SOLENOIDALIZATION AND DENJOIDS

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Abstract. We describe a method (solenoidalization) of obtaining flows on κ–solenoids from a given flow on a κ–torus. When we apply this process to the Denjoy flows on $\mathbf{T}^2$, we obtain flows whose minimal sets we call denjoids. We give a topological classification of these indecomposable, one-dimensional continua.

1. Introduction

We begin by providing a method (solenoidalization) of obtaining flows on κ–solenoids from a given flow on a torus of the same dimension. We examine some general properties of such flows and then examine in detail the solenoidalization of the aperiodic flows on $\mathbf{T}^2$ which are not topologically equivalent to linear flows, the minimal sets of which we call denjoids. We calculate the Čech cohomology of denjoids and provide a topological classification of these denjoids. This generalizes the classification of the classical Denjoy continua found in [Fok]. M. Barge and R. F. Williams, using an entirely different method, have given a complete proof of the classification of the classical Denjoy continua [BW].

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2. The Topological Pullback of Flows

By a flow on a space X we shall always mean a continuous group action of $(\mathbb{R},+)$ on X mapping $\mathbb{R}\times X \to X$, and by ϕ_x we denote the ϕ–orbit of $x \in X$. Throughout we use the terminology of Spanier [S], Chapt 2, and we assume that $p : \left(\widetilde{X},\widetilde{x}\right) \to (X,x)$ is a fibration with unique path lifting and that $\widetilde{X}$ is connected and locally path connected. The notion of the pullback $p^*\phi$ of ϕ is

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usually considered in the context of differentiable flows (see, e.g., [CN] p. 177), but we shall give a purely topological construction which makes no requirement on the differentiability of ϕ.

We have the following diagram

$$\begin{array}{ccc} \left(\mathbb{R}\times\widetilde{X},(0,\widetilde{x})\right) & \dashrightarrow & \left(\widetilde{X},\widetilde{x}\right) \\ {\scriptstyle(id\times p)}\downarrow & & \downarrow^{p} \\ \left(\mathbb{R}\times X,(0,x)\right) & \overset{\phi}{\to} & (X,x) \end{array},$$

where $\dashrightarrow$ may be filled in uniquely by a map $p^*\phi$ making the diagram commute provided that

$$\phi_\# \circ (id \times p)_\# \pi_1 \left(\mathbb{R}\times\widetilde{X},(0,\widetilde{x})\right) \subset p_\#\pi_1\left(\widetilde{X},\widetilde{x}\right),$$

where $\pi_1(Z,z)$ denotes the fundamental group of Z based at z. Let

$$\left[\psi : \left(S^1,0\right) \to \left(\mathbb{R}\times\widetilde{X},(0,\widetilde{x})\right)\right] \in \pi_1\left(\mathbb{R}\times\widetilde{X},(0,\widetilde{x})\right)$$

and let

$$C : \left(\mathbb{R}\times\widetilde{X}\times[0,1], \{0\}\times\widetilde{X}\times[0,1]\right) \to \left(\mathbb{R}\times\widetilde{X},\{0\}\times\widetilde{X}\right)$$

be the strong deformation retraction given by

$$(t,z,s) \mapsto (st,z).$$

Then $C\circ\psi$ provides a homotopy of ψ with a map $g : \left(S^1,0\right) \to \left(\mathbb{R}\times\widetilde{X},(0,\widetilde{x})\right)$ whose image is contained in $\{0\}\times\widetilde{X}$. Via the natural homeomorphism $\{0\}\times\widetilde{X} \to \widetilde{X};\ (0,z)\mapsto z$ we may identify g with a map $g' : \left(S^1,0\right)\to\left(\widetilde{X},\widetilde{x}\right)$. Then

$$\phi_\# \circ (id\times p)_\# [\psi] = \phi_\# \circ (id\times p)_\# [g] = p_\# [g']$$

since

$$\phi\circ(id\times p)(0,z) = p(z),$$

demonstrating the inclusion needed to obtain $p^*\phi$. It is routine to verify that $p^*\phi$ is in fact a flow.

Notice that in the above X need not be locally path connected.

Lemma 2.1. *Let $q : \widetilde{Y} \to \widetilde{X}$ be a fibration with unique path lifting with $\widetilde{Y}$ connected and locally path connected. Then $q^*(p^*\phi) = (p\circ q)^*\phi$.*

PROOF. With $x = p \circ q(\widetilde{y})$ for a given $\widetilde{y} \in \widetilde{Y}$, we have

$$p \circ q \circ (q^* (p^* \phi)(t, \widetilde{y})) = p \circ (p^* \phi)(t, q(\widetilde{y})) = \phi(t, p \circ q(\widetilde{y})) = \phi_x(t)$$

and

$$(p \circ q) \circ (p \circ q)^* \phi(t, \widetilde{y}) = \phi(t, p \circ q(\widetilde{y})) = \phi_x(t),$$

and so $q^* (p^* \phi)_{\widetilde{y}}$ and $(p \circ q)^* \phi_{\widetilde{y}}$ both provide $(p \circ q)$–lifts $(\mathbb{R}, 0) \to (\widetilde{Y}, \widetilde{y})$ of the map $\phi_x : (\mathbb{R}, 0) \to (X, x)$, and so they are equal since this lift is uniquely determined. □

We provide the proof of some basic lemmas that do not seem to appear in the literature.

Lemma 2.2. *If $x \in X$ has a periodic $\phi-$orbit and if p is a covering map from a compact space $\widetilde{X}$, then each point in $p^{-1}(x)$ has a periodic orbit under the flow $p^* \phi$.*

PROOF. Each point of $p^{-1}(x)$ is a closed and hence compact subset of $\widetilde{X}$ consisting of isolated points when p is a covering map, implying that $p^{-1}(x) = \{y_1, ..., y_k\}$ is a finite set. Let $T \neq 0$ be such that $\phi(T, x) = x$. Let $i \in \{1, ..., k\}$ and let ℓ be any integer, then $p \circ p^* \phi(\ell T, y_i) = \phi(\ell T, p(y_i)) = \phi(\ell T, x) = x$. Hence, $p^* \phi(\ell T, y_i) \in \{y_1, ..., y_k\}$ for each integer ℓ, and so the orbit of y_i is periodic since it must eventually take on a value it has already taken on. □

Notice that the hypothesis that $\widetilde{X}$ be compact is needed, as the example $\phi : \mathbb{R} \times S^1 \to S^1$; $(t, x) \to \pi(t) + x$ and $p = \pi : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ shows. (Throughout, we shall use π to denote this map.)

Lemma 2.3. *If $x \in X$ has an aperiodic orbit, then the trajectories of the points of $p^{-1}(x)$ under the flow $p^* \phi$ are pairwise disjoint and aperiodic, and p restricted to any such trajectory is one–to–one.*

PROOF. Suppose that for $\{y, y'\} \subset p^{-1}(x)$ we have that for some $T \neq 0$ that $p^* \phi(T, y) = y'$. Then $x = p(y') = p \circ (p^* \phi)(T, y) = \phi(T, p(y)) = \phi(T, x)$, contrary to the hypothesis that x not have a periodic orbit. Hence, the trajectories of the points of $p^{-1}(x)$ are pairwise disjoint. Suppose then that $p^* \phi(T, y) = y$ for some $T \neq 0$. Then $x = p(y) = p \circ (p^* \phi)(T, y) = \phi(T, p(y)) = \phi(T, x)$, again a contradiction, and so any $y \in p^{-1}(x)$ is aperiodic. Suppose that $p \circ p^* \phi(T, y) = p \circ p^* \phi(T', y)$ for $T \neq T'$. Then

$$\phi(T, x) = \phi(T, p(y)) = p \circ p^* \phi(T, y) = p \circ p^* \phi(T', y) = \phi(T', p(y)) = \phi(T', x)$$

contrary to our hypothesis, and so p is one$-$to$-$one on the orbit of y . □

3. Solenoidalization

Recall that the κ–solenoid $\sum_{\overline{M}}$ is the inverse limit of a sequence of $\mathbf{T}^{\kappa}$ with the epimorphic bonding maps f_i^{i+1} represented by matrices $M_i : (\mathbb{R}^{\kappa}, \mathbf{0}) \rightarrow (\mathbb{R}^{\kappa}, \mathbf{0})$ [Cl].

Definition 1. Given a flow ϕ on $\mathbf{T}^{\kappa}$ and given a κ-solenoid $\sum_{\overline{M}}$ with bonding maps f_i^j, we define the $\overline{M}$ *-solenoidalization* of ϕ :

$$\phi_{\overline{M}} : \mathbb{R} \times \sum_{\overline{M}} \rightarrow \sum_{\overline{M}}$$

$$\left(t, (\mathbf{x}_i)_{i\in\mathbb{N}}\right) \longmapsto \left(\phi\left(t, \mathbf{x}_1\right), \left(f_1^2\right)^* \phi\left(t, \mathbf{x}_2\right), \left(f_1^3\right)^* \phi\left(t, \mathbf{x}_3\right), ..\right).$$

This is a well-defined flow: since

$$\left(f_1^2\right) \circ \left(f_1^2\right)^* \phi\left(t, \mathbf{x}_2\right) = \phi\left(t, f_1^2\left(\mathbf{x}_2\right)\right) = \phi\left(t, \mathbf{x}_1\right),$$

$$\begin{aligned} f_2^3 \circ \left(f_1^3\right)^* \phi\left(t, \mathbf{x}_3\right) &= f_2^3 \circ \left(\left(f_2^3\right)^* \left(f_1^2\right)^* \phi\right)\left(t, \mathbf{x}_3\right) \\ &= \left(f_1^2\right)^* \phi\left(t, \left(f_2^3\left(\mathbf{x}_3\right)\right)\right) = \left(f_1^2\right)^* \phi\left(t, \mathbf{x}_2\right), ..., \end{aligned}$$

we have that

$$\left(\phi\left(t, \mathbf{x}_1\right), \left(f_1^2\right)^* \phi\left(t, \mathbf{x}_2\right), \left(f_1^3\right)^* \phi\left(t, \mathbf{x}_3\right), ..\right) \in \sum_{\overline{M}}.$$

And $\phi_{\overline{M}}$ is continuous since the flows $\left(f_1^n\right)^* \phi$ are continuous; also: $\phi_{\overline{M}}\left(0, (\mathbf{x}_i)_{i\in\mathbb{N}}\right) = (\mathbf{x}_i)_{i\in\mathbb{N}}$ and

$$\begin{aligned} \phi_{\overline{M}}\left(s+t, (\mathbf{x}_i)_{i\in\mathbb{N}}\right) &= \left(\phi\left(s+t, \mathbf{x}_1\right), \left(f_1^2\right)^* \phi\left(s+t, \mathbf{x}_2\right), ..\right) \\ &= \left(\phi\left(s, \phi\left(t, \mathbf{x}_1\right)\right), \left(f_1^2\right)^* \phi\left(s, f_1^* \phi\left(t, \mathbf{x}_2\right)\right), ...\right) \\ &= \phi_{\overline{M}}\left(s, \phi_{\overline{M}}\left(t, (\mathbf{x}_i)_{i\in\mathbb{N}}\right)\right). \end{aligned}$$

Definition 2. Given a flow ϕ on $\mathbf{T}^{\kappa}$ and $\mathcal{M} = \overline{\phi_e\left(\mathbb{R}\right)}$ and given a κ -solenoid $\sum_{\overline{M}}$ we define the $\overline{M}$*−solenoidalization* of $\mathcal{M}$, denoted $\sum_{\overline{M}}(\mathcal{M})$, to be the space

$$\overline{\phi_{\overline{M}}\left(\mathbb{R}\times \{e_{\overline{M}}\}\right)} \subset \sum_{\overline{M}}.$$

We now try to get some idea of the structure of the solenoidalization of a minimal set and begin by recording a basic result from [N] that makes it easy to identify $\sum_{\overline{M}}(\mathcal{M})$ as a subset of $\sum_{\overline{M}}$.

Lemma 3.1. *Let* $\left\{X_n, f_n^{n+1}\right\}_{n\in\mathbb{N}}$ *be an inverse sequence of metric spaces with inverse limit* X_∞ *and let* $f_n : X_\infty \to X_n \quad (n \in \mathbb{N})$ *be the projection maps. Let* A *be a compact subset of* X_∞*. Then,* $\left\{f_n(A), f_n^{n+1}|_{(f_{n+1})(A)}\right\}_{n\in\mathbb{N}}$ *is an inverse sequence with onto bonding maps and*

$$\varprojlim \left\{f_n(A), f_n^{n+1}|_{(f_{n+1})(A)}\right\}_{n\in\mathbb{N}} = A = \left[\prod_{n=1}^{\infty} f_n(A)\right] \cap X_\infty \text{ [N, p.20].}$$

□

Lemma 3.2. *For any* $n \in \mathbb{N}$*,* $f_n\left(\sum_{\overline{M}}(\mathcal{M})\right) = \overline{(f_1^n)^* \phi_e(\mathbb{R})}$*, and in particular*

$$f_1\left(\sum\nolimits_{\overline{M}}(\mathcal{M})\right) = \mathcal{M}.$$

Proof. We have for each $n \in \mathbb{N}$ that $f_n \circ (\phi_{\overline{M}})_{e_{\overline{M}}}(t) = (f_1^n)^* \phi_e(t)$, and so

$$f_n\left((\phi_{\overline{M}})_{e_{\overline{M}}}(\mathbb{R})\right) = (f_1^n)^* \phi_e(\mathbb{R}).$$

The continuity of f_n yields that

$$f_n\left(\sum\nolimits_{\overline{M}}(\mathcal{M})\right) = f_n\left(\overline{(\phi_{\overline{M}})_{e_{\overline{M}}}(\mathbb{R})}\right) \subset \overline{f_n\left((\phi_{\overline{M}})_{e_{\overline{M}}}(\mathbb{R})\right)} = \overline{(f_1^n)^* \phi_e(\mathbb{R})}.$$

But we also have that

$$f_n\left(\sum\nolimits_{\overline{M}}(\mathcal{M})\right) \supset f_n\left((\phi_{\overline{M}})_{e_{\overline{M}}}(\mathbb{R})\right) = (f_1^n)^* \phi_e(\mathbb{R}),$$

and $f_n\left(\sum_{\overline{M}}(\mathcal{M})\right)$ is closed since $\sum_{\overline{M}}(\mathcal{M})$ is compact. Therefore,

$$f_n\left(\sum\nolimits_{\overline{M}}(\mathcal{M})\right) \supset \overline{(f_1^n)^* \phi_e(\mathbb{R})}.$$

□

While we shall only consider solenoidalization, it should be noted that a similar construction is possible for any inverse sequence with bonding maps consisting of covering maps.

4. Denjoids and their Semiconjugate Linear Flows

Given an orientation-preserving homeomorphism $f : S^1 \to S^1$ with unique lift $F : \mathbb{R} \to \mathbb{R}$ (with $F(0) \in [0,1)$), we have the equivalence relation $\approx_f$ on $\mathbb{R}\times X$ given by

$$[(s,x) \approx_f (t,y)] \Leftrightarrow \left[\text{there is an } n \in Z \text{ with } t = s+n \text{ and } y = f^{-n}(x)\right]$$

and we have the space $\mathcal{S}_h = (\mathbb{R}\times X) / \approx_f$. The $\approx_f$ class of (s,x) will be denoted $[s,x]_f$. There is then the flow

$$\sigma_f : \mathbb{R}\times \mathcal{S}_f \to \mathcal{S}_f \ ; \ \left(t, [s,x]_f\right) \mapsto [t+s,x]_f \, .$$

We refer to both $\mathcal{S}_f$ and σ_f as the *suspension* of f.

