Answer Key, hw. #14

§4.2 ① a) $P(T>t)=e^{-\lambda t}$ where $\lambda = \frac{\ln 2}{1} = \ln 2$

$\therefore\ P(T>t) = e^{-(\ln 2)t} = \left(e^{\ln 2}\right)^{-t} = 2^{-t}$ and $P(T>5) = 2^{-5} = \frac{1}{32}$.

b) Let n = number of atoms present originally, $N(t)$ = number of atoms remaining at time t. Then $N(t) \sim Bin(n,p)$ with $p = P(T>t)$ $= 2^{-t}$. So $E[N(t)] = np = n\cdot 2^{-t} = \frac{1}{10}n \Rightarrow 2^{-t} = \frac{1}{10} \Rightarrow 2^{t} = 10$

$\Rightarrow t = \frac{\ln 10}{\ln 2} = 3.32$ years.

c) Now $n = 1024$, so $E[N(t)] = 1024\cdot 2^{-t} = 1 \Rightarrow 2^{t} = 1024 \Rightarrow t = 10$ years

d) $P(N(10)=0) = (1-p(10))^{1024} = (1-2^{-10})^{1024} = \left(1-\frac{1}{1024}\right)^{1024} \approx e^{-1}$

$= 0.3679$, since $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = e^{-1}$.

§4.2 ④ a) Let X be a particular component's lifetime. Since $E(X)=10$, we have $\frac{1}{\lambda} = 10$, and $\lambda = \frac{1}{10}$. Thus, $P(X \geq 20) = e^{-\lambda\cdot 20} = e^{-\cancel{\frac{1}{2}}2}$

$\doteq 0.6065$

b) Let m be the median lifetime. Then $P(X>m) = \frac{1}{2} \Rightarrow e^{-\lambda m} = \frac{1}{2}$

$\Rightarrow \lambda m = \log 2 \Rightarrow m = \frac{\log 2}{\lambda} = 10\log 2 \doteq 6.931$

c) $SD(X) = E(X) = 10$

d) Let $\bar{T}_{100} = (T_1 + \cdots + T_{100})/100$ be the average lifetime, where $T_1, \ldots, T_{100}$ are independent exponential ($\lambda = \frac{1}{10}$). Unfortunately, $\bar{T}_{100}$ does <u>not</u> have an exponential distribution! However, $\bar{T}_{100} = 100\,S_{100}$, where $S_{100} = T_1 + \cdots + T_{100}$. Since $T_1, \ldots, T_{100}$ are independent with the same distribution, we can use the <u>normal approximation</u>: $E(S_{100}) = 100\,E(X) = 1000$, $SD(S_{100}) = \sqrt{100}\,SD(X) = 100$.

So:

$$P(\bar{T}_{100} > 11) = P(S_{100} > 1100) \approx 1 - \Phi\left(\frac{1100-1000}{100}\right) = 1 - \Phi(1) \doteq .1587.$$

⑩ a) Clearly the range of $\text{int}(T)$ is $\{0,1,2,\ldots\}$, and for k in this set,

$P(\text{int}(T)=k) = P(k \leq T < k+1) = P(T \geq k) - P(T \geq k+1)$

$= e^{-\lambda k} - e^{-\lambda(k+1)} = e^{-\lambda k}(1-e^{-\lambda}) = (e^{-\lambda})^{k}(1-e^{-\lambda})$, so

$\text{int}(T)$ has a (shifted!) geometric distribution on $\{0,1,2,\ldots\}$ with $p = 1-e^{-\lambda}$.

(13) a) Let T be a particular component's lifetime. We expect that $E(T) \approx 20$.
Thus, $\frac{1}{\lambda} \approx 20$, and $\lambda \approx \frac{1}{20} = 0.05$, or 5% per day.

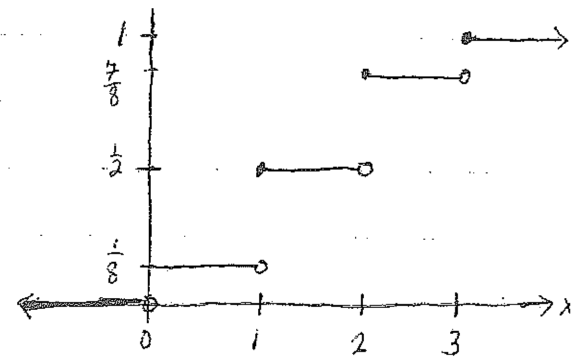
b) $N_d \sim$ binomial (n, p_d) where $n = 10{,}000$, and $p_d = P(T > d) = e^{-\lambda d}$.
For $d = 10$: $p_d = e^{-10/20} = e^{-1/2} = .6065$, so $E(N_d) = (10{,}000)(.6065) = 6065$,
and $SD(N_d) = \sqrt{np_d(1-p_d)} = 48.85$.
The cases $d=20$ and $d=30$ are similar

§4.5 (2) a)

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\Rightarrow$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{7}{8}, & 2 \le x < 3 \\ 1, & 3 \le x \end{cases}$$



b)

x	1	2	3	4	$\cdots$
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\cdots$

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{3}{4}, & 2 \le x < 3 \\ \frac{7}{8}, & 3 \le x < 4 \\ \vdots & \end{cases}$$