THE DYNAMICS OF MAPS OF SOLENOIDS HOMOTOPIC TO THE IDENTITY

ALEX CLARK

Abstract. Given a map of a solenoid homotopic to the identity, we use its difference from the identity to study its rotation and the nature of its fixed point set. With the aid of hyperspace theory, we show that a certain class of solenoids admits no expansive homeormorphism.

1. Introduction

Corresponding to a given sequence of natural numbers $N = (n_1, n_2, ...)$ with $n_j \geq 2$ for each j there is the solenoid Σ_N, the inverse limit

$$S^1 \xleftarrow{n_1} S^1 \xleftarrow{n_2} S^1 \xleftarrow{n_3} \cdots \qquad \Sigma_N = \left\{ \langle z_j \rangle_{j=1}^{\infty} \in \prod_{i=1}^{\infty} S^1 \mid z_j = z_{j+1}^{n_j} \text{ for } j = 1, 2, ... \right\}.$$

If $\prod_{i=1}^{\infty} S^1$ is given the group operation (denoted " $+$ ") of component-wise multiplication, then each solenoid Σ_N is a topological subgroup. Letting ρ denote the metric on S^1 inherited from the Euclidean metric on $\mathbb{R}^2$, we obtain a translation invariant metric d for $\prod_{i=1}^{\infty} S^1$ and hence Σ_N

$$d\left(\langle x_j \rangle_{j=1}^{\infty}, \langle y_j \rangle_{j=1}^{\infty} \right) = \sum_{j=1}^{\infty} \frac{1}{2^j} \rho\left(x_j, y_j \right).$$

Also, each solenoid Σ_N admits a fibration with unique path lifting $\pi_N : \mathbb{R} \to \Sigma_N$ onto the arc component of the identity $e = (1, 1, ...)$ given by

$$t \mapsto \left\langle \exp\left(2\pi i t\right), \exp\left(\frac{2\pi i t}{n_1}\right), ..., \exp\left(\frac{2\pi i t}{n_1 \cdots n_j}\right), ... \right\rangle, \text{[C1]}.$$

Given any map $f : \Sigma_N \to \Sigma_N$ homotopic to the identity, the map $\delta(x) = f(x) - x$ is homotopic to the map with constant value e and so is lifted by a map $\widetilde{\delta}$ making the following diagram commute

$$\begin{array}{ccc} & & \mathbb{R} \\ & \widetilde{\delta} \nearrow & \downarrow \pi_N \\ \Sigma_N & \xrightarrow[\delta]{} & \Sigma_N \end{array}.$$

A *character* of Σ_N is a continuous homomorphism $\chi : \Sigma_N \to S^1 \subset \mathbb{C}$. Any continuous map $\Sigma_N \to \mathbb{C}$ may be uniformly approximated by linear combinations of finitely many *characters* of Σ_N, see, e.g., [P],[Wa]. Thus for $x \in \Sigma_N$ the induced map $\Delta_x : \mathbb{R} \to \mathbb{R}$, $t \mapsto \widetilde{\delta} \circ (x + \pi_N(t))$ is a *limit periodic function*, meaning that it can be uniformly approximated to within any prescribed $\varepsilon > 0$ by a periodic

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function (depending on ε). In our case we may use trigonometirc polynomials with rational periods for these approximations, (see, e.g., [C2]). In what follows we will use the map $\widetilde{\delta}$ to describe the dynamics of f.

Since any homeomorphism of Σ_N which does not permute its composants is homotopic to the identity (see [F] 1 Theorem 2.6) and since Σ_N admits a minimal almost periodic flow which has its composants as orbits, it follows from Theorem 3.8 of [CHM] that any closed subset of Σ_N may be realized as the fixed point set of a homeomorphism of Σ_N homotopic to the identity. Our analysis will show, however, that any isolated continuum of fixed points is necessarily unstable under perturbations of the map. In fact, this result generalizes to maps homotopic to the identity map of any one-dimensional compact space (other than S^1) that admits a minimal flow.