With $\lfloor t \rfloor$ denoting the greatest integer less than or equal to t and $\rfloor t \lfloor \overset{\text{def}}{=} t - \lfloor t \rfloor$, we have the homeomorphism $\mu_f : \mathcal{S}_f \to \mathbf{T}^2$

$$[s, \pi(r)]_f \overset{\mu_f}{\mapsto} \left\langle \pi \left((1 - \rfloor s \lfloor) F^{\lfloor s \rfloor}(r) + \rfloor s \lfloor F^{\lfloor s \rfloor + 1}(r) \right), \pi(s) \right\rangle,$$

and the equivalent flow $\sum(f) = \mu_f \circ \sigma_f \circ \left(id_{\mathbb{R}} \times (\mu_f)^{-1} \right)$ on $\mathbf{T}^2$ given by

$$\begin{array}{rcl} \sum(f) & : & \mathbb{R} \times \mathbf{T}^2 \to \mathbf{T}^2 \\ (t, \langle \pi(r), \pi(s) \rangle) & \mapsto & \left\langle \pi \left((1 - \rfloor t \lfloor) F^{\lfloor t \rfloor}(r) + \rfloor t \lfloor F^{\lfloor t \rfloor + 1}(r) \right), \pi(s+t) \right\rangle. \end{array}$$

Definition 3. Let $\mathfrak{F}$ be the collection of non-transitive orientation-preserving circle homeomorphisms f which have an irrational rotation number and which have $\pi(0)$ as an endpoint of the minimal Cantor set M_f.

The set $\mathcal{D}_f \overset{\text{def}}{=} \{\sum(f)(\mathbb{R}, \langle x_1, x_2 \rangle) \mid x_1 \in M_f\}$ consisting of the suspended trajectories of the minimal set M_f is then a minimal set of the suspension $\sum(f)$, see, e.g., [Schw].

Definition 4. We define the *generalized Denjoy continua* as the collection

$$\{\mathcal{D}_f \mid f \in \mathfrak{F}\},$$

and we define a flow $\sum(f)$ for $f \in \mathfrak{F}$ to be a *generalized Denjoy flow.*

Any aperiodic C^1 flow on $\mathbf{T}^2$ is topologically equivalent to the suspension of an orientation-preserving circle diffeomorphism with irrational rotation number, see, e.g., [KH] 14.2.3, 0.3 and 11.1.4, and so any minimal set occurring in an aperiodic C^1 flow on $\mathbf{T}^2$ which is a proper subset of $\mathbf{T}^2$ is homeomorphic with some $\mathcal{D}_f$. Gutierrez has shown [G] that any continuous flow on $\mathbf{T}^2$ is topologically equivalent with a C^1 flow, and so any minimal set of a continuous aperiodic $\mathbf{T}^2$-flow is homeomorphic with some $\mathcal{D}_f$.

Definition 5. We refer to the solenoidalization $\sum_{\overline{M}}(\mathcal{D}_f)$ of a generalized Denjoy continuum $\mathcal{D}_f$ as a *denjoid.*

We shall begin to explore the degree to which known facts about Denjoy continua carry over to denjoids. To do so, we start by seeing how maps of $\mathbf{T}^2$ can be pulled back by epimorphisms.

Lemma 4.1. *Suppose* $g : (\sum_{\overline{M}}, e_{\overline{M}}) \to (\sum_{\overline{M}}, e_{\overline{M}})$ *is a map of the* κ*-solenoid* $\sum_{\overline{M}}$ *and that* $h : \sum_{\overline{M}} \to \sum_{\overline{M}}$ *is the unique homomorphism homotopic to* g*, see* [Sch]*. If* $\tilde{g}, \tilde{h} : (\mathbb{R}^\kappa, \mathbf{0}) \to (\mathbb{R}^\kappa, \mathbf{0})$ *are the* $\pi_{\overline{M}}$*–lifts of* $g \circ \pi_{\overline{M}}$ *and* $h \circ \pi_{\overline{M}}$ *respectively, then for all* $\mathbf{x} \in \mathbb{R}^\kappa$ *and for all* $\mathbf{k} \in \ker \pi_{\overline{M}}$ *we have*

$$\tilde{g}(\mathbf{k} + \mathbf{x}) = \tilde{h}(\mathbf{k}) + \tilde{g}(\mathbf{x}).$$

Proof. Here, $\pi_{\overline{M}} : (\mathbb{R}^\kappa, \mathbf{0}) \to \sum_{\overline{M}}$ denotes the fibration with unique path lifting onto the arc component of $e_{\overline{M}}$ as in [Cl], which generalizes the standard fibration

$$\pi^\kappa : (\mathbb{R}^\kappa, \mathbf{0}) \to (\mathbf{T}^\kappa, e), \ (t_i)_{i=1}^\kappa \mapsto (\pi(t_i))_{i=1}^\kappa .$$

Fix $\mathbf{x} \in \mathbb{R}^\kappa$ and $\mathbf{k} \in \ker \pi_{\overline{M}}$ and let $H : (\sum_{\overline{M}} \times [0,1], e_{\overline{M}} \times [0,1]) \to (\sum_{\overline{M}}, e_{\overline{M}})$ be a homotopy between $h = H(\_, 0)$ and $g = H(\_, 1)$ and let $\widetilde{H} : (\mathbb{R}^\kappa \times [0,1], (\mathbf{0}, 0)) \to (\mathbb{R}^\kappa, \mathbf{0})$ be the $\pi_{\overline{M}}$–lift of the homotopy

$$H \circ (\pi_{\overline{M}} \times id) : (\mathbb{R}^\kappa \times [0,1], (\mathbf{0}, 0)) \to \left(\sum_{\overline{M}}, e_{\overline{M}}\right),$$

which then provides a homotopy between $\tilde{h}$ and $\tilde{g}$. Then let

$$\mathfrak{p} \overset{\text{def}}{=} H \circ (\pi_{\overline{M}} \times id) \mid_{\{\mathbf{x}\} \times [0,1]} : [0,1] \to \sum_{\overline{M}}$$

be the path $t \mapsto H(\pi_{\overline{M}}(\mathbf{x}), t)$ which goes from $h(\pi_{\overline{M}}(\mathbf{x}))$ to $g(\pi_{\overline{M}}(\mathbf{x}))$. Then $\mathfrak{p}$ lifts uniquely to the path

$$\tilde{p} \overset{\text{def}}{=} \widetilde{H} \mid_{\{\mathbf{x}\} \times [0,1]} : ([0,1], 0) \to \left(\mathbb{R}^\kappa, \tilde{h}(\mathbf{x})\right)$$

from $\tilde{h}(\mathbf{x})$ to $\tilde{g}(\mathbf{x})$. Since $\pi_{\overline{M}}(\mathbf{k} + \mathbf{x}) = \pi_{\overline{M}}(\mathbf{x})$, we have that

$$\mathfrak{p} = H \circ (\pi_{\overline{M}} \times id) \mid_{\{\mathbf{x}\} \times [0,1]} = H \circ (\pi_{\overline{M}} \times id) \mid_{\{\mathbf{x}+\mathbf{k}\} \times [0,1]},$$

and so the path

$$q \overset{\text{def}}{=} \widetilde{H} \mid_{\{\mathbf{x}+\mathbf{k}\} \times [0,1]} : ([0,1], 0) \to \left(\mathbb{R}^\kappa, \tilde{h}(\mathbf{k} + \mathbf{x})\right)$$

from $\tilde{h}(\mathbf{k} + \mathbf{x})$ to $\tilde{g}(\mathbf{k} + \mathbf{x})$ provides the unique lift $([0,1], 0) \to \left(\mathbb{R}^\kappa, \tilde{h}(\mathbf{k} + \mathbf{x})\right)$ of $\mathfrak{p}$. But we also have the path $p'(t) \overset{\text{def}}{=} \tilde{h}(\mathbf{k}) + \tilde{p}(t)$ and

$$p'(0) = \tilde{h}(\mathbf{k}) + \tilde{p}(0) = \tilde{h}(\mathbf{k}) + \tilde{h}(\mathbf{x}) = \tilde{h}(\mathbf{k} + \mathbf{x}) \text{ and}$$

$$\begin{aligned}\pi_{\overline{M}} \circ p'(t) &= \pi_{\overline{M}}\left(\tilde{h}(\mathbf{k}) + \tilde{p}(t)\right) = \pi_{\overline{M}} \circ \tilde{h}(\mathbf{k}) + \pi_{\overline{M}} \circ \tilde{p}(t) \\ &= h\left(\pi_{\overline{M}}(\mathbf{k})\right) + \mathfrak{p}(t) = \mathfrak{p}(t),\end{aligned}$$

and so p' is also a lift of $\mathfrak{p}$ beginning at $\tilde{h}(\mathbf{k}+\mathbf{x})$. By the uniqueness of the path-lifting of $\pi_{\overline{M}}$, we must have that $p' = q$, and so $\tilde{h}(\mathbf{k}) + \tilde{g}(\mathbf{x}) = p'(1) = q(1) = \tilde{g}(\mathbf{k}+\mathbf{x})$. □

Corollary 4.2. *If $g : (\mathbf{T}^\kappa, e) \to (\mathbf{T}^\kappa, e)$ is homotopic to $id_{\mathbf{T}^\kappa}$ and if $\tilde{g} : (\mathbb{R}^\kappa, \mathbf{0}) \to (\mathbb{R}^\kappa, \mathbf{0})$ is the lift of $g \circ \pi^\kappa$ and if $H : \mathbb{R}^\kappa \to \mathbb{R}^\kappa$ is any automorphism of $\mathbb{R}^\kappa$ which is the lift of an endomorphism h of $\mathbf{T}^\kappa$, then $H^{-1} \circ \tilde{g} \circ H$ determines a map $g^{(H)} : (\mathbf{T}^\kappa, e) \to (\mathbf{T}^\kappa, e)$*

$$x \longmapsto \pi^\kappa \circ H^{-1} \circ \tilde{g} \circ H\left(\left(\pi^\kappa\right)^{-1}(x)\right),$$

and $g^{(H)}$ is homotopic to $id_{\mathbf{T}^\kappa}$.

Proof. We need to show that $g^{(H)}$ is well-defined, that it is independent of the choice of $\left(\pi^\kappa\right)^{-1}(x)$. If $y \in \left(\pi^\kappa\right)^{-1}(x)$, any element of $\left(\pi^\kappa\right)^{-1}(x)$ can be represented as $\mathbf{k} + y$ for some $\mathbf{k} \in \ker \pi^\kappa$. Since H is a lift of the homomorphism, say h, we have $\pi^\kappa \circ H(\mathbf{k}) = h(\pi^\kappa(\mathbf{k})) = e$ and so $H(\mathbf{k}) \in \ker \pi^\kappa$. And since g is homotopic to $id_{\mathbf{T}^\kappa}$, we have by the preceding Lemma:

$$\begin{aligned}\pi^\kappa \circ H^{-1} \circ \tilde{g} \circ H(\mathbf{k}+y) &= \pi^\kappa \circ H^{-1} \circ \tilde{g}\left(H(\mathbf{k}) + H(y)\right) \\ &= \pi^\kappa \circ H^{-1}\left(H(\mathbf{k}) + \tilde{g} \circ H(y)\right) \\ &= \pi^\kappa\left(\mathbf{k} + H^{-1} \circ \tilde{g} \circ H(y)\right) = \pi^\kappa \circ H^{-1} \circ \tilde{g} \circ H(y)\end{aligned}$$

as desired. And if $G : (\mathbf{T}^\kappa \times I, \{e\} \times I) \to (\mathbf{T}^\kappa, e)$ is a homotopy between $G_0 = G(\_, 0) = g$ and $G_1 = G(\_, 1) = id_{\mathbf{T}^\kappa}$, then $G^{(H)} : (\mathbf{T}^\kappa \times I, \{e\} \times I) \to ((\mathbf{T}^\kappa, e))$;

$$(x,t) \mapsto \pi^\kappa \circ H^{-1} \circ \tilde{G}_t \circ H\left(\left(\pi^\kappa\right)^{-1}(x)\right)$$

is a homotopy between $g^{(H)}$ and $id_{\mathbf{T}^\kappa}$, where $\tilde{G}_t$ is the lift $(\mathbb{R}^\kappa, \mathbf{0}) \to (\mathbb{R}^\kappa, \mathbf{0})$ of $G(\_, t) \circ \pi^\kappa$. □

We shall use this for maps of $\mathbf{T}^2$, and so we will go through the details of the construction of $g^{(H)}$ in this case. There are $P, Q \in GL(\mathbb{Z}, 2)$ such that $H = P\Delta Q$ for some diagonal integer matrix $\Delta = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. And so H^{-1} induces the following isomorphisms of lattices

$$\mathbb{Z}\oplus\mathbb{Z}\overset{P^{-1}}{\to}\mathbb{Z}\oplus\mathbb{Z}\overset{\Delta^{-1}}{\to}\frac{1}{d_1}\mathbb{Z}\oplus\frac{1}{d_2}\mathbb{Z}\overset{Q^{-1}}{\to}\mathcal{L},$$

where $\mathcal{L}$ is some lattice in the plane. For $(i,j)\in\{1,...,d_1\}\times\{1,...,d_2\}$, define $R_{(i,j)}$ to be the rectangle $\left[\frac{i-1}{d_1},\frac{i}{d_1}\right]\times\left[\frac{j-1}{d_2},\frac{j}{d_2}\right]$ and define $\mathcal{P}_{(i,j)}$ to be the parallelogram $Q^{-1}\left(R_{(i,j)}\right)$. Then, since $\cup R_{(i,j)}$ is the unit square and Q^{-1} is the lift of a homeomorphism of $\mathbf{T}^2$, $\pi^2\left(\cup\mathcal{P}_{(i,j)}\right)=\mathbf{T}^2$. Now

$$\begin{aligned} g\left(H\right)\left(\pi^2\left(\mathcal{P}_{(i,j)}\right)\right) &= \pi^2\circ H^{-1}\circ\tilde{g}\circ H\left(\mathcal{P}_{(i,j)}\right)\\ &= \pi^2\circ H^{-1}\circ\tilde{g}\left(P\left([i-1,i]\times[j-1,j]\right)\right)\end{aligned}$$

and since g is homotopic to $id_{\mathbf{T}^2}$ and

$$P\left([i-1,i]\times[j-1,j]\right)=P\left((i-1,j-1)\right)+P\left([0,1]\times[0,1]\right)$$

and $P\left((i-1,j-1)\right)\in\ker\pi^2$, we have that $\tilde{g}$ maps

$$P\left([i-1,i]\times[j-1,j]\right)\to P\left([i-1,i]\times[j-1,j]\right)$$

just as $\tilde{g}$ maps

$$P\left([0,1]\times[0,1]\right)\to P\left([0,1]\times[0,1]\right)$$

(see Lemma 4.1), and $P\left([0,1]\times[0,1]\right)$ represents $\mathbf{T}^2$ since $P\in GL\left(2,\mathbb{Z}\right)$. And so

$$g^{(H)}\left(\pi^2\left(\mathcal{P}_{(i,j)}\right)\right)=\pi^2\circ H^{-1}\left(P\left([i-1,i]\times[j-1,j]\right)\right)=\pi^2\left(\mathcal{P}_{(i,j)}\right)$$

and $g^{(H)}$ maps the neighborhood $\pi^2\left(\mathcal{P}_{(i,j)}\right)$ onto its image $\pi^2\left(\mathcal{P}_{(i,j)}\right)$ just as g maps $\mathbf{T}^2$.

