A CLASSIFICATION OF ORDINALS UP TO BOREL ISOMORPHISM

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ABSTRACT. We consider the Borel structures on ordinals generated by their order topologies and provide a complete classification of all ordinals up to Borel isomorphism in ZFC. We also consider the same classification problem in the context of AD and give a partial answer for ordinals $\leq \omega_2$.

1. INTRODUCTION

Recall that the order topology on a linearly ordered set X is generated by the subbase of open rays $(x, \rightarrow) = \{y \in X : x < y\}$ and $(\leftarrow, y) = \{x \in X : x < y\}$ for $x, y \in X$. It is the most natural topology on ordinals. When we speak of an ordinal as a topological space we always assume that it has the order topology.

A complete classification of ordinals up to homeomorphism was known ([1]; an independent proof was given in [5]). Specifically, given an arbitrary ordinal a complete homeomorphism invariant for its order topology can be computed from its Cantor normal form. Benedikt Löwe proposed to study the similar classification problem for ordinals up to Borel isomorphism. He asked whether the Cantor normal form still provides a complete invariant. Since for example all countable ordinals are Borel isomorphic, Borel isomorphism is a genuinely more general notion of equivalence than homeomorphism.

In this paper we give a complete classification of all ordinals up to Borel isomorphism. It turns out that the computation of the complete invariants is not related to the Cantor normal form of the ordinals and is in fact somewhat simpler. To state our main theorem precisely, we define a cardinal $\kappa(\alpha)$ for any given ordinal α as follows. For an ordinal α, let $\kappa(\alpha) = 0$ if $|\alpha|$ is singular or countable, and otherwise let $\kappa(\alpha)$ be the largest cardinal such that $|\alpha| \cdot \kappa(\alpha) \leq \alpha$.

Theorem 1.1. *Let α and β be ordinals. Then α is Borel isomorphic to β iff $|\alpha| = |\beta|$ and $\kappa(\alpha) = \kappa(\beta)$.*

Note that the above main theorem will be proved in ZFC, in particular with essential use of AC in the proof. This is in contrast with the classification of ordinals up to homeomorphism, which can be done in ZF only (this is easier to see from the presentation of [5]). Not much of our discussions on the Borel structures of ordinals can survive in ZF. It thus seems to be very interesting to consider the same classification problem in the context of AD and to see how different the complete invariants would be. On this we have the following partial result.

Theorem 1.2 (ZF + AD)**.** *All uncountable ordinals $< \omega_2$ are Borel isomorphic.*

2000 *Mathematics Subject Classification.* Primary 03E10, 54H05; Secondary 03E60, 06F30.
Key words and phrases. Order topology; ordinals; Borel isomorphism; Borel σ-algebra.
The first author would like to acknowledge the support of U.S. NSF grant DMS-0501039.

This paper is organized as follows. In Section 2 we review some preliminaries on the Borel structures generated by the order topologies on ordinals. In particular we give a characterization of Borelness for subsets of ordinals which will be useful in further research. In Section 3 we give the proof of Theorem 1.1. In Section 4 we work under determinacy and prove Theorem 1.2.

We would like to thank Benedikt Löwe and Dan Mauldin for useful discussions on the subject.

2. Preliminaries

Recall that the Borel structure of any topological space is the σ-algebra generated by its open sets, i.e., the smallest σ-algebra that contains all the open sets and is closed under complements and countable unions. Also all Borel sets appear in the Borel hierarchy, which is defined by induction on $\alpha < \omega_1$ as follows:

$$\begin{aligned}
\mathbf{\Sigma}^0_1 &= \text{all open sets},\\
\mathbf{\Pi}^0_\alpha &= \text{all complements of } \mathbf{\Sigma}^0_\alpha \text{ sets},\\
\mathbf{\Sigma}^0_\alpha &= \text{all countable unions} \bigcup_{n\in\mathbb{N}} A_n, \text{where } A_n \in \mathbf{\Pi}^0_{\alpha_n} \text{ for some } \alpha_n < \alpha.\\
\mathbf{\Delta}^0_\alpha &= \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha.
\end{aligned}$$

The following proposition records the basic facts about these levels of the Borel hierarchy which are true in any topological space.

Proposition 2.1. *In any topological space the following hold.* $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta$ *for* $\alpha < \beta$, *and similarly,* $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$. $\mathbf{\Sigma}^0_\alpha$ *is closed under countable unions and* $\mathbf{\Pi}^0_\alpha$ *under countable intersections.* $\mathbf{\Sigma}^0_\alpha$ *is closed under finite intersections and* $\mathbf{\Pi}^0_\alpha$ *under finite unions for all* $\alpha \neq 3$. *If* $2 \leq \alpha \leq \beta$, *then* $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$ *and* $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta$.

Proof. All of the statements are immediate from the definitions except perhaps the closure of $\mathbf{\Sigma}^0_\alpha$ under finite intersections for $\alpha \neq 3$. To see this, suppose $A, B \in \mathbf{\Sigma}^0_\alpha$, say $A = \bigcup_n A_n$, $B = \bigcup_m B_m$, where $A_n \in \mathbf{\Pi}^0_{\alpha_n}$, $B_m \in \mathbf{\Pi}^0_{\beta_m}$ and $\alpha_n, \beta_m < \alpha$. Then $A \cap B = \bigcup_{n,m}(A_n \cap B_m)$. If $\alpha \geq 4$, then A_{α_n} B_{β_m} both lie in $\mathbf{\Pi}^0_\delta$ where $\delta = \max\{\alpha_n, \beta_m, 3\}$. This is because $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta$ for $2 \leq \alpha \leq \beta$ and $\mathbf{\Pi}^0_1 \subseteq \mathbf{\Sigma}^0_2 \subseteq \mathbf{\Pi}^0_3$. Since $\mathbf{\Pi}^0_\delta$ is closed under intersections, $A_{\alpha_n} \cap B_{\beta_m} \in \mathbf{\Pi}^0_\delta$, and so $A \cap B \in \mathbf{\Sigma}^0_\alpha$. If $\alpha = 1$, the result is immediate from the definition of a topology, and if $\alpha = 2$ the result follows from the fact that each A_{α_n}, B_{β_m} will be $\mathbf{\Pi}^0_1$, and thus so will be $A_{\alpha_n} \cap B_{\beta_m}$. □

If the underlying space is Polish (completely metrizable and separable) or even just metrizable, then the Borel hierarchy has the usual additional properties such as $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$ for $\alpha < \beta$ (and similarly on the $\mathbf{\Pi}$-side) and $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ are closed under finite unions and finite intersections. In particular, every $\mathbf{\Sigma}^0_{\alpha+1}$ set is a countable union $\bigcup_n A_n$ where each $A_n \in \mathbf{\Pi}^0_\alpha$.

However, these additional facts are no longer true for ordinal spaces. If the underlying space is an uncountable ordinal, then there are always open sets which are not F_σ. Thus in general $\mathbf{\Sigma}^0_1 \not\subseteq \mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_1 \not\subseteq \mathbf{\Pi}^0_2$. The following observation shows that $\mathbf{\Sigma}^0_3$ is not closed under finite intersections if the underlying space is an ordinal $\geq \omega_2$.