The structure of the group of homeomorphisms of the solenoid Σ_N has been described topologically in [K] and is known to depend crucially on N. Independent of N there is always the subgroup of homeomorphisms E_N consisting of all homeomorphisms which are homotopic to a composition of one or more of the following: the involution $x \mapsto -x$ and a (possibly trivial) translation. We shall show that no homeomorphism from E_N can be expansive. For a certain class of solenoids, E_N is the entire group of homeomorphisms, and so no solenoid from this class admits an expansive homeomorphism. Hence, while the dyadic solenoid is one of the primary examples of a hyperbolic invariant set, there is an entire class of solenoids no member of which can occur as a hyperbolic invariant set, see, e.g, [Rob].

2. Rotation Number

As in [C2], with $p_j : \Sigma_N \to S^1$ denoting the projection onto the j^{th} coordinate and $\widehat{N}$ denoting the subgroup of $(\mathbb{Q}, +)$ generated by

$$\left\{ \frac{1}{n_1}, \frac{1}{n_1 n_2}, ..., \frac{1}{n_1 n_2 \cdots n_j}, ... \mid j = 1, 2, ... \right\},$$

$X(N) = \left\{ \chi_r : \Sigma_N \to S^1 \mid r \in \widehat{N} \right\}$ is the group of characters of Σ_N, where for

$$r = \frac{m}{n_1 n_2 \cdots n_j} \in \widehat{N} \ , \ \chi_r(x) = (p_{j+1}(x))^m .$$

Given a map $f : \Sigma_N \to \Sigma_N$ homotopic to the identity, the associated map $\widetilde{\delta}$ has an expansion

$$\widetilde{\delta}(x) \sim \delta_0 + \sum_{r \in \widehat{N} - \{0\}} \delta_r \chi_r(x) .$$

The constant term

$$\delta_0 \text{ is } \int_{\Sigma_N} \widetilde{\delta}(x) \, d\mu_N,$$

where μ_N is the Haar measure, which is the unique invariant normalized measure of the strictly ergodic almost periodic standard flow

$$(t, x) \mapsto \pi_N(t) + x$$

(see, [NS] V, 9.34, p.510). Hence, δ_0 is also the mean value of each function Δ_x:

$$\delta_0 = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Delta_x(t) \, dt.$$

Hence, δ_0 provides a measure of the average amount a point is moved along its composant by the map f. In general, this simple–minded rotation number is not as telling as the rotation number of an orientation–preserving homeomorphism of S^1, but the following general result does hold.

Theorem 2.1. *If $\delta_0 = 0$, then the associated map f has uncountably many fixed points.*

Proof. It is easy to construct functions with mean value 0 which do not take on the value 0, but we shall show that this is not possible for limit periodic maps. After doing so, it follows that each function Δ_x has at least one zero. Since Σ_N has uncountably many composants, this yields the desired result.

Suppose then that $g : \mathbb{R} \to \mathbb{R}$ is limit periodic with mean value 0 and that g is never 0. By continuity we may assume without loss of generality that g is positive. Let $m = \min \{g(t) \mid t \in [0,1]\} > 0$ and let $P \geq 1$ be a period of a periodic function which uniformly approximates g to within $m/2$. Then on each interval $I_n = [(n-1)P, nP]$, $n = 1, 2, \ldots$ we have $\int_{I_n} g(t)\, dt \geq m/2$. For large enough n,

$$\frac{1}{nP} \int_0^{nP} g(t)\, dt \geq \frac{m}{2P}$$

will be within $\frac{m}{4P}$ of the mean value of g. This contradicts g having mean value 0. In fact, we can conclude that if g is not identically 0 it must have both positive and negative values. □

3. Fixed Point Sets

While every closed subset of Σ_N occurs as the fixed point set of some homeomorphism $\Sigma_N \to \Sigma_N$ homotopic to id_{Σ_N} [CHM], the most significant fixed point sets dynamically are those which persist under small perturbations of f. Interestingly, the topological structure of the set of fixed points is sufficient to determine that certain fixed points disappear under arbitrarily small perturbations. In particular, it will be shown that isolated fixed points are unstable under perturbations of the map. In fact, the theorem applies to any one–dimensional compact metric space $M \neq S^1$ which is the minimal set of a flow $\phi_M : \mathbb{R} \times M \to M$. Any such space M is an example of an orientable matchbox manifold (see [AM]). In what follows, we consider a map f' to be a *perturbation* of f if f' is homotopic to f, and we consider a collection of fixed points of a map f *unstable under perturbation* if the collection vanishes under arbitrarily small perturbations of f.

