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On a Homoclinic Group that is not Isomorphic to the Character Group *

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We exhibit an example of Hendrik Lenstra of an expansive automorphism on a compact connected abelian group K such that its homoclinic group is not isomorphic to the Pontryagin dual $\hat{K}$.

Key Words: expansive automorphism, homoclinic group, Pontryagin duality.

1. INTRODUCTION

We consider the action of a continuous automorphism h on a locally compact abelian group K. In most cases K will be compact. The automorphism is *expansive* if there exists a neighborhood $0 \in U \subset K$ of the unit element such that

$$\bigcap_{n \in \mathbf{Z}} h^n(U) = \{0\},$$

in which case U is a called a *separating* neighborhood. So an automorphism is expansive if and only if it admits a separating neighborhood. The subgroup

$$H = \{x \in X \colon \lim_{|n| \to \infty} h^n(x) = 0\}.$$

is called the *homoclinic group.*

An element $x \in H$ is called a *fundamental homoclinic point* if its orbit generates H. For a compact K Lind and Schmidt [4] showed that if such a fundamental homoclinic point exists, then the Pontryagin dual $\hat{H}$ is isomorphic to K. Even more so, in this case the dynamical system $(\hat{H}, \hat{h})$ is dynamically equivalent to (K, h); i.e., there exists an isomorphism between $\hat{H}$ and K that conjugates $\hat{h}$ to h.

It is known that $\hat{H}$ and K need not be dynamically equivalent if H has no fundamental homoclinic point [3]. Even more so, if K is disconnected, then it is easy to construct an example such that $\hat{H}$ and K are not algebraically equivalent: let C be compact with expansive automorphism a with a fundamental homoclinic point and let F be finite, then $h = a \times \mathrm{Id}$ is expansive on $C \times F$, but the homoclinic group H is contained in the component of the identity. So, $\hat{H}$ is isomorphic to C instead of $C \times F$. For a connected group K it is much more difficult to construct an expansive automorphism such that $\hat{H}$ is not isomorphic to K. We present such an example in this paper. The example was kindly provided by Hendrik Lenstra through private communication and is used with his permission.

2. HYPERBOLIC AUTOMORPHISMS

LEMMA 1. *Suppose that (X, h) is an expansive automorphism on a locally compact group and that $L \subset X$ is a discrete invariant subgroup. Then the induced automorphism on X/L is expansive.*

Proof. Let U be a separating neighborhood of h. By choosing U sufficiently small, we may suppose that all translates $U + L$ are disjoint and that both $h(U)$ and $h^{-1}(U)$ intersect $U + L$ in U only. So if a point leaves U under iteration of h, then it leaves $U + L$. In other words, U projects onto a separating neighborhood of X/L. ▮

An automorphism is *contracting* if all points are forward asymptotic to 0. Suppose that C is compact and that V is a neighborhood of 0. Since h is contracting C can be covered by finitely many $h^{-n}(V)$. In particular $h^n(C) \subset V$ for large enough n. Hence, $\{0\}$ is the only invariant compact set. This implies that $\bigcap_{n \in \mathbf{Z}} h^n(U) = \{0\}$,and so contracting automorphisms are expansive. An automorphism is *expanding* if all points are backward asymptotic to 0. A product of an expanding and a contracting system is *hyperbolic.* Both are expansive.

LEMMA 2. *Suppose that (X, h) is hyperbolic and that $L \subset X$ is an invariant lattice; i.e., L is discrete and co-compact. Then the homoclinic group H of the induced automorphism on X/L is isomorphic to L.*

Proof. Let $X = V \times W$ with V expanding and W contracting. Let $f: X \to X/L$ be the composition $(v, w) \mapsto (0, w) \mapsto (0, w) \bmod L$. Suppose that $(v, w) \in L$. Then $f(v, w) = (0, w) \bmod L$ is forward asymptotic to $(0, 0)$ in X/L and $f(v, w) = (-v, 0) \bmod L$ is backward asymptotic to $(0, 0)$ in X/L. It follows that $f(L) \subset H$. We prove that $f: L \to H$ is in fact an isomorphism.

L is invariant and discrete, so non-zero elements of L do not converge to 0. Hence $L \cap \{0\} \times W = \{(0, 0)\}$, which implies that $f(v, w) \in L$ only if $w = 0$. By the same argument $L \cap V \times \{0\} = \{(0, 0)\}$ and $(v, 0) \in L$ only if $v = 0$. It follows that $f: L \to H$ is injective.