Proposition 2.2. *There exists an open $U \subseteq \omega_2$ and a closed $F \subseteq \omega_2$ such that $U \cap F$ is not $\mathbf{\Sigma}^0_3$.*

Proof. Without loss of generality assume the underlying space is ω_2. Let $U = \omega_2 - \{\omega_1 \cdot \alpha \colon \alpha < \omega_2\}$ and let F be the set of all limit ordinals below ω_2. Clearly, U is open and F is closed. Suppose $U \cap F$ is $\mathbf{\Sigma}^0_3$, say

$$U \cap F = \bigcup_{n\in\omega} A_n \cup \bigcup_{n\in\omega} B_n,$$

where each A_n is $\mathbf{\Pi}^0_1$ and each B_n is $\mathbf{\Pi}^0_2$. Since U misses a club in ω_2, $U \cap F$ is nonstationary, which in turn implies that each A_n is bounded in ω_2. Now that the union $\bigcup_{n\in\mathbb{N}} A_n$ is also bounded in ω_2, let β be an upper bound. Let α_0 be the least ordinal such that $\omega_1 \cdot \alpha_0 \geq \beta$.

Now consider the copy of ω_1 consisting of ordinals in the interval $I = (\omega_1 \cdot \alpha_0, \omega_1 \cdot \alpha_0 + \omega_1)$. Our assumption implies that the limit ordinals in I can be written as $\bigcup_n (B_n \cap I)$. It follows that the limit ordinals in ω_1 can be written as $\bigcup_n C_n$ where each C_n is $\mathbf{\Pi}^0_2$. Since the limit ordinals in ω_1 form a club, one of the C_n must be stationary. We claim that a stationary $\mathbf{\Pi}^0_2$ subset of ω_1 must contain a tail, and this is a contradiction.

Suppose $G = \bigcap_{n\in\omega} G_n$ is a stationary $\mathbf{\Pi}^0_2$ in ω_1, with all G_n open. Each G_n is also stationary, and therefore it contains a tail. Since $\mathrm{cof}(\omega_1) > \omega$, a countable intersection of tails is still a tail. Hence, G contains a tail. □

The Borel structures on ordinals have been studied before, e.g., in [8] and [6]. We summarize the known results as well as present the techniques used in the study of this topic. For the convenience of the reader we include some proofs of previously known results here.

Lemma 2.3 (Rao-Rao [8])**.** *Every Borel subset of a limit ordinal either contains or misses a club.*

Proof. Every subset of a limit ordinal of cofinality ω either contains or misses a club. In case of uncountable cofinality, a countable intersection of clubs is still a club. Hence, the collection of all sets which contain or miss a club is a σ-algebra containing all closed sets and therefore contains all the Borel sets. □

In particular, a stationary and costationary subset of a limit ordinal is not Borel. For ω_1, a subset is Borel if and only if it either contains or misses a club [8]. Another characterization of Borel subsets of ω_1 was also given in [8], and it was completely generalized by Mauldin in [6], as follows.

Theorem 2.4 (Mauldin [6])**.** *Every Borel subset of an ordinal can be expressed as a union of countably many sets, each of which is the intersection of an open set and a closed set.*

Mauldin's theorem shows that the Borel hierarchy on any ordinal collapses to a rather low level, and every Borel subset of an ordinal is in fact $\mathbf{\Delta}^0_4$. In view of Proposition 2.2 this estimate is optimal.

Below we give another characterization of Borelness of subsets of ordinals. We state the result in a way that encompasses the results in [8] and [6], and provide a self-contained proof. It should be noted, however, that the main ideas and techniques used in the proof are the same as those presented in [8] and [6].

We will use the following simple lemma repeatedly throughout the paper.

Lemma 2.5. *Let X be an arbitrary topological space. Suppose $X = \bigcup_{i\in I} U_i$, where $\{U_i\}_{i\in I}$ is a family of pairwise disjoint open subsets. Let $\xi < \omega_1$ and $B \subseteq X$. Then B is $\mathbf{\Sigma}^0_\xi$ (or $\mathbf{\Pi}^0_\xi$) iff for every $i \in I$, $B \cap U_i$ is $\mathbf{\Sigma}^0_\xi$ (respectively $\mathbf{\Pi}^0_\xi$) in U_i.*

Proof. A simple induction on ξ. □

Theorem 2.6. *Let α be an ordinal. Then the following are equivalent:*

(1) *$B \subseteq \alpha$ is Borel.*
(2) *$B = \bigcup_{n\in\mathbb{N}}(U_n \cap F_n)$, where each U_n is open and each F_n is closed.*
(3) *For every limit ordinal $\beta \leq \alpha$, B contains or misses a club in β.*
(4) *For every limit ordinal $\beta \leq \alpha$ and every club C in β, B contains or misses a club of C.*

Proof. The implication (1) ⇒ (4) is immediate from Theorem 2.3. The implications (2) ⇒ (1) and (4) ⇒ (3) are trivial. It suffices to show (3) ⇒ (2). We use induction on α. For the base case and the successor case there is nothing to do. We assume that α is a limit. By (3) B contains or misses a club in α. For definiteness assume that B misses a club C in α. In this case let α_i, $i < \eta = \mathrm{cof}(\alpha)$, enumerate the elements of C in the increasing order. Without loss of generality assume $\alpha_0 = 0$. Then let $U_i = (\alpha_i, \alpha_{i+1})$ for $i < \eta$. Thus we get that $\alpha - C = \bigcup_{i<\eta} U_i$. Note that clause (3) is still true for each interval U_i. Since each U_i is a copy of an ordinal $< \alpha$, the inductive hypothesis gives that $B \cap U_i$ is a union of countably many sets, each of which is the intersection of an open set with a closed set. Now the proof of Lemma 2.5 implies that

$$B = B \cap (\alpha - C) = \bigcup_{n\in\mathbb{N}}(U_n \cap F_n)$$

for relatively open U_n in $\alpha - C$ and relatively closed F_n in $\alpha - C$. Let C_n be the closure of F_n in α, $U_{-1} = \alpha - C$ and $V_n = U_n \cap U_{-1}$. Then each V_n is open in α, C_n is closed in α, $F_n = C_n \cap U_{-1}$ and

$$B = \bigcup_{n\in\mathbb{N}}(U_n \cap F_n) = \bigcup_{n\in\mathbb{N}}(U_n \cap C_n \cap U_{-1}) = \bigcup_{n\in\mathbb{N}}(V_n \cap C_n).$$

This finishes the proof of the case that B misses a club C in α. Suppose alternatively B contains a club C in α, then $B - C$ continues to satisfy (3) and the same argument shows that $B - C$ is a union of the form in (2). It follows that B is of the same form since $B = (B - C) \cup C$. □

As another application of the same technique we note below that every Borel subset of ω_1 is $\mathbf{\Delta}^0_3$.

Proposition 2.7. *Every Borel subset of ω_1 is $\mathbf{\Delta}^0_3$.*

Proof. It suffices to show that every Borel subset of ω_1 is $\mathbf{\Sigma}^0_3$, and in view of Theorem 2.4 it is enough to show that the intersection of an open set U and a closed set F is $\mathbf{\Sigma}^0_3$. If $U \cap F$ is bounded then it is countable and easily seen to be $\mathbf{\Sigma}^0_3$. Assume $U \cap F$ is unbounded. In particular both U and F are unbounded. If $\omega_1 - U$ is bounded, then the bounded part of $U \cap F$ is relatively $\mathbf{\Sigma}^0_2$, the unbounded part is relatively closed, thus relatively $\mathbf{\Sigma}^0_2$, hence by Lemma 2.5, $U \cap F$ is $\mathbf{\Sigma}^0_2$ in ω_1.