For the remainder of this section, f represents an element of $\mathfrak{F}$ with rotation number θ. By the Poincaré Classification Theorem (see, e.g., [KH] 11.2.7), there is a monotone map $m:S^1\to S^1$ with $m\circ f=R_\theta\circ m$, where $R_\theta\left(x\right)=x+\pi\left(\theta\right)$. We may then use this m to construct a map $g:\mathbf{T}^2\to\mathbf{T}^2$ providing a semiconjugacy of $\sum\left(f\right)$ with the linear flow $\Phi^{(\theta,1)}$ on $\mathbf{T}^2$: $g\circ\sum\left(f\right)=\Phi^{(\theta,1)}\circ\left(id\times g\right)$, denoted $g:\sum\left(f\right)\overset{sc}{\succeq}\Phi^{(\theta,1)}$. While g may not map $e\mapsto e$, we can follow g with translation by $-g\left(e\right)$ to obtain a map which still provides the semiconjugacy since translations equate $\Phi^{(\theta,1)}$ with itself. Hence, we may assume that $g(e)=e$, and hereafter we shall make this assumption. This map g glues together the suspended trajectories of the endpoints of the minimal Cantor set M_f and is homotopic to $id_{\mathbf{T}^2}$, as can be seen by beginning the homotopy with $id_{\mathbf{T}^2}$ and gradually gluing these trajectories together until we arrive at the map g. Thus, our above results apply to this map g.

So then let h be an epimorphism $\mathbf{T}^2 \to \mathbf{T}^2$ which lifts to the automorphism H of $\mathbb{R}^2$. Given any flow ψ on $\mathbf{T}^2$, we have the flow $\tilde{\psi} \stackrel{\text{def}}{=} \left(\pi^2\right)^* \psi$ on $\mathbb{R}^2$. And with $\tilde{g}$ denoting the π^2–lift of g, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}\times\mathbb{R}^2 & \stackrel{H^*\tilde{\phi}}{\to} & \mathbb{R}^2 \\ (id\times H)\downarrow & & \downarrow H \\ \mathbb{R}\times\mathbb{R}^2 & \stackrel{\tilde{\phi}}{\to} & \mathbb{R}^2 \\ (id\times\tilde{g})\downarrow & & \downarrow\tilde{g} \\ \mathbb{R}\times\mathbb{R}^2 & \stackrel{\widetilde{\Phi^{(\theta,1)}}}{\to} & \mathbb{R}^2 \\ \left(id\times H^{-1}\right)\downarrow & & \downarrow H^{-1} \\ \mathbb{R}\times\mathbb{R}^2 & \stackrel{\widetilde{\Phi^{(\omega_1,\omega_2)}}}{\to} & \mathbb{R}^2 \end{array}, \text{ where } \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = H^{-1}\begin{pmatrix}\theta\\ 1\end{pmatrix}.$$

The map $g^{(H)} = \pi^2 \circ H^{-1} \circ g \circ H \circ \left(\pi^2\right)^{-1}$ as in Corollary 4.2 then induces the diagram

$$\begin{array}{ccc} \mathbb{R}\times\mathbf{T}^2 & \stackrel{h^*\phi}{\to} & \mathbf{T}^2 \\ \left(id\times g^{(H)}\right)\downarrow & & \downarrow g^{(H)} \\ \mathbb{R}\times\mathbf{T}^2 & \stackrel{\Phi^{(\omega_1,\omega_2)}}{\to} & \mathbf{T}^2 \end{array}.$$

We are then led to the following commutative diagram:

$$(\maltese)\quad \begin{array}{ccccccccc} \mathbf{T}^2 & \stackrel{f_1^2}{\longleftarrow} & \mathbf{T}^2 & \stackrel{f_2^3}{\longleftarrow} & \cdots & \stackrel{f_{n-1}^n}{\longleftarrow} & \mathbf{T}^2 & \cdots & \sum_{\overline{M}} \\ g\downarrow & & g^{(M_1)}\downarrow & & & & g^{(M_1\circ\cdots\circ M_{n-1})}\downarrow & & \Gamma\downarrow \\ \mathbf{T}^2 & \stackrel{f_1^2}{\longleftarrow} & \mathbf{T}^2 & \stackrel{f_2^3}{\longleftarrow} & \cdots & \stackrel{f_{n-1}^n}{\longleftarrow} & \mathbf{T}^2 & \cdots & \sum_{\overline{M}} \end{array}.$$

It can then be seen that the induced map $\Gamma : \sum_{\overline{M}} \to \sum_{\overline{M}}$ provides the semiconjugacy $\sum_{\overline{M}} \left(\sum (f)\right) \stackrel{sc}{\succeq} \Phi_{\overline{M}}^{(\theta,1)}$, where $\Phi_{\overline{M}}^{(\theta,1)}$ is a linear flow on $\sum_{\overline{M}}$ [Cl]. Also, the map Γ is homotopic to the identity of $\sum_{\overline{M}}$: if g_t is a homotopy of g with $id_{\mathbf{T}^2}$, then

$$\begin{array}{cccccc} \mathbf{T}^2 & \stackrel{f_1^2}{\leftarrow} & \mathbf{T}^2 & \stackrel{f_2^3}{\leftarrow} & \cdots & \sum_{\overline{M}} \\ g_t\downarrow & & g_t^{(M_1)}\downarrow & & & \Gamma_t\downarrow \\ \mathbf{T}^2 & \stackrel{f_1^2}{\leftarrow} & \mathbf{T}^2 & \stackrel{f_2^3}{\leftarrow} & \cdots & \sum_{\overline{M}} \end{array}$$

provides a homotopy of Γ with the identity. And also $\Gamma\left(e_{\overline{M}}\right) = e_{\overline{M}}$ since $g(e) = e$. And since the $\sum (f)$–trajectory of $e_{\overline{M}}$ is dense in $\mathcal{D}_f$, Γ is uniquely determined on $\sum_{\overline{M}}(\mathcal{D}_f)$ by the condition that it map $e_{\overline{M}} \mapsto e_{\overline{M}}$ and that it provides a semiconjugacy.

Lemma 4.3. *If h is a k−to−one epimorphism of $\mathbf{T}^\kappa$ whose lift to $\mathbb{R}^\kappa$ is H and if $g : \mathbf{T}^\kappa \to \mathbf{T}^\kappa$ is a surjection homotopic to the identity, then*

$$\left[g^{-1}(g(x)) = \{x\}\right] \Rightarrow \left[\left(g^{(H)}\right)^{-1}\left(g^{(H)}(y)\right) = \{y\} \text{ for all } y \in h^{-1}(x)\right].$$

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{T}^\kappa & \overset{h}{\to} & \mathbf{T}^\kappa \\ g^{(H)} \downarrow & & \downarrow g \\ \mathbf{T}^\kappa & \overset{h}{\to} & \mathbf{T}^\kappa \end{array},$$

and h is k-to-one. Let x be a point with $g^{-1}(g(x)) = \{x\}$ and let

$$\{y_1, ..., y_k\} = h^{-1}(x) = (g \circ h)^{-1}(g(x))$$

and let $\{z_1, ..., z_k\} = h^{-1}(g(x))$. Then

$$\{y_1, ..., y_k\} = \left(h \circ g^{(H)}\right)^{-1}(g(x)) = \left(g^{(H)}\right)^{-1}(\{z_1, ..., z_k\}).$$

Now $g^{(H)}$ is a surjection and

$$\left(g^{(H)}\right)^{-1}(\{z_1, ..., z_k\}) \subset \{y_1, ..., y_k\},$$

and so $g^{(H)}$ provides a bijection between $\{y_1, ..., y_k\}$ and $\{z_1, ..., z_k\}$ and

$$\left(g^{(H)}\right)^{-1}\left(g^{(H)}(y_i)\right) = y_i$$

for each $i = 1, ..., k$. □

Lemma 4.4. *There are points ξ of $\sum_{\overline{M}}$ with $\Gamma^{-1}(\Gamma(\xi)) = \xi$.*

Proof. Let $\mathfrak{X} \in \mathbf{T}^2$ be a point satisfying $g^{-1}(g(\mathfrak{X})) = \{\mathfrak{X}\}$, as will be the case with any $\mathfrak{X} = \mu_f\left([0,x]_f\right)$, where x is an element of the Cantor set M_f which is not an endpoint. Then let $y \in \left(\pi^2\right)^{-1}(\mathfrak{X})$. We then have the point $\pi_{\overline{M}}(y) \overset{\text{def}}{=} \langle \mathfrak{X}_n \rangle_{n\in\mathbb{N}} \in \sum_{\overline{M}}$ and we claim that $\Gamma^{-1}\left(\Gamma\left(\langle \mathfrak{X}_n \rangle_{n\in\mathbb{N}}\right)\right) = \left\{\langle \mathfrak{X}_n \rangle_{n\in\mathbb{N}}\right\}$. Suppose $\Gamma\left(\langle \mathfrak{X}_n \rangle_{n\in\mathbb{N}}\right) = \Gamma\left(\langle \mathbf{z}_n \rangle_{n\in\mathbb{N}}\right)$. Then $g(\mathfrak{X}) = g(\mathfrak{X}_1) = g(\mathbf{z}_1)$ and so $\mathfrak{X}_1 = \mathbf{z}_1$ by our choice of $\mathfrak{X}$. Then for any $n \in \mathbb{N}$ we have $\{\mathbf{z}_n, \mathfrak{X}_n\} \subset (f_1^n)^{-1}(\mathfrak{X}_1)$ and $g^{(M_1\circ\cdots\circ M_n)}(\mathbf{z}_n) = g^{(M_1\circ\cdots\circ M_n)}(\mathfrak{X}_n)$ and so by the above, $\mathbf{z}_n = \mathfrak{X}_n$. Hence $\langle \mathbf{z}_n \rangle_{n\in\mathbb{N}} = \langle \mathfrak{X}_n \rangle_{n\in\mathbb{N}}$ and we have our claim. □

It is perhaps worth noting the following corollary of this property of Γ.

Corollary 4.5. *The orbit of any point* $\xi \in \sum_{\overline{M}}$ *with* $\Gamma^{-1}(\Gamma(\xi)) = \{\xi\}$ *as above is* N*-almost periodic.*

Proof. This follows directly from [LZ] Chapt 7.5, Thm 2, p. 109; see pp. 53-4 for a definition of N-almost periodic. □

Nemytskii showed that any space which supports a minimal almost periodic flow has the structure of a topological group [NS] Chapt V, 8.16 and is thus homogeneous, meaning that given any two points x and y of the space there is a homeomorphism of the space sending x to y. We will see later that these denjoids are not homogeneous and so cannot support a minimal almost periodic flow.

Definition 6. For a given set $X \subset S^1$ define $\mathfrak{D}_X \subset \mathbf{T}^2$ to be the set

$$\left\{\mu_f\left([t,x]_f\right) \in \mathbf{T}^2 \mid t \in \mathbb{R} \text{ and } x \in X\right\}.$$

Lemma 4.6. *Let* $J \subset S^1 - M_f$ *be an interval which is a component of* $S^1 - M_f$*. Then for any* k*-to-one epimorphism* $h : \mathbf{T}^2 \to \mathbf{T}^2$*, we have that* $h^{-1}(\mathfrak{D}_J)$ *is the disjoint union of* k *open disks, each of which* h *maps homeomorphically onto* $\mathfrak{D}_J$*.*

Proof. For small enough $\varepsilon > 0$ and an adequately large ℓ, we have that the open set $\mathcal{O} = \left\{\mu_f\left([s,m]_f\right) \in \mathbf{T}^2 \mid s \in (\ell-\varepsilon, \ell+\varepsilon) \text{ and } m \in J\right\}$ is as small in diameter as needed to ensure that it is evenly covered by the covering map h; that is, $h^{-1}(\mathcal{O})$ is the disjoint union of k open sets $\mathcal{O}_1, ..., \mathcal{O}_k$ each of which is mapped homeomorphically onto $\mathcal{O}$ by h. For $i \in \{1, ..., k\}$ and $t \in \mathbb{R}$, let $\mathcal{O}_i^t = h^* \sum(f)(t, \mathcal{O}_i)$ and let $\mathcal{O}^t = \sum(f)(t, \mathcal{O})$. Then we have $\mathfrak{D}_J = \cup_{t\in\mathbb{R}}\mathcal{O}^t$ and $h(\mathcal{O}_i^t) = \mathcal{O}^t$ since $h \circ h^* \sum(f)(t,x) = \sum(f)(t, h(x))$, and so with $D_i = \cup_{t\in\mathbb{R}}\mathcal{O}_i^t$ we have $h(D_i) = \mathfrak{D}_J$. And if $x \in h^{-1}(\mathfrak{D}_J)$, we must have that $x \in h^{-1}(\mathcal{O}^{t_0})$ for some t_0 . Hence, $h(x) \in \mathcal{O}^{t_0}$ and

$$h^* \sum(f)(-t_0, x) \in h^{-1}\left(\sum(f)(-t_0, h(x))\right) \subset h^{-1}(\mathcal{O})$$

and so $h^* \sum(f)(-t_0, x) \in \mathcal{O}_i$ for some i and $x \in \mathcal{O}_i^{t_0} \subset D_i$ and we conclude that $h^{-1}(\mathfrak{D}_J) = \cup_{i=1}^k D_i$.

Suppose that $x \in D_i \cap D_j$ for $i \neq j$. Then x would be on the $h^* \sum(f)$ −trajectory T_i of a point $x_i \in \mathcal{O}_i \cap \left\{h^{-1}\left(\mu_f\left([\ell,m]_f\right)\right) \mid m \in J\right\}$ and on the $h^* \sum(f)$ −trajectory T_j of a point $x_j \in \mathcal{O}_j \cap \left\{h^{-1}\left(\mu_f\left([\ell,m]_f\right)\right) \mid m \in J\right\}$ since D_i and D_j are the union of such trajectories. But any two such trajectories are disjoint: if $h(T_i) = h(T_j)$, then $h(x_i) = h(x_j)$ since the trajectories of the points of

$$\left\{\mu_f\left([\ell,m]_f\right) \in \mathbf{T}^2 \mid m \in J\right\} \subset \mathcal{O}$$

are pairwise disjoint and so $T_i \cap T_j$ is empty by Lemma 2.3, and if $h(T_i) \neq h(T_j)$, then $h(T_i)$ and $h(T_j)$ are disjoint since h maps $h^* \sum(f)$ −trajectories to $\sum(f)$− trajectories and distinct trajectories are disjoint, and so $h(x) \in h(T_i \cap T_j) \subset h(T_i) \cap h(T_j) = \emptyset$ − a contradiction. Hence, the sets $D_1, ..., D_k$ are pairwise disjoint.

Since h provides a one-to-one correspondence between

$$\mathcal{O}_i \cap \left\{ h^{-1}\left(\mu_f\left([\ell, m]_f\right)\right) \mid m \in J \right\}$$

and $\left\{ \mu_f\left([\ell, m]_f\right) \mid m \in J \right\}$, h maps each $h^* \sum(f)$-trajectory T_x of each point

$$x \in \mathcal{O}_i \cap \left\{ h^{-1}\left(\mu_f\left([\ell, m]_f\right)\right) \mid m \in J \right\}$$

injectively onto the $\sum(f)$-trajectory $T_{h(x)}$ of $h(x)$ by Lemma 2.3. The $\sum(f)$ -trajectories of distinct points of $\left\{ \mu_f\left([\ell, m]_f\right) \mid m \in J \right\}$ are distinct, and so for

$$x, x' \in \mathcal{O}_i \cap \left\{ h^{-1}\left(\mu_f\left([\ell, m]_f\right)\right) \mid m \in J \right\}$$

and $x \neq x'$, we must have that the $\sum(f)$ -trajectories $T_{h(x)}$ and $T_{h(x')}$ are distinct since $h(x) \neq h(x')$, implying that the $h^* \sum(f)$-trajectories T_x and $T_{x'}$ are distinct, for they would otherwise map onto the same $\sum(f)$-trajectory. Thus, h maps the union of all the orbits of all the points of

$$\mathcal{O}_i \cap \left\{ h^{-1}\left(\mu_f\left([\ell, m]_f\right)\right) \mid m \in J \right\},$$

which is D_i, injectively and thus homeomorphically onto its image, which is $\mathfrak{D}_J$. And so we may conclude that h maps each D_i homeomorphically onto its image $\mathfrak{D}_J$. □

Thus, the number of trajectories "blown up" in $h^* \sum(f)$ is k times the number of trajectories blown up in $\sum(f)$. Then one would expect to find countably infinitely many holes in a denjoid, provided that the bonding maps of $\sum_{\overline{M}}$ are not eventually isomorphisms. And this is indeed what we shall find when we calculate $\check{H}^1\left(\sum_{\overline{M}}(\mathcal{D}_f)\right)$ in the next section.

