

# THE LACZKOVICH–KOMJÁTH PROPERTY FOR COANALYTIC EQUIVALENCE RELATIONS

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ABSTRACT. Let  $E$  be a coanalytic equivalence relation on a Polish space  $X$  and  $(A_n)_{n \in \omega}$  a sequence of analytic subsets of  $X$ . We prove that if  $\limsup_{n \in K} A_n$  meets uncountably many  $E$ -equivalence classes for every  $K \in [\omega]^\omega$ , then there exists a  $K \in [\omega]^\omega$  such that  $\bigcap_{n \in K} A_n$  contains a perfect set of pairwise  $E$ -inequivalent elements.

## 1. INTRODUCTION

Let  $(A_n)_{n \in \omega}$  be a sequence of sets and  $K \in [\omega]^\omega$  an infinite subset of  $\omega$ . The limit superior  $\limsup_{n \in K} A_n$  is the set of all elements which belong to  $A_n$  for infinitely many  $n \in K$ . Laczkovich [6] showed that for every sequence  $(A_n)_{n \in \omega}$  of Borel sets in a Polish space, if  $\limsup_{n \in K} A_n$  is uncountable for every  $K \in [\omega]^\omega$ , then there exists a  $K \in [\omega]^\omega$  such that  $\bigcap_{n \in K} A_n$  is uncountable. Komjáth [5] generalized this result to the case where the sets  $(A_n)_{n \in \omega}$  are analytic. Note that by the perfect set property of analytic sets, if  $\bigcap_{n \in K} A_n$  is uncountable, then it contains a perfect set.

Balcerzak and Głab [1] extended these results to  $F_\sigma$  equivalence relations in the following way.

**Definition.** An equivalence relation  $E$  on a Polish space  $X$  is has the *Laczkovich–Komjáth property* if for every sequence  $(A_n)_{n \in \omega}$  of analytic subsets of  $X$  such that  $\limsup_{n \in K} A_n$  meets uncountably many  $E$ -equivalence classes for every  $K \in [\omega]^\omega$ , there exists a  $K \in [\omega]^\omega$  such that  $\bigcap_{n \in K} A_n$  contains a perfect set of pairwise  $E$ -inequivalent elements.

In this terminology, Komjáth has shown that the identity relation  $=$  has the Laczkovich–Komjáth property. Balcerzak and Głab [1] proved that every  $F_\sigma$  equivalence relation has the Laczkovich–Komjáth property. In this paper, we generalize this to coanalytic equivalence relations.

**Theorem 1.** *Every coanalytic equivalence relation on a Polish space has the Laczkovich–Komjáth property.*

A fundamental result on coanalytic equivalence relations is Silver’s theorem: a coanalytic equivalence relation either has only countably many equivalence classes, or else there exists a perfect set of pairwise inequivalent elements. Silver’s original proof [9] used forcing. Harrington (unpublished) later gave a simpler (forcing) proof

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using effective descriptive set theory, which nowadays is usually cast in terms of the Gandy–Harrington topology. We will use similar methods and assume familiarity with effective descriptive set theory throughout the paper.

An introduction to effective descriptive set theory is given in [7], where the reader can also find the topological version of Harrington’s proof. The review in [4] provides details on the Gandy–Harrington topology and strong Choquet games. Instead of strong Choquet games, we will make use of the set of low elements, which is a Polish space in the Gandy–Harrington topology. We will summarize the technical facts we use later on. Further details can be found in [2], which also provides another source on effective descriptive set theory.

This paper is organized as follows. In Section 2 we review a well-known coding mechanism for  $\Pi_1^1$  and  $\Delta_1^1$  sets, mainly to fix notation and establish the uniformity of a diagonal intersection operator. In Section 3, we provide details on canonical cofinal sequences as developed in [3]. We use these sequences in Section 4 to prove our main technical result. Finally, we derive our main theorem in Section 5, where we also derive a parametric version of the theorem, as was done by Balcerzak and Głab [1].

