ENDOMORPHISMS OF COMPACT ABELIAN GROUPS MINIMIZE ENTROPY

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Abstract. The entropy conjecture for tori has been settled a long time ago by Misiurewicz and Przytycki. We show that their result extends from tori to compact and connected abelian groups. We give stronger results for hyperbolic automorphisms.

Throughout this paper G denotes a compact and connected metric abelian group. The group operation on G is written additively.

1. Introduction

Shub [14] conjectured that the topological entropy $\mathrm{h_{top}}(f)$ of a smooth map on a differentiable manifold M is bounded below by the spectral radius of the induced map f_* on the direct sum of the homology groups $H_i(M,\mathbb{R})$. This is known as the Entropy Conjecture and it has been proved for many classes of spaces and maps [6]. Misiurewicz and Przytycki proved in [11] that the Entropy Conjecture holds for maps on a torus. The idea of the proof is that every homotopy class contains a unique endomorphism E that minimizes the entropy and it turns out that $\mathrm{h_{top}}(E)$ is equal to the spectral radius of E_*. In particular, if M is a torus, then the Entropy Conjecture holds for maps that are not necessarily smooth.

In this paper we extend Misiurewicz and Przytycki's result from tori to compact connected abelian groups. According to a theorem of Gorin and Scheffer [7, 13] each pointed homotopy class of a compact and connected abelian group G, with base-point $0 \in G$, contains one and only one endomorphism E. Our main result is that E minimizes the entropy within its homotopy class.

Our main result implies that the Entropy Conjecture holds for compact abelian groups, if homology is replaced by cohomology. It is well known that the entropy of an endomorphism E can be recovered from its induced action E^* on the first Čech-cohomology group $H^1(G,\mathbb{Q})$. More specifically, if G is finite-dimensional, then the entropy can be calculated from Yuzvinskii's formula [16], as follows[1]. Let $p(x)$ be the characteristic polynomial of the induced action E^* on $H^1(G,\mathbb{Q})$. Let a be the least positive integer such that $a \cdot p(x)$ is integral. Then the entropy of E is equal to

$$\mathrm{h_{top}}(E) = a \cdot \sum_{|\lambda|>1} \log|\lambda|, \tag{1}$$

where the summation runs over all roots of $p(x)$ in $\mathbb{C}$. It turns out that this is equal to the Mahler measure of $a \cdot p(x)$ and that there is an interesting interplay between dynamics and number theory, see [4]. It follows from our main result that $\mathrm{h_{top}}(f)$ is bounded below by the Mahler measure of the characteristic polynomial of the induced action f^* on $H^1(G,\mathbb{Q})$.

[1] Yuzvinskii's original result extends to infinite-dimensional non-abelian groups, but it is usually stated in the form that is given here.

A result of Franks [5] on Anosov diffeomorphisms implies that any map f that is homotopic to a hyperbolic endomorphism E is actually semi-conjugate to E. The existence of a semi-conjugacy implies that $\mathrm{h_{top}}(f) \geq \mathrm{h_{top}}(E)$. So for hyperbolic automorphisms, Franks' result implies the Entropy Conjecture. In the first part of the paper we show that Franks' proof extends from tori to compact abelian groups.

We prove our main result by lifting endomorphisms to a restricted product of locally compact fields, following ideas of Lind and Ward [10].

2. Hyperbolic automorphisms

Franks [5] proved that a diffeomorphism f that is homotopic to an Anosov diffeomorphism A is actually semi-conjugate to A: there exists a surjection h such that $A \circ h = h \circ f$. This implies that $\mathrm{h_{top}}(f) \geq \mathrm{h_{top}}(A)$. In particular, if A is a hyperbolic toral automorphism, then it minimizes the topological entropy in its homotopy class. We show that this semi-conjugation property extends from tori to general G.

The defining property of an Anosov diffeomorphism is that it respects the direct sum of the tangent bundle of a manifold into the stable and unstable bundle. If the diffeomorphism has the additional structure of an automorphism on a toral group, then its tangent space at the origin suffices: an automorphism A of $\mathbb{T}^n$ is hyperbolic (or Anosov) if the induced map $T(A)$ on the tangent space $\mathbb{R}^n = T_0(\mathbb{T}^n)$ is a hyperbolic map, i.e., none of its eigenvalues has modulus 1. Lin [9] pointed out that it is possible to define the tangent space at the origin of a general compact abelian group G, so it is possible to extend the hyperbolic notion to topological groups.

Definition 1. *The tangent space $L(G) = \mathrm{Hom}(\mathbb{R}, G)$ is the collection of one-parameter subgroups of G endowed with the compact-open topology.*

The tangent space can be conveniently described in terms of the Pontryagin dual group (or character group) $\hat{G}$.

Lemma 2. *$L(G)$ is isomorphic to $\mathbb{R}^{\dim G}$ with the product topology. To be more specific, let $S \subset \hat{G}$ be a maximal algebraically independent subset. Then $L(G) \cong \mathbb{R}^S$ endowed with the product topology.*

Proof. Since $\mathbb{R}$ is self-dual, under Pontryagin duality $\mathrm{Hom}(\mathbb{R}, G)$ is topologically isomorphic to $\mathrm{Hom}(\hat{G}, \mathbb{R})$ with the compact-open topology. Since $\mathbb{R}$ is divisible and S is independent, any map $f\colon S \to \mathbb{R}$ extends to a homomorphism $f\colon \hat{G} \to \mathbb{R}$ and uniquely so since S is maximal. □

The exponential map $\exp\colon L(G) \to G$ is defined by $\exp(f) = f(1)$. The image of exp is equal to the path-component of 0. The exponential map has the following lifting properties, see [9].

Lemma 3. *Let $f\colon X \to G$ be a null-homotopic map. Then there exists a map $\widetilde{f}\colon X \to L(G)$ such that $\exp \circ \widetilde{f} = f$.*

Lemma 4. *Let $f\colon G \to H$ be a continuous map such that $f(0)$ is in the path-component of $0 \in H$. Then there exists a map $L(f)\colon L(G) \to L(H)$ such that $\exp \circ L(f) = f \circ \exp$.*

We call $L(f)$ the *linearization* of f. The linearization $L(E)$ of an endomorphism E is a linear map on $L(G)$. If G is finite-dimensional, then we say that an automorphism A is *hyperbolic* if $L(A)$ has no eigenvalues of modulus 1.