While we have been dealing with a specific representation of the topological equivalence class of $\mathcal{D}_f$ to construct $\sum_{\overline{M}}(\mathcal{D}_f)$, $\sum_{\overline{M}}(\mathcal{D}_f)$ is independent of this choice to a degree, as the following shows.

Lemma 4.7. *With g, h and $g^{(H)}$ as above; if g is a homeomorphism, then $g^{(H)}$ is also a homeomorphism.*

Proof. By Lemma 4.3, $g^{(H)}$ will be one-to-one when g is. □

Corollary 4.8. *If ϕ and ψ are flows on $\mathbf{T}^{\kappa}$ and if $g:(\mathbf{T}^{\kappa},e)\to(\mathbf{T}^{\kappa},e)$ is a homeomorphism homotopic to the identity with $g:\phi\overset{top}{\approx}\psi$ and if $\sum_{\overline{M}}$ is any $\kappa-$solenoid, then*

$$\begin{array}{ccccccccc}
\mathbf{T}^{\kappa} & \overset{f_1^2}{\longleftarrow} & \mathbf{T}^{\kappa} & \overset{f_2^3}{\longleftarrow} & \cdots & \overset{f_{n-1}^n}{\longleftarrow} & \mathbf{T}^{\kappa} & \cdots & \sum_{\overline{M}} \\
g\downarrow & & g^{(M_1)}\downarrow & & & & g^{(M_1\circ\cdots\circ M_{n-1})}\downarrow & & \Gamma\downarrow \\
\mathbf{T}^{\kappa} & \overset{f_1^2}{\longleftarrow} & \mathbf{T}^{\kappa} & \overset{f_2^3}{\longleftarrow} & \cdots & \overset{f_{n-1}^n}{\longleftarrow} & \mathbf{T}^{\kappa} & \cdots & \sum_{\overline{M}}
\end{array}$$

maps $\sum_{\overline{M}}\left(\overline{\phi_{\mathbf{0}}(\mathbb{R})}\right)$ homeomorphically onto $\sum_{\overline{M}}\left(\overline{\psi_{\mathbf{0}}(\mathbb{R})}\right)$. □

5. The Čech Cohomology of Denjoids

In the interest of space, we merely sketch the calculation of the Čech cohomology (with integer coefficients) of $\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$. We know that $\mathbf{T}^2-\mathcal{D}_{\mathfrak{f}}=\cup_{i=1}^{\kappa}D_i$ ($\kappa\leq\infty$), where each D_i is a component of $\mathbf{T}^2-\mathcal{D}_{\mathfrak{f}}$ and is homeomorphic to an open disk. We will break the calculation into two cases: (**1**) $\kappa<\infty$ and (**2**) $\kappa=\infty$.

Case (**1**): $\kappa<\infty$.

For each $i\in\{1,...,\kappa\}$ choose $D_i^1\subset D_i$ homeomorphic to an open disk and so that

$$\left(\pi^2\right)^{-1}\left(D_i^1\right)\cap[\mathbb{R}\times\mathbb{Z}\cup\mathbb{Z}\times\mathbb{R}]=\emptyset.$$

Then for each $i\in\{1,...,\kappa\}$ we represent D_i as a union of increasing subsets D_i^j, $j\in\mathbb{N}$, with each D_i^j homeomorphic to an open disk, chosen so that each D_i^j is a strong deformation retract of D_i^{j+1}: $D_i=\cup_{j=1}^{\infty}D_i^j$. We then have

$$\mathcal{D}_{\mathfrak{f}}=\cap_{j=1}^{\infty}\left[\mathbf{T}^2-\cup_{i=1}^{\kappa}D_i^j\right]\overset{\text{def}}{=}\cap_{j=1}^{\infty}K_j.$$

Using standard CW-complex techniques (see, e.g., [M]) we find that the singular cohomology of each K_j is isomorphic to $\oplus_{i=1}^{\kappa+1}\mathbb{Z}$, and the continuity of Čech cohomology leads to the calculation

$$\check{H}^1\left(\mathcal{D}_{\mathfrak{f}}\right)\cong\oplus_{i=1}^{\kappa+1}\mathbb{Z}.$$

And we also know that for a d-to-one bonding map $h:\mathbf{T}^2\to\mathbf{T}^2$ in the sequence defining $\sum_{\overline{M}}$, $\mathbf{T}^2-h^{-1}\left(\mathcal{D}_{\mathfrak{f}}\right)=\cup_{i=1}^{d\kappa}\mathfrak{D}_i$, where each $\mathfrak{D}_i$ is a component of $\mathbf{T}^2-h^{-1}\left(\mathcal{D}_{\mathfrak{f}}\right)$ and is mapped homeomorphically by h onto some D_j. Again using the continuity of Čech cohomology and the above property stated property of h to calculate its induced map on cohomology, we find that

$$\check{H}^1\left(\sum\nolimits_{\overline{M}}\left(\mathcal{D}_{\mathfrak{f}}\right)\right)\cong\check{H}^1\left(\sum\nolimits_{\overline{M}}\right)\oplus\left(\oplus_{i=1}^{\infty}\mathbb{Z}\right),$$

provided that $\sum_{\overline{M}}$ is not isomorphic to $\mathbf{T}^2$ (if $\sum_{\overline{M}} \cong \mathbf{T}^2$, $\check{H}^1\left(\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right)$ is isomorphic to $\bigoplus_{i=1}^{\lambda+1}\mathbb{Z}$, where λ is the number of complementary disks).

Case (**2**): $\kappa = \infty$.

In this case we must be careful since we cannot give the torus with infinitely many holes a CW-complex structure – it would not have the "weak topology" required for this. However, by punching out the holes one at a time and using the continuity of Čech cohomology, we again find

$$\check{H}^1\left(\sum\nolimits_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right) \cong \check{H}^1\left(\sum\nolimits_{\overline{M}}\right) \oplus \left(\oplus_{i=1}^{\infty}\mathbb{Z}\right)$$

(even when the bonding maps are eventually isomorphisms).

6. The Classification of Denjoids

With $\cong$ denoting homeomorphism of spaces, the known classification of the Denjoy continua

$$\mathbb{D} = \left\{\mathcal{D}_f \mid \sum(f) \text{ has only one blown up trajectory}\right\}$$

may be summarized as follows. For $\mathcal{D}_f$ and $\mathcal{D}_g$ in $\mathbb{D}$:

$$[\mathcal{D}_f \cong \mathcal{D}_g\ ] \Leftrightarrow \left[\Phi^{(\alpha,1)} \overset{equiv}{\approx} \Phi^{(\beta,1)}\right],$$

where Φ^{ω} is the $\omega-$linear flow on $\mathbf{T}^2$ and where α and β are the rotation numbers of f and g respectively. We shall show that this result generalizes to denjoids as specified in Theorem 6.6. But before doing so, we shall first need to establish some background results using some techniques similar to those Fokkink employed in his treatment of $\mathbb{D}$ [Fok], Chapter 2§2. The above classification providing an interesting inverse limit representation was proven in [BW]. It should also be noted that aperiodic flows on $\mathbf{T}^2$ have been classified up to topological equivalence, see [ABZ], 6.1.7.

Definition 7. Two maps $\phi_i : \mathbb{R} \to X$ $(i \in \{1,2\})$ are *asymptotic* if

$$\lim_{t\to\pm\infty} d\left(\phi_1(t), \phi_2(t)\right) = 0.$$

Lemma 6.1. *If $\phi_i : \mathbb{R} \to \mathcal{T}_i \subset \mathbf{T}^2$ $(i \in \{1,2\})$ are asymptotic one-to-one maps onto trajectories of a flow $\sum(f)$ on $\mathbf{T}^2$, then there exists points x and y in S^1 with $\mu_f\left([\mathbb{R},x]_f\right) = \mathcal{T}_1$ and $\mu_f\left([\mathbb{R},y]_f\right) = \mathcal{T}_2$ satisfying* $\lim_{n\to\infty} d_1\left(f^n(x), f^n(y)\right) = 0.$

PROOF. Since the sets $\mathcal{T}_i$ $(i \in \{1,2\})$ are trajectories of the flow $\sum(f)$, we have that $\phi_i(t) = \mu_f\left([s_i(t), w_i]_f\right)$ for points $w_i \in S^1$ and maps $s_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2\}$. As the maps ϕ_i are asymptotic, there is a T such that

$$d_1(\pi_2(\phi_1(t)), \pi_2(\phi_2(t))) = d_1(\pi(s_1(t)), \pi(s_2(t))) < \frac{1}{4}$$

for all t with $|t| \geq T$, where $\pi_2 : \mathbf{T}^2 \to S^1$; $\langle x_1, x_2\rangle \mapsto x_2$ (recall that μ_f "switches coordinates"). Then for all $t \geq T$, we have an integer n_t such that $s_1(t) = n_t + s_2(t) + \varepsilon(t)$, where $\varepsilon : [T, +\infty) \to \left(-\frac{1}{4}, \frac{1}{4}\right)$ and $\lim_{t\to+\infty} \varepsilon(t) = 0$. By continuity, n_t must be constant and so for all $t \geq T$, and so $s_1(t) = n + s_2(t) + \varepsilon(t)$ for a fixed integer n. Similarly, there is a fixed integer m such that $s_1(t) = m + s_2(t) + \varepsilon(t)$ for all $t \leq -T$, where $\varepsilon : (-\infty, -T] \to \left(-\frac{1}{4}, \frac{1}{4}\right)$ and $\lim_{t\to-\infty} \varepsilon(t) = 0$. Then for all $t \geq T$

$$\phi_1(t) = \mu_f\left([n + s_2(t) + \varepsilon(t), w_1]_f\right) = \mu_f\left(\left[s_2(t) + \varepsilon(t), f^{-n}(w_1)\right]_f\right)$$

and for all $t \leq -T$

$$\phi_1(t) = \mu_f\left(\left[s_2(t) + \varepsilon(t), f^{-m}(w_1)\right]_f\right).$$

Then if (**1**) s_2 reverses orientation, we have $s_2([T, +\infty))$ contains all but finitely many elements of $-\mathbb{N} = \{-1, -2, ...\}$, and if (**2**) s_2 preserves orientation, $s_2((-\infty, T])$ contains all but finitely many elements of $-\mathbb{N}$. Then with

$$x = \begin{cases} f^{-n}(w_1) & \text{in case } (\mathbf{1}) \\ f^{-m}(w_1) & \text{in case } (\mathbf{2}) \end{cases} \quad \text{and with } y = w_2,$$

we have for all but finitely many elements k of $-\mathbb{N}$

$$\begin{aligned} d_2(\phi_1(k), \phi_2(k)) &= d_2\left(\mu_f\left(\left[k + \varepsilon\left(s_2^{-1}(k)\right), x\right]_f\right), \mu_f\left([k, y]_f\right)\right) \\ &= d_2\left(\mu_f\left(\left[\varepsilon\left(s_2^{-1}(k)\right), f^{-k}(x)\right]_f\right), \mu_f\left(\left[0, f^{-k}(y)\right]_f\right)\right) \end{aligned}$$

and so $\lim_{\ell\to+\infty} d_1\left(f^\ell(x), f^\ell(y)\right) = \lim_{k\to-\infty} d_1\left(f^{-k}(x), f^{-k}(y)\right) = 0$ since the ϕ_i are asymptotic and $\varepsilon(t) \to 0$. □

See also [Fok], 2.2 p. 47.

Definition 8. If ϕ is a flow on X and if ψ is a flow on Y, then we call a surjective map $g : X \to Y$ a *trajectory map*, denoted $g : \phi \overset{traj}{\to} \psi$, if $g$maps ϕ–orbits onto ψ–orbits.

Corollary 6.2. *If $\phi_i : \mathbb{R} \to \mathcal{T}_i \subset \mathbf{T}^2$ $(i \in \{1,2\})$ are asymptotic one-to-one maps onto trajectories of an aperiodic flow ψ on $\mathbf{T}^2$, then given any trajectory map $g : \psi \overset{traj}{\to} \Phi^{(\omega_1,\omega_2)}$, we have $g(\mathcal{T}_1) = g(\mathcal{T}_2)$.*

Proof. : We have $(\widetilde{\omega_2} \times id_{\mathbf{T}^2}) : \Phi^{(\omega_1,\omega_2)} \overset{equiv}{\approx} \Phi^{(\omega_1/\omega_2,1)} \overset{equiv}{\approx} \sum \left(R_{(\omega_1/\omega_2)}\right)$, and since the translation map $R_{(\omega_1/\omega_2)}$ is an isometry, no two distinct trajectories of $\Phi^{(\omega_1/\omega_2,1)}$ admit asymptotic parameterizations by the above. But the uniform continuity of g guarantees that $g \circ \phi_i$ are asymptotic parameterizations of $g(\mathcal{T}_1)$ and $g(\mathcal{T}_2)$. □

Lemma 6.3. *Given any generalized Denjoy flow $\phi = \sum(\mathfrak{f})$ and given any $2-$solenoid $\sum_{\overline{M}}$, the pairs of trajectories of the flow $\phi_{\overline{M}}$ contained in $\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ admitting asymptotic parameterizations are precisely those pairs $(\mathcal{T}_1,\mathcal{T}_2)$ which contain points $p_i \in \mathcal{T}_i$ whose $\phi_{\overline{M}}-$orbits $\mathcal{O}_i : (\mathbb{R},0) \to (\mathcal{T}_i, p_i)$ are projected by f_n onto asymptotic orbits $f_n \circ \mathcal{O}_i$ of the flow $(f_1^n)^* \phi$ for each $n \in \mathbb{N}$.*

Proof. Clearly, for a pair of trajectories as described the $\phi_{\overline{M}}-$orbits $\mathcal{O}_i$ provide asymptotic parameterizations, since we then have for each $n \in \mathbb{N}$

$$\lim_{t\to\pm\infty} d_2\left(f_n \circ \mathcal{O}_1(t), f_n \circ \mathcal{O}_2(t)\right) = 0$$

in this case.

Now suppose that $(\mathcal{T}_1,\mathcal{T}_2)$ is any pair of distinct $\phi_{\overline{M}}-$trajectories admitting asymptotic parameterizations λ_1 and λ_2. Then $f_1(\mathcal{T}_1)$ and $f_1(\mathcal{T}_2)$ are trajectories of the flow ϕ which, by uniform continuity, admit the asymptotic parameterizations $f_1 \circ \lambda_i$. We then have by Lemma 6.1 that $f_1(\mathcal{T}_1) = \mu_{\mathfrak{f}}\left([\mathbb{R},x]_{\mathfrak{f}}\right)$ and $f_1(\mathcal{T}_2) = \mu_{\mathfrak{f}}\left([\mathbb{R},y]_{\mathfrak{f}}\right)$ for some points x and y of S^1 with $\lim_{n\to\infty} d_1\left(\mathfrak{f}^n(x), \mathfrak{f}^n(y)\right) = 0$.

