THE DYNAMICS OF TILING SPACES

ALEX CLARK

Abstract. We examine some questions on the dynamics of tiling spaces, providing a brief survey explaining the less familiar of these questions.

Introduction

In order to provide a topological structure for spaces of tilings, one uses a metric defined in analogy with a commonly used metric on symbolic spaces. Given a finite alphabet $\mathcal{A}$, one can define a metric on $\mathcal{A}^{\mathbb{Z}^d}$ by setting

$$d\left((x_{\mathbf{n}}),(y_{\mathbf{n}})\right)=\frac{1}{1+\min\{\ |\mathbf{n}|:x_{|\mathbf{n}|}\neq y_{|\mathbf{n}|}\}}\ ,$$

where $|\mathbf{n}|$ denotes the norm of $\mathbf{n}\in\mathbb{Z}^d$. With this metric $\mathcal{A}^{\mathbb{Z}^d}$ is a Cantor set that supports the continuous *shift* dynamical system: $\mathbb{Z}^d$ acts continuously by translation on index

$$\mathbf{m}\ .\ (x_{\mathbf{n}})=(x_{\mathbf{m}+\mathbf{n}})$$

A *subshift* is the restriction of this action to a closed, shift-invariant subset.

A *tile* in $\mathbb{R}^d$ is subset of $\mathbb{R}^d$ homeomorphic to a closed d-dimensional ball, and a *tiling* of $\mathbb{R}^d$ is a covering by tiles that only intersect in their boundary. Given a finite set of polyhedral tiles $\mathcal{P}$, consider the collection $X_{\mathcal{P}}$ of all tilings of $\mathbb{R}^d$ by elements of $\mathcal{P}$ that meet only full edge to full edge (provided such exist). Then for two tiles T and T'

$$d\left(T,T'\right)=\inf\left\{\left\{\frac{1}{\sqrt{2}}\right\}\cup\left\{\varepsilon>0:\begin{array}{l}\text{For some } u\in\mathbb{R}^d \text{ with } |u|<\varepsilon,\\ T+u \text{ and } T' \text{ agree on } B\left(\mathbf{0},\frac{1}{\varepsilon}\right)\end{array}\right\}\right\}$$

This metric provides $X_{\mathcal{P}}$ a compact topology with respect to which the translation action $u\,.\,T\ =\ T-u$ is continuous. A *tiling space* is a closed subset of $X_{\mathcal{P}}$ that is invariant under this action. We shall focus on the dynamics of a particular type of tiling space: the tiling space $\mathcal{T}$ of a single tiling T, formed by taking the closure of the orbit of T. For a general survey, see, e.g., [25].

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Topological Rigidity

An especially well-behaved class of tilings are the self-similar tilings, see, e.g., [25, Section 4]. If T and T' are self-similar tilings with homeomorphic tiling spaces $\mathcal{T}$ and $\mathcal{T}'$, one should not expect the typical homeomorphism to be a conjugacy. But the structure of such tilings is so rigid, one might expect that this could almost be so. By considering a tiling T and the tiling T' obtained from T by inflating all tiles by a factor $\lambda > 1$, one obtains homeomorphic tiling spaces for which (in general) there can be no conjugacy of actions in the strictest sense.

Definition $\mathbb{R}^d$ actions on X and Y are *linearly equivalent* if there is a homeomorphism $h : X \to Y$ and a linear map L of $\mathbb{R}^d$ satisfying

$$h\,(u\,.\,x) \;=\; L(u)\,.\,h(x)$$

In general, not all homeomorphic tiling spaces are linearly equivalent, see [10, 11].

? 1001 **Question.** *If T and T' are self-similar tilings with homeomorphic tiling spaces $\mathcal{T}$ and $\mathcal{T}'$, are $\mathcal{T}$ and $\mathcal{T}'$ linearly equivalent?*

Should the answer turn out to be negative, one might modify the question so as to apply to other classes of tilings; for example, to tilings with pure point discrete spectrum. Should the answer turn out to be positive, one may then ask whether any homeomorphism $\mathcal{T} \to \mathcal{T}'$ is homotopic to a homeomorphism that induces a linear equivalence.

The Topological Structure of Tiling Spaces

While any compact metric space is homeomorphic to the inverse limit of a sequence of compact polyhedra with PL-bonding maps (see, e.g., [20, Ch I, §5.2]), the structure of tiling spaces leads to some especially natural inverse sequences. For substitution tiling spaces see [1], and for more general spaces see [26, 9, 8, 16]. There has been extensive use of cohomology in the study tiling spaces, and in many cases well-chosen inverse sequences allow one to calculate the cohomology, see [1]. The occurrence of torsion in cohomology is still a bit mysterious, see [16].