## 2. CODING $\Pi_1^1$ AND $\Delta_1^1$ SETS

In this section we review a well-known coding mechanism for  $\Pi_1^1$  and  $\Delta_1^1$  sets, mainly to fix notation. A good introduction can be found in [4, Section 3.2], where the notion of uniformity is also discussed. We will need the uniformity of a diagonal intersection operation. Since this operation is not canonical, we provide a little more of the details.

A *product space* is any  $X = X_0 \times \cdots \times X_n$  (with the product topology), where each factor is either  $\omega$  or  $\omega^\omega$ . For every product space  $X$  there is a  $U^X \subseteq \omega \times X$  such that  $U^X \in \Pi_1^1$  and for any  $A \subseteq X$ ,  $A \in \Pi_1^1$  iff  $\exists n(A = U_n^X)$ . Such a set  $U^X$  is called a *universal  $\Pi_1^1$  set*. A  $\Pi_1^1$  *code* for  $A \subseteq X$  is any  $n \in \omega$  such that  $A = U_n^X$ .

There exists a collection  $\{U^X\}$  of universal  $\Pi_1^1$  sets with the following additional property: for any  $m \in \omega$  and any product space  $X$  there is a recursive function  $S^{m,X} : \omega^{m+1} \rightarrow \omega$  such that

$$(e, k_1, \dots, k_m, x) \in U^{\omega^m \times X} \Leftrightarrow (S^{m,X}(e, k_1, \dots, k_m), x) \in U^X.$$

Such a collection is called a *good universal system*. For the rest of this paper, fix a good universal system  $\{U^X\}$  for  $\Pi_1^1$ . This good universal system can be used to code  $\Delta_1^1$  subsets, which we now describe. This coding is always relative to a particular product space  $X$ . When there is no danger of confusion, we will drop the superscript in  $U^X$ . For every  $k \in \omega$ , fix a recursive bijection  $(n_1, \dots, n_k) \mapsto \langle n_1, \dots, n_k \rangle$  between  $\omega^k$  and  $\omega$ . Define

$$\begin{aligned} (\langle m, n \rangle, x) \in U_0 &\Leftrightarrow (m, x) \in U, \\ (\langle m, n \rangle, x) \in U_1 &\Leftrightarrow (n, x) \in U. \end{aligned}$$

Then  $U_0, U_1 \in \Pi_1^1$ . By the reduction property for  $\Pi_1^1$  sets, there are  $\Pi_1^1$  sets  $U_0^*, U_1^* \subseteq \omega \times X$  such that  $U_0^* \cup U_1^* = U_0 \cup U_1$  and  $U_0^* \cap U_1^* = \emptyset$ . Let  $P = U_0^*$  and  $S = (\omega \times X) \setminus U_1^*$ . Define

$$\langle m, n \rangle \in C \Leftrightarrow \forall x \in X ((\langle m, n \rangle, x) \in U_0^* \vee (\langle m, n \rangle, x) \in U_1^*)$$

Then  $C \in \Pi_1^1$ . For all  $n \in C$ ,

$$P_n = S_n := D_n.$$

A  $\Delta_1^1$  code for  $A \subseteq X$  is any  $n \in C$  such that  $A = D_n$ . In that case,  $(n)_0$  is a  $\Pi_1^1$  code for  $A$  and  $(n)_1$  is a  $\Pi_1^1$  code for  $X \setminus A$ . Conversely, if  $m, n \in \omega$  are  $\Pi_1^1$  codes for  $A$  and  $X \setminus A$ , respectively, then  $\langle m, n \rangle$  is a  $\Delta_1^1$  code for  $A$ . It is important that the set  $C$  of  $\Delta_1^1$  codes is  $\Pi_1^1$  and that operations hold effectively in the codes, in the following way.