Adopting the notation of 3 and the diagram (✠) contained therein, we have that for each $n \in \mathbb{N}$

$$id \times g^{(M_1\circ\cdots\circ M_{n-1})} : (f_1^n)^* \phi \overset{sc}{\succeq} \Phi^{\omega_n}$$

for some linear flow Φ^{ω_n} (the map $g^{(M_1\circ\cdots\circ M_{n-1})}$ is understood to be g when $n = 1$). The uniform continuity of f_n then guarantees that $f_n \circ \lambda_i$ are asymptotic parameterizations of the $(f_1^n)^* \phi-$trajectories $f_n(\mathcal{T}_1)$ and $f_n(\mathcal{T}_2)$. Hence, Corollary 6.2 implies that for each $n \in \mathbb{N}$ $g^{(M_1\circ\cdots\circ M_{n-1})}$ maps $f_n(\mathcal{T}_1)$ and $f_n(\mathcal{T}_2)$ to the same trajectory $\mathfrak{T}_n$ of Φ^{ω_n}.

We now proceed to show that $f_1(\mathcal{T}_1) \neq f_1(\mathcal{T}_2)$. Suppose to the contrary $z \in f_1(\mathcal{T}_1) \cap f_1(\mathcal{T}_2)$ with $(z = x_1, x_2, ...) \in \mathcal{T}_1$ and $(z = y_1, y_2, ...) \in \mathcal{T}_2$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are distinct trajectories of the flow $\phi_{\overline{M}}$, they are disjoint, and so there is a least $n>1$ with $x_n \neq y_n$. Now for $m<n$ we have $f_m(\mathcal{T}_1) = (f_1^m)^* \phi_{x_m}(\mathbb{R}) = (f_1^m)^* \phi_{y_m}(\mathbb{R}) = f_m(\mathcal{T}_2)$. By construction, $f_{n-1}^n(x_n) = f_{n-1}^n(y_n)$ while $x_n \neq y_n$, and so Lemma 2.3 yields that

$$\left(f_{n-1}^n\right)^* \left(f_1^{n-1}\right)^* \phi_{x_n}(\mathbb{R}) = (f_1^n)^* \phi_{x_n}(\mathbb{R}) = f_n(\mathcal{T}_1)$$

and

$$\left(f_{n-1}^n\right)^* \left(f_1^{n-1}\right)^* \phi_{y_n}(\mathbb{R}) = (f_1^n)^* \phi_{y_n}(\mathbb{R}) = f_n(\mathcal{T}_2)$$

are disjoint. At the same time, $f_1^n \circ f_n(\mathcal{T}_1) = f_1(\mathcal{T}_1)$ and $f_1^n \circ f_n(\mathcal{T}_2) = f_1(\mathcal{T}_2)$, and as $\phi-$trajectories with the point z in common, we must have $f_1^n \circ f_n(\mathcal{T}_1) = f_1^n \circ f_n(\mathcal{T}_2)$. Suppose f_1^n is $k-$to$-$one. Then by Lemma 2.3 $(f_1^n)^{-1}(f_1(\mathcal{T}_1))$ is the disjoint union of k $(f_1^n)^* \phi-$trajectories $\{T_1, ..., T_k\}$ (two of which are $f_n(\mathcal{T}_1)$ and $f_n(\mathcal{T}_2)$), and $(f_1^n)^{-1}(\mathfrak{T}_1)$ is the disjoint union of k $\Phi^{\omega_n}-$trajectories $\{\tau_1, ..., \tau_k\}$ (one of which is $\mathfrak{T}_n$). Under the composition

$$g \circ f_1^n = f_1^n \circ g^{(M_1 \circ \cdots \circ M_{n-1})},$$

each T_i is mapped to $\mathfrak{T}_1$ since

$$\mathfrak{T}_1 = g(f_1(\mathcal{T}_1)) = g \circ f_1^n(T_i).$$

Hence, for each i

$$T_i \subset \left(g^{(M_1 \circ \cdots \circ M_{n-1})}\right)^{-1} \left((f_1^n)^{-1}(\mathfrak{T}_1)\right),$$

which is to say that each T_i is the $g^{(M_1 \circ \cdots \circ M_{n-1})}-$preimage of some τ_j. But $g^{(M_1 \circ \cdots \circ M_{n-1})}$ maps $\mathbf{T}^2$ onto itself and it maps two distinct T_i's ($f_n(\mathcal{T}_1)$ and $f_n(\mathcal{T}_2)$) onto a single τ_j (namely, $\mathfrak{T}_n$). Hence, there must be a $(f_1^n)^* \phi-$trajectory $T \notin \{T_1, ..., T_k\}$ which $g^{(M_1 \circ \cdots \circ M_{n-1})}$ maps to some τ_j. But then

$$g \circ f_1^n(T) = f_1^n \circ g^{(M_1 \circ \cdots \circ M_{n-1})}(T) = f_1^n(\tau_j) = \mathfrak{T}_1,$$

which is to say that $f_1^n(T) \neq f_1(\mathcal{T}_1) \subset g^{-1}(\mathfrak{T}_1)$. This means that there are the two distinct $\phi-$trajectories $f_1^n(T)$ and $f_1(\mathcal{T}_1)$ which g maps to $\mathfrak{T}_1$. Hence, there is a disk $\mathfrak{D}_J$ for some interval $J \subset S^1$ which g maps onto $\mathfrak{T}_1$ and with $g^{-1}(\mathfrak{T}_1) = \mathfrak{D}_J \cup f_1^n(T) \cup f_1(\mathcal{T}_1)$ (g is one$-$to$-$one on

$$\mathcal{M}_{\mathfrak{f}} - \{\text{suspended trajectories of the endpoints of the Cantor set } M_{\mathfrak{f}}\}).$$

We have the flow on the plane $\left(\pi^2\right)^* \phi$ and $\left(\pi^2\right)^{-1}(\mathfrak{D}_J)$ is the disjoint union of strips, each of which is homeomorphic to an open disk and is bordered by asymptotic $\left(\pi^2\right)^* \phi-$trajectories which are mapped by π^2 to $f_1^n(T)$ and $f_1(\mathcal{T}_1)$. In this sense, $f_1^n(T)$ and $f_1(\mathcal{T}_1)$ border $\mathfrak{D}_J$. By Lemma 4.6, $(f_1^n)^{-1}(\mathfrak{D}_J)$ is the disjoint union of k disks $\{\mathfrak{D}_1, ..., \mathfrak{D}_k\}$, each of which is mapped homeomorphically by f_1^n onto $\mathfrak{D}_J$. We must then have that each $\mathfrak{D}_i$ is bordered by two $(f_1^n)^* \, \phi-$trajectories L_i and R_i which are mapped by f_1^n onto $f_1^n(T)$ and $f_1(\mathcal{T}_1)$ respectively. We then have for each i

$$f_1^n \circ g^{(M_1 \circ \cdots \circ M_{n-1})}(\mathfrak{D}_i) = g \circ f_1^n(\mathfrak{D}_i) = g(\mathfrak{D}_J) = \mathfrak{T}_1,$$

which is to say that for each i $g^{(M_1 \circ \cdots \circ M_{n-1})}(\mathfrak{D}_i) \subset \cup_{j=1}^k \tau_j$. And since $g^{(M_1 \circ \cdots \circ M_{n-1})}(\mathfrak{D}_i)$ is arc-connected and contains a $\Phi^{\omega_n}-$trajectory since $\mathfrak{D}_i$ contains a $(f_1^n)^* \, \phi-$trajectory, we must have $g^{(M_1 \circ \cdots \circ M_{n-1})}(\mathfrak{D}_i) = \tau_j$ for some j. Since the preimage of an arc contained in τ_j must be a closed subset of $\mathbf{T}^2$, we also have $g^{(M_1 \circ \cdots \circ M_{n-1})}(L_i \cup R_i) = g^{(M_1 \circ \cdots \circ M_{n-1})}(\mathfrak{D}_i) = \tau_j$. And for each j we have

$$\left(g^{(M_1 \circ \cdots \circ M_{n-1})}\right)^{-1}(\tau_j) \subset (f_1^n)^{-1} g^{-1}(\mathfrak{T}_1) = (f_1^n)^{-1}(\mathfrak{D}_J \cup f_1^n(T) \cup f_1(\mathcal{T}_1)).$$

As $g^{(M_1 \circ \cdots \circ M_{n-1})}$ is onto, for each j we must then have

$$\left(g^{(M_1 \circ \cdots \circ M_{n-1})}\right)^{-1}(\tau_j) = \mathfrak{D}_i \cup L_i \cup R_i \text{ for exactly one } i,$$

and in particular, there is an i with $\mathfrak{D}_i \cup L_i \cup R_i = \left(g^{(M_1 \circ \cdots \circ M_{n-1})}\right)^{-1}(\mathfrak{T}_n)$. Now $f_n\left(\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right) \cap \mathfrak{D}_i = \emptyset$ since $f_1^n(\mathfrak{D}_i) = \mathfrak{D}_J \subset \mathbf{T}^2 - \mathcal{D}_{\mathfrak{f}} = \mathbf{T}^2 - f_1\left(\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right)$ (see Lemma 3.2: if $x \in f_n\left(\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right) \cap \mathfrak{D}_i$, $f_1^n(x) \in f_1\left(\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right) \cap \mathfrak{D}_J = \emptyset$). At the same time, $(f_n(\mathcal{T}_1) \cup f_n(\mathcal{T}_2)) \subset f_n\left(\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right) \cap \left(g^{(M_1 \circ \cdots \circ M_{n-1})}\right)^{-1}(\mathfrak{T}_n)$. Hence, $(f_n(\mathcal{T}_1) \cup f_n(\mathcal{T}_2)) \subset L_i \cup R_i$. Since $(f_n(\mathcal{T}_1), f_n(\mathcal{T}_2))$ and (L_i, R_i) are pairs of distinct $(f_1^n)^* \, \phi-$trajectories, we must have $\{f_n(\mathcal{T}_1), f_n(\mathcal{T}_2)\} = \{L_i, R_i\}$. But L_i and R_i are mapped by f_1^n to the distinct $\phi-$trajectories $f_1^n(T)$ and $f_1(\mathcal{T}_1)$, while

$$f_1^n \circ f_n(\mathcal{T}_1) = f_1(\mathcal{T}_1) = f_1(\mathcal{T}_2) = f_1^n \circ f_n(\mathcal{T}_2).$$

We thus arrive at a contradiction and must therefore have that $f_1(\mathcal{T}_1)$ and $f_1(\mathcal{T}_2)$ are disjoint.

Thus, the points x and y in S^1 as above with $f_1(\mathcal{T}_1) = \mu_{\mathfrak{f}}\left([\mathbb{R}, x]_{\mathfrak{f}}\right)$ and $f_1(\mathcal{T}_2) = \mu_{\mathfrak{f}}\left([\mathbb{R}, y]_{\mathfrak{f}}\right)$ must be distinct and are the endpoints of an open interval $J \subset S^1$ with $g(\mathfrak{D}_J) = \mathfrak{T}_1$. Since f_1 is one-to-one when restricted to any $\phi_{\overline{M}}-$ trajectory [the proof of Lemma 2.3 remains valid for $\phi_{\overline{M}}$ viewed as the f_1-pullback of ϕ; $f_1 \circ \phi_{\overline{M}}(t, x) = \phi(t, f_1(x))$], we may choose p_1 and p_2 to be the points which f_1 maps to x and y respectively. We then let $\mathcal{O}_i$ be the $\phi_{\overline{M}}-$ orbit of the point p_i. Then by construction we have that $f_1 \circ \mathcal{O}_i$ are asymptotic orbits of the flow ϕ.

We now have that $f_1(\mathcal{T}_1)$ and $f_1(\mathcal{T}_2)$ border the disk $\mathfrak{D}_J \stackrel{\text{def}}{=} D_1$ and

$$g(D_1 \cup f_1(\mathcal{T}_1) \cup f_1(\mathcal{T}_2)) = \mathfrak{T}_1.$$

Adopting the above notation, for each $n \in \mathbb{N}$, f_1^n maps $\mathfrak{T}_n$ one-to-one onto the trajectory $\mathfrak{T}_1$. And since $f_1(\mathcal{T}_1) = f_1^n \circ f_n(\mathcal{T}_1)$ and $f_1(\mathcal{T}_2) = f_1^n \circ f_n(\mathcal{T}_2)$, we must have that $f_n(\mathcal{T}_1)$ and $f_n(\mathcal{T}_2)$ are disjoint for each $n \in \mathbb{N}$, while both are mapped to $\mathfrak{T}_n$. Hence, $f_n(\mathcal{T}_1)$and $f_n(\mathcal{T}_2)$ border an open disk D_n in $\mathbf{T}^2 - \overline{f_n(\mathcal{T}_i)}$ formed by the blowing up of $\mathfrak{T}_n$ and D_n is mapped homeomorphically by f_1^n onto D_1. Since $f_1 \circ \mathcal{O}_i$ are asymptotic orbits of the flow ϕ for $i = 1$ and 2, for T sufficiently large in absolute value there are neighborhoods U_T of $\{f_1 \circ \mathcal{O}_1(T), f_1 \circ \mathcal{O}_2(T)\}$ which are evenly covered by f_1^n. Now for $i = 1$ and 2 and T as above, we have $f_1^n \circ f_n \circ \mathcal{O}_i(T) = f_1 \circ \mathcal{O}_i(T)$, and so $f_n \circ \mathcal{O}_1(T)$ and $f_n \circ \mathcal{O}_2(T)$ are contained in a single component of $(f_1^n)^{-1}(U_T)$ since $f_n \circ \mathcal{O}_i(\mathbb{R})$ border D_n, which is mapped homeomorphically by f_1^n to D_1. And for points sufficiently close, f_1^n increases the distance between points or leaves the distance fixed, and so the distances $d_2(f_n \circ \mathcal{O}_1(T), f_n \circ \mathcal{O}_2(T)) \to 0$ as $T \to \pm\infty$ at at least the same rate as $d_2(f_1 \circ \mathcal{O}_1(T), f_1 \circ \mathcal{O}_2(T)) \to 0$. □

With notation as above, $f_n(\mathcal{T}_1)$ and $f_n(\mathcal{T}_2)$ border the open disk D_n for each $n \in \mathbb{N}$ with $D_1 = \mathfrak{D}_J$, and f_1^n maps D_n homeomorphically onto $\mathfrak{D}_J$. We then have that $\mathcal{T}_1$ and $\mathcal{T}_2$ together with the disk

$$\Delta(\mathcal{T}_1, \mathcal{T}_2) = (\mathfrak{D}_J \times D_2 \times \cdots \times D_n \times \cdots) \cap \sum\nolimits_{\overline{M}}$$

represent a blown up trajectory of $\Phi_{\overline{M}}^{(\theta,1)}$ in the sense that

$$\Gamma(\Delta(\mathcal{T}_1, \mathcal{T}_2)) = \Gamma(\mathcal{T}_1) = \Gamma(\mathcal{T}_2) \subset \sum\nolimits_{\overline{M}}.$$

That $\Delta(\mathcal{T}_1, \mathcal{T}_2)$ is a disk follows from the fact that f_1^n maps D_n homeomorphically onto $\mathfrak{D}_J$.