Let U be a compact neighborhood of $(0, 0)$ such that all translates $U + L$ are disjoint and such that $h(U)$ intersects $U + L$ in U only. Suppose that (v, w) is homoclinic in X/L. Then there exists an N such that $h^n(v, w) \in U + L$ for $n \geq N$. By translating (v, w) over L we may assume that $h^N(v, w) \in U$. Since $h(U)$ intersects $U + L$ in U only, it follows that $h^n(v, w) \in U$ for $n \geq N$. We see that the forward orbit of $(v, 0)$ remains within a compact set and so does its backward orbit since h is contracting on $V \times \{0\}$. By the observation on contracting maps above, we find that $v = 0$. So any homoclinic point in X/L is the image of some $(0, w)$. By symmetry, it is the image of some $(v, 0)$ as well. This implies that $(-v, w) \in L$, and since $f(-v, w) = f(0, w)$ we find that f is surjective. ∎

These lemmas show that if $L \subset X$ is an invariant lattice, then the factor $(X/L, h)$ is expansive with homoclinic group isomorphic to L. In fact, there is a much stronger result in [3]: any expansive (K, h) on a compact group is a factor of a hyperbolic automorphism on a self-dual locally compact group X. The self-duality of X can be expressed by a pairing to the circle group $\varphi: X \times X \to \mathbf{T}$. The dual group of K is isomorphic to the annihilator of L with respect to φ.

3. LENSTRA'S EXAMPLE

We want to construct an expansive automorphism on a compact abelian group such that the homoclinic group is not isomorphic to the Pontryagin dual. So we have to find a lattice L that is not isomorphic to its annihilator.

The self-duality of $\mathbf{R}$ can be expressed by the pairing $\pi_0(x, y) = xy \bmod 1$. Any character on $\mathbf{R}$ is equivalent to $x \mapsto \pi_0(x, y)$ for some $y \in \mathbf{R}$. A similar pairing exists for other locally compact rings. For a natural number g, a g-adic number is a one-sided formal power series $\sum_{n \geq k} a_n g^n$ with coefficients $a_n \in \{0, 1, \ldots, g-1\}$, as described for instance in [5]. If $k \geq 0$ then the series is a g-adic integer. Let $\mathbf{Q}_g$ denote the ring of g-adic numbers and $\mathbf{Z}_g$ its subring of g-adic integers. Since $\mathbf{Q}_g/\mathbf{Z}_g \cong \mathbf{Z}[\frac{1}{g}]/\mathbf{Z}$, it embeds into the circle group $\mathbf{T}$. The g-adic numbers are self-dual with pairing $\pi_g(x, y) = xy \bmod \mathbf{Z}_g$.

Note that if X is self-dual with pairing φ and Y is self-dual with pairing ψ, then $X \times Y$ is self-dual with pairing $\varphi + \psi$. The following lemma can be found in [1, page 510].

Lemma 3. *Let* $\mathbf{Z}[\frac{1}{g}] \subset \mathbf{R} \times \mathbf{Q}_g$ *be canonically embedded along the diagonal. Its annihilator under the pairing* $\pi_0 + \pi_g$ *is* $A_g = \left\{(x, -x) \colon x \in \mathbf{Z}[\frac{1}{g}]\right\}$, *which is isomorphic to* $\mathbf{Z}[\frac{1}{g}]$.

Lemma 3 shows that $\mathbf{Z}[\frac{1}{g}] \subset \mathbf{R} \times \mathbf{Q}_g$ is isomorphic to its annihilator. So $\mathbf{Z}[\frac{1}{g}] \times \mathbf{Z}[\frac{1}{h}] \subset \mathbf{R} \times \mathbf{Q}_g \times \mathbf{R} \times \mathbf{Q}_h$ is isomorphic to its annihilator as well. However, it is possible to find an $L \subset \mathbf{Z}[\frac{1}{g}] \times \mathbf{Z}[\frac{1}{h}]$ that is not isomorphic to its own annihilator and that is invariant under a hyperbolic automorphism.

If d is coprime to g, then g is invertible in $\mathbf{Z}/d\mathbf{Z}$ and we have a natural homomorphism $\mathbf{Z}[\frac{1}{g}] \to \mathbf{Z}/d\mathbf{Z}$. We say that $x = j \bmod d$ if this is the image of x under the natural homomorphism. So if d is coprime to gh, then for any integer a the following group is well defined:

$$L_a = \{(x, y) \in \mathbf{Z}[1/g] \times \mathbf{Z}[1/h] \colon\ x = ay \bmod d\}.$$

Lemma 4. *If* $x = j \bmod d$ *then the pairing of* (x, x) *and* $(\frac{1}{d}, -\frac{1}{d})$ *in* $\mathbf{R} \times \mathbf{Q}_g$ *is equal to* $\frac{j}{d} \bmod \mathbf{Z}$.