The following theorem is a generalization of Franks' conjugation result.

Theorem 5. *Suppose that* $\dim G < \infty$ *and that* A *is a hyperbolic automorphism. If a homeomorphism* $f : G \to G$ *is homotopic to* A*, then there is a map* $h : G \to G$ *homotopic to* id_G *satisfying* $A \circ h = h \circ f$*.*

Proof. Given A and f we are trying to solve the functional equation $A \circ h = h \circ f$ for an h homotopic to id_G. Since h has to be homotopic to id and since f is homotopic to A we may write $h = id + \gamma$ and $f = A + \varphi$ for null-homotopic maps γ, φ so the functional equation becomes $A \circ \gamma = \varphi + \gamma \circ f$. By Lemma 3 the maps γ, φ lift to $\widetilde{\gamma}, \widetilde{\varphi}\colon G \to L(G)$. By Lemma 4 the endomorphism admits a linearization $L(A)$, so we may lift the functional equation to the tangent space and we may write

$$\widetilde{\gamma} \;=\; L(A)^{-1} \circ \widetilde{\varphi} \;+\; L(A)^{-1} \circ \widetilde{\gamma} \circ f. \tag{2}$$

Here we use that A is invertible, hence so is $L(A)$. One verifies that the original equation $A \circ h = h \circ f$ is solved by $h = id_G + \gamma$ for $\gamma = \exp \circ \widetilde{\gamma}$.

The map $\widetilde{\gamma}$ that we are looking for is a fixed point under an operator on the Banach space $C(G, L(G))$. A formal solution to this fixed point problem is

$$\sum_{n=1}^{\infty} L(A)^{-n} \circ \widetilde{\varphi} \circ f^{n-1}. \tag{3}$$

This series converges in the stable coordinate of $L(G)$. Another formal solution is given by

$$-\sum_{n=0}^{\infty} L(A)^{n} \circ \widetilde{\varphi} \circ f^{-1-n}, \tag{4}$$

which converges in the unstable coordinate. Here we use that f is invertible.

Since A is hyperbolic the tangent space decomposes into $W_A^s \oplus W_A^u$, the direct sum of the stable and unstable space. In particular we may write $\widetilde{\varphi} = \widetilde{\varphi}^s + \widetilde{\varphi}^u$. The solution to the fixed point problem is

$$\widetilde{\gamma} = \sum_{n=1}^{\infty} L(A)^{-n} \circ \widetilde{\varphi}^u \circ f^{n-1} + L(A)^{n-1} \circ \widetilde{\varphi}^s \circ f^{-n} \tag{5}$$

□

Corollary 6. *An expansive automorphism minimizes the entropy in its isotopy class.*

Proof. According to Theorem 5 there exists a semi-conjugation $A \circ h = h \circ f$ for an h that is homotopic to id. If we prove that h is surjective, then $\mathrm{h_{top}}(f) \geq \mathrm{h_{top}}(A)$, see, e.g., [15, Theorem 7.2]. To prove surjectivity, it suffices to show that $L(h)$ is surjective since the path-component of 0 is dense in G. We have already seen that $L(h) = id + L(\gamma)$ for a bounded map $L(\gamma)$. Such a map on $\mathbb{R}^n$ extends to the one-point compactification S^n where it has Brouwer degree 1. So $L(h)$ is surjective. □

Corollary 7. *Suppose that* $\dim G = 1$*. Then every automorphism on* G *minimizes the entropy in its isotopy class.*

Proof. Let $f\colon G \to G$ be a homeomorphism. By the theorem of Gorin and Scheffer there exists a unique automorphism A that is homotopic to f. Since $\dim G = 1$ the tangent space is equal to $\mathbb{R}$ and $L(A)(x) = cx$ for some constant c. If $c = \pm 1$, then A is $\pm$ identity and has zero entropy. If $c \neq \pm 1$, then A is hyperbolic and f is semi-conjugate to A. □

This corollary was previously obtained by Kwapisz [8]. We shall prove below that the restriction of the dimension is not necessary.

It is possible that $L(E)$ is invertible even if E is not. For instance, the doubling map on the circle $x \mapsto 2x$ is not invertible but $L(E)$ is. In formal solution 3 it is necessary only that $L(E)$ is invertible, and if $L(G)$ is equal to the unstable space of $L(E)$, the solution is convergent. We say that a surjective endomorphism E is *expanding* if all eigenvalues of $L(E)$ are of modulus > 1. For instance, the doubling map on the circle is forward expanding.

Corollary 8. *Suppose that* $\dim G < \infty$*. If a continuous map* $f : G \to G$ *is homotopic to a forward expanding endomorphism* E*, then there is a map* $h : G \to G$ *homotopic to* id_G *satisfying* $E \circ h = h \circ f$*. In particular, forward expanding endomorphisms minimize the entropy in their homotopy class.*

In the remainder of this section we consider G that are infinite-dimensional.

Definition 9. *An automorphism* A *is hyperbolic if* $L(G)$ *decomposes as* $W^s \oplus W^u$*, where* $W^s = \{v \in L(G)\colon L(A)^n(v) \to 0 \text{ if } n \to \infty\}$ *and* W^u *is defined likewise. Note that if* A *is hyperbolic, then* $L(A)$ *admits no homoclinic points other than* 0*.*

The proof of Theorem 5 has only one technical problem in the infinite-dimensional case: we have to show that the formal solutions converge in the stable and in the unstable subspace. To get around this difficulty, we analyze the algebraic properties of a group that admits a hyperbolic automorphism. G has the algebraic structure of a $\mathbb{Z}[x, x^{-1}]$-module, with $x \cdot g = A(g)$. The Pontryagin dual $\hat{G}$ inherits this structure under the adjoint action $\chi \mapsto \chi \circ A$ for a character $\chi \in A$.