**Example.** Given  $\Delta_1^1$  codes  $m, n \in C$  for  $A, B \subseteq X$ , we can effectively compute a  $\Delta_1^1$  code for  $A \setminus B$ . To see this, define

$$\begin{aligned} (m, n, x) \in Z_0 &\Leftrightarrow x \in D_m \wedge x \notin D_n, \\ (m, n, x) \in Z_1 &\Leftrightarrow x \notin D_m \vee x \in D_n. \end{aligned}$$

Clearly,  $Z_0, Z_1 \in \Pi_1^1$ . Let  $e_0, e_1$  be their respective  $\Pi_1^1$  codes. Then for  $i = 0, 1$ ,

$$(m, n, x) \in Z_i \Leftrightarrow (e_i, m, n, x) \in U^{\omega^2 \times X} \Leftrightarrow (S^{2,X}(e_i, m, n), x) \in U^X.$$

Also,  $Z_0 = (\omega^2 \times X) \setminus Z_1$ . Thus,

$$\langle S^{2,X}(e_0, m, n), S^{2,X}(e_1, m, n) \rangle$$

is a  $\Delta_1^1$  code for  $A \setminus B$ .

Similar uniformities hold for all basic set-theoretic operations. We will need the uniformity of a diagonal intersection operator, which we define next. Recall that when  $H, K \in [\omega]^\omega$ ,  $H \subseteq^* K$  denotes that  $H$  is almost contained in  $K$ , i.e.  $K \setminus H$  is finite.

**Definition.** For a (finite or infinite) sequence  $(K_n)$  of infinite subsets of  $\omega$  with  $K_n \subseteq^* K_m$  for  $n > m$ , define  $\Delta K_n$  by  $m \in \Delta K_n$  iff there exists  $m_0 < m_1 < \dots < m_k = m$  such that  $m_0$  is the least element of  $K_0$ ,  $m_1$  is the least element of  $K_0 \cap K_1$  such that  $m_1 > m_0$ ,  $\dots$ ,  $m_k$  is the least element of  $K_0 \cap \dots \cap K_k$  such that  $m_k > m_{k-1}$ .

Note that  $\Delta K_n \subseteq^* K_m$  for all  $m$ . To obtain the desired uniformity for this diagonal intersection operation, we need to assume that the sequence of  $\Delta_1^1$  codes for  $(K_n)$  is effective. One way to formalize this is to let  $n \in C^*$  iff

- (1)  $n \in C^\omega$ ,
- (2)  $D_n^\omega$  is infinite,
- (3)  $\forall m (m \in D_n^\omega \Rightarrow (m)_1 \in C^\omega)$ , and
- (4)  $\forall i \exists! m (m \in D_n^\omega \wedge (m)_0 = i)$ .

Informally,  $n \in C^*$  iff  $n$  is a  $\Delta_1^1$  code for an infinite subset of  $\omega$  of the form  $\{\langle i, n_i \rangle : i \in \omega, n_i \in C\}$ . Clearly,  $C^* \in \Pi_1^1$ .

**Lemma 2.** *There is a function  $\text{Diag}: \omega \rightarrow \omega$  which is  $\Delta_1^1$  on  $C^*$  such that whenever  $n \in C^*$  is a code for an infinite  $\Delta_1^1$  subset  $\{\langle i, n_i \rangle : i \in \omega, n_i \in C\}$  of  $\omega$ ,  $\text{Diag}(n)$  is a  $\Delta_1^1$  code for  $\Delta D_{n_i}^\omega$ .*

*Proof.* It suffices to find  $\Pi_1^1$  codes  $e_0$  and  $e_1$  for  $\Delta D_{n_i}^\omega$  and  $\omega \setminus \Delta D_{n_i}^\omega$ , respectively, because  $\langle e_0, e_1 \rangle$  will then be a  $\Delta_1^1$  code for  $\Delta D_{n_i}^\omega$ . We need the following 3 auxiliary functions:

- (1) There is a recursive function  $u: \omega \rightarrow \omega$  such that whenever  $n = \langle n_0, \dots, n_k \rangle$  is a finite sequence of  $\Delta_1^1$  codes,  $u(n)$  is a  $\Delta_1^1$  code for  $D_{n_0}^\omega \cap \dots \cap D_{n_k}^\omega$ .