If $\omega_1 - U$ is unbounded, write $U = \bigcup I_\gamma$, where the I_γ are maximal disjoint open intervals. Each I_γ is homeomorphic to a countable ordinal, hence $U \cap F$ is $\mathbf{\Pi}^0_2$ in

I_γ, thus in U. Hence, $U \cap F$ is the intersection of an open and a $\mathbf{\Pi}^0_2$ set in ω_1, hence $\mathbf{\Pi}^0_2$. □

In view of the collapse of the Borel hierarchy our basic Lemma 2.5 can be restated as the following convenient fact for subsets of ordinals. For obvious reasons we will refer to it as the gluing lemma.

Corollary 2.8 (The gluing lemma)**.** *Let α be an ordinal, $\{U_i\}_{i\in I}$ be a family of pairwise disjoint open sets in α, and let C be the closed set with $\alpha - C = \bigcup_{i\in I} U_i$. Then a subset B of $\alpha - C$ is Borel in α iff $B \cap U_i$ is Borel in U_i for every $i \in I$.* □

We now turn to a review of Borel isomorphisms. Let X and Y be arbitrary topological spaces. A map $f : X \to Y$ is called *Borel measurable* (or simply *Borel*) if for any open set U in Y, $f^{-1}(U)$ is a Borel subset of X. f is called a *Borel isomorphism* if it is a bijection so that both f and f^{-1} are Borel. If there is a Borel isomorphism from X onto Y then we say that X and Y are *Borel isomorphic*, and denote it by $X \cong_B Y$.

Recall again that if both X and Y are Polish spaces then $X \cong_B Y$ iff there is a Borel injection from X into Y and also a Borel injection from Y into X. Here a Borel injection is merely an injective Borel map. The proof is a repetition of that of the classical Cantor-Bernstein theorem. However, we should remark that the reason it runs smoothly in this context is because of the important theorem of Luzin-Suslin that a Borel injection from a Polish space to another preserves Borelness of subsets.

In our context the following definition is needed. A Borel injection $f\colon X \to Y$ is called a *Borel embedding* if the image of a Borel set under f is Borel. Now the proof of the classical Cantor-Bernstein theorem can be repeated to show that if there exist Borel embeddings $f\colon X \to Y$ and $g\colon Y \to X$, then X and Y are Borel isomorphic. We also adopt the notation $f : X \hookrightarrow_B Y$ to denote that f is a Borel embedding from X into Y, and write $X \hookrightarrow_B Y$, or simply $X \hookrightarrow Y$ if there is no danger of confusion, if there exists $f : X \hookrightarrow_B Y$.

The following simple observations on Borel isomorphism and embeddability of ordinals will be useful. Let $\alpha < \beta$ be ordinals. Note that the canonical injection (namely the identity map) from α into β is a Borel embedding (in fact a homeomorphic one). If follows that for $\alpha < \beta$ we have $\alpha \cong_B \beta$ iff $\beta \hookrightarrow \alpha$. The following lemma is our main tool to show that β Borel embeds into $\alpha < \beta$.

Lemma 2.9. *Let $\alpha < \beta$ be ordinals, $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be respectively pairwise disjoint open sets in α and in β, and let C and D be closed subsets of α and β respectively with $\alpha - C = \bigcup_{i\in I} U_i$ and $\beta - D = \bigcup_{j\in J} V_j$. Suppose that there is a $k \in I$ such that $\psi\colon D \hookrightarrow_B U_k$ and there is an injection $\pi\colon J \to I - \{k\}$ such that for every $j \in J$ there is $\psi_j\colon V_j \hookrightarrow_B U_{\pi(j)}$. Then β Borel embeds into $\alpha - C$, thus into α, and $\beta \cong_B \alpha$.*

Proof. Let $\phi\colon \beta \to \alpha - C$ be the piecewise defined map from ψ and the ψ_j's. Clearly, ϕ is injective. If $B \subseteq \beta$ is Borel, then $B \cap D$ is Borel in D and $B \cap V_j$ is Borel for each $j \in J$. Hence, ϕ“B is Borel in each U_j. By the gluing lemma, ϕ“B is Borel in $\alpha - C$. Similarly, if $B \subseteq \alpha - C$ is Borel, then $\phi^{-1}(B \cap U_k)$ is Borel in D and for any $l \in J - \{k\}$, $\phi^{-1}(B \cap U_l)$ is Borel in $V_{\pi^{-1}(l)}$, hence $\phi^{-1}B$ is Borel in β again by the gluing lemma. □

Under the hypothesis of the above lemma a particularly easy way to guarantee $D \hookrightarrow U_k$ for some k is to make sure that $\mathrm{ot}(D) \leq \mathrm{ot}(U_k)$. Note that the lemma is still meaningful even if the ordinals are the same. Specifically, if $\alpha \geq \omega$ and $C \subseteq \kappa\cdot\alpha$ is closed with order type $\leq \kappa$, then the lemma gives that $\kappa \cdot \alpha - C \cong_B \kappa \cdot \alpha$.

3. The Classification

In this section we classify all ordinals up to Borel isomorphism. Since all countable ordinals are Borel isomorphic, and $\alpha \ncong_B \beta$ whenever $|\alpha| \neq |\beta|$, we can restrict ourselves to ordinals α and β so that $\kappa \leq \alpha < \beta < \kappa^+$ for some uncountable cardinal κ. As remarked before, in order to show that $\alpha \cong_B \beta$, it suffices to find a Borel embedding of β into α.

We split the proof of the classification Theorem 1.1 into three parts. First, we show that all ordinals $\geq \kappa \cdot \mathrm{cof}(\kappa)$ are Borel isomorphic to $\kappa \cdot \mathrm{cof}(\kappa)$. Second, we show that for singular κ, $\kappa \cdot \mathrm{cof}(\kappa)$ is Borel isomorphic to κ. Finally, we identify the Borel isomorphism types between κ and κ^2 for regular κ.

For the first part, we need the following lemma.

Lemma 3.1. *If $\omega \leq \alpha \leq \kappa$, then $\kappa \cdot \alpha^2 \cong_B \kappa \cdot \alpha$.*

Proof. We first show $\kappa \cdot \alpha^2 \hookrightarrow \kappa \cdot \alpha \cdot 2$. Let $C = \{\kappa \cdot \xi \colon \xi < \alpha^2\}$. Then C is a club in $\kappa \cdot \alpha^2$ and $\kappa \cdot \alpha^2 - C$ consists of $|\alpha^2| = |\alpha|$ many maximal disjoint open intervals each of which is a copy of the ordinal κ. We refer to these maximal open intervals as *κ-blocks*.

For $\kappa \cdot \alpha \cdot 2$ we let $D = \{\kappa \cdot \alpha + \kappa \cdot \xi \colon \xi < \alpha\}$. Then $\kappa \cdot \alpha \cdot 2 - D$ consists of a copy of $\kappa \cdot \alpha$ and $|\alpha|$ many κ-blocks. Now since $\mathrm{ot}(C) \leq \kappa \cdot \alpha$, C can be Borel embedded into the copy of $\kappa \cdot \alpha$. Since there are the same number of κ-blocks in the remaining parts of the two ordinals, they can be paired off. Lemma 2.9 gives the desired Borel embedding.