Lemma 10. *If* A *is a hyperbolic automorphism on* G *then* $\hat{G}$ *is a torsion module.*

Proof. We argue by contradiction. Suppose that $\hat{G}$ is not a torsion module. Then there exists a character χ_0 such that $S_0 = \{\chi_0 \circ A^n \colon n \in \mathbb{Z}\}$ is a set of linearly independent characters. By Zorn's lemma S_0 extends to a maximal algebraically independent set $S \subset \hat{G}$. By Lemma 2 we know that $L(G)$ is isomorphic to $\mathbb{R}^S$ and it contains $\mathbb{R}^{S_0}$ as an invariant subspace, on which $L(A)$ acts as a shift. Therefore $L(G)$ has non-trivial homoclinic points, for instance the sequence $(\delta_\chi)_{\chi \in S}$ for which $\delta_\chi = 0$ if $\chi \neq \chi_0$ and $\delta_{\chi_0} = 1$. This is ridiculous since A is hyperbolic. □

Lemma 11. *If* $\hat{G}$ *is a torsion module, then* $L(G)$ *embeds as a submodule into a direct product of finite-dimensional hyperbolic modules.*

Proof. Since $L(G)$ is a vector space over $\mathbb{R}$ it has a basis S of independent characters χ_s. Each χ_s generates a torsion module $M_s \subset \hat{G}$. The direct sum over all M_s maps surjectively onto $\hat{G}$. Therefore $\mathrm{Hom}(\hat{G}, \mathbb{R})$ maps injectively into the direct product over all $\mathrm{Hom}(M_s, \mathbb{R})$. The projection of $\mathrm{Hom}(\hat{G}, \mathbb{R})$ onto $\mathrm{Hom}(M_s, \mathbb{R})$ is surjective since a set of independent characters in M_s extends to a set of independent characters of $\hat{G}$. Therefore $\mathrm{Hom}(M_s, \mathbb{R})$ is equal to the sum of its stable and unstable module. Since it is finite-dimensional, it cannot contain homoclinic points. It follows that $\mathrm{Hom}(M_s, \mathbb{R})$ is a finite-dimensional hyperbolic module. □

As a consequence of these lemmata, we find that if A is hyperbolic and if $v \in L(G)$ is restricted to a bounded set, then the series $\sum_{n=0}^{\infty} L(A)^n(v)$ converges, since this is true for finite-dimensional hyperbolic modules. So the formal solutions of the finite-dimensional case can be used as well in the infinite-dimensional case.

Theorem 12. *We may drop the restriction that* $\dim G < \infty$ *in Theorem 5.*

It follows that for a general compact and connected abelian group a hyperbolic automorphism minimizes the entropy in its isotopy class. As the neutral eigenvalues do not contribute to Yuzvinskii's formula (1), the entropy of an endomorphism depends only on its hyperbolic part and it is natural to expect that general endomorphisms have the same property.

3. Lifting endomorphisms

We consider an endomorphism E on a compact, connected, metrizable abelian group G and a continuous map $f\colon G \to G$ that is base-point preserving $f(0) = 0$ and homotopic to E. Following ideas of Lind and Ward we lift E to an endomorphism of a 'nice' covering group, a product of full solenoids, preserving the entropy of E and of any f in its homotopy class.

The Pontryagin dual $\hat{G}$ is torsion-free of countable, possibly infinite, rank r. It embeds as a subgroup into $\mathbb{Q}^r = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{G}$, the direct sum of r copies of $\mathbb{Q}$. We say that $\hat{\mathbb{Q}}$ is the *full solenoid* and denote it by $\mathcal{S}$. Since $\hat{G}$ embeds in $\mathbb{Q}^r$, G is a quotient group of $\mathcal{S}^r$ and the projection $p\colon \mathcal{S}^r \to G$ is a bundle projection with a totally disconnected fiber. An endomorphism on $\hat{G}$ extends to $\mathbb{Q}^r$, so a homomorphism on G lifts to $\mathcal{S}^r$. The bundle $p\colon \mathcal{S}^r \to G$ has the homotopy lifting property so the homotopy class of E lifts to $\mathcal{S}^r$. In this section we show that lifting preserves the entropy, by subdividing the bundle projection into a sequence of bundles with finite fibers.

Lemma 13. *Suppose that* $p\colon \tilde{G} \to G$ *is a surjective homomorphism with finite kernel. Then* f *lifts to a unique base-point preserving map* $\tilde{f}$ *and* $\mathrm{h_{top}}(\tilde{f}) = \mathrm{h_{top}}(f)$.

Proof. Let $\tilde{d}$ be a translation-invariant metric on $\tilde{G}$ and let $d(x, y) = \tilde{d}(p^{-1}(x), p^{-1}(y))$ be the induced metric on G. Then p is a local isometry. If $\tilde{S}$ is (n, ϵ)-spanning then the projection $p(\tilde{S})$ is (n, ϵ)-spanning under f, so lifting cannot decrease the entropy. To prove that it does not increase the entropy either, it suffices to show that $\tilde{S} = p^{-1}(S)$ is (n, ϵ)-spanning if S is (n, ϵ)-spanning and ϵ is sufficiently small. For each $\tilde{x} \in \tilde{G}$ there exists an $\tilde{s} \in \tilde{S}$ such that $\tilde{d}(\tilde{x}, \tilde{s}) < \epsilon$ and such that $\dot{(}f^j(x), f^j(s)) < \epsilon$ for all $0 \leq j \leq n$. In other words, for each such j there exists an element $\tilde{n}_j \in p^{-1}(0)$ such that $\tilde{d}(\tilde{f}^j(\tilde{x}), \tilde{f}^j(\tilde{s}) + \tilde{n}_j) < \epsilon$ and we have to show that $\tilde{n}_j = 0$. The map $\tilde{g} \to \tilde{g} + \tilde{n}_j$ is a deck-transformation, so it commutes with $\tilde{f}$: $\tilde{f}(\tilde{g} + \tilde{n}_j) = \tilde{f}(\tilde{g}) + \tilde{n}_j$. If ϵ is sufficiently small, much smaller than the distance between points in $p^{-1}(0)$, then the only point in the fiber over $f^{j+1}(s)$ that is ϵ-close to $\tilde{f}^{j+1}(\tilde{x})$ is $\tilde{f}(\tilde{f}^j(s) + \tilde{n}_j) = \tilde{f}^{j+1}(s) + \tilde{n}_j$. So $\tilde{n}_{j+1} = \tilde{n}_j$ and by induction it is equal to 0. □

If $G_j \to G_{j-1}$ is a sequence of finite coverings then the inverse limit $\lim_{\leftarrow} G_j$ is a bundle over G_0 with a totally disconnected fiber. A base-point preserving map on G_0 lifts uniquely to a base-point preserving map on the inverse limit. The following lemma is similar to Brown's inverse limit theorem [3, Corollary 4.4.1], but here we deal with a mapping f that may not be surjective, as required in Brown's theorem.