Given any map $f : M \to M$ homotopic to id_M we are no longer able to construct the difference map as before, but we are able to construct the following analogous map $\widetilde{\delta} : M \to \mathbb{R}$

$$\widetilde{\delta}(x) = \text{the unique time } t \text{ satisfying } \phi_M(t, x) = f(x)$$

since $f(x)$ will be on the trajectory of x and such a t is uniquely determined when $M \neq S^1$. Unless otherwise stated, all maps $M \to M$ considered in this section are homotopic to id_M.

Definition 3.1. A fixed point p of $f : M \to M$ is *proper* if there is a sequence $\{t_n\} \subset \mathbb{R}$ satisfying:

(1) $t_n \to 0$
(2) $t_{2n-1} < 0$ and $t_{2n} > 0$ for $n = 1, 2, \ldots$

(3) $\operatorname{sign}\Big(\widetilde{\delta}\left(\phi_M\left(t_{2n-1},p\right)\right)\cdot\widetilde{\delta}\left(\phi_M\left(t_{2n},p\right)\right)\Big)=-1,\ n=1,2,\ldots$.

For example, $p\in\Sigma_N$ is proper when the graph of Δ_p intersects the horizontal axis transversely at $(0,0)$.

Theorem 3.2. *If p is a proper fixed point of f, then p is a limit point of the set of fixed points of f.*

Proof. The minimality of ϕ_M guarantees the existence of a sequence of real numbers $\{s_i\}\to\infty$ satisfying $p=\lim_i\{\phi_M(s_i,p)\}=\lim_i\{p_i\}$. Since ϕ_M is a continuous group action, for any t we also have $\phi_M(t,p)=\lim_i\{\phi_M(t,p_i)\}$. Thus, for $n=1,2,\ldots$ the continuity of $\widetilde{\delta}$ guarantees the existence of a corresponding $i_n\geq n$ with

$$\operatorname{sign}\left(\widetilde{\delta}\left(\phi_M\left(t_{2n-1},p_{i_n}\right)\right)\cdot\widetilde{\delta}\left(\phi_M\left(t_{2n},p_{i_n}\right)\right)\right)=-1.$$

Then the arc

$$A_n\stackrel{\text{def}}{=}\phi_M\left(\left[t_{2n-1},t_{2n}\right],p_{i_n}\right)$$

must contain a zero of $\widetilde{\delta}$ and thus a fixed point q_n of f. Since the diameter of the arcs A_n goes to zero as $n\to\infty$ and since $\{p_{i_n}\}_n\to p$, $q_n\to p$. □

Theorem 3.3. *If p is a fixed point of f which is isolated from all other fixed points of f, then there is a neighborhood U of p and arbitrarily small perturbations f_ε of f which have no fixed points in U.*

Proof. There is a local basis for M at p of the form

$$\{Z\times[0,1]\mid Z\text{ is zero–dimensional}\}\ \text{(see [AM])},$$

and since p is isolated, there is such a basis element $Z_0\times[0,1]$ containing no fixed points other than p. With the exception of the arc containing p, the arcs in this basis element contain no fixed points, and so $\widetilde{\delta}$ will be of constant sign on each such arc. If in each neighborhood of p consisting of arcs from $Z_0\times[0,1]$ there were arcs of different signs, there would be a sequence of "positive" and a sequence of "negative" arcs converging to the arc containing p. Then $\widetilde{\delta}$ would be constantly 0 on the arc containing p, contrary to the isolation of p. Hence, there must be a neighborhood U of p consisting of arcs on which $\widetilde{\delta}$ is either (a) non-negative or (b) non-positive, and without loss of generality we assume case (a). Let $\varepsilon(x):M\to[0,\varepsilon)$ be a map satisfying:

$$\varepsilon(x)=\begin{cases}0 & \text{if } x\notin U\\ \text{positive} & \text{if } x=p\end{cases}.$$