Second, we show $(\kappa \cdot \alpha) \cdot 2 \hookrightarrow \kappa \cdot \alpha$. Let $C_1 = \{\kappa \cdot \xi \colon \xi < \alpha\}$ and let $C_2 = \{\kappa \cdot \alpha + \kappa \cdot \xi \colon \xi < \alpha\}$. Since $\mathrm{ot}(C_1) = \mathrm{ot}(C_2) = \alpha \leq \kappa$, we can embed C_1 into the first κ-block of $\kappa \cdot \alpha$, and C_2 into the second κ-block of $\kappa \cdot \alpha$. Now we are in a position to apply Lemma 2.9 again, since there are again the same number $|\alpha \cdot 2| = |\alpha|$ of κ-blocks in the remaining part of the two ordinals. □

Theorem 3.2. *If $\kappa \cdot \mathrm{cof}(\kappa) \leq \alpha < \kappa^+$, then $\alpha \cong_B \kappa \cdot \mathrm{cof}(\kappa)$.*

Proof. We prove by induction that α can be partitioned into countably many Borel subsets $A_0, A_1, \dots$ such that each A_n embeds into $\kappa \cdot \mathrm{cof}(\kappa)$. This gives a Borel embedding of α into $\kappa\cdot\mathrm{cof}(\kappa)\cdot\omega$, which embeds into $\kappa\cdot\mathrm{cof}(\kappa)^2$ and hence in $\kappa\cdot\mathrm{cof}(\kappa)$ by the preceding lemma.

The statement is certainly true for $\alpha = \kappa\cdot\mathrm{cof}(\kappa)$. The successor case is also easy. We assume α is a limit ordinal. Let $C = \{x_\beta \colon \beta < \mathrm{cof}(\alpha)\}$ be a club in α, with $x_0 = 0$. Since $\mathrm{cof}(\alpha) \leq \kappa$ (because $\alpha < \kappa^+$), C can be embedded into κ and thus in $\kappa \cdot \mathrm{cof}(\kappa)$. For each $\beta < \mathrm{cof}(\alpha)$ let $I_\beta = (x_\beta, x_{\beta+1})$. The I_β's are pairwise disjoint open subsets of α such that $\alpha - C = \bigcup_{\beta<\mathrm{cof}(\alpha)} I_\beta$. Also for each $\beta < \mathrm{cof}(\alpha)$, I_β is a copy of an ordinal $< \alpha$. Thus by the inductive hypothesis, for every $\beta < \mathrm{cof}(\alpha)$ there is a pairwise disjoint family $\{A'_{\beta,n} \colon n \in \mathbb{N}\}$ such that $I_\beta = \bigcup_{n<\omega} A'_{\beta,n}$, every $A'_{\beta,n}$ is Borel in I_β and there is a Borel embedding $\varphi_{\beta,n} \colon A'_{\beta,n} \to \kappa \cdot \mathrm{cof}(\kappa)$.

Define $A'_n := \bigcup_{\beta<\mathrm{cof}(\alpha)} A'_{\beta,n}$. Since each $A'_n \cap I_\beta = A'_{\beta,n}$ is Borel in I_β, A'_n is Borel in α by the gluing lemma. Also for every $n < \omega$, $A'_n = \bigcup_{\beta<\mathrm{cof}(\alpha)} A'_{\beta,n}$ is Borel embeddable in $\kappa \cdot \mathrm{cof}(\kappa) \cdot \mathrm{cof}(\kappa)$, and thus A'_n embeds into $\kappa \cdot \mathrm{cof}(\kappa)$ by the preceding lemma. Then $A_0 = C$, $A_{n+1} = A'_n$ is the required decomposition of α. $\square$

Thus between any cardinal κ and its successor κ^+ there are no new isomorphism types after $\kappa \cdot \mathrm{cof}(\kappa)$. For singular κ, there is in fact only one isomorphism type after all.

Theorem 3.3. *If κ is singular and $\kappa \leq \alpha < \kappa^+$, then $\alpha \cong_B \kappa$.*

Proof. In view of Theorem 3.2 it suffices to prove that $\kappa \cdot \mathrm{cof}(\kappa) \cong_B \kappa$. Fix a club-in-$\kappa$ sequence $\langle \lambda_\zeta \colon \zeta < \mathrm{cof}(\kappa)\rangle$ of cardinals such that $\mathrm{cof}(\kappa)^2 \leq \lambda_\zeta < \kappa$. Let

$$C = \{\kappa \cdot \xi \colon \xi < \mathrm{cof}(\kappa)\} \cup \bigcup_{\xi < \mathrm{cof}(\kappa)} \{\kappa \cdot \xi + \lambda_\zeta \colon \zeta < \mathrm{cof}(\kappa)\}.$$

This is a club in $\kappa \cdot \mathrm{cof}(\kappa)$ of order type $\mathrm{cof}(\kappa)^2$. Again $\kappa \cdot \mathrm{cof}(\kappa) - C$ can be written as a union of $|\,\mathrm{cof}(\kappa)^2| = \mathrm{cof}(\kappa)$ many maximal disjoint open intervals, or *blocks*, each of which is a copy of some λ_ζ. Moreover, for each $\zeta < \mathrm{cof}(\kappa)$ there are exactly $\mathrm{cof}(\kappa)$ many λ_ζ-blocks.

On the other hand, $D = \{\lambda_\zeta \colon \zeta < \mathrm{cof}(\kappa)\}$ is a club in κ of order type $\mathrm{cof}(\kappa)$, and $\kappa - D$ is the union of $\mathrm{cof}(\kappa)$ many blocks each of which is a copy of some λ_ζ. However, for each $\zeta < \mathrm{cof}(\kappa)$ there is exactly one λ_ζ-block in $\kappa - D$, which we denote by B_ζ.

We now define a Borel embedding from $\kappa \cdot \mathrm{cof}(\kappa)$ into κ in view of Lemma 2.9. First note that C embed into B_0 since $\lambda_0 \geq \mathrm{cof}(\kappa)^2$. Then for each $\zeta < \mathrm{cof}(\kappa)$ we let all $\mathrm{cof}(\kappa)$ many λ_ζ-blocks in $\kappa \cdot \mathrm{cof}(\kappa)$ embed into the $\lambda_{\zeta+1}$-block $B_{\lambda_{\zeta+1}}$ of κ. This is possible since $\lambda_{\zeta+1} > \lambda_\zeta, \mathrm{cof}(\kappa)$ is a cardinal. $\square$

Finally, we consider ordinals between κ and κ^2 when κ is a regular uncountable cardinal. Any such ordinal can be written as $\kappa \cdot \alpha + \beta$ with $0 < \alpha \leq \kappa$ and $0 \leq \beta < \kappa \cdot \alpha$.