- (2) There is a  $\Delta_1^1$  on the codes function  $i: \omega \times \omega \rightarrow \omega$  such that whenever  $n \in C^*$ ,  $i(n, j)$  is the (unique)  $m \in \omega$  such that  $\langle j, m \rangle \in D_n^\omega$ .
- (3) There is a  $\Delta_1^1$  on the codes function  $\mu: \omega \times \omega \rightarrow \omega$  such that whenever  $n$  is a  $\Delta_1^1$  code for an infinite subset of  $\omega$ ,  $\mu(n, j)$  is the least element of  $D_n^\omega$  greater than or equal to  $j$ .

Now define

$$(n, m) \in Z_0 \Leftrightarrow n \in C^* \wedge \exists \langle m_0, \dots, m_k \rangle (m_0 < \dots < m_k \wedge m_k = m \wedge \\ m_0 = \mu(u(\langle i(n, 0) \rangle), 0) \wedge m_1 = \mu(u(\langle i(n, 0), i(n, 1) \rangle), m_0 + 1) \wedge \\ \dots \wedge m_k = \mu(u(\langle i(n, 0), \dots, i(n, k) \rangle), m_{k-1} + 1)).$$

Then  $Z_0 \in \Pi_1^1$ . Pick a  $\Pi_1^1$  code  $e_0$  for  $Z_0$ . Similarly, we can write down a  $\Pi_1^1$  definition for  $Z_1 = C^* \setminus Z_0$  and pick a  $\Pi_1^1$  code  $e_1$ . The rest of the argument is as in the Example.  $\square$

Now that we have established the uniformity of this diagonal intersection operator, we will use it implicitly. Finally, for codes  $h, k \in C^\omega$ , we write  $h \subseteq^* k$  iff the set coded by  $h$  is almost contained in the set coded by  $k$ . Writing out the definitions, we see that  $h \subseteq^* k$  is  $\Delta_1^1$  on the set  $C^\omega$  of codes.

### 3. CANONICAL COFINAL SEQUENCES

For  $w \in 2^\omega$ , define a binary relation  $<_w$  on a subset of  $\omega$  by

$$m <_w n \Leftrightarrow w(\langle m, n \rangle) = 1.$$

The domain of  $<_w$  is the set

$$\text{dom}(<_w) = \{n \in \omega : \exists m \in \omega (m <_w n \text{ or } n <_w m)\}.$$

Let  $\text{LO}$  denote the set of all  $w \in 2^\omega$  such that  $<_w$  is a linear order, and let  $\text{LO}^*$  denote the set of all  $w \in \text{LO}$  such that  $<_w$  has a least element and every  $n \in \text{dom}(<_w)$  has an immediate successor  $n_{<_w}^+$ . For  $w \in \text{LO}$ , let  $|\langle <_w \rangle|$  denote the order type of  $<_w$ . The next lemma shows that in a uniform way, we can effectively obtain a canonical cofinal sequence in  $<_w$  given  $w \in \text{LO}^*$ .

**Lemma 3** (Gao–Jackson–Laczkovich–Mauldin [3]). *There is a  $\Delta_1^1$  function*

$$\text{Cof} : \{(w, n, j) \in \text{LO}^* \times \omega^2 : n \in \text{dom}(<_w)\} \rightarrow \omega$$

such that

- (1) if  $w \in \text{LO}^*$ ,  $n \in \text{dom}(<_w)$  and  $j \in \omega$ , then  $\text{Cof}(w, n, j) \in \text{dom}(<_w)$  and  $\text{Cof}(w, n, j) <_w n$ , unless  $n$  is the  $<_w$ -least element;
- (2) if  $w \in \text{LO}^*$  and  $n \in \text{dom}(<_w)$  has an immediate predecessor in  $<_w$ , then  $\text{Cof}(w, n, j)_w^+ = n$  for all  $j \in \omega$ ;
- (3) if  $w \in \text{LO}^*$ ,  $n \in \text{dom}(<_w)$  is not  $<_w$ -least and  $n$  does not have an immediate predecessor in  $<_w$ , then
  - (a) if  $j < j'$ , then  $\text{Cof}(w, n, j) <_w \text{Cof}(w, n, j')$ , and
  - (b) for any  $q \in \text{dom}(<_w)$  with  $q <_w n$  there is a  $j \in \omega$  such that  $q <_w \text{Cof}(w, n, j)$ .  $\square$