However, much less is known about the role of homotopy and shape theory in tiling spaces. A well-studied class of sequences are the Sturmian sequences (see e.g. [13]). Tiling spaces derived from Sturmian sequences (in other contexts known as Denjoy continua [7]) are homeomorphic to the inverse limit of a sequence $\{K_i, f_i\}$, where each K_i is a wedge of two circles and each f_i induces an isomorphism of fundamental groups. It follows that these tiling spaces have the shape of the wedge of two circles [20]. A natural generalization of this type of tiling space are the quasiperiodic tiling spaces formed by the cut and project technique, including the Penrose tiling space, see, e.g., [25, Section 8].

? 1002 **Question.** *Does the Penrose tiling space have the shape of a polyhedron?*

? 1003 **Question.** *Is there a natural class $\mathcal{Q}$ of quasiperiodic tiling spaces (metrically equivalent to toral Kronecker actions) so that each $\mathcal{T} \in \mathcal{Q}$ has the shape of a polyhedron?*

An answer to these questions could likely be revealed by understanding the homomorphisms on the homotopy groups induced by the bonding maps in the same inverse sequences used to calculate cohomology (when available).

Sadun and Williams [29] have shown that any tiling space of the type under consideration fibers over a torus with a totally disconnected fiber. Williams [30, Conjecture 2.4] conjectured that up to homotopy the fiber bundle of the Penrose tiling could be given in 5 different ways. Robinson has calculated the discrete spectrum of the Penrose tiling space $\mathbf{P}$ [25, Section 8] and found the group of eigenvalues to be isomorphic to $\mathbf{Z}^4$. To an element of this group there corresponds a map $g_i : \mathbf{P} \to S^1$ that factors the action on $\mathbf{P}$ onto a Kronecker action of S^1 (one for which all maps $x \mapsto t \, . \, x$ are translations). Any choice of two distinct such maps leads to a bundle projection $g_i \times g_j : \mathbf{P} \to \mathbf{T}^2$. It is not difficult to show that different choices of (i, j) lead to homotopically distinct bundle projections. In fact, there will be infinitely many homotopically distinct bundle projections, but the spirit of the conjecture can be conveyed by the following.

Question. *Is every bundle map $\mathbf{P} \to \mathbf{T}^2$ homotopic to a map that factors the action on $\mathbf{P}$ to a Kronecker action of $\mathbf{T}^2$?* 1004 ?

Question. *If $\mathcal{T}$ has pure point discrete spectrum and $p : \mathcal{T} \to \mathbf{T}^d$ is a bundle projection with totally disconnected fiber, is p homotopic to a map that factors the action on $\mathcal{T}$ to a Kronecker action of $\mathbf{T}^d$?* 1005 ?

In their topological classification of one-dimensional tiling spaces, Barge and Diamond [2] made critical use of the asymptotic orbits of the tiling spaces. A homeomorphism carries a pair of topologically asymptotic orbits to a pair of topologically asymptotic orbits. Barge and Diamond have proved the coincidence conjecture for Pisot substitutions of two letters [3], and in the course of trying to construct a proof for the general case the weaker notion of proximality has proven key. The orbits of T and T' are *proximal* if there exists a sequence $u_n \in \mathbb{R}^d$ with $|u_n| \to \infty$ and $d\,(u_n \, . \, T, u_n \, . \, T') \to 0$.

Question. *(Barge and Diamond) If $h : \mathcal{T} \to \mathcal{T}'$ is a homeomorphism of one-dimensional tiling spaces, does h necessarily map a pair of proximal orbits to a pair of proximal orbits?* 1006 ?

Deformations of Tiling Spaces

It the tiling T' is obtained from the tiling T by adjusting the size and shape of the tiles in T without changing the combinatorics of the tiling (which tiles border which others), the respective tiling spaces $\mathcal{T}$ and $\mathcal{T}'$ are homeomorphic [29]. However, the actions may not be linearly equivalent. We will refer to $\mathcal{T}'$ as a *deformation* of $\mathcal{T}$. In [10, 11] there are general results that allow one to determine when deformations change the dynamics. For large classes of substitution tiling spaces, these results suffice to completely determine how deformations effect the dynamics. However, the results are difficult to apply to tiling spaces that do not arise from substitutions.

