\frac{0-(\mu-\nu)}{\sigma\sqrt{2}}\right) = \Phi\left(\frac{-20{,}000}{10{,}000\sqrt{2}}\right)$$
$$= 1 - \Phi(\sqrt{2}) \simeq 1 - \Phi(1.41) = 1 - .9207 = .0793.$$

§5.4 (3) a) Let X = waiting time in first queue, Y = w.t. in second queue. Then X, Y indep., $X \sim \exp(\alpha)$, $Y \sim \exp(\beta)$, so

$$f_X(x) = \alpha e^{-\alpha x} \quad (x>0), \quad f_Y(y) = \beta e^{-\beta y} \quad (y>0).$$

Therefore, for $z>0$,

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx = \int_0^z \alpha e^{-\alpha x} \beta e^{-\beta(z-x)}\,dx$$
$$= \alpha\beta e^{-\beta z} \int_0^z e^{(\beta-\alpha)x}\,dx \quad (*)$$
$$= \frac{\alpha\beta}{\beta-\alpha} e^{-\beta z}\left(e^{(\beta-\alpha)z} - 1\right) = \frac{\alpha\beta}{\beta-\alpha}\left(e^{-\alpha z} - e^{-\beta z}\right)$$

if $\alpha \neq \beta$. If $\alpha = \beta$, then same calculation up to $(*)$ gives

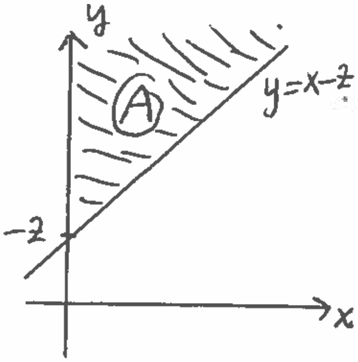
$$f_{X+Y}(z) = \alpha^2 e^{-\alpha z} \int_0^z 1\,dx = \alpha^2 z e^{-\alpha z},$$ in which case $X+Y \sim$ gamma $(2, \alpha)$.

b) $E(X+Y) = E(X) + E(Y) = \frac{1}{\alpha} + \frac{1}{\beta}$

c) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$\Rightarrow SD(X+Y) = \sqrt{\frac{1}{\alpha^2} + \frac{1}{\beta^2}}$. $\left(\neq \frac{1}{\alpha} + \frac{1}{\beta} \text{ !!}\right)$

§5.4 (13) Range$(Z) = (-\infty, \infty)$. For $z < 0$,



$$F_Z(z) = P(Z \le z) = P(X - Y \le z) = P(Y \ge X - z)$$

$$= \iint_A f(x,y)\,dx\,dy = \int_0^\infty \int_{x-z}^\infty \lambda^2 e^{-\lambda x} e^{-\lambda y}\,dy\,dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} P(Y > x - z)\,dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} e^{-\lambda(x-z)}\,dx = \int_0^\infty \lambda e^{\lambda z} e^{-2\lambda x}\,dx$$

$$= \frac{1}{2} e^{\lambda z} \int_0^\infty \underbrace{2\lambda e^{-2\lambda x}}_{\text{exp }(2\lambda)\text{ density!}}\,dx = \frac{1}{2} e^{\lambda z}$$

$$\therefore f_Z(z) = F_Z'(z) = \frac{1}{2}\lambda e^{\lambda z} = \frac{1}{2}\lambda e^{-\lambda |z|}.$$

By symmetry, $X - Y \sim Y - X$ so $Z \sim -Z$, and so
$f_Z(z) = \frac{1}{2}\lambda e^{-\lambda |z|}$ also for $z > 0$.