Lemma 3.4. *If $\beta < \kappa \cdot \alpha$, then $\kappa \cdot \alpha + \beta \cong_B \kappa \cdot \alpha$.*

Proof. This is immediate when β is finite, so assume β is infinite. In this case $\kappa \cdot \alpha + \beta = \kappa \cdot \alpha + 1 + \beta$ is the disjoint union of the open sets $[0, \kappa \cdot \alpha + 1)$ and $(\kappa \cdot \alpha, \kappa \cdot \alpha + \beta)$. In other words, $\kappa \cdot \alpha + \beta$ is homeomorphic to the direct sum $(\kappa \cdot \alpha + 1) \oplus \beta$. Replacing β with the Borel isomorphic $\beta + 1$, we are allowed to transpose the disjoint open parts:

$$(\kappa \cdot \alpha + 1) \oplus \beta \cong_B (\kappa \cdot \alpha + 1) \oplus (\beta + 1) \cong (\beta + 1) \oplus (\kappa \cdot \alpha + 1).$$

Finally, $(\beta + 1) \oplus (\kappa \cdot \alpha + 1) \cong \beta + 1 + \kappa \cdot \alpha + 1 = \kappa \cdot \alpha + 1 \cong_B \kappa \cdot \alpha$. $\square$

We can therefore restrict our attention to ordinals of the form $\kappa \cdot \alpha$ for $0 < \alpha \leq \kappa$. It follows immediately from Lemma 2.9 that $\kappa \cdot \alpha \cong_B \kappa \cdot \beta$ whenever $|\alpha| = |\beta|$. To motivate the converse, suppose towards a contradiction that θ is a Borel isomorphism between $\omega_1 \cdot 2$ and ω_1. The larger ordinal $\omega_1 \cdot 2$ consists of two copies B_1, B_2 of ω_1 (and a limit point), while the smaller ordinal ω_1 has only one block. Each of the copies is Borel in $\omega_1 \cdot 2$ and therefore so are their images $\theta``B_1$ and $\theta``B_2$. By Lemma 2.3, both images either contain or omit a club. Since $\theta``B_1$ and $\theta``B_2$ are disjoint, and any two clubs meet, one of the images, say $\theta``B_1$ must omit a club $C$. This closed set splits $\omega_1$ into open blocks. One can construct a stationary and costationary $S \subseteq B_1$ such that $\theta``S$ contains at most one point in each block. Hence, $\theta``S$ is Borel in ω_1 by the gluing lemma, but S is not Borel in B_1 and hence not in $\omega_1 \cdot 2$, a contradiction. The argument in the proof of the following theorem is a generalization of this idea.

Theorem 3.5. *Let κ be a regular uncountable cardinal and let $\alpha < \beta \leq \kappa$. If $|\alpha| \neq |\beta|$, then $\kappa \cdot \alpha \not\cong_B \kappa \cdot \beta$.*

Proof. We may assume without loss of generality that $|\alpha| < |\beta|$. Let $C_0 = \{\kappa \cdot \xi \colon \xi < \alpha\}$ and $D_0 = \{\kappa \cdot \xi \colon \xi < \beta\}$. It follows from Lemma 2.9 that $\kappa \cdot \alpha \cong_B \kappa \cdot \alpha - C_0$ and $\kappa \cdot \beta \cong_B \kappa \cdot \beta - D_0$. Thus it suffices to show that $\kappa \cdot \alpha - C_0 \not\cong_B \kappa \cdot \beta - D_0$.

Toward a contradiction we assume that $\theta \colon \kappa \cdot \beta - D_0 \to \kappa \cdot \alpha - C_0$ is a Borel isomorphism. As before $\kappa \cdot \alpha - C_0$ consists of $|\alpha|$ many κ-blocks, which we denote in increasing order by A_ζ for $\zeta < \alpha$. Similarly $\kappa \cdot \beta - D_0$ consists of $|\beta|$ many κ-blocks, which we denote in increasing order by B_ξ for $\xi < \beta$.

Claim 1. *There is a $\xi < \beta$ such that for every $\zeta < \alpha$, $A_\zeta \cap \theta``B_\xi$ is nonstationary in A_ζ.*

Proof. Note that the κ-blocks A_ζ, B_ξ are open. For every $\xi < \beta$, $\theta``B_\xi$ is Borel in $\kappa \cdot \alpha$, thus for every $\zeta < \alpha$, $A_\zeta \cap \theta``B_\xi$ is Borel in $\kappa \cdot \alpha$ and thus in A_ζ. But A_ζ is a copy of the regular κ, hence $A_\zeta \cap \theta``B_\xi$ must either contain or miss a club in $A_\zeta$ by Lemma 2.3. Since two clubs necessarily meet, for every $\zeta < \alpha$ there can be at most one $\xi < \beta$ such that $A_\zeta \cap \theta``B_\xi$ contains a club in A_ζ. Because $|\alpha| < |\beta|$, there must be a $\xi < \beta$ such that for every $\zeta < \alpha$, $A_\zeta \cap \theta``B_\xi$ is nonstationary in A_ζ. □

Claim 2. *There is a stationary $S \subseteq B_\xi$ such that $\theta``S \subseteq A_\zeta$ for some $\zeta < \alpha$.*

Proof. For each $\zeta < \alpha$ let $B_{\xi,\zeta} = B_\xi \cap \theta^{-1}(A_\zeta)$. Then $B_\xi = \bigcup_{\zeta < \alpha} B_{\xi,\zeta}$. Since B_ξ is a copy of κ and $\alpha < \kappa$, it follows from the regularity of κ that B_ξ is not the union of $|\alpha|$ many nonstationary sets. Hence, there must be a $\zeta < \alpha$ such that $B_{\xi,\zeta}$ is stationary. This stationary set $S = B_{\xi,\zeta}$ has the required property. □

We now have a stationary set $S \subseteq B_\xi$ such that $\theta``S$ is entirely contained in $A_\zeta$. Since $\theta``B_\xi$ is nonstationary on every κ-block of $\kappa \cdot \alpha$, $\theta``S$ is nonstationary in A_ζ. Note that both B_ξ and A_ζ are copies of the regular cardinal κ.

Let C be a club in A_ζ such that $\theta``S \cap C = \emptyset$. Then $A_\zeta - C$ can be written as the disjoint union of maximal open intervals, say $A_\zeta - C = \bigcup_{i \in \kappa} U_i = \bigcup_{i \in \kappa} (\gamma_i, \gamma_{i+1})$. Note that $\theta``S \subseteq \bigcup_{i \in \kappa} U_i$.

Claim 3. *There is an $S_1 \subseteq S$ which is stationary and costationary in B_ξ such that $\theta``S_1 \cap U_i$ is Borel in U_i for every $i \in \kappa$.*

Proof. For any $x \in S$, denote by $\mathrm{block}(x) \in \kappa$ the index of the block that $\theta(x)$ is in, that is, $\theta(x) \in U_{\mathrm{block}(x)}$. We will construct a club D such that for $x, y \in D \cap S$ with $x \neq y$, $\mathrm{block}(x) \neq \mathrm{block}(y)$. Then $S_0 := D \cap S$ is a stationary set such that $|\theta``S_0 \cap U_i| \leq 1$. This trivially implies that $\theta``S_0 \cap U_i$ is Borel in U_i for every $i \in \kappa$. Furthermore, let $S_1 \subseteq S_0$ be any stationary and costationary subset. Then $\theta``S_1 \cap U_i$ is Borel in U_i for every $i \in \kappa$.