Lemma 14. *Suppose that* $\lim_{\leftarrow} G_j$ *is the inverse limit over a sequence of finite coverings by compact connected abelian groups* G_j*. Lifting a base-point preserving map from* G_0 *to* $\lim_{\leftarrow} G_j$ *preserves the entropy.*

Proof. Lifting cannot decrease the entropy, by the same argument as in the proof above. To prove that it does not increase the entropy either, we bound the cardinality of an $(n, 2\epsilon)$-spanning subset of $\lim_{\leftarrow} G_j$ by the cardinality of an (n, ϵ)-spanning subset of G_0. By the previous lemma, we may just as well take an (n, ϵ)-spanning subset of some G_k. There exists a k such that the projection $p\colon \lim_{\leftarrow} G_j \to G_k$ has fibers of diameter $< \varepsilon$. Hence, if S_k is (n, ϵ)-spanning in S_k and if $\tilde{S}_k$ is a subset of the inverse limit that intersects the fiber of each element of S_k in a single point, then $\tilde{S}_k$ is $(n, 2\epsilon)$-spanning. □

Lemma 15. *A base-point preserving map f of G lifts to $\mathcal{S}^r$ and lifting preserves the entropy. Hence we may restrict our attention to endomorphisms of products of full solenoids*

Proof. The product of full solenoids $\mathcal{S}^r$ is an inverse limit of finite coverings. To see this, note that there exists a chain of groups $\hat{G} = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_\infty =\subset \mathbb{Q}^r$ such that each quotient Γ_j/Γ_{j-1} is finite. Indeed, there exists such a chain from $\mathbb{Z}^r$ to $\mathbb{Q}^r$ and $\hat{G}$ is of rank r so it contains a copy of $\mathbb{Z}^r$. The result now follows from the previous lemma. □

$\mathcal{S}^r$ has a relatively easy topological structure:

Lemma 16. *Let $H \subset \mathcal{S}^r$ be a connected closed subgroup and $k = \dim H$. Then H is a factor, i.e., $S^r \cong H \times \mathcal{S}^{r-k}$ and $H \cong \mathcal{S}^k$.*

Proof. The Pontryagin dual $\hat{H}$ is a quotient group of $\mathbb{Q}^r$, hence it is divisible. It is torsion-free since H is connected. It follows that $\hat{H}$ itself is a rational vector space. In particular, there exists a right-inverse to the quotient map $\mathbb{Q}^r \to \hat{H}$ and $\hat{H}$ embeds as a factor into $\mathbb{Q}^r$ and we may write $\mathbb{Q}^r = G \times H$. □

We shall use this lemma in the following way. If $\mathcal{S}^k \subset \mathcal{S}^r$ is an invariant subgroup, then we can project the homotopy class of E from $\mathcal{S}^r$ to its restriction on $\mathcal{S}^k$. Since the projection $p\colon \mathcal{S}^r \to \mathcal{S}^k$ is a contraction under the product metric on $\mathcal{S}^k$ and its cofactor, this projection does not increase the entropy. In particular, if restricting E to $\mathcal{S}^k$ preserves the entropy, then we may replace $\mathcal{S}^n$ by the subgroup $\mathcal{S}^k$.

There exists an ascending chain of groups $\hat{G} = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_\infty = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{G}$ such that each quotient Γ_{j+1}/Γ_j is finite. Indeed, there exists such a chain from $\mathbb{Z}^r$ to $\mathbb{Q}^r$ and $\hat{G}$ is torsion-free of rank r so it contains a copy of $\mathbb{Z}^r$. The dual of the chain is a sequence of projections $G = G_0 \leftarrow G_1 \leftarrow \ldots \leftarrow G_\infty = \mathcal{S}^r$ such that each $G_j \leftarrow G_{j+1}$ is a finite covering. More specifically, each projection $G_j \leftarrow G_{j+1}$ is a local isometry. The inverse limit over the sequence of projections is isomorphic to $\mathcal{S}^r$.

Theorem 17. *To prove that endomorphisms minimizes entropy in their homotopy class, we may restrict our attention to surjective endomorphisms on $\mathcal{S}^r$. In particular, if r is finite, then we may restrict our attention to automorphisms.*

Proof. Let E be an arbitrary endomorphism on $\mathcal{S}^r$. The sequence $E^j(\mathcal{S}^r)$ is descending and $S = \bigcap E^j(\mathcal{S}^r)$ is a, possibly trivial, compact connected subgroup of $\mathcal{S}^r$. Indeed, S is the maximal invariant subgroup on which E is surjective. By the lemma we may write $\mathcal{S}^r = S' \times S$ and $\mathrm{h_{top}}(E)$ is the sum of the entropies of the restrictions to S and S'. All elements of S' converge to zero under iteration of E, so E has zero entropy on S'. Therefore, we may project the homotopy class of E from $\mathcal{S}^r$ to S. If $r < \infty$ and E is surjective, then the adjoint $\hat{E}$ is an injective linear map on a finite dimensional vector space $\mathbb{Q}^r$. Hence $\hat{E}$ is surjective as well. □

The natural extension of a surjective endomorphism E is an automorphism A and $\mathrm{h_{top}}(E) = \mathrm{h_{top}}(A)$ by Brown's inverse limit theorem. It follows that the calculation of the topological entropy of an endomorphism can always be reduced to that of an automorphism. For instance, Peters' entropy formula for automorphisms [12] extends to endomorphisms.[2]

[2]Peters' formula involves sums of inverse iterations. Forward iterations of the automorphism give the same result and in this way the formula extends.