These disks $\Delta(\mathcal{T}_1, \mathcal{T}_2)$ are easy to see: each arc component $\mathbf{C}$ of $\sum_{\overline{M}}$ admits a fibration of the form $\pi_{\overline{M}} + c : \mathbb{R}^2 \to \mathbf{C}$ and $(\pi_{\overline{M}} + c)^{-1}(\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}}))$ is a translation of $(\pi^2)^{-1}(\mathcal{D}_{\mathfrak{f}})$, as can be seen by the following commutative diagram

$$(\diamond) \quad \begin{array}{ccccc} \left(\mathbb{R}^2, \widetilde{f_1 c}\right) & \stackrel{-\widetilde{f_1 c}}{\longrightarrow} & \left(\mathbb{R}^2, \mathbf{0}\right) & \stackrel{\pi_{\overline{M}}+c}{\longrightarrow} & \left(\sum_{\overline{M}}, c\right) \\ & \searrow^{\pi^2} & & \swarrow^{f_1} & \\ & & \left(\mathbf{T}^2, f_1 c\right) & & \end{array},$$

where $\widetilde{f_1 c} \in \left(\pi^2\right)^{-1}(f_1 c)$. This follows from

$$f_1 \circ (\pi_{\overline{M}} + c) \circ \left(-\widetilde{f_1 c}\right)(x) = f_1\left(\pi_{\overline{M}}\left(-\widetilde{f_1 c}\right) + \pi_{\overline{M}}(x) + c\right)$$
$$= \pi^2\left(-\widetilde{f_1 c}\right) + \pi^2(x) + f_1(c) = \pi^2(x).$$

We then have that $(f_1)^{-1}(\mathcal{D}_{\mathfrak{f}}) = \sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ [see Lemmas 3.1 and 3.2], and so

$$(\pi_{\overline{M}} + c)^{-1}\left(\sum\nolimits_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right) + \widetilde{f_1 c} = \left(-\widetilde{f_1 c}\right)^{-1}(\pi_{\overline{M}} + c)^{-1}\left((f_1)^{-1}(\mathcal{D}_{\mathfrak{f}})\right)$$
$$= \left(\pi^2\right)^{-1}(\mathcal{D}_{\mathfrak{f}}).$$

Notice that this holds for any choice of $\widetilde{f_1 c} \in \left(\pi^2\right)^{-1}(f_1 c)$. Setting $c = e_{\overline{M}}$, with $\mathbf{k}$ any element of $\mathbb{Z}^2 = \left(\pi^2\right)^{-1}(f_1 e_{\overline{M}})$, we have $\left(\pi^2\right)^{-1}(\mathcal{D}_{\mathfrak{f}}) + \mathbf{k} = \left(\pi^2\right)^{-1}(\mathcal{D}_{\mathfrak{f}})$.

Provided that $\sum_{\overline{M}}$ is not $\mathbf{T}^2$, $\pi_{\overline{M}} + c$ maps countably infinitely many of the strips forming the complement $\mathbb{R}^2 - (\pi_{\overline{M}} + c)^{-1}\left(\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right)$ to distinct disks in $\sum_{\overline{M}}$, and $\pi_{\overline{M}} + c$ maps the borders of these strips in $\mathbb{R}^2$ to asymptotic pairs $(\mathcal{T}_1, \mathcal{T}_2)$. Provided that $\sum_{\overline{M}}$ is not $\mathbf{T}^2$, there are uncountably many arc components of $\sum_{\overline{M}}$, and so there will be uncountably many such disks, with a countable infinity in each arc component of $\sum_{\overline{M}}$. In view of Lemmas 3.1 and 3.2, $\sum_{\overline{M}} - \sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ is the union of these disks Δ.

Definition 9. For a given flow ϕ, a pair of distinct $\phi-$trajectories $(\mathcal{T}_1, \mathcal{T}_2)$ is be *asymptotic* if there are corresponding $\phi-$ orbits for $\mathcal{T}_1$ and $\mathcal{T}_2$ which are asymptotic, and a $\phi-$trajectory is *non-asymptotic* if it is not asymptotically paired with any other $\phi-$trajectory (under any parameterization).

Notice that if $h : \sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}}) \to \sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ is a homeomorphism, then h is uniformly continuous and so maps asymptotic pairs of trajectories $(\mathcal{T}_1, \mathcal{T}_2)$ to asymptotic pairs $(\mathcal{T}_1', \mathcal{T}_2')$. In particular, $\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ is not homogeneous since some trajectories are not paired asymptotically with another trajectory (see also page 10 and [Fok], p. 48).

We now determine a consistent way to choose arcs joining points on distinct asymptotic trajectories $\mathcal{T}_1$ and $\mathcal{T}_2$, the interiors of which are contained in $\Delta(\mathcal{T}_1, \mathcal{T}_2)$.

Definition 10. For a given orientation-preserving homeomorphism $\mathfrak{f} : S^1 \to S^1$, with lift $F : \mathbb{R} \to \mathbb{R}$ ($F(0) \in [0, 1)$)

$$\widetilde{\mu}_{\mathfrak{f}} : \left(\mathbb{R}^2, \mathbf{0}\right) \to \left(\mathbb{R}^2, \mathbf{0}\right) \text{ by}$$
$$(s, t) \mapsto \left([1 - \rfloor s \lfloor]\ F^{\lfloor s \rfloor}(t) + \rfloor s \lfloor F^{\lfloor s \rfloor + 1}(t), s\right)$$

Definition 11. We define the map

$$\begin{array}{rcl} p_{\mathfrak{f}} & : & \mathbb{R}^2 \stackrel{id\times\pi}{\longrightarrow} \mathbb{R}\times S^1 \to \mathcal{S}_{\mathfrak{f}} \text{ by} \\ (s,t) & \mapsto & (s,\pi(t)) \mapsto [s,\pi(t)]_{\mathfrak{f}}. \end{array}$$

We then have the following commutative diagram

$$\begin{array}{ccc} \left(\mathbb{R}^2,\mathbf{0}\right) & \stackrel{\widetilde{\mu_{\mathfrak{f}}}}{\longrightarrow} & \left(\mathbb{R}^2,\mathbf{0}\right) \\ {\scriptstyle p_{\mathfrak{f}}}\downarrow & & \downarrow^{\pi^2} \\ \left(\mathcal{S}_{\mathfrak{f}},[0,\pi(0)]_{\mathfrak{f}}\right) & \stackrel{\mu_{\mathfrak{f}}}{\longrightarrow} & \left(\mathbf{T}^2,\mathbf{0}\right) \end{array}$$

and $\widetilde{\mu_{\mathfrak{f}}}(\mathbb{R}\times J) \subset \left(\pi^2\right)^{-1}(\mathfrak{D}_J)$.

Lemma 6.4. *The map $\widetilde{\mu_{\mathfrak{f}}}$ is a homeomorphism.*

Proof. Suppose that $\widetilde{\mu_{\mathfrak{f}}}((s,t)) = \widetilde{\mu_{\mathfrak{f}}}((s',t'))$. Then we must have that $s = s'$. Since $\mathfrak{f}$ is orientation-preserving, F is monotone increasing, and so if, say, $t < t'$ we would have

$$[1-\rfloor s\lfloor]\, F^{\lfloor s\rfloor}(t) + \rfloor s\lfloor F^{\lfloor s\rfloor+1}(t) < [1-\rfloor s\lfloor]\, F^{\lfloor s\rfloor}(t') + \rfloor s\lfloor F^{\lfloor s\rfloor+1}(t').$$

Hence we must have that $t = t'$, implying that $\widetilde{\mu_{\mathfrak{f}}}$ is one-to-one. By invariance of domain, $\widetilde{\mu_{\mathfrak{f}}}$ is an open map. Since $\widetilde{\mu_{\mathfrak{f}}}$ is clearly onto, $\widetilde{\mu_{\mathfrak{f}}}$ is a homeomorphism. □

Roughly speaking, we now develop a linear structure for $\sum_{\overline{M}}$ induced by the linear structure on $\mathbb{R}^2$ and the maps $(\pi_{\overline{M}} + c)$. For each $c \in \sum_{\overline{M}}$ making the definite choice

$$\widetilde{f_1 c} = \left(\pi^2\right)^{-1}(f_1(c)) \cap [0,1) \times [0,1),$$

we have the commutative diagram ($\diamond$). We now know that two distinct asymptotic trajectories $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\phi_{\overline{M}}$ will lie in the same path component $\mathbf{C}$ of $\sum_{\overline{M}}$, for the disk

$$\Delta(\mathcal{T}_1,\mathcal{T}_2) = (\mathfrak{D}_J \times D_2 \times \cdots \times D_n \times \cdots) \cap \sum\nolimits_{\overline{M}}$$

spans the two trajectories. And so for any $c \in \mathbf{C}$ we have that the fibration $\pi_{\overline{M}}+c$ maps $\mathbb{R}^2$ onto $\mathbf{C} \supset \Delta(\mathcal{T}_1,\mathcal{T}_2)$. The translating factor $\widetilde{f_1 c}$ then puts

$$(\pi_{\overline{M}} + c)^{-1}\left(\sum\nolimits_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\right)$$

into the standardized form $\left(\pi^2\right)^{-1}(\mathcal{D}_{\mathfrak{f}})$.

Now

$$\left(\pi^2\right)^{-1}(\mathfrak{D}_J) = \widetilde{\mu_{\mathfrak{f}}}\left((p_{\mathfrak{f}})^{-1}(\mu_{\mathfrak{f}})^{-1}(\mathfrak{D}_J)\right)$$

since $\pi^2 \circ \widetilde{\mu_{\mathfrak{f}}} = \mu_{\mathfrak{f}} \circ p_{\mathfrak{f}}$ and $(\mu_{\mathfrak{f}})^{-1}(\mathfrak{D}_J) = [\mathbb{R}, J]_{\mathfrak{f}}$, and so

$$\left(\pi^2\right)^{-1}(\mathfrak{D}_J) = \widetilde{\mu_{\mathfrak{f}}}\left((p_{\mathfrak{f}})^{-1}\left([\mathbb{R}, J]_{\mathfrak{f}}\right)\right) = \widetilde{\mu_{\mathfrak{f}}}\left(\bigcup_{n\in\mathbb{Z}}\left[\mathbb{R} \times \pi^{-1}(\mathfrak{f}^n(J))\right]\right).$$

We thus have that $\left(\pi^2\right)^{-1}(\mathfrak{D}_J)$ is the union of the strips in $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R} \times \pi^{-1}(\mathfrak{f}^n(J))\right)$. For a fixed n , $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R} \times \pi^{-1}(\mathfrak{f}^n(J))\right)$ contains translates of a strip by $(k, 0)$, with k any integer. If we agree to let π^{-1} take on values in a specific range (e.g., $[0, 1)$) for choosing the left endpoint of $\pi^{-1}(\mathfrak{f}^n(J))$ on the 0 level, then $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R} \times \pi^{-1}(\mathfrak{f}^n(J))\right)$ is translation of $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R} \times \pi^{-1}(J)\right)$ by $(0, n)$.

By the diagram $(\diamond)$ we have

$$\left(-\widetilde{f_1 c}\right)^{-1}(\pi_{\overline{M}} + c)^{-1}(x) \subset \left(\pi^2\right)^{-1}(f_1(x)),$$

and so the preimages

$$\left(-\widetilde{f_1 c}\right)^{-1}(\pi_{\overline{M}} + c)^{-1}(x) \text{ and } \left(-\widetilde{f_1 c}\right)^{-1}(\pi_{\overline{M}} + c)^{-1}(y)$$

are contained in the borders of the strips $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R} \times \pi^{-1}\left(\mathfrak{f}^n\left(\overline{J}\right)\right)\right)$. We then have the question: to what extent do these preimages depend on our choice of c? And: what if $\pi_{\overline{M}} + c$ is not one-to-one, as happens when $\sum_{\overline{M}}$ has an S^1 factor? In general, these preimages depend on the choice of c and there may be countably infinitely many elements in these sets. But there is something we can say. Since all elements of $\left(-\widetilde{f_1 c}\right)^{-1}(\pi_{\overline{M}} + c)^{-1}(x)$ are mapped by π^2 to $f_1(x)$, the possible candidates for the preimages $\left(-\widetilde{f_1 c}\right)^{-1}(\pi_{\overline{M}} + c)^{-1}(x)$ are separated by translations by elements of $\mathbb{Z}^2$. And we have already seen that $\left(\pi^2\right)^{-1}(\mathcal{D}_{\mathfrak{f}})$ is invariant under such translations. So while these preimages are not uniquely determined, the way the π^2 preimages of $f_1(x)$ and $f_1(y)$ are "situated" in $\left(\pi^2\right)^{-1}(\mathcal{D}_{\mathfrak{f}})$ is determined.

We have that the borders of the strips $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R} \times \pi^{-1}\left(\mathfrak{f}^n\left(\overline{J}\right)\right)\right)$ contain the preimages $\left(\pi^2\right)^{-1}(f_1(x))$ and $\left(\pi^2\right)^{-1}(f_1(y))$ and these borders are trajectories of the pullback flow

$$\left(-\widetilde{f_1 c}\right)^{*}(\pi_{\overline{M}} + c)^{*}\phi_{\overline{M}} = \left(\pi^2\right)^{*}\phi$$

and so are mapped by $(\pi_{\overline{M}} + c) \circ \left(-\widetilde{f_1 c}\right)$ one-to-one onto $\phi_{\overline{M}}-$ trajectories and by π^2 one-to-one onto $\phi-$trajectories (Lemma 2.3). Thus, the sets $\left(\pi^2\right)^{-1}(f_1(x))$ and $\left(\pi^2\right)^{-1}(f_1(y))$ are grouped in pairs, with a pair on the borders of each strip in

$\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R}\times\pi^{-1}\left(\mathfrak{f}^{n}\left(\overline{J}\right)\right)\right)$. Enumerate the set of component strips of $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R}\times\pi^{-1}\left(\mathfrak{f}^{n}\left(\overline{J}\right)\right)\right)$ as $\{S_m\}_{m\in\mathbb{N}}$. For $n\in\mathbb{N}$ let $x^n=(x_1^n,x_2^n)$ and $y^n=(y_1^n,y_2^n)$ be the point of S_n in $\left(\pi^2\right)^{-1}(f_1(x))$ and $\left(\pi^2\right)^{-1}(f_1(y))$ respectively. There is no extremely simple way of specifying in a consistent way an arc joining x^n and y^n: while the sets $\mathbb{R}\times\mathfrak{f}^n\left(\overline{J}\right)$ are convex, the map $p_{\mathfrak{f}}$ distorts distances, and the sets $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R}\times\pi^{-1}\left(\mathfrak{f}^{n}\left(\overline{J}\right)\right)\right)$ are not convex $-$ but they are "piecewise convex" : they are bent along the lines $\mathbb{R}\times k$, where k is an integer. We shall exploit the piecewise convex structure of $\widetilde{\mu_{\mathfrak{f}}}\left(\mathbb{R}\times\pi^{-1}\left(\mathfrak{f}^{n}\left(\overline{J}\right)\right)\right)$ to form our arc and begin by forming a (broken) segment between x^0 and y^0, $A\left[x^0,y^0\right]$.