Then the perturbed map $f_\varepsilon(x)=\phi_M\left(\widetilde{\delta}(x)+\varepsilon(x),x\right)$ contains no fixed point in U. In fact, the set of fixed points is unchanged but for the loss of p. □

In much the same way as was done with a point, we could define an arc of fixed points to be proper when it is flanked by sequences converging to the endpoints, where the sign of $\widetilde{\delta}$ for terms in the sequences vary in the same way as above. It would then follow that there would be a sequence of fixed points not on the arc which converges to some point of the arc. If there is an isolated arc of fixed points, ideas similar to those above can be used to show that the entire arc is unstable under perturbation. Only the identity has M as its fixed point set, and this vanishes under any time–ε map of the flow. Since the only subcontinua of M are points, arcs and

M itself, we are led to the conclusion that any isolated continuum of fixed points is unstable under perturbation.

It is important to realize that these results are special to maps homotopic to the identity. For example, e is an isolated but stable fixed point of the shift automorphism and the involution $x \mapsto -x$ of the dyadic solenoid.

4. Prime Solenoids Admit no Expansive Homeomorphism

If each prime p is a factor of only finitely many of the n_i from the sequence $N = (n_1, n_2, ...)$, we refer to the corresponding solenoid Σ_N as *prime*, and all other solenoids will be referred to as *composite*. Using the terminology of the introduction, it follows from [K] that E_N is the entire group of homeomorphisms of a prime solenoid Σ_N. After we have demonstrated that no homeomorphism from E_N (independent of N) can be expansive, it follows that no prime solenoid admits an expansive homeomorphism; that is, a homeomorphism h for which there is a constant $c > 0$ satisfying the condition that for all $x \neq y$ there is an $n \in \mathbb{Z}$ with

$$d\left(h^n\left(x\right), h^n\left(y\right)\right) > c.$$

This provides a partial answer to D. Bellamy's question of which solenoids admit an expansive homeomorphism. If $N = (n, n, ...)$, it follows from the proof of [W] that the shift automorphism on Σ_N is expansive. All composite solenoids admit a shift–like automorphism; however, while this automorphism is continuum–wise expansive (see [Ka] for an explanation of this term), it is not generally expansive. To provide a complete characterization of the solenoids that admit expansive homeomorphisms, it remains to determine which composite solenoids admit expansive homeomorphisms.

Theorem 4.1. *Independent of N, no $h \in E_N$ is expansive.*

Proof. Given any two points x, y in the same composant of Σ_N, the map $t \mapsto x + \pi_N\left(t\right)$ provides a consistent orientation: $x < y$ if $x + \pi_N\left(t\right) = y$ for $t > 0$ and $x > y$ if $x + \pi_N\left(t\right) = y$ for $t < 0$. If a map $\Sigma_N \to \Sigma_N$ preserves this orientation, so will any iteration of this map; but if $\Sigma_N \to \Sigma_N$ reverses this orientation, any even-numbered iteration of this map will preserve orientation. Now we have the homeomorphism

$$E_N \approx \Sigma_N \times G_0\left(N\right) \times \mathbb{Z}_2$$

as described in [K], where $G_0\left(N\right)$ is the contractible subgroup consisting of all homeomorphisms $\left(\Sigma_N, e\right) \to \left(\Sigma_N, e\right)$ isotopic to id_{Σ_N} and $\Sigma_N \times \{id_{\Sigma_N}\} \times \{0\}$ corresponds to translations. While the above homeomorphism $\approx$ does not respect the group structure of E_N, consideration of the above remarks on orientation shows that some amount of the structure of $\mathbb{Z}_2$ is reflected in the group of homeomorphisms: with $id_{\Sigma_N} \sim (e, id_{\Sigma_N}, 0)$ and $-id_{\Sigma_N} \sim (e, id_{\Sigma_N}, 1)$, any homeomorphism $h \sim (x, f, 0)$ composed with a homeomorphism of the same form will again be of the same form; while an even number of iterations of a homeomorphism $h \sim (x, f, 1)$ will be of the form $(y, g, 0)$.