4. Automorphisms of finite products $\mathcal{S}^n$

In this section $\dim G = n$ is finite. By Theorem 17 we may restrict our attention to automorphisms on $\mathcal{S}^n$. Following [10] we lift the automorphism from $\mathcal{S}^n$ to a locally compact group $\mathcal{Q}^n$ that is analogous to a covering space. Let $\mathbb{Q}_p$ be the p-adic completion of $\mathbb{Q}$ and let $\mathbb{Q}_\infty$ be the archimedean completion, i.e., the real line. $\mathcal{Q}$ is the restricted direct product $\prod' \mathbb{Q}_p$ with respect to $\mathbb{Z}_p$. It is known that $\mathcal{Q}$ is a locally compact abelian group, that the embedding of $\mathbb{Q}$ along the diagonal is discrete and that the quotient group $\mathcal{Q}/\mathbb{Q}$ is isomorphic to $\mathcal{S}$.

Lemma 18. *Let A be an automorphism of $\mathcal{S}^n$ for $n \in \mathbb{N}$ and let f be homotopic to $\mathcal{A}$. Then A lifts to an automorphism $\mathcal{A}$ of $\mathcal{Q}^n$ and f lifts to a uniformly continuous map that is homotopic to $\mathcal{A}$.*

Proof. Since $p\colon \mathcal{Q}^n \to \mathcal{S}^n$ is a fiber bundle with a discrete fiber, it has the homotopy lifting property, so we only have to lift A. The adjoint $\hat{A}$ extends from $\mathbb{Q}^n$ to the p-adic completion $\mathbb{Q}_p^n$ for all p, finite or infinite. The direct product of all these completions gives a continuous automorphism α of $\prod' \mathbb{Q}_p^n$ that acts as $\hat{A}$ on the subgroup $\mathbb{Q}^n \subset \prod' \mathbb{Q}_p^n$. Now $\mathcal{Q}^n$ is self-dual, so the adjoint of α is an automorphism $\mathcal{A}$ that acts as A on the quotient group $\mathcal{S}^n$. □

In fact, $\mathcal{Q}^n$ is the product of $\mathbb{R}^n$, which is the tangent space of the previous section, and an adic component $\prod'_{p \text{ finite}} \mathbb{Q}_p^n$, which is totally disconnected. In the adic component the lifting of f is identical to the lifting of A. We denote $\mathcal{Q}_P = \prod'_{p \text{ finite}} \mathbb{Q}_p$ so $\mathcal{Q} = \mathbb{R} \times \mathcal{Q}_P$. The only homomorphism between $\mathbb{R}$ and $\mathcal{Q}_P$ is the trivial one. To see this, observe that $\mathbb{R}$ is connected and contains no non-trivial bounded subgroups, whereas $\mathbb{Q}_P$ is totally disconnected and contains a union of bounded subgroups that is dense. It follows that the lifted automorphism $\mathcal{A}$ is a product map $\mathcal{A}_\mathbb{R} \times \mathcal{A}_P$ on $\mathbb{R}^n \times \mathcal{Q}_P^n$.

Lemma 19. *The lifting of f to $\mathbb{R}^n \times \mathcal{Q}_P^n$ in the previous lemma is of the form $(x, y) \mapsto (\mathcal{A}_\mathbb{R}(x) + b(x, y), \mathcal{A}_P(y))$ for a bounded map b that is invariant under translation over $\mathbb{Q}^n$.*

Proof. The lifting of f is of the form $(x, y) \mapsto (g(x, y), \mathcal{A}_P(y))$. In particular, the lifting of a null-homotopic map is of the form $(x, y) \mapsto (g(x, y), 0)$. The map $f - \mathcal{A}$ is null-homotopic under a homotopy H_t such that $H_1 = f - \mathcal{A}$ and H_0 is the null map. Let B_t be the lifting of H_t to $\mathcal{Q}^n$. For any $q \in \mathbb{Q}^n$ and any $z \in \mathcal{Q}^n$, $B_t(z + q) - B_t(z) \in \mathbb{Q}^n$ is invariant under t. Since H_0 is the null map, $B_t(z+q) - B_t(z) = 0$, so B_t is invariant under translation over $\mathbb{Q}^n$. Now put $b = B_1$. □

Denote the standard p-adic valuation on $\mathbb{Q}$ by $|\cdot|_p$. It is well known that the subset

$$\mathcal{Z}_0 = \{(x_p) \in \mathcal{Q}\colon |x_p|_p \leq 1 \text{ for all finite } p \text{ and } 0 \leq x_\infty < 1\} \subset \mathcal{Q}$$

is a fundamental domain of $\mathbb{Q}$. To see this, note that no two elements of $\mathcal{Z}_0$ differ by a translation over $\mathbb{Q}$, so it suffices to translate any $(x_p) \in \mathcal{Q}$ into $\mathcal{Z}_0$. Take $n = \lfloor x_\infty \rfloor$. There are at most finitely many primes such that $|x_p - n|_p > 1$. If p is such a prime, then $x_p - n = \frac{n_k}{p^k} + \ldots + \frac{n_1}{p} + x$ for representatives $n_i \in \{0, \ldots, p-1\}$ and some $x \in \mathcal{Z}_p$. Either $x_\infty - \frac{n_k}{p^k} \in [0, 1)$ or $x_\infty + 1 - \frac{n_k}{p^k} \in [0, 1)$. Hence by translation over $\frac{n_k}{p^k}$ or $1 - \frac{n_k}{p^k}$ we can reduces the norm at the p-th coordinate and this does not affect the value at the other finite primes. One of the two translations keeps the coordinate at infinity in the interval $[0, 1)$. In finitely many steps, x translates into $\mathbb{Z}_0$.

For $x = (x_p) \in \mathcal{Q}$, $||x|| = \sup\{|x_p|_p \colon p \text{ finite or infinite}\}$ is a translation invariant metric on $\mathcal{Q}$. It induces is a translation-invariant metric on $\mathcal{S} = \mathcal{Q}/\mathbb{Q}$ so the projection $p\colon \mathcal{Q}^n \to \mathcal{S}^n$ is a local isometry. The metric extends to $\mathcal{Q}^n$ by $||(x_1, \ldots, x_n)|| = \max\{||x_i||\colon i = 1, \ldots, n\}$.

Let $\mathcal{Z}$ be the closure of $\mathcal{Z}_0$. In particular $\mathcal{Z}^n$ is a compact set that projects almost $1-1$ onto $\mathcal{S}^n$.