We assume without loss of generality that $x_1^0<y_1^0$ and $x_2^0<y_2^0$, the construction being made in an exactly analogous way in the other cases. If $\lfloor x_2^0\rfloor=\lfloor y_2^0\rfloor$ or if $\lfloor x_2^0\rfloor+1=y_2^0$, then we take

$$\begin{aligned} A\left[x^0,y^0\right] \quad &:\quad [0,1]\to\mathbb{R}^2 \text{ to be} \\ t \quad &\mapsto\quad (1-t)x^0+ty^0, \end{aligned}$$

which is contained in S_0 since $S_0\cap\mathbb{R}\times\left[\lfloor x_2^0\rfloor,\lfloor x_2^0\rfloor+1\right]$ is convex (in fact, a trapezoid). We assume then that $\lfloor y_2^0\rfloor=\lfloor x_2^0\rfloor+k$ with $k>0$ and $y_2^0\neq\lfloor x_2^0\rfloor+1$. Then for $i=1,...,k$ we set ℓ_i and r_i to be the points on the trajectories of x^0 and y^0 respectively which meet $\mathbb{R}\times\left\{\lfloor x_2^0\rfloor+i\right\}$, and we set $s=y_2^0-x_2^0$. We then define for $i=1,...,k$

$$\begin{aligned} p_i &= \left(\frac{y_2^0-\lfloor x_2^0\rfloor-i}{s}\right)\ell_i+\left(\frac{\lfloor x_2^0\rfloor+i-x_2^0}{s}\right)r_i \\ &= \left(\frac{y_2^0-\lfloor x_2^0\rfloor-i}{s}\right)\ell_i+\left[1-\left(\frac{y_2^0-\lfloor x_2^0\rfloor-i}{s}\right)\right]r_i \end{aligned}$$

and define

$$A\left[x^0,y^0\right]:[0,1]\to\mathbb{R}^2 \text{ to be the map}$$

mapping $\left[0,\frac{\lfloor x_2^0\rfloor+1}{s}\right]$ linearly onto the segment joining x^0 and p_1, $\left[\frac{\lfloor x_2^0\rfloor+1}{s},\frac{\lfloor x_2^0\rfloor+2}{s}\right]$ linearly onto the segment joining p_1 and p_2, ..., and $\left[\frac{\lfloor x_2^0\rfloor+k}{s},1\right]$ linearly onto the segment joining p_k and y^0. We know that each of the segments is contained in S_0 since each intersection $S_0\cap\mathbb{R}\times[m,m+1]$ is convex for any integer m. We then have for any $n\in\mathbb{N}$ a $\mathbf{k}_n\in\mathbb{Z}^2$ with $x^n=x^0+\mathbf{k}_n$ and $y^n=y^0+\mathbf{k}_n$ and we define the path $[0,1]\to\mathbb{R}^2$

$$A\left[x^n,y^n\right](t)=A\left[x^0,y^0\right](t)+\mathbf{k}_n.$$

Then for any $n \in \mathbb{N}$ $\pi^2 \circ A[x^n, y^n]$ defines the same arc $[0,1] \to \mathbf{T}^2$ sending 0 to $f_1(x)$ and 1 to $f_1(y)$, and for each n we have the arc $[0,1] \to \sum_{\overline{M}}$ defined by $(\pi_{\overline{M}} + c) \circ \left(-\widetilde{f_1 c}\right) \circ A[x^n, y^n]$. At least one of these defines an arc joining x and y which maps $(0,1)$ into $\Delta(\mathcal{T}_1, \mathcal{T}_2)$, but it is possible that $(\pi_{\overline{M}} + c)$ is not one-to-one and so there may be many such arcs. But if we compose two such arcs

$$(\pi_{\overline{M}} + c) \circ \left(-\widetilde{f_1 c}\right) \circ A[x^n, y^n] \text{ and } (\pi_{\overline{M}} + c) \circ \left(-\widetilde{f_1 c}\right) \circ A[x^m, y^m]$$

with f_1 we get the same arc $\pi^2 \circ A\left[x^0, y^0\right]$ by $(\diamond)$, and since f_1 has unique path lifting (see [Cl]), we know that the two arcs into $\sum_{\overline{M}}$ are the same. Thus, we get a uniquely determined arc

$$A[x,y] : ([0,1], (0,1)) \to \left(\sum\nolimits_{\overline{M}}, \Delta(\mathcal{T}_1, \mathcal{T}_2)\right)$$

with endpoints x and y as desired.

Lemma 6.5. *A homeomorphism $\mathfrak{h} : \sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}}) \to \sum_{\overline{N}}(\mathcal{D}_{\mathfrak{g}})$ can be extended to a homeomorphism $\overline{\mathfrak{h}} : \sum_{\overline{M}} \to \sum_{\overline{N}}$.*

Proof. We let $\phi_{\overline{M}} = \sum(\mathfrak{f})_{\overline{M}}$ and $\phi_{\overline{N}} = \sum(\mathfrak{g})_{\overline{N}}$ and we let

$$\{(\mathcal{T}_{1_a}, \mathcal{T}_{2_a}) \mid a \in A\} \text{ and } \{(\mathfrak{T}_{1_b}, \mathfrak{T}_{2_b}) \mid b \in B\}$$

be the collection of pairs of asymptotic $\phi_{\overline{M}}-$ (and $\phi_{\overline{N}}$)$-$trajectories respectively. To the pairs $(\mathcal{T}_{1_a}, \mathcal{T}_{2_a})$ and $(\mathfrak{T}_{1_b}, \mathfrak{T}_{2_b})$ there correspond the disks $\Delta(\mathcal{T}_{1_a}, \mathcal{T}_{2_a})$ and $\Delta(\mathfrak{T}_{1_b}, \mathfrak{T}_{2_b})$. We need to define $\overline{\mathfrak{h}}$ on $\sum_{\overline{M}} - \sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$, which we now know to be the union of the disks $\Delta(\mathcal{T}_{1_a}, \mathcal{T}_{2_a})$.

Since $\mathfrak{h}$ maps arc components to arc components and is uniformly continuous, it maps $\phi_{\overline{M}}-$trajectories to $\phi_{\overline{N}}-$trajectories and an asymptotic pair $(\mathcal{T}_{1_a}, \mathcal{T}_{2_a})$ to an asymptotic pair $(\mathfrak{T}_{1_b}, \mathfrak{T}_{2_b})$. By Lemma 6.3, there are points p_1 and p_2 of $\mathcal{T}_{1_a}$ and $\mathcal{T}_{2_a}$ respectively, such that the corresponding $\phi_{\overline{M}}-$orbits $\mathcal{O}_i$ are asymptotic. We extend $\mathfrak{h}$ to $\overline{\mathfrak{h}}$ on $\Delta(\mathcal{T}_{1_a}, \mathcal{T}_{2_a})$ as follows:

$$\text{for } (s,t) \in \mathbb{R} \times [0,1],\ A[\mathcal{O}_1(s), \mathcal{O}_2(s)](t) \mapsto A[\mathfrak{h} \circ \mathcal{O}_1(s), \mathfrak{h} \circ \mathcal{O}_2(s)](t)\ .$$

Since $\mathfrak{h}$ provides a monotonic (with respect to orbits) correspondence between $\mathcal{O}_1(\mathbb{R})$ and $\mathfrak{h} \circ \mathcal{O}_1(\mathbb{R})$ and between $\mathcal{O}_2(\mathbb{R})$ and $\mathfrak{h} \circ \mathcal{O}_2(\mathbb{R})$, by the construction of the arcs $A[x,y]$ we get that $\overline{\mathfrak{h}}$ maps $\Delta(\mathcal{T}_{1_a}, \mathcal{T}_{2_a})$ one-to-one onto $\Delta(\mathfrak{T}_{1_b}, \mathfrak{T}_{2_b})$, and so $\overline{\mathfrak{h}}$ is one-to-one and onto. Since these disks are neither open nor closed, this does not guarantee the continuity of $\overline{\mathfrak{h}}$, but since these arcs depend continuously on their endpoints and $\mathfrak{h}$ is continuous, we get that $\overline{\mathfrak{h}}$ is continuous and so a homeomorphism. □

Given any generalized Denjoy flow $\sum(\mathfrak{f})$ and any $2-$ solenoid $\sum_{\overline{M}}$, there is a map $\Gamma_{\mathfrak{f}} : (\sum_{\overline{M}}, e_{\overline{M}}) \to (\sum_{\overline{M}}, e_{\overline{M}})$ homotopic to the identity with $(id \times \Gamma_{\mathfrak{f}}) : \phi_{\overline{M}} \overset{sc}{\succeq} \Phi_{\overline{M}}^{(\theta,1)}$ (see page 10), where θ is the rotation number of $\mathfrak{f}$. The $\phi_{\overline{M}}-$trajectories corresponding to asymptotically paired $\phi_{\overline{M}}-$orbits are mapped to the same $\Phi_{\overline{M}}^{(\theta,1)}-$ trajectory. Each $\Phi_{\overline{M}}^{(\theta,1)}-$trajectory which is not the $\Gamma_{\mathfrak{f}}-$image of asymptotically paired $\phi_{\overline{M}}-$ trajectories is the $\Gamma_{\mathfrak{f}}-$image of exactly one $\phi_{\overline{M}}-$trajectory. We shall generalize this slightly.

Definition 12. Given a flow ϕ on X and a flow ψ on Y and a trajectory map $\Gamma : \phi \overset{traj}{\to} \psi$, we define a ψ–trajectory to be a Γ–*asymptotic* trajectory if it is the $\Gamma-$ image of an asymptotic pair of ϕ–trajectories.

Theorem 6.6. *Let $\mathcal{D}_{\mathfrak{f}}$ and $\mathcal{D}_{\mathfrak{g}}$ be generalized Denjoy continua with*

$$\Gamma_{\mathfrak{f}} : \left(\sum\nolimits_{\overline{M}}, e_{\overline{M}}\right) \to \left(\sum\nolimits_{\overline{M}}, e_{\overline{M}}\right);\ (id \times \Gamma_{\mathfrak{f}}) : \phi_{\overline{M}} \overset{def}{=} \sum(\mathfrak{f})_{\overline{M}} \overset{sc}{\succeq} \Phi_{\overline{M}}^{(\alpha,1)}$$

and

$$\Gamma_{\mathfrak{g}} : \left(\sum\nolimits_{\overline{N}}, e_{\overline{N}}\right) \to \left(\sum\nolimits_{\overline{N}}, e_{\overline{N}}\right);\ (id \times \Gamma_{\mathfrak{g}}) : \phi_{\overline{N}} \overset{def}{=} \sum(\mathfrak{g})_{\overline{N}} \overset{sc}{\succeq} \Phi_{\overline{N}}^{(\beta,1)}.$$

Then

$$\left[\sum\nolimits_{\overline{M}}(\mathcal{D}_{\mathfrak{f}}) \cong \sum\nolimits_{\overline{N}}(\mathcal{D}_{\mathfrak{g}})\right] \Leftrightarrow$$

[There is a translation of an isomorphism $\mathfrak{j} : \Phi_{\overline{M}}^{(\alpha,1)} \overset{top}{\approx} \Phi_{\overline{N}}^{(\beta,1)}$ which provides a one-to-one correspondence between Γ –asymptotic trajectories and $\Gamma_{\mathfrak{g}}$–asymptotic trajectories, where $\Gamma : \phi_{\overline{M}} \overset{traj}{\to} \Phi_{\overline{M}}^{(\alpha,1)}$ is homotopic to $id_{\sum_{\overline{M}}}$ and is such that the Γ –preimage of a Γ –asymptotic trajectory contains only two trajectories of $\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ and is such that the Γ –preimage of a trajectory which is not Γ –asymptotic is a single $\phi_{\overline{M}}$–trajectory.]

PROOF. ($\Rightarrow$) Let $\mathfrak{h} : \sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}}) \cong \sum_{\overline{N}}(\mathcal{D}_{\mathfrak{g}})$ be a homeomorphism. We then have that $\mathfrak{h}$ extends to a homeomorphism $\overline{\mathfrak{h}} : \sum_{\overline{M}} \to \sum_{\overline{N}}$. There is an isomorphism $\mathfrak{i} : \sum_{\overline{M}} \to \sum_{\overline{N}}$ homotopic to $\overline{\mathfrak{h}} - \overline{\mathfrak{h}}(e_{\overline{M}})$ which then lifts to an automorphism $i : \mathbb{R}^2 \to \mathbb{R}^2$, see [Sch] and [Cl]. Since $\Gamma_{\mathfrak{g}}$ is homotopic to $id_{\sum_{\overline{N}}}$, we have that $\overline{\mathfrak{h}}(e_{\overline{M}})$ and $\Gamma_{\mathfrak{g}}\left(\overline{\mathfrak{h}}(e_{\overline{M}})\right)$ are in the same path component of $\sum_{\overline{N}}$. Hence, $\overline{\mathfrak{h}} = \left(\overline{\mathfrak{h}} - \overline{\mathfrak{h}}(e_{\overline{M}})\right) + \overline{\mathfrak{h}}(e_{\overline{M}})$ and $\mathfrak{j} \overset{def}{=} \mathfrak{i} + \Gamma_{\mathfrak{g}}\left(\overline{\mathfrak{h}}(e_{\overline{M}})\right)$ are homotopic since translations by elements of the same path component of $\sum_{\overline{N}}$ are isotopic, see [Cl]. Then with $i((\alpha,1)) = (\eta,\delta)$ and with $\widetilde{\delta}$ multiplication by δ and with $\gamma = \eta/\delta$ (we know that

$\delta \neq 0$ since α is irrational), we have

$$\left(\widetilde{\delta}\right) \times \mathfrak{j} : \Phi_{\overline{M}}^{(\alpha,1)} \overset{equiv}{\approx} \Phi_{\overline{N}}^{(\gamma,1)}$$

since translations preserve linear flows, see [Cl], Lemma 3.3.

We now set out to show that $\beta = \gamma$. Now the maps $\overline{\mathfrak{h}}$ and are $\mathfrak{j}$ are homotopic, and $\Gamma_{\mathfrak{f}}$ and $\Gamma_{\mathfrak{g}}$ are homotopic to their respective identities, and so $\Gamma_{\mathfrak{g}} \circ \overline{\mathfrak{h}}$ and $\mathfrak{j} \circ \Gamma_{\mathfrak{f}}$ are homotopic as maps $\sum_{\overline{M}} \to \sum_{\overline{N}}$ and both map $e_{\overline{M}} \mapsto \Gamma_{\mathfrak{g}}\left(\overline{\mathfrak{h}}\left(e_{\overline{M}}\right)\right)$ since $\Gamma_{\mathfrak{f}}\left(e_{\overline{M}}\right) = e_{\overline{M}}$. With $\mathcal{T}$ the $\phi_{\overline{M}}-$trajectory of $e_{\overline{M}}$, we have by construction that $\Gamma_{\mathfrak{g}} \circ \overline{\mathfrak{h}}\left(\mathcal{T}\right)$ is the $\Phi_{\overline{N}}^{(\beta,1)}-$trajectory of $\Gamma_{\mathfrak{g}}\left(\overline{\mathfrak{h}}\left(e_{\overline{M}}\right)\right)$ and that $\mathfrak{j} \circ \Gamma_{\mathfrak{f}}\left(\mathcal{T}\right)$ is the $\Phi_{\overline{N}}^{(\gamma,1)}-$trajectory of $\Gamma_{\mathfrak{g}}\left(\overline{\mathfrak{h}}\left(e_{\overline{M}}\right)\right)$. We then also have that $\mathfrak{d} = \Gamma_{\mathfrak{g}} \circ \overline{\mathfrak{h}} - \mathfrak{j} \circ \Gamma_{\mathfrak{f}}$ is homotopic to the constant map sending $\sum_{\overline{M}}$ to $e_{\overline{N}}$. Hence, there is a map $\widetilde{\mathfrak{d}}$ making the following diagram commute

$$\begin{array}{ccc} & & \left(\mathbb{R}^2, \mathbf{0}\right) \\ & \widetilde{\mathfrak{d}} \nearrow & \downarrow^{\pi_{\overline{N}}} \\ \left(\sum_{\overline{M}}, e_{\overline{M}}\right) & \overset{\mathfrak{d}}{\to} & \left(\sum_{\overline{N}}, e_{\overline{N}}\right) \end{array}$$

since $\pi_{\overline{N}}$ is a fibration and the constant map sending $\sum_{\overline{M}}$ to $e_{\overline{N}}$ lifts.

Suppose then that $\beta \neq \gamma$. We then choose a sequence $\left\{\phi_{\overline{M}}\left(t_n, e_{\overline{M}}\right)\right\}_{n\in\mathbb{N}}$ which converges in $\sum_{\overline{M}}$, where $\left\{t_n\right\}_{n\in\mathbb{N}}$ is an unbounded sequence of the reals. We then arrive at a contradiction just as in the proof of Theorem 3.9 in [Cl] since $\left\{\widetilde{\mathfrak{d}} \circ \phi_{\overline{M}}\left(t_n, e_{\overline{M}}\right)\right\}_{n\in\mathbb{N}}$ must converge, while $\left\{\widetilde{\mathfrak{d}} \circ \phi_{\overline{M}}\left(t_n, e_{\overline{M}}\right)\right\}_{n\in\mathbb{N}}$ represents at the same time the differences of points in the plane which are unbounded in the plane and which lie on lines which, having different slopes when lifted to $\mathbb{R}^2$ via $\pi_{\overline{N}} + \Gamma_{\mathfrak{g}}\left(\overline{\mathfrak{h}}\left(e_{\overline{M}}\right)\right)$, diverge. We then must have that $\beta = \gamma$, and so $id \times \mathfrak{j}$: $\Phi_{\overline{M}}^{(\alpha,1)} \overset{equiv}{\approx} \Phi_{\overline{N}}^{(\eta,\delta)}$ with $\eta/\delta = \beta$ and $\mathfrak{j} : \Phi_{\overline{M}}^{(\alpha,1)} \overset{top}{\approx} \Phi_{\overline{N}}^{(\beta,1)}$.