Suppose then that $h \in E_N$ and that h is expansive. Since $h \circ h$ is expansive when h is (see, e.g., [Wa] Corollary 5.22.1), by the above considerations we may assume without loss of generality that h is homotopic to translation by some $\xi \in \Sigma_N$, from which it follows that for $n \in \mathbb{Z}$ h^n is homotopic to translation by $n\xi$. To reach a contradiction, we examine the consequences for the induced map

$\widehat{h}:C(\Sigma_N)\to C(\Sigma_N)$ on the hyperspace of subcontinua Σ_N. Rogers showed in [R] that $C(\Sigma_N)$ is homeomorphic to the cone over Σ_N, $Cone(\Sigma_N)$. The proper non-degenerate subcontinua of Σ_N are arcs A of the form $x+\pi_N(I)$ for some interval $I\subset\mathbb{R}$, and if one defines the *length* $\ell(A)$ to be the length of any such associated interval I, one sees that any two arcs of the same length are isometric since any two such arcs are translates of each other and the metric for Σ_N is translation invariant. One can construct a homeomorphism $C(\Sigma_N)\approx Cone(\Sigma_N)$ as follows: the singletons are mapped to the bottom of the cone, all arcs of a given length are mapped to the same level of the cone, with the center of the arcs mapping to the corresponding point on the corresponding level of the cone, and Σ_N is mapped to the top of the cone. Viewed differently, one can construct a Whitney function $w:C(\Sigma_N)\overset{\text{onto}}{\to}[0,1]$ which has the same value on any two isometric subcontinua (see, e.g., [IN], p. 108). If an arc $A\subset\Sigma_N$ is shorter than an arc $B\subset\Sigma_N$, there is a translation $A+x\subsetneqq B$, and so $w(A)=w(A+x)<w(B)$, see [IN], p. 105. Thus, for proper non–degenerate subcontinua, w is a function of length and any Whitney level $w^{-1}(t)$ for $t\in(0,1)$ is the solenoid $\Sigma_N(t)$ of arcs of the same length (see also [KN]).

We proceed to show that for $f:\Sigma_N\to\Sigma_N$ homotopic to translation by γ, $\widehat{f}:C(\Sigma_N)\to C(\Sigma_N)$ "preserves the Whitney level on average." In much the same way as before, there is a map $\widetilde{\delta}:\Sigma_N\to\mathbb{R}$ lifting the map $f(x)-(x+\gamma)$. Fix a level $w^{-1}(t)$, $t\in(0,1)$, with associated length λ and define $d:\Sigma_N\to\mathbb{R}$ by

$$d(x)=\widetilde{\delta}(x+\pi_N(\lambda))-\widetilde{\delta}(x).$$

For a given x and corresponding arc $A_x=x+\pi_N([0,\lambda])$, the value $d(x)$ is the difference $\ell\left(\widehat{f}(A_x)\right)-\ell(A_x)$. For each $x\in\Sigma_N$ the maps Δ_x and $\Delta_{x+\pi_N(\ell)}$ both have mean value $\delta_0=\int_{\Sigma_N}\widetilde{\delta}(x)\,d\mu_N$, and so d has mean value 0:

$$\int_{\Sigma_N}d(x)\,d\mu_N=0.$$

As shown in Theorem 2.1, d attains the value 0 uncountably many times. To each zero x of d the corresponding arc $A_x\in w^{-1}(t)$ satisfies $w(A_x)=w\left(\widehat{f}(A_x)\right)$.

By Corollary 3.4 of [Ka] there is a $\gamma>0$ so that one of the following holds for any given $\varepsilon>0$

1. If $A\in C(\Sigma_N)$ and $0<$diam$(A)<\gamma$, then there is a positive integer n_0 so that diam$(h^n(A))<\varepsilon$ for all $n\geq n_0$ or
2. If $A\in C(\Sigma_N)$ and $0<$diam$(A)<\gamma$, then there is a positive integer n_0 so that diam$(h^{-n}(A))<\varepsilon$ for all $n\geq n_0$.