We also need a variation of this lemma for  $\Pi_1^1$  norms, whose proof uses the same ideas. Recall that a  $\Pi_1^1$ -norm on a pointset  $P \in \Pi_1^1$  is a function  $\varphi$  from  $P$  into

the ordinals  $\text{On}$  such that there exist binary relations  $<_\varphi^*$  and  $\leq_\varphi^*$  in  $\Pi_1^1$  with the following properties:

$$\begin{aligned} x \leq_\varphi^* y &\Leftrightarrow P(x) \wedge (\neg P(y) \vee \varphi(x) \leq \varphi(y)), \\ x <_\varphi^* y &\Leftrightarrow P(x) \wedge (\neg P(y) \vee \varphi(x) < \varphi(y)). \end{aligned}$$

Recall that  $\text{WO}$  denotes the set of all  $w \in \text{LO}$  such that  $<_w$  is a well-order. Every  $\Pi_1^1$  set  $P \subseteq \omega$  admits a  $\Pi_1^1$ -norm  $\varphi: P \rightarrow \omega_1^{\text{CK}}$ , where

$$\omega_1^{\text{CK}} = \sup\{|\langle w \rangle| : w \in \text{WO is recursive}\},$$

see for example [8, Section 4B].

**Lemma 4.** *Let  $\varphi$  be a  $\Pi_1^1$ -norm on a  $\Pi_1^1$  set  $P \subseteq \omega$ . There is a  $\Pi_1^1$  function  $\text{Cof}: \omega \rightarrow \omega$  such that*

- (1) for all  $j \in \omega$ ,  $\text{Cof}(j) \in P$ ;
- (2) if  $j < j'$ , then  $\text{Cof}(j) <_\varphi^* \text{Cof}(j')$  unless  $\text{Cof}(j)$  is  $<_\varphi$ -maximal.
- (3) for any  $q \in P$ , there is an  $j \in \omega$  such that  $q <_\varphi^* \text{Cof}(j)$  unless  $q$  is  $<_\varphi$ -maximal.

*Proof.* We define the function  $\text{Cof}$  by induction on  $j$ . Let  $p_0 = \text{Cof}(0)$  be the least integer in  $P$ . Assume we have defined  $p_j = \text{Cof}(j)$ . If  $p_j$  is  $<_\varphi$ -maximal, let  $p_{j+1} = p_j$ . Otherwise, let  $p_{j+1} = \text{Cof}(j+1)$  be the smallest integer in  $P$  such that  $p_j < p_{j+1}$  and  $p_j <_\varphi^* p_{j+1}$ . Since  $n = p_{j+1}$  iff  $n \in P$  and  $p_j < n$  and  $p_j <_\varphi^* n$  and  $\forall m(p_j < m < n \Rightarrow m \leq_\varphi^* p_j)$ , this defines a  $\Pi_1^1$  function. To see that (3) holds, let  $q \in P$  be a nonmaximal element. Since the sequence  $(p_j)_{j \in \omega}$  is strictly increasing in the natural order  $<$  on  $\omega$ , there is a least integer  $j$  such that  $p_j \leq q < p_{j+1}$ . Because  $p_{j+1}$  is the least integer larger than  $p_j$  such that  $p_j <_\varphi^* p_{j+1}$ , we cannot have  $p_j <_\varphi^* q$ . Hence,  $q \leq_\varphi^* p_j <_\varphi^* p_{j+1}$ .  $\square$

#### 4. A COMPLETELY GOOD PAIR

Suppose  $E$  is a  $\Pi_1^1$  equivalence relation on  $\omega^\omega$ . A key idea in Harrington's proof of Silver's dichotomy is to consider the set

$$W = \{x \in \omega^\omega : \text{there is no } \Delta_1^1 \text{ set } D \text{ such that } x \in D \subseteq [x]_E\}.$$

A computation shows that  $W$  is  $\Sigma_1^1$ . Moreover, when  $E$  has uncountably many equivalence classes,  $W \neq \emptyset$  and every nonempty  $\Sigma_1^1$  subset  $X \subseteq W$  meets uncountably many  $E$ -equivalence classes. In fact, a nonempty  $\Sigma_1^1$  subset  $X \subseteq \omega^\omega$  meets uncountably many  $E$ -equivalence classes iff  $X \cap W \neq \emptyset$ .