Proof. There exists $y \in \mathbf{Z}[\frac{1}{g}]$ such that $x = j + dy$, so $\frac{x}{d} = \frac{j}{d} + y$. Note that $\frac{j}{d} \in \mathbf{Z}_g$, so $\frac{x}{d} = y \bmod \mathbf{Z}_g$ and we find that $\pi_g(\frac{1}{d}, x) = y \bmod \mathbf{Z}$. Therefore $\pi_0(\frac{1}{d}, x) + \pi_g(-\frac{1}{d}, x) = \frac{x}{d} - y = \frac{j}{d} \bmod \mathbf{Z}$. ∎

Lemma 5. *For any integer* a *that is coprime to* d, *the annihilator of* $L_a \subset \mathbf{R} \times \mathbf{Q}_g \times \mathbf{R} \times \mathbf{Q}_h$ *is isomorphic to* L_b *for* $ab = -1 \bmod d$.

Proof. Let A be the annihilator of $\mathbf{Z}[\frac{1}{g}] \times \mathbf{Z}[\frac{1}{h}]$. Let A_a be the annihilator of L_a. Then $A_a \supset A$ has index d since $L_a \subset \mathbf{Z}[\frac{1}{g}] \times \mathbf{Z}[\frac{1}{h}]$ has index d. Lemma 3 implies that

$$A = \{(x, -x, y, -y) \colon x \in \mathbf{Z}[1/g],\ y \in \mathbf{Z}[1/h]\} \subset \mathbf{R} \times \mathbf{Q}_g \times \mathbf{R} \times \mathbf{Q}_h.$$

Lemma 4 implies that $w = (\frac{b}{d}, -\frac{b}{d}, \frac{1}{d}, -\frac{1}{d})$ annihilates L_a. Now w and A generate the group

$$\{(x, -x, y, -y) \colon dx \in \mathbf{Z}[1/g],\ dy \in \mathbf{Z}[1/h],\ dx = b(dy) \bmod d\},$$

which contains A as a subgroup of index d. So this group is equal to A_a. Upon dividing the coordinates by d, we find that A_a is isomorphic to L_b. ▮

LEMMA 6. *Suppose that there exist primes p, q such that $p \mid g$ and $q \mid h$ but p does not divide h and q does not divide g. Let $J \subset (\mathbf{Z}/d\mathbf{Z})^*$ be generated by -1 and all the primes that divide gh. Then $L_a \cong L_b$ if and only if $a = b \bmod J$.*

Proof. For any prime $r \mid g$ the map $(x, y) \mapsto (rx, y)$ induces an isomorphism between L_{ra} and L_a and for any prime $s \mid h$ the map $(x, y) \mapsto (x, sy)$ induces an isomorphism between L_a and L_{sa}. By transitivity we find that $L_a \cong L_b$ if $a = b \bmod J$.

For both L_a and L_b the characteristic subgroup of elements of infinite p-height is equal to $d\mathbf{Z}[\frac{1}{g}] \times \{0\}$. Similarly, the characteristic subgroup of elements of infinite q-height is $\{0\} \times d\mathbf{Z}[\frac{1}{h}]$. Since $\mathbf{Z}[\frac{1}{g}]$ and $\mathbf{Z}[\frac{1}{h}]$ are torsion-free groups of rank 1, any homomorphism between L_a and L_b is of the form $(x, y) \mapsto (ux + wy, vx + zy)$ for rational numbers u, v, w, z. Since isomorphisms have to preserve characteristic groups, we conclude that $w = z = 0$ and that $u \in \mathbf{Z}[\frac{1}{g}]$ and $v \in \mathbf{Z}[\frac{1}{h}]$ are units. In particular, the primes that divide uv divide gh, so $a = b \bmod J$. ▮

Suppose that $u \in \mathbf{Z}[\frac{1}{g}]$ and $v \in \mathbf{Z}[\frac{1}{h}]$ are units and that $u \bmod d = v \bmod d$ in $(\mathbf{Z}/d\mathbf{Z})^*$. Then L_a is invariant under the transformation $(x, y) \mapsto (ux, vy)$, which is hyperbolic if every prime that divides g divides u and every prime that divides h divides v. The induced transformation on the cokernel of L_a in $\mathbf{R} \times \mathbf{Q}_g \times \mathbf{R} \times \mathbf{Q}_h$ is an expansive automorphism with homoclinic group isomorphic to L_a and dual group isomorphic to $L_{-1/a}$. The previous lemmas imply that we get an expansive automorphism for which the dual group is not isomorphic to the homoclinic group if we choose: $g = u = 13, h = -v = 29, d = 7, a = 2$.

EXAMPLE 7. The map on $((\mathbf{R} \times \mathbf{Q}_{13}) \times (\mathbf{R} \times \mathbf{Q}_{29}))/L_2$ induced by $(x, y) \mapsto (13x, -29y)$ has homoclinic group isomorphic to L_2, which is not isomorphic to the dual group L_3.

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