Lemma 20. *Suppose that F is the lifting of f to $\mathcal{Q}^n$. Then $\mathrm{h_{top}}(f) = \mathrm{h_{top}}(F, \mathcal{Z}^n) = \mathrm{h_{top}}(F)$.*

Proof. The topological entropy $\mathrm{h_{top}}(F)$ of a uniformly continuous map F on a locally compact space X has been defined by Bowen [1] as the supremum of $\mathrm{h_{top}}(f, K)$ over all compact subsets $K \subset X$. Any compact subset can be covered by a finite number of translates $q + \mathcal{Z}^n$ for $q \in \mathbb{Q}^n$, so we need only consider the supremum of $\mathrm{h_{top}}(F, \bigcup q + \mathcal{Z}^n)$ over all finite unions. By Lemma 19 $F(z+q) = F(x) + \mathcal{A}_Q(q)$ for all $q \in \mathbb{Q}^n$. Suppose that $N \subset \mathcal{Z}^n$ is an (n, ε)-separated subset for F. Then the translation $q + N$ is (n, ε)-separated as well, since

$$||F^k(q+x) - F^k(q+y)|| = ||F^k(x) + \mathcal{A}_Q^n(q) - F^n(y) - \mathcal{A}_Q^n(q)|| = ||F^k(n) - F^k(m)||.$$

It follows that $\mathrm{h_{top}}(F, q + \mathcal{Z}^n)$ is the same for all $q \in \mathbb{Q}^n$. Hence $\mathrm{h_{top}}(F, \bigcup q + \mathcal{Z}^n) = \mathrm{h_{top}}(F, \mathcal{Z}^n)$.

The projection $\mathcal{Q}^n \to \mathcal{S}^n$ is a local isometry and the elements of $\mathbb{Q}^n$ have mutual distance ≥ 1. If $\mathrm{diam}(V) < \varepsilon$ and if ε is sufficiently small, then V has a pre-image $\mathbb{Q}^n + V \subset \mathcal{Q}^n$ of translates that have mutual distance $> 1 - 2\varepsilon$. By the uniform continuity of F there exists an $\varepsilon_0 > 0$ such that $||F(x) - F(y)|| < 1 - 2\varepsilon_0$ if $||x - y|| < \varepsilon_0$ for all $x, y \in \mathcal{S}^n$. We claim that if $N \subset \mathcal{S}^n$ is (n, ε)-spanning, then $p^{-1}(N) \cap \mathcal{Z}^n$ is also (n, ε)-spanning. To see this, note that for all any $x \in \mathcal{Z}^n$, there exists a $y \in p^{-1}(N) \cap \mathcal{Z}^n$ such that the iterates $F^k(x), F^k(y)$ remain ε_0 close modulo $\mathbb{Q}^n$. Now if x, y do not remain ε_0 close there is a first iterate $k \leq n$ such that $||F^k(x) - F^k(y)|| > \varepsilon_0$. Since $||f^k(p(x)) - f^k(p(y))|| < \varepsilon_0$ in $\mathcal{S}^n$ we must have that $||F^k(x) - F^k(y)|| > 1 - 2\varepsilon_0$, but this is absurd. It follows that $\mathrm{h_{top}}(F, \mathcal{Z}^n) \leq \mathrm{h_{top}}(f)$. The opposite inequality is obvious. □

A finite dimensional real vector space V is normed by $||v|| = ||(v_1, \ldots, v_n)|| = \sup\{|v_i| \colon 1 \leq i \leq n\}$. where $|v_i|$ denotes the valuation of $v_i \in k$. If A is a linear map on a V then we say that two different eigenvalues λ and μ are *distinct* if they are not complex conjugate. If $|\lambda| > 1$ then we say that the eigenvalue is *unstable.*

Lemma 21. *Let A be a linear map on a finite-dimensional real vector space V with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Then V decomposes into a direct sum of subspaces V_j for $j = 1, \ldots, m$. If λ_j is unstable then $||A(v)||/||v|| \to |\lambda_j|$ if $v \in V_j$ and $||v|| \to \infty$. If λ_j is not unstable then $||A^n(v)|| \leq n||v||$ for all $v \in V_j$.*

Proof. According to the Jordan decomposition theorem a complex vector space W decomposes as a direct sum of spaces W_j such that A restricted to W_j acts as $(x_1, x_2 \ldots, x_m) \mapsto (\lambda_j x_1, x_1 + \lambda_j x_2, \ldots, x_{m-1} + \lambda_j x_m)$. V is a real vector space, so its complexification $W = \mathbb{C} \otimes_{\mathbb{R}} V$ is a direct sum of spaces W_j. Take $V_j = W_j \bigcap V$ if λ_j is real and take $V_j = \left(W_j + \overline{W}_j\right) \bigcap V$ if λ_j is complex. □

We now consider the analogous case in which V is a finite-dimensional vector space over a separable field k that is complete with respect to a non-archimedean valuation. We norm V in the same way. A linear map A on V has eigenvalues in a finite extension $K \supset k$. By extending K if necessary, we may assume that K is a finite Galois extension with group $\mathfrak{g} = \mathrm{Gal}(K, k)$. Again, we say that two eigenvalues λ, λ' of A are *distinct* if there does not exist a $\sigma \in \mathfrak{g}$ such that $\lambda' = \sigma(\lambda)$.

Lemma 22. *If V is a finite-dimensional vector space over k and A is a linear map on V with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, then V decomposes as a direct sum of V_j such that $||A(v_j)|| = |\lambda_j| \cdot ||v_j||$ if $v_j \in V_j$ and λ_j is unstable. If λ_j is not unstable then $||A(v_j)|| \leq ||v_j||$*

Proof. The linear map A extends to $V(K) = K \otimes_k V$ and $V(K)$ decomposes as a direct sum of Jordan blocks $V_j(K)$ on which A acts as

$$v = (x_1, x_2 \ldots, x_m) \mapsto (\lambda x_1, x_1 + \lambda x_2, \ldots, x_{m-1} + \lambda x_m).$$

By our definition of the norm $||A(v)|| = \max\{|\lambda x_i + x_{i-1}| \colon 1 \leq i \leq m\}$. Since the valuation is non-archimedean this is bounded by $\max(|\lambda x_i|, |x_{i-1}|)$. If $|\lambda| \leq 1$, then it follows that $||A(v)|| \leq ||v||$. If $|\lambda| > 1$, then $||A(v)|| = |\lambda| \cdot |x_k|$ if the maximum of $|x_i|$ is attained at the k-th coordinate. The action of $\mathfrak{g}$ on K extends to $V(K)$ by $\sigma(\kappa \otimes v) = \sigma(\kappa) \otimes v$. We can retrieve V as the subspace of elements of $V(K)^{\mathfrak{g}}$ that are invariant under $\mathfrak{g}$. Define V_j as $V_j(K)^{\mathfrak{g}}$. Since the valuation of K is invariant under the action of $\mathfrak{g}$, so is the norm on V. It follows that V_j has the desired properties. □

Theorem 23. *Suppose that* $\dim G < \infty$ *and that* f *is a base-point preserving continuous map on* G. *Let* E *be the endomorphism that is homotopic to* f. *Then* $\mathrm{h_{top}}(E) \leq \mathrm{h_{top}}(f)$.