? 1007 **Question.** *If $\mathcal{T}'$ is a deformation of a Sturmian tiling space $\mathcal{T}$, are $\mathcal{T}'$ and $\mathcal{T}$ linearly equivalent?*

When $\mathcal{T}$ is Sturmian and a substitution tiling, then deformations are linearly equivalent [24]. But the general case is not as clear. For example, whether the irrational number α associated to the Sturmian has a bounded continued fraction expansion might be relevant. This leads naturally to the following.

? 1008 **Question.** *If $\mathcal{T}'$ is a deformation of a quasiperiodic tiling space $\mathcal{T}$, when are $\mathcal{T}'$ and $\mathcal{T}$ linearly equivalent?*

Again, the focus is on those tiling spaces that do not arise from substitutions. Recently, Harriss and Lamb [17] have found conditions that allow one to determine when a cut and project tiling is also a substitution tiling.

Mixing Properties

A tiling space is (topologically) weakly mixing if it has no non-constant continuous eigenfunction, meaning it has no Kronecker action on a circle as a continuous factor. A tiling space is *topologically mixing* if for any pair of nonempty open sets U and V, there is a corresponding M so that if $|u| > M$, then $(u \,.\, U) \cap V \neq \emptyset$.

? 1009 **Question.** [19] *If a primitive substitution has an associated matrix with no eigenvalues of modulus one, is topological mixing equivalent to weak mixing?*

This question applies to symbolic systems as well as to (one-dimensional) tiling spaces based on substitutions, and it is shown to have a positive answer in the case of substitutions on two letters in [19].

Much less is known about the mixing properties of tiling spaces that do not derive from substitutions.

? 1010 **Question.** *Can a tiling space based on a Sturmian sequence be weakly mixing? If so, is topological mixing equivalent to weak mixing?*

It is highly unlikely that a tiling space as we are currently considering could be (strong) mixing in the measure theoretic sense. However, it is still unknown whether more general tiling spaces with a larger group than the translation group acting on the tiling space, such as the pinwheel tiling investigated by Radin in [23], could be strongly mixing. As pointed out in [19], it is not even known whether the pinwheel tiling is topologically mixing.

? 1011 **Question.** *Is the pinwheel tiling topologically mixing?*

? 1012 **Question.** [23] *Is the pinwheel tiling mixing?*

Tiling Spaces that are not Locally Finite

To this point we have been considering tiling spaces arising from tilings by polyhedra meeting full edge to full edge. Given that there are well known tilings by fractals, this would seem to be a very restrictive class of tilings. However, Priebe [22] has shown with a Voronoi cell construction that any tiling space arising from a tiling with finite local complexity is conjugate to a tiling space with polyhedral tiles meeting full edge to full edge. A tiling has *finite local complexity* if up to translation there is a finite number of patches of two tiles. Solomyak [28] found arithmetic conditions for the weak mixing of self-similar tilings of $\mathbb{R}^2$ with finite local complexity. Little is known about tilings without finite local complexity. Some easy to digest examples of such tilings may be found in [14].

Question. *Is there an arithmetic condition for the weak mixing of self-similar tilings of $\mathbb{R}^2$ without finite local complexity?* **1013 ?**

In general, one may consider which of the known results can be generalized to tilings without finite local complexity.

Pisot Conjecture

The Pisot conjecture is one of the most hotly pursued open problems in the theory of tiling spaces. It has connections to symbolic substitution systems, graph directed systems, β-shifts, and automorphisms of compact connected abelian groups. As a result, it has drawn the attention of a wide range of people. A survey of what is known and how the conjecture relates to tilings may be found in [6]. There are various formulations of the conjecture corresponding to the different perspectives.

Question. *If $\mathcal{T}$ is a tiling space associated to an irreducible, unimodular Pisot substitution, does $\mathcal{T}$ have pure point discrete spectrum?* **1014 ?**

There are various finiteness conditions on the associated Pisot number that ensure the conjecture holds. The first such condition seems to have been introduced in [15]. The most general conditions under which the conjecture is now known to hold are given in [4, 5].

Some results of Siegel [13, 27] indicate that it may not be necessary to assume that the substitution is unimodular.

Question. *If $\mathcal{T}$ is a tiling space associated to an irreducible, Pisot substitution, does $\mathcal{T}$ have pure point discrete spectrum?* **1015 ?**

New Directions

In his thesis, Peach [21] gave a way of constructing an algebra associated to a tiling of the plane by rhombi. The questions he was most interested in were purely algebraic, and there is no apparent connection between the structure of this algebra and the dynamics of the tiling. However, by introducing quiver relations that reflect the nature of a substitution, it might possible to construct powerful invariants that reflect the dynamics.

1016 ? **Question.** *Is it possible to construct a quiver algebra for a self-similar tiling of the plane that provides an important dynamical invariant?*

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