Replacing h with h^{-1} if necessary, we assume without loss of generality that 1 holds. Choose

$$\sigma<\min\{\gamma,\operatorname{diam}(\Sigma_N)\}\ \text{, let } \varepsilon=\sigma/2 \text{ and}$$

let K be the compact set $\{A\in C(\Sigma_N)\mid \operatorname{diam}(A)\leq\sigma\}$. Then $K=w^{-1}([0,t])$ for some $t\in(0,1)$ and $\{A\in C(\Sigma_N)\mid\operatorname{diam}(A)\leq\varepsilon\}=w^{-1}([0,t'])$ for some $t'\in(0,t)$. By 1 and the continuity of diam$\circ\hat{f}$ for any map $f:\Sigma_N\to\Sigma_N$ (see [IN]), for each $A\in K$ there is the least number $n_A\in\{1,2,...\}$ with diam$(h^{n_A}(B))<\varepsilon$ for all B in some neighborhood U_A of A. By the compactness of K there is a finite subcover of $\{U_A\mid A\in K\}$:

$$K\subset U_{A_1}\cup\cdots\cup U_{A_m}.$$

With $M = \max\{n_{A_1}, ..., n_{A_m}\}$, for all $A \in K$ there is an $n \in \{1, 2, ..., M\}$ with $\operatorname{diam}(h^n(A)) < \varepsilon$. Since $\widehat{h^n}$ is a homeomorphism sending $\Sigma_N \mapsto \Sigma_N$ and since ℓ is continuous, the compactness of K guarantees the existence of

$$L = \max\{\ell(h^n(A)) \mid A \in K \text{ and } n \in \{1, 2, ..., M\}\}.$$

Lemma 4.2. *For all $A \in K$ and for all $n \in \{1, 2, ...\}$, $\ell(h^n(A)) \leq L$.*

Proof. Let $A \in K$ and $k \in \{1, 2, ...\}$. Then $B = h^{n_A}(A) \in K$ and $\ell(h^n(A)) \leq L$ for all $n \leq n_A$ since $n_A \leq M$. Applying the same reasoning to B, we have that $h^{n_A+n_B}(A) \in K$ and $\ell(h^n(A)) \leq L$ for all $n \leq n_A + n_B$ and $n_A < n_A + n_B$. Proceeding in this manner a total of k times, we can conclude $\ell(h^n(A)) \leq L$ for all $n \leq k$. □

Let λ and λ' be the lengths of arcs in $w^{-1}(t)$ and $w^{-1}(t')$ respectively and for $x \in \Sigma_N$ let $A_x = x + \pi_N([0, \lambda])$. For each $n \in \{1, 2, ...\}$ let $F_n : \Sigma_N \to \mathbb{R}$ be the measurable function given by

$$F_n(A_x) = \sup_{k \geq n} \left\{\ell\left(h^k(A_x)\right)\right\},$$

which is well–defined by 1. Then with $E_n = \{x \in \Sigma_N \mid F_n(A_x) \geq \lambda'\}$, we have $E_{n+1} \subset E_n$ and $\cap E_n = \emptyset$, again by 1. Then for the Haar measure μ_N we must have

$$\lim_{n \to \infty} \mu_N(E_n) = 0.$$

Choose n_0 so that $\mu_N(E_{n_0}) < \min\left\{\frac{\lambda - \lambda'}{2L}, \frac{1}{2}\right\}$. With $f = h^{n_0}$ (homotopic to a translation) we have the associated map

$$d(x) = \widetilde{\delta}(x + \pi_N(\lambda)) - \widetilde{\delta}(x)$$

and $\int_{\Sigma_N} d(x)\, d\mu_N = 0$ as shown above. Notice that if $h^{n_0}(A_x) \in w^{-1}([0, t'])$, $d(x) \leq \lambda' - \lambda < 0$ and so

$$\begin{aligned}\int_{\Sigma_N} d(x)\, d\mu_N &= \int_{E_{n_0}} d(x)\, d\mu_N + \int_{\Sigma_N - E_{n_0}} d(x)\, d\mu_N \\ &< L\left(\frac{\lambda - \lambda'}{2L}\right) + (\lambda' - \lambda)\frac{1}{2} = 0,\end{aligned}$$

a contradiction. Hence, no such expansive h can exist. □

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Department of Mathematics, University of North Texas, Denton, TX 76203-1430
E-mail address: alexc@unt.edu