To construct this club D, we define a function $g \colon B_\xi \to B_\xi$ and then let D be the set of closure points of g, that is, $D = \{\alpha \in B_\xi \colon \forall \beta < \alpha \; (g(\beta) < \alpha)\}$. Let $x \in B_\xi$ be arbitrary. Let $B = \{\mathrm{block}(z) \colon z \in S \wedge z \leq x\}$. Since κ is regular, B is bounded in κ. Let $g(x) = \sup\{x' \in B_\xi \cap S \colon \mathrm{block}(x') \in B\}$. Since κ is regular and θ is one-to-one, $g(x) \in B_\xi$. To see this works, suppose $x, y \in S_0 = D \cap S$ with

$x < y$. Since $y \in D$, $g(x) < y$. Thus, $\text{block}(y) \notin \{\text{block}(z) \colon z \leq x \wedge z \in S\}$, a set which includes $\text{block}(x)$. □

Since $\theta``S_1 \cap U_i$ is Borel in $U_i$ for every $i < \kappa$, $\theta``S_1$ is Borel in $\bigcup_{i<\kappa} U_i$ by the gluing lemma, and hence $\theta``S_1$ is Borel in A_ζ and also in $\kappa \cdot \alpha$. But S_1 is not Borel in B_ξ by Lemma 2.3 and thus not Borel in $\kappa \cdot \beta$. This contradicts the assumption that θ is a Borel embedding. □

This completes the proof of Theorem 1.1: if κ is singular or countable, all ordinals between κ and κ^+ are Borel isomorphic by Theorems 3.2 and 3.3, and if κ is regular and uncountable, the Borel isomorphism types are precisely $\kappa \cdot \lambda$ for cardinals $1 \leq \lambda \leq \kappa$ by Theorems 3.2 and 3.5.

4. The Situation Assuming Determinacy

In this section we consider the Borel isomorphism question assuming now $\mathsf{ZF} + \mathsf{AD}$. The results of §3 were proved assuming AC, and no longer hold in this context. We first consider the question of which sets of ordinals are Borel. Under AD, the club filter on ω_1 is a measure, that is, there are no stationary, costationary subsets of ω_1 (which would be non-Borel sets). This suggests the possibility of the following theorem. This theorem can be proved either using the "simple set" type analysis occurring in the analysis of measures on ω_1, or by an indiscernibility argument. We give the proof following the indiscernibility argument. We will use the basic theory of the Silver indiscernibles. The reader can consult §30 of [4] or §8H of [7] for a presentation of this theory. The measure analysis can be found in [2] or [3] (we make a few comments at the end about how the following proof can be modified along those lines).

Theorem 4.1 ($\mathsf{ZF} + \mathsf{AD}$). *Every subset of ω_1 is Borel, and can be written in the form $\bigcup_n (F_n \cap U_n)$ where F_n is closed and U_n is open.*

Proof. Let $A \subseteq \omega_1$. The only AD fact we use is that $A \in L[x]$ for some real x. Let $C \subseteq \omega_1$ be the canonical set of Silver indiscernibles for $L[x]$, so C is club in ω_1. Let C' denote the set of limit points of C, so C' is also a club set of indiscernibles for $L[x]$.

There are finitely many indiscernibles $\alpha_0 < \cdots < \alpha_a < \omega_1$, and finitely many $\omega_1, \ldots, \omega_b$ and a term u such that $A = u^{L[x]}(\alpha_0, \ldots, \alpha_a, \omega_1, \ldots, \omega_b)$. For notational simplicity we suppress writing the $\omega_1, \ldots, \omega_b$ as well as the superscript $L[x]$, and just write $A = u(\vec{\alpha})$.

By the *type* of an ordinal $\alpha < \omega_1$ we mean the specification of:

(1) Finitely many $\beta_1, \ldots, \beta_c \leq \max\{\alpha_1, \ldots, \alpha_a\}$ in C'.
(2) An $L[x]$ term $t = t^{L[x]}(x_1, \ldots x_c, y_1, \ldots y_n)$ for some $n \in \omega$.

There are clearly only countably many types, we enumerate them as $T_1, T_2, \ldots$. For a type T as above, we say an ordinal $\alpha < \omega_1$ has type T provided $\alpha = t^{L[x]}(\vec{\beta}, \gamma_1, \ldots, \gamma_n)$ for some indiscernibles $\gamma_1 < \cdots < \gamma_n < \omega_1$ in C with $\gamma_1 > \max(\vec{\alpha})$. Every countable ordinal is represented by some type, since C is a generating set of indiscernible for $L[x]$.

For a type T as above, we say T is *normal* if the following statements are in $x^\#$:

(a) $t(\vec{\beta}, \gamma_1, \ldots, \gamma_n) \geq \gamma_n$.

(b) $t(\vec{\beta}, \gamma_1, \ldots, \gamma_i, \ldots, \gamma_k) \neq t(\vec{\beta}, \gamma_1, \ldots, \gamma_i', \ldots, \gamma_k)$ for $\gamma_1 < \ldots \gamma_i < \gamma_i' < \gamma_{i+1} < \cdots < \gamma_n$.

Every countable ordinal is represented by a normal type. This follows from two observations. First, if $t(\vec{\beta}, \gamma_1, \dots, \gamma_n) < \gamma_n$ is in $x^{\#}$ then there is a term $u = u(\vec{x}, y_1, \dots, y_{n-1})$ such that $t(\vec{\beta}, \gamma_1, \dots, \gamma_n) = u(\vec{\beta}, \gamma_1, \dots, \gamma_{n-1})$ for all indiscernibles $\gamma_1 < \cdots < \gamma_n$ (we again suppress the dependence on $\omega_1, \dots, \omega_l$). Secondly, if $t(\vec{\beta}, \gamma_1, \dots, \gamma_i, \dots, \gamma_n) = t(\vec{\beta}, \gamma_1, \dots, \gamma_i', \dots, \gamma_n)$ is in $x^{\#}$, then for some term $t' = t'(\vec{x}, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$ we have

$$t(\vec{\beta}, \gamma_1, \dots, \gamma_n) = t'(\vec{\beta}, \gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n).$$

Thus, we may eliminate variables until (b) is satisfied. From (b), note that we must actually have $t(\vec{\beta}, \gamma_1, \dots, \gamma_n) > \gamma_n$ unless $n = 1$ and $t(\vec{\beta}, \gamma_1) = \gamma_1$. We henceforth assume that all the T_n are normal.

To show A is Borel, it suffices to show that $A \cap X_n$ is Borel, where X_n is the set of countable ordinals of type T_n. We also show that $A \cap X_n$ is of the form $\bigcup_n (F_n \cap U_n)$ as required. Henceforth, fix a type T corresponding to a term $t = t(\vec{x}, y_1, \dots, y_n)$, and we show $A \cap X$ is Borel (X being the set of ordinals of type T). By indiscernibility, we either have $X \subseteq A$ or $X \subseteq \omega_1 - A$. So, either $A \cap X = X$ or $A \cap X = \emptyset$. It suffices therefore to show that X is Borel, and in fact $X = \bigcup_n (F_n \cap U_n)$ for F_n closed, U_n open.

For the rest of the argument we suppress writing the fixed ordinals $\vec{\alpha}$ and $\vec{\beta}$, and consider only $X - (\max(\vec{\alpha}) + 1)$ (any countable set is clearly of the required form).

If $t(\gamma_1, \dots, \gamma_n) = \gamma_n$, then $n = 1$ and $X = C$, which is closed. So we may henceforth assume $t(\gamma_1, \dots, \gamma_n) > \gamma_n$.

Consider first the simple case $n = 1$. By indiscernibility, $t(\gamma_1) < \gamma_2$ for all indiscernibles $\gamma_1 < \gamma_2$. So, there is exactly one element of X between any indiscernible $\gamma \in C$ and the next indiscernible. Since $X \cap C = \emptyset$, it follows that X is closed in $\omega_1 - C$. Thus, X is the intersection of a closed set (the closure of X) and an open set ($\omega_1 - C$).