We now define $\Gamma \overset{\text{def}}{=} \mathfrak{j}^{-1} \circ \Gamma_{\mathfrak{g}} \circ \overline{\mathfrak{h}}$. Since $\Gamma_{\mathfrak{g}}$ is homotopic to the identity and $\overline{\mathfrak{h}}$ and $\mathfrak{j}$ are homotopic, we have that Γ is homotopic to the identity. Now $\Gamma_{\mathfrak{g}} \circ \overline{\mathfrak{h}} : \phi_{\overline{M}} \overset{traj}{\to} \Phi_{\overline{N}}^{(\beta,1)}$ since $\Gamma_{\mathfrak{g}} : \phi_{\overline{N}} \overset{traj}{\to} \Phi_{\overline{N}}^{(\beta,1)}$ and $\overline{\mathfrak{h}}$ maps $\phi_{\overline{M}}-$ trajectories contained in $\sum_{\overline{M}}\left(\mathcal{D}_{\mathfrak{f}}\right)$ to $\phi_{\overline{N}}-$trajectories in $\sum_{\overline{N}}\left(\mathcal{D}_{\mathfrak{g}}\right)$ and maps the $\phi_{\overline{M}}-$trajectories of $\sum_{\overline{M}} - \sum_{\overline{M}}\left(\mathcal{D}_{\mathfrak{f}}\right)$ into a disk contained in $\sum_{\overline{N}} - \sum_{\overline{N}}\left(\mathcal{D}_{\mathfrak{g}}\right)$, which disk is in turn mapped to a single $\Phi_{\overline{N}}^{(\beta,1)}-$trajectory by $\Gamma_{\mathfrak{g}}$. By the above, we have that $\mathfrak{j}^{-1}$: $\Phi_{\overline{N}}^{(\beta,1)} \overset{traj}{\to} \Phi_{\overline{M}}^{(\alpha,1)}$, and so $\Gamma : \phi_{\overline{M}} \overset{traj}{\to} \Phi_{\overline{M}}^{(\alpha,1)}$. By construction, $\Gamma_{\mathfrak{g}}$ maps asymptotic pairs of $\phi_{\overline{N}}-$trajectories to a single $\Phi_{\overline{M}}^{(\beta,1)}-$trajectory and maps non-asymptotic $\phi_{\overline{N}}-$ trajectories to a $\Phi_{\overline{M}}^{(\beta,1)}-$ trajectory which has a single $\phi_{\overline{N}}-$trajectory as its $\Gamma_{\mathfrak{g}}-$preimage. And so, since $\overline{\mathfrak{h}}$ only maps asymptotic pairs of $\phi_{\overline{M}}-$ trajectories to asymptotic pairs of $\phi_{\overline{N}}-$trajectories and since $\mathfrak{j}^{-1}$ provides a one-to-one correspondence between $\Phi_{\overline{M}}^{(\beta,1)}-$trajectories and $\Phi_{\overline{M}}^{(\alpha,1)}-$trajectories, Γ maps asymptotic pairs of $\phi_{\overline{M}}-$ trajectories to a $\Phi_{\overline{M}}^{(\alpha,1)}-$ trajectory having this pair as its $\Gamma-$preimage in $\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ and Γ maps non-asymptotic $\phi_{\overline{M}}-$trajectories to a $\Phi_{\overline{M}}^{(\alpha,1)}-$trajectory which has a single $\phi_{\overline{M}}-$trajectory as its $\Gamma-$preimage. Notice that the following diagram of trajectory maps commutes by construction

$$\begin{array}{ccc} \sum_{\overline{M}} & \overset{\overline{\mathfrak{h}}}{\rightarrow} & \sum_{\overline{N}} \\ \Gamma\downarrow & & \downarrow\Gamma_{\mathfrak{g}} \\ \sum_{\overline{M}} & \overset{\mathfrak{j}}{\rightarrow} & \sum_{\overline{N}} \end{array},$$

and so a $\Gamma-$asymptotic trajectory $\mathfrak{T}$ with the asymptotic $\phi_{\overline{M}}-$trajectories $\mathcal{T}$ and $\mathcal{T}'$ contained in $\Gamma^{-1}(\mathfrak{T})\cap\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ is mapped by $\mathfrak{j}$ to $\Gamma'\left(\overline{\mathfrak{h}}(\mathcal{T})\right)=\Gamma'\left(\overline{\mathfrak{h}}(\mathcal{T}')\right)$, which is then a $\Gamma_{\mathfrak{g}}-$asymptotic trajectory. This is what we required.

($\Leftarrow$) Under the given assumptions we need to find a homeomorphism $\mathfrak{h}:\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})\rightarrow\sum_{\overline{N}}(\mathcal{D}_{\mathfrak{g}})$. Let $x\in\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$. There are two possibilities for the $\phi_{\overline{M}}-$trajectory $\mathcal{T}$ of x: (1) $\mathcal{T}$ is asymptotically paired with another $\phi_{\overline{M}}-$trajectory $\mathcal{T}'$ and (2) $\mathcal{T}$ is non-asymptotic.

(1) In this case, since there is exactly one other trajectory $\mathcal{T}'\subset\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ with which $\mathcal{T}$ is asymptotically paired, we have that $\Gamma^{-1}(\Gamma(x))\cap\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ is a pair of points $\{x,x'\}$, with one point from each of $\mathcal{T}$ and $\mathcal{T}'$. Now $\mathfrak{j}(\Gamma(x))$ is by assumption a point on a $\Gamma_{\mathfrak{g}}-$asymptotic trajectory which will similarly have two $\Gamma_{\mathfrak{g}}-$preimages $\{y,y'\}$ in $\sum_{\overline{N}}(\mathcal{D}_{\mathfrak{g}})$, with $y\in\mathfrak{T}$ and $y'\in\mathfrak{T}'$, where $\mathfrak{T}$ and $\mathfrak{T}'$ are asymptotically paired $\phi_{\overline{N}}-$trajectories.

Our analysis of the maps $\pi_{\overline{M}}+x$ (see the diagram ($\diamond$) and the construction of arcs joining points on the borders of the disks $\Delta(\mathcal{T},\mathcal{T}')$) shows that the relative position of the points

$$\left(-\widetilde{f_1 x}\right)^{-1}(\pi_{\overline{M}}+x)^{-1}(\{x,x'\})$$

in the plane is determined up to translation by an element of $\mathbb{Z}^2$. And so, it is well-defined to say that one of the pair $(\mathcal{T},\mathcal{T}')$ lies to the left of the other, meaning that in any connected strip $S=\mathbb{R}\times J$ which $(\pi_{\overline{M}}+x)\circ\left(-\widetilde{f_1 x}\right)\circ\widetilde{\mu_{\mathfrak{f}}}$ maps onto $\Delta(\mathcal{T},\mathcal{T}')$, the so-called left trajectory of the pair $(\mathcal{T},\mathcal{T}')$ forms the left border of S (see Definition 11) . Similar considerations hold for the map $\pi_{\overline{N}}+y$ and the pair $\left(\mathfrak{T},\mathfrak{T}'\right)$. We assume then without loss of generality that $\mathcal{T}$ lies to the left of $\mathcal{T}'$ and that $\mathfrak{T}$ lies to the left of $\mathfrak{T}'$.

Since the maps Γ and $\Gamma_{\mathfrak{g}}$ are homotopic to the respective identities by assumption, we have that $(\pi_{\overline{M}} + \Gamma(x))(\mathbb{R}^2) \supset \{x, x'\} \cup \{\Gamma(x)\}$ and that

$$(\pi_{\overline{N}} + \Gamma_{\mathfrak{g}}(y))(\mathbb{R}^2) \supset \{y, y'\} \cup \{\Gamma_{\mathfrak{g}}(y)\}.$$

And since $\mathbb{R}^2$ is contractible, there is a map j making the following diagram commute

$$\begin{array}{ccc} (\mathbb{R}^2, \mathbf{0}) & \overset{j}{\to} & (\mathbb{R}^2, \mathbf{0}) \\ (\pi_{\overline{M}} + \Gamma(x)) \downarrow & & \downarrow (\pi_{\overline{N}} + \Gamma_{\mathfrak{g}}(y)) \\ (\sum_{\overline{M}}, \Gamma(x)) & \overset{\mathfrak{j}}{\to} & (\sum_{\overline{N}}, \Gamma_{\mathfrak{g}}(y)) \end{array}.$$

It is a routine matter to show that j is a homeomorphism of $\mathbb{R}^2$ by looking at a similar diagram for $\mathfrak{j}^{-1}$ and composing the two diagrams to get an identity on the bottom row and the composition of the two lifts on the top row, which must then be the identity by the uniqueness of lifting for path connected spaces. In fact, the additivity of the maps $(\pi_{\overline{M}} + \Gamma(x))$, $\mathfrak{j}$ and $(\pi_{\overline{N}} + \Gamma_{\mathfrak{g}}(y))$ can be used to show that j is an automorphism which can be represented by a matrix, but we do not really need this. We then define $\mathfrak{h}(x) = y$ and $\mathfrak{h}(x') = y'$ (or $\mathfrak{h}(x) = y'$ and $\mathfrak{h}(x') = y$), according as j preserves (reverses) the orientation of the plane [which could be determined by examining the sign of the determinant of the matrix representing j]. In other words, if j preserves orientation the left point is mapped to the left point, and if j reverses orientation the left point is mapped to the right point.

(2) In this case, there is in some sense no choice. By assumption, $\mathfrak{j}(\Gamma(x))$ is contained in a trajectory which is not $\Gamma_{\mathfrak{g}}-$asymptotic trajectory, and so its $\Gamma_{\mathfrak{g}}-$preimage is a single point y. We then define $\mathfrak{h}(x) = y$.

To see that the so defined function $\mathfrak{h}$ is a homeomorphism, first notice that it is one-to-one and onto since $\mathfrak{j}$ provides a one-to-one correspondence between $\Gamma-$asymptotic and $\Gamma_{\mathfrak{g}}-$ asymptotic trajectories and $\mathfrak{j}$ itself is a homeomorphism. It then only remains to show that $\mathfrak{h}$ is continuous. To see this, we examine a point $y \in \sum_{\overline{N}}(\mathcal{D}_{\mathfrak{g}})$ and a neighborhood

$$(g_n)^{-1}(U) \cap \sum\nolimits_{\overline{N}}(\mathcal{D}_{\mathfrak{g}})$$

of y, where $g_n : \sum_{\overline{N}} \to \mathbf{T}^2$ is the projection onto the n^{th} factor and U is an open neighborhood of $g_n(y)$ in $\mathbf{T}^2$ (we use g_j^i for the bonding maps of $\sum_{\overline{N}}$). Now g_n projects the flow $\phi_{\overline{N}}$ to the flow $(g_1^n)^* \sum(\mathfrak{g})$, and the neighborhood U of $g_n(y)$ in $\mathbf{T}^2$ then contains a "rectangular" neighborhood $\mathcal{N}$ of $g_n(y)$ which is homeomorphic to the product of an arc contained in the $(g_1^n)^* \sum(\mathfrak{g})-$ trajectory of $g_n(y)$ and a transverse segment $\ell \subset \mathcal{N}$ containing $g_n(y)$ in its interior and which is such that the n^{th} factor map of $\Gamma_{\mathfrak{g}}\quad g^{(N_1\circ\cdots\circ N_{n-1})}$ maps $\mathcal{N}$ to a set which is homeomorphic to the product of the image of these two line segments, the segment of the trajectory being mapped homeomorphically by $g^{(N\circ\cdots\circ_1 N_{n-1})}$ while ℓ is mapped monotonically [here we are using the specific construction of $\Gamma_{\mathfrak{g}}$, see page 10; see page 8 for a description of the action of the map $g^{(N\circ\cdots\circ_1 N_{n-1})}$ – the described behaviour of $g^{(N_1\circ\cdots\circ N_{n-1})}$]. And since y is on a non-asymptotic trajectory, $g^{(N\circ\cdots\circ_1 N_{n-1})}(\ell)$ will contain $g_n(\Gamma_{\mathfrak{g}}(y))$ in its interior and $g^{(N\circ\cdots\circ_1 N_{n-1})}(\mathcal{N})$ is a neighborhood $\mathcal{N}'$ of $g_n(\Gamma_{\mathfrak{g}}(y))$. Thus, $\Gamma_{\mathfrak{g}}\left((g_n)^{-1}(U)\right)$ contains the neighborhood $V\overset{\text{def}}{=}(g_n)^{-1}(\mathcal{N}')$ of $\Gamma_{\mathfrak{g}}(y)$. But then

$$\Gamma^{-1}\left(\mathfrak{j}^{-1}(V)\right)\cap\sum\nolimits_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$$

is a neighborhood of $\mathfrak{h}^{-1}(y)$ which $\mathfrak{h}$ maps into $(g_n)^{-1}(U)$ by construction.

And if y is on a non-asymptotic trajectory and if $(g_n)^{-1}(U)\cap\sum_{\overline{N}}(\mathcal{D}_{\mathfrak{g}})$ is a neighborhood of y, we still have a rectangular neighborhood $\mathcal{N}\subset U$ of $g_n(y)$ as described above – only now the image of ℓ under $g(N_{n-1}\circ\cdots\circ N_1)$ might contain $g_n(\Gamma_{\mathfrak{g}}(y))$ as an endpoint. In this case,

$$V\overset{\text{def}}{=}(g_n)^{-1}(g(N_{n-1}\circ\cdots\circ N_1)(\mathcal{N}))$$

is a neighborhood and it contains $\Gamma_{\mathfrak{g}}(y)$, but it might not contain $\Gamma_{\mathfrak{g}}(y)$ in its interior. Now $\mathfrak{j}^{-1}$ might "flip" V, but our construction takes the possible orientation reversal of $\mathfrak{j}$ into account and $\Gamma^{-1}\left(\mathfrak{j}^{-1}(V)\right)\cap\sum_{\overline{M}}(\mathcal{D}_{\mathfrak{f}})$ is a neighborhood of $\mathfrak{h}^{-1}(y)$ which $\mathfrak{h}$ maps into $(g_n)^{-1}(U)$ by construction. $\square$

Previously we noted that any minimal set occurring in an aperiodic continuous flow on $\mathbf{T}^2$ which is a proper subset of $\mathbf{T}^2$ is homeomorphic with some $\mathcal{D}_f$. We are thus led to the following conjecture:

Conjecture. *If $\mathcal{M}$ is a minimal set occurring in a flow on $\sum_{\overline{M}}$ which has no orbit whose closure is homogeneous, $\mathcal{M}$ is homeomorphic with some denjoid $\sum_{\overline{N}}(\mathcal{D}_f)$.*

We cannot require $\overline{N}$ to be $\overline{M}$ as it is easy to construct denjoids $\sum_{\overline{N}}(\mathcal{D}_f)\subset\sum_{\overline{M}}$ which are not realized as a denjoid $\sum_{\overline{M}}(\mathcal{D}_g)$ and which have corresponding minimal flows on $\sum_{\overline{M}}$ which are the "pullbacks" by the n^{th} $(n>1)$ projection maps f_n of flows on $\mathbf{T}^2$.

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