Proof. By the previous lemmas we may lift f all the way to $\mathbb{R}^n \times \mathcal{Q}_P^n$ where it has the form $F(x, y) = (\mathcal{A}_R(x) + b(x, y), \mathcal{A}_P(y))$. The automorphism $\mathcal{A}_R$ is the archimedean completion of a $\mathbb{Q}$-vector space automorphism and $\mathcal{A}_P$ is the product of its non-archimedean completions. For all but finitely many primes p the eigenvalues have value $|\lambda_1|_p = \ldots = |\lambda_m|_p = 1$. For these primes the completion acts as an isometry and this does not contribute to the entropy. So we may consider $\mathcal{A}_P$ as a finite product.

Suppose that p is a finite prime for which some of the λ_i are unstable. Denote the p-adic completion of the adjoint by $\mathcal{A}_p$. According to the previous lemma there exists a decomposition into invariant subspaces $\mathbb{Q}_p^n = U_p \oplus S_p$ such that $\mathcal{A}_p$ expands the Haar measure on U_p by the product of all $|\lambda_i|_p > 1$, while it is non-expansive on S. Since $\mathcal{A}_P$ is a finite product of all these primes $\mathcal{A}_p$ it acts as a product map on $U_P \oplus S_P$ and it expands U by the product of all $|\lambda_i|_p > 1$ over all finite primes.

The archimedean component decomposes into $\mathcal{A}_R$ invariant subspaces V_j such that for unstable eigenvalues $||\mathcal{A}_R(v_j)|| > (|\lambda_j| - \varepsilon)||v||$ if $||v||$ is sufficiently large. Let U_∞ be the direct sum of all the unstable spaces V_j. Then for sufficiently large R the volume of a ball $B(0, R) \subset U_\infty$ under $\mathcal{A}_R$ grows by a factor that is larger than the product over all unstable $|\lambda_j| - \varepsilon$. Since b is bounded, the same is true for the projection of the lifting $F(B_R)$ along the sum of the stable subspaces onto U. By the same growth argument as in [2] the entropy of F is bounded below by the sum of $\log |\lambda_i|_p$ over all i and all primes, finite or infinite, such that $|\lambda_i|_p > 1$. This is Yuzvinskii's formula as interpreted in [10]. □

We have given our result in terms of endomorphisms. We could have also given it in terms of affine transformations, by the same proof:

Theorem 24. *Suppose that* $\dim G < \infty$ *and that* f *is a continuous map on* G. *Let* T *be the affine map* $x \mapsto f(0) + E(x)$ *that is homotopic to* f. *Then* $\mathrm{h_{top}}(T) \leq \mathrm{h_{top}}(f)$.

5. Endomorphisms of infinite products $\mathcal{S}^\infty$

In this section we consider endomorphisms on infinite-dimensional groups, so $\dim G = \infty$. By the results above, we may assume that the endomorphism is surjective and that the group is a direct product of full solenoids $\mathcal{S}^\infty$. Its Pontryagin dual is a direct sum $\mathbb{Q}^\infty$.

Lemma 25. *Let* E *be an endomorphism on* G. *Suppose that the Pontryagin dual* $\hat{G}$ *is a torsion module under the* $\mathbb{Z}[x]$*-action induced by the adjoint* $\hat{E}$. *Then* E *minimizes the entropy in its homotopy class.*

Proof. Note that if $\hat{G}$ is a torsion module, then so is $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{G}$. Therefore we may assume that E is surjective and that G is an infinite product of full solenoids. Since $\mathbb{Q}^\infty$ is a torsion module, it is the union of an increasing chain of finite-dimensional submodules $\mathbb{Q}^n \subset \mathbb{Q}^r$. By Lemma 16 all of these $\mathbb{Q}^n$ are factors. So $\mathcal{S}^\infty$ is a union of an increasing chain of finite-dimensional invariant subgroups S^n, all of which are factors. If f is homotopic to E, then we know that its projection $p_n \circ f$ onto a finite-dimensional invariant subgroup S^n has entropy $\mathrm{h_{top}}(p_n \circ f) \geq \mathrm{h_{top}}(E, S^n)$. The result now follows from $\mathrm{h_{top}}(E) = \lim_{n\to\infty} \mathrm{h_{top}}(E, S^n)$ and $\mathrm{h_{top}}(f) \geq \mathrm{h_{top}}(p \circ f)$. □

The case of a torsion-free $\mathbb{Z}[x]$–module $\mathbb{Q}^\infty$ remains, since $\mathbb{Q}^\infty$ decomposes into a sum of a torsion module and a torsion-free submodule. One example of such an endomorphism is the one-sided shift on $\mathcal{S}^{\mathbb{N}}$. In fact, this is the only relevant example:

Lemma 26. *Suppose that E is a surjective endomorphism on $\mathcal{S}^\infty$ and that the adjoint action on $\mathbb{Q}^\infty$ is torsion-free. Then the one-sided shift σ on $\mathcal{S}^{\mathbb{N}}$ is a quotient of E. Furthermore, there exists an embedding $\iota\colon \mathcal{S}^{\mathbb{N}} \hookrightarrow \mathcal{S}^\infty$ such that $p \circ \iota = id$. If f is a map on $\mathcal{S}^\infty$ then $\mathrm{h_{top}}(p \circ f \circ \iota) \leq \mathrm{h_{top}}(f)$.*