Consider now the general case $n > 1$. By indiscernibility and wellfoundedness, $t(\vec{\gamma})$ is monotonically increasing in each argument. From (b), $t(\vec{\gamma})$ is actually strictly increasing in each argument. We need the following rather standard claim.

Claim 4. *There is permutation $\pi = (i_1, \dots, i_n)$ of $\{1, 2, \dots, n\}$ with $i_1 = n$ such that for all indiscernibles $\gamma_1 < \cdots < \gamma_n$, $\delta_1, \cdots < \delta_n$, we have $t(\vec{\gamma}) < t(\vec{\delta})$ iff*

$$(\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_n}) <_{lex} (\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_n}).$$

For t and π as in the claim, we say the term t has type π. We adopt the practice of writing the arguments to t in any order, which causes no confusion as we actually only evaluate $t(\gamma_1, \dots, \gamma_n)$ for $\gamma_1 < \dots, \gamma_n$. For example, we may write $t(\gamma_{i_1}, \dots, \gamma_{i_n})$. Also, we say the sequence $(\gamma_{i_1}, \dots, \gamma_{i_n})$ is of type π if it is order-isomorphic to $\pi = (i_1, \dots, i_n)$.

We use the following technical result.

Claim 5. *Let $t = t(y_1, \dots, y_n)$ be an $L[x]$ term of type $\pi = (i_1, \dots, i_n)$. Then one of the following holds:*

(1) For every $\gamma_1 < \cdots < \gamma_n$ in C with $\gamma_{i_n} \in C'$ we have

$$t(\vec{\gamma}) = \sup\{t(\gamma_{i_1}, \dots, \gamma_{i_{n-1}}, \gamma') \colon \gamma' < \gamma_{i_n} \wedge \gamma' \in C\}.$$

or

(2) For every $\gamma_1 < \cdots < \gamma_n$ in C with $\gamma_{i_n} \in C'$ we have

$$t(\vec{\gamma}) > \sup\{t(\gamma_{i_1}, \ldots, \gamma_{i_{n-1}}, \gamma') \colon \gamma' < \gamma_{i_n} \wedge \gamma' \in C\}.$$

Proof. First note that for all $\gamma_1 < \cdots < \gamma_n$ in C with $\gamma_{i_n} \in C'$ that

$$\sup\{t(\gamma_{i_1}, \ldots, \gamma_{i_{n-1}}, \gamma') \colon \gamma' < \gamma_{i_n} \wedge \gamma' \in C\} =$$
$$\sup\{t(\gamma_{i_1}, \ldots, \gamma_{i_{n-1}}, \gamma') \colon \gamma' < \gamma_{i_n} \wedge t(\gamma_{i_1}, \ldots, \gamma_{i_{n-1}}, \gamma') < t(\gamma_{i_1}, \ldots, \gamma_{i_{n-1}}, \gamma_{i_n})\}$$

The set in the first supremum is contained in the set of the second supremum, so the first supremum is less than or equal to the second supremum. For the other direction note that if $\gamma' < \gamma_{i_n}$, then $\gamma' = w(\vec{\delta})$ for some term w and indiscernibles $\vec{\delta}$ below γ_{i_n}. So, if $t(\gamma_{i_1}, \ldots, \gamma') = t(\gamma_{i_1}, \ldots, w(\vec{\delta})) < t(\gamma_{i_1}, \ldots, \gamma_{i_n})$, then by indiscernibility for large enough $\eta < \gamma_{i_n}$ in C we have $t(\gamma_{i_1}, \ldots, \gamma') \leq t(\gamma_{i_1}, \ldots, \gamma_{i_{n-1}}, \eta)$. Thus, the two suprema above are equal. It follows that there is an $L[x]$ term v such that for all $\vec{\gamma}$ in C with $\gamma_{i_n} \in C'$ we have $v(\vec{\gamma}) = \sup\{t(\gamma_{i_1}, \ldots, \gamma_{i_{n-1}}, \gamma') \colon \gamma' < \gamma_{i_n} \wedge \gamma' \in C\}$. The claim then follows by indiscernibility. □

If the first alternative in claim 5 holds, then we say t is of continuous type, and otherwise of discontinuous type.

We also require the following result.

Claim 6. *Let $t(y_1, \ldots, y_n)$ be a term of type $\pi = (i_1, \ldots, i_n)$. Suppose $j < n$ and $\gamma_1 < \cdots < \gamma_n$ are in C with $\gamma_{i_j} \in C'$. Then*

$$t(j)(\gamma_{i_1}, \ldots, \gamma_{i_j}) := \sup\{t(\gamma_{i_1}, \ldots, \gamma_{i_{j-1}}, \delta_{i_j}, \ldots, \delta_{i_n}) \colon \delta_{i_j} < \gamma_{i_j}, \delta_{i_j}, \ldots, \delta_{i_n} \in C, \text{ and } (\gamma_{i_1}, \ldots, \delta_{i_n}) \text{ is of type } \pi\}$$

is not in the range of $t \restriction C$ (that is, is not in X).

Proof. Note that as in claim 5, the function $t(j)$ defined above is given by an $L[x]$ term $v(\gamma_{i_1}, \ldots, \gamma_{i_j})$. Suppose that

$$t(j)(\gamma_{i_1}, \ldots, \gamma_{i_j}) = v(\gamma_{i_1}, \ldots, \gamma_{i_j}) = t(\delta_{i_1}, \ldots, \delta_{i_n}),$$

where all the ordinals are in C. By indiscernibility, we may move the ordinals so that all of them lie in C', and this equation is still satisfied. Since $j < n$, there is some δ_{i_k} which is not equal to any of the $\gamma_{i_1}, \ldots, \gamma_{i_j}$. Since all the ordinals are in C', we can move δ_{i_k} to a new value $\delta'_{i_k} \in C$ keeping the same relative ordering of the ordinals. By indiscernibility the equation still holds, which contradicts the fact that t is strictly increasing in each argument. □

We now show that X is the intersection of a closed and an open set in ω_1. If we are in case (1) of claim 5, then $\overline{X} - X = \bigcup_{j<n} A_j$ where A_j is the set of ordinals of the form $t(j)(\gamma_{i_1}, \ldots, \gamma_{i_j})$, where all of the ordinals are in C, and $\gamma_{i_j} \in C'$. To see this, suppose $\eta \in \overline{X} - X$. Then η is the increasing limit of a sequence $\eta_n = t(\gamma^n_{i_1}, \ldots, \gamma^n_{i_n})$. Let j be least such that $\{\gamma^n_{i_j}\}_{m\in\omega}$ is not eventually constant. For $k < j$, let γ_{i_k} be the eventual value of the $\gamma^n_{i_k}$, and let $\gamma_{i_j} = \sup_n \gamma^n_{i_j}$. Thus, $\gamma_{i_j} \in C'$. Since t is of type π it follows from the definition of $t(j)$ that $\eta = t(j)(\gamma_{i_1}, \ldots, \gamma_{i_j})$. If $j = n$, then from case (1) we have that $t(n)(\gamma_{i_1}, \ldots, \gamma_{i_n}) = \sup_{\gamma' < \gamma_{i_n}} t(\gamma_{i_1}, \ldots, \gamma_{i_n}) = t(\gamma_{i_1}, \ldots, \gamma_{i_n}) \in X$, a contradiction. So, $\overline{X} - X \subseteq \bigcup_{j<n} A_j$. The reverse inclusion follows from claim 6. Thus, $X = \overline{X} - F$, where $F = \bigcup_{j<n} A_j$. Finally, note that

