Answer Key, Hw. #15

§4.4 (3) ~~Let $Y = U^2$. For $0<y<1$, $F_Y(y) = P(U^2 \le y) = P(U \le \sqrt{y}) = \sqrt{y}$, so $f_Y(y) = F_Y'(y) = \frac{1}{2\sqrt{y}}$ for $0\le y<1$, and $f_Y(y)=0$ elsewhere.~~

(6) $f_\Phi(\phi) = \frac{1}{\pi}$ for $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, and $=0$ elsewhere. Thus, for $y \in \mathbb{R}$,

$$F_Y(y) = P(Y\le y) = P(\tan \Phi \le y) = P(\Phi \le \arctan y)$$
$$= \frac{1}{\pi}\left(\arctan y - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{\pi}\arctan y + \frac{1}{2},$$

and

$$f_Y(y) = F_Y'(y) = \frac{1}{\pi}\cdot\frac{1}{1+y^2}.$$

Since $f_Y(-y) = \frac{1}{\pi}\cdot\frac{1}{1+(-y)^2} = \frac{1}{\pi}\cdot\frac{1}{1+y^2} = f_Y(y)$, this density is symmetric about 0. However, the integral

$$\int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{-\infty}^{\infty} \frac{y}{\pi(1+y^2)}\,dy$$

fails to converge, since both integrals $\int_{-\infty}^{0} \frac{y}{\pi(1+y^2)}\,dy$ and $\int_0^{\infty} \frac{y}{\pi(1+y^2)}\,dy$ diverge (e.g. by the integral test).

(9) a) Let $Y = T^\alpha$. Then, for $y>0$, $F_Y(y) = P(T^\alpha \le y) = P(T \le y^{1/\alpha})$
$= F_T(y^{1/\alpha})$. So $f_Y(y) = F_T'(y^{1/\alpha})\cdot \frac{1}{\alpha} y^{\frac{1}{\alpha}-1} = f(y^{1/\alpha})\cdot\frac{1}{\alpha}y^{\frac{1}{\alpha}-1}$
$= \lambda\alpha (y^{1/\alpha})^{\alpha-1} e^{-\lambda(y^{1/\alpha})^\alpha}\cdot\frac{1}{\alpha}y^{\frac{1}{\alpha}-1} = \lambda y^{(1-\frac{1}{\alpha})} y^{\frac{1}{\alpha}-1} e^{-\lambda y} = \lambda e^{-\lambda y}$.

§4.5 (6) a) $P(X\ge \frac{1}{2}) = 1 - P(X<\frac{1}{2}) = 1 - \lim_{a\to(\frac{1}{2})^-} F(a) = 1 - \lim_{a\to(\frac{1}{2})} a^3 = 1 - (\frac{1}{2})^3 = \frac{7}{8}$.

b) $f(x) = F'(x) = \begin{cases} 3x^2, & 0<x<1 \\ 0, & \text{otherwise.}\end{cases}$

c) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$; or $E(X) = \int_0^1 (1-F(x))\,dx = \left[x - \frac{x^4}{4}\right]_0^1 = \frac{3}{4}$

d) $X = \max(Y_1, Y_2, Y_3)$, so for $0<x<1$, $F_X(x) = P(X\le x) = P(Y_1\le x, Y_2 \le x, Y_3\le x)$
$= P(Y_1\le x)P(Y_2\le x)P(Y_3\le x) = x\cdot x\cdot x = x^3 = F(x)$

§4.5 9 a. $E(X) = E[F^{-1}(U)] = \int_0^1 F^{-1}(u) \cdot f_U(u)\, du = \int_0^1 F^{-1}(u)\, du$

$=$ (area between the y-axis and the curve $x = F^{-1}(y)$ from $y=0$ to $y=1$)

$=$ (area between the curve $y = F(x)$ and the line $y = 1$)

$= \int_0^\infty [1 - F(x)]\, dx = \int_0^\infty P(X > x)\, dx.$