Proof. Take any nonzero element $m \in \mathbb{Q}^\infty$ and let $M = \mathbb{Q}[x]\cdot m$ be the submodule generated by m. In particular $M \cong \mathbb{Q}^{\mathbb{N}}$ and x acts on M as the shift $(q_i)_{i=1}^\infty \mapsto (q_{i-1})_{i=1}^\infty$ with $q_0 = 0$. Hence the dual group of M is $\mathcal{S}^{\mathbb{N}}$, which is a quotient group of $\mathcal{S}^\infty$, and E acts on it as the shift $(s_i) \mapsto (s_{i+1})$. M embeds as a factor group of $\mathbb{Q}^\infty$, so $\mathcal{S}^{\mathbb{N}}$ embeds $\mathcal{S}^\infty$. If we endow $\mathcal{S}^\infty$ with the product metric on $\mathcal{S}^{\mathbb{N}}$ and its cofactor, then the projection $p\colon \mathcal{S}^\infty \to \mathcal{S}^{\mathbb{N}}$ is a contraction and so it is entropy-decreasing. □

Theorem 27. *Suppose that f is a base-point preserving continuous map on G. Let E be the endomorphism that is homotopic to f. Then $\mathrm{h_{top}}(E) \leq \mathrm{h_{top}}(f)$.*

Proof. By the results above, it only remains to show that this is true if the endomorphism E is the one-sided shift σ. The entropy of σ is infinite. So, we have to derive the pathological result that any f in the homotopy class of σ has infinite entropy.
The solenoid $\mathcal{S}$ is covered by $\mathcal{Q} = \mathbb{R} \times \mathcal{Q}_P$, so $\mathcal{S}^{\mathbb{N}}$ is covered by $\mathcal{Q}^{\mathbb{N}} = \mathbb{R}^{\mathbb{N}} \times \mathcal{Q}_P^{\mathbb{N}}$. The component $\mathbb{R}^{\mathbb{N}}$ is the tangent space that we encountered in the first section and $\mathcal{Q}_P^{\mathbb{N}}$ is the adic component. The shift on $\mathcal{S}^{\mathbb{N}}$ lifts to the shift on $\mathcal{Q}^{\mathbb{N}}$, hence f lifts to $\mathcal{Q}^{\mathbb{N}}$ by the homotopy lifting property of the tangent space and on the adic component, the homotopy is trivial. We denote the lifting by F. It acts as a shift on the adic component.
The compact subset

$$\mathcal{Z}^{\mathbb{N}} = \{(x, y)\colon x \in [0,1]^{\mathbb{N}},\ y \in \mathcal{Z}_P^{\mathbb{N}}\}$$

projects almost $1-1$ onto $\mathcal{S}^{\mathbb{N}}$. Let $||\cdot||$ be the metric on $\mathcal{Q}$, then $\mathbb{Q}^{\mathbb{N}}$ is metrizable by the product metric $||(q_i) - (r_i)||_\infty = \sum_{i\in\mathbb{N}} \frac{||q_i - r_i||}{2^i}$. The projection $p\colon \mathcal{Z}^{\mathbb{N}} \to \mathcal{S}^{\mathbb{N}}$ is a local isometry. Hence, by the same argument as in Lemma 20 we find that $\mathrm{h_{top}}(F, \mathcal{Z}^{\mathbb{N}}) = \mathrm{h_{top}}(f)$. Now F picks up infinite entropy on the adic component $\mathcal{Z}_P^{\mathbb{N}}$, since it acts as the shift on this component and each finite prime contributes $\log p$. □

6. Final remark

We only consider compact abelian groups G that are connected. If G is not connected, then there are homotopy classes that do not contain an endomorphism (nor an affine map), so our main result is no longer valid. However, if a homotopy class does contain an endomorphism E, then it minimizes entropy. To prove this one verifies that the entropy of E is equal to the sum of $\mathrm{h_{top}}(E, G_0)$, where G_0 denotes the component of 0, and $\mathrm{h_{top}}(E, G/G_0)$. A base-point preserving map f that is homotopic to E induces the same action on G/G_0 and this can be used to show that $\mathrm{h_{top}}(f) \geq \mathrm{h_{top}}(E)$.

References

[1] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414; *Erratum*, ibid, **181** (1973), pp. 509-510

[2] R. Bowen, *Entropy and the fundamental group,* Lecture Notes in Math. 668, Springer, Berlin (1978), 21–29.

[3] J.R. Brown, *Ergodic theory and topological dynamics*, Academic Press, 1976.

[4] G. Everest and T. Ward, *Heights of polynomials and entropy in algebraic dynamics*, Universitext, Springer, New York, 1999.

[5] J. Franks, *Anosov diffeomorphisms*, in Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) Amer. Math. Soc. (1970), 61–93.

[6] J. Franks and M. Misiurewicz, *Topological methods in dynamics*, in Handbook of Dynamical Systems 1A, eds. B. Hasselblat and A. Katok, North-Holland, Amsterdam, 2002.

[7] E.A. Gorin, *A function-algebra variant of the Bohr-van Kampen theorem*, (Russian) Mat. Sb. (N.S.) **82** (124), 1970, 260–272.

[8] J. Kwapisz, *Homotopy and dynamics for homeomorphisms of solenoids and Knaster continua,* Fund. Math. 168 (2001), no. 3, 251–278.

[9] V. Ja. Lin, *Semi-invariant integration with values in a group, and some applications of it,* (Russian) Mat. Sb. (N.S.) **82** (124), 1970, 233–259.

[10] D. Lind and T. Ward, *Automorphisms of solenoids and p-adic entropy,* Ergodic Theory Dynam. Systems 8 (1988), no. 3, 411–419.

[11] M. Misiurewicz and F. Przytycki, *Entropy conjecture for tori*, Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys. 25 (1977), no. 6, 575–578.

[12] J. Peters, *Entropy on discrete abelian groups*, Adv. Math. **33** (1979), 1–13.

[13] W. Scheffer, *Maps between topological groups that are homotopic to homomorphisms,* Proc. Amer. Math. Soc. 33 (1972), 562–567.

[14] M. Shub, *Dynamical systems, filtrations and entropy*, Bull. Amer. Math. Soc. **80** (1974), 27–41.

[15] P. Walters, *An introduction to ergodic theory,* Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

[16] S.A. Yuzvinskii, *Calculation of the entropy of a group-endomorphism*, Sibirsk. Mat. Z. **8** (1967), 230–239.

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