$\bigcup_{j<n} A_j$ is closed by a similar argument. If we are in case (2) of claim 5, the argument is similar, except we have $\overline{X} - X = \bigcup_{j\leq n} A_j$. □

As we mentioned above, the above proof can be given using the measure analysis on ω_1. Say that a set $A \subseteq \omega_1$ is *very simple* if there is a club $C \subseteq \omega_1$, an $n \in \omega$, and an $h\colon (C)^n \to \omega_1$ with $A = h``(C)^n$ where h satisfies:

(1) $h \restriction (C)^n$ is strictly increasing in each argument.
(2) There is a permutation $\pi = (i_1, \dots, i_n)$ of $\{1, \dots, i_n\}$ with $i_1 = n$ such that for all $\alpha_1 < \cdots < \alpha_n$ in C and all $\beta_1 < \cdots < \beta_n$ in C, $h(\vec{\alpha}) < h(\vec{\beta})$ iff $(\alpha_{i_1}, \dots, \alpha_{i_n}) <_{\text{lex}} (\beta_{i_1}, \dots, \beta_{i_n})$.
(3) The conclusion of claim 5 holds, using the club set C instead of the set of indiscernibles.
(4) The conclusion of claim 6 holds, again using the current club set C.

The analysis of measures on ω_1 (c.f. [2] or [3]) shows that for any measure ν on ω_1, there is an $n \in \omega$ and an $h\colon (\omega_1)^n \to \omega_1$ such that for any $A \subseteq \omega_1$, $\nu(A) = 1$ iff there is a club $C \subseteq \omega_1$ such that $h``(C)^n \subseteq A$. However, for any function $h\colon (\omega_1)^n \to \omega_1$, straightforward partition arguments show that there is an $n' \leq n$ such that h only depends on a subset of its arguments of size n', and that as a function of n' arguments it satisfies the above 4 properties on a club set. Thus, for every measure ν on ω_1, there is a very simple set A such that $\nu(A) = 1$. Given this, the usual argument (due to Kunen) using the fact that every countably additive ideal on a $\lambda < \Theta$ can be extended to a measure (i.e., every set in the ideal has measure zero) shows that every subset of ω_1 is a countable union of very simple sets. The proof of Theorem 4.1 now shows that every very simple set is an intersection of a closed and an open set (the four properties above were all that was used in this argument).

Theorem 4.1 can be pushed a little higher, which we do in the next theorem. We can again either prove this result using the measure analysis or using indiscernibles; we again give the proof using indiscernibles. The proof is a slight extension of the proof of Theorem 4.1.

Theorem 4.2 ($\mathsf{ZF}+\mathsf{AD}$)**.** *For every $\lambda < \omega_2$, every subset of λ is Borel and can be written as $\bigcup_n (F_n \cap U_n)$ where F_n is closed, U_n is open.*

Proof. Fix $\lambda < \omega_2$ and $A \subseteq \lambda$. Let $\prec$ be a wellordering of ω_1 of length λ. Fix a real x such that $\prec$ lies in $L[x]$. We may write $\prec = u(\alpha_1, \dots, \alpha_a, \omega_1, \dots, \omega_b)$ for some $\alpha_1 < \cdots < \alpha_a$ in C (the set of indiscernibles for $L[x]$) and some $L[x]$ term u. Let $A' = \{\gamma < \omega_1 \colon |\gamma|_\prec \in A\}$. Increasing the set $\vec{\alpha}$ and the value of b if necessary, we may assume that A' is definable from $\vec{\alpha}$ and $\omega_1, \dots, \omega_b$. Thus, $A = w(\vec{\alpha}, \omega_1, \dots, \omega_b)$ for some term w.

We have the notions of type and normal type for ordinal below ω_1 from the proof of Theorem 4.1. We extend these to ordinals $\omega_1 < \alpha < \lambda$ as follows: we say α has type T if $\gamma < \omega_1$ has type T, where γ is the unique ordinal such that $|\gamma|_\prec = \alpha$. Every α between ω_1 and λ has type T for some normal type T. Also, if $\omega_1 < \alpha_1 < \alpha_2 < \lambda$ both have type T, then $\alpha_1 \in A$ iff $\alpha_2 \in A$ by indiscernibility [Say $\alpha_1 = |\gamma_1|_\prec$, where $\gamma_1 = t(\vec{\beta}, \vec{\epsilon})$, $\alpha_2 = |\gamma_2|_\prec$, where $\gamma_2 = t(\vec{\beta}, \vec{\rho})$, where $\vec{\beta} < \max(\vec{\alpha})$ and t corresponds to the type T. Then $\alpha_1 \in A$ iff the rank of $t(\vec{\beta}, \vec{\epsilon})$ in $u(\vec{\alpha}, \omega_1, \dots, \omega_b)$ is in $w(\vec{\alpha}, \omega_1, \dots, \omega_b)$ iff the rank of $t(\vec{\beta}, \vec{\rho})$ in $u(\vec{\alpha}, \omega_1, \dots, \omega_b)$ is in $w(\vec{\alpha}, \omega_1, \dots, \omega_b)$ iff $\alpha_2 \in A$.]

It suffices therefore to fix a normal type T (and corresponding term t and ordinals $\vec{\beta} < \max(\vec{\alpha})$) and show that X is the intersection of a closed and an open set, where X is the set of α between ω_1 and ω_2 of type T. For the rest of the argument we again suppress writing the fixed ordinals $\vec{\alpha}$, $\vec{\beta}$ and the $\omega_1, \ldots, \omega_b$. Corresponding to the term t we define the term t' by

$$t'(\gamma_1, \ldots, \gamma_n) = |t(\gamma_1, \ldots, \gamma_n)|_{\prec}.$$

So, t' defines a function from $(C)^n$ to λ. Since the map $\gamma \mapsto |\gamma|_{\prec}$ is one-to-one, and since T is normal (so t is increasing in each argument), an easy argument shows that t' is also increasing in each argument. An analog of claim 4 holds, namely, there is a permutation $\pi = (i_1, \ldots, i_n)$ of $(1, \ldots, n)$ such that for all $\alpha_1 < \cdots < \alpha_n \in C$, $\beta_1 < \cdots < \beta_n \in C$, we have $t'(\vec{\alpha}) < t'(\vec{\beta})$ iff $(\alpha_{i_1}, \ldots, \alpha_{i_n}) <_{\text{lex}} (\beta_{i_1}, \ldots, \beta_{i_n})$ (the difference is that now we do not necessarily have that $i_1 = n$). The proofs of claims 5 and 6 carry over to t' as well. We then define the A_j and F exactly as in the proof of Theorem 4.1, and the same proof gives that $X = \overline{X} - F$. □

As an immediate corollary we have the following.

Corollary 4.3 ($\mathsf{ZF} + \mathsf{AD}$). *For any ordinals $\alpha, \beta < \omega_2$, α and β are Borel isomorphic iff $|\alpha| = |\beta|$.*

Theorem 4.2 does not hold for any $\alpha \geq \omega_2$ (assuming again AD). This is because there are stationary, costationary subsets of ω_2, for example, the set of ordinals of cofinality ω.

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