We will establish the following corresponding result in our context.

**Proposition 5.** *Let  $E$  be a  $\Pi_1^1$  equivalence relation on  $\omega^\omega$  and  $(A_n)_{n \in \omega}$  a sequence of uniformly  $\Sigma_1^1$  subsets of  $\omega^\omega$ . If  $\limsup_{n \in K} A_n$  meets uncountably many  $E$ -equivalence classes for every  $K \in [\omega]^\omega$ , then there exists a nonempty  $\Sigma_1^1$  set  $V \subseteq \omega^\omega$  and a  $\Delta_1^1$  set  $H \in [\omega]^\omega$  such that for every nonempty  $\Sigma_1^1$  set  $X \subseteq V$  and every  $\Delta_1^1$  set  $K \in [H]^\omega$  the set  $X \cap \limsup_{n \in K} A_n$  meets uncountably many  $E$ -equivalence classes.*

We call such a pair  $(V, H)$  *completely good*. The rest of this section is devoted to the proof of Proposition 5 and a further refinement. In contrast with Harrington's proof, we need a recursive construction of transfinite length, in which we remove all possible 'bad pairs' one by one.

**Definition.** We say that  $n = \langle y, k \rangle \in \omega$  is a *bad pair* if the following properties hold:

- (1)  $y \in C^{\omega^\omega}$  and  $k \in C^\omega$ ,
- (2)  $D_k^\omega \in [\omega]^\omega$ , and
- (3)  $D_y^{\omega^\omega} \cap \limsup_{n \in D_k^\omega} A_n$  meets only countably many  $E$ -equivalence classes, i.e.  $D_y^{\omega^\omega} \cap W \cap \limsup_{n \in D_k^\omega} A_n = \emptyset$ .

It is clear from this definition that the set  $P \subseteq \omega$  of all bad pairs is  $\Pi_1^1$ . Let  $\varphi: P \rightarrow \omega_1^{\text{CK}}$  be a  $\Pi_1^1$ -norm on  $P$ . Define a well-order on  $P$  by

$$m <_\varphi n \Leftrightarrow \varphi(m) < \varphi(n) \vee (\varphi(m) = \varphi(n) \wedge m < n)$$

and let  $\leq_\varphi^*$  be the  $\Pi_1^1$  relation given by

$$m \leq_\varphi^* n \Leftrightarrow P(m) \wedge (\neg P(n) \vee \varphi(m) \leq \varphi(n)).$$

For the rest of the paper, let  $\text{Cof}: \omega \rightarrow \omega$  be  $\Pi_1^1$  function related to  $\varphi$  and  $P$  as given by Lemma 4.

Denote by  $C_\infty^\omega$  the set of all  $n \in C^\omega$  such that  $D_n^\omega \in [\omega]^\omega$ . Then  $C_\infty^\omega$  is  $\Pi_1^1$ . Given an  $h \in C_\infty^\omega$ , we define the *next bad pair relative to  $h$*  to be the  $<_\varphi$ -least  $\langle y, k \rangle \in P$  such that  $k \subseteq^* h$ . Set  $R(h, \langle y, k \rangle)$  iff  $\langle y, k \rangle$  is the next bad pair relative to  $h$ .

**Lemma 6.** *The relation  $R \subseteq \omega \times \omega$  is  $\Pi_1^1$ . Moreover,  $R$  is a  $\Delta_1^1$  function on the set  $B = \{h \in \omega: h \in C_\infty^\omega \wedge \exists n(R(h, n))\}$ .*

*Proof.* We have  $R(h, \langle y, k \rangle)$  iff

$$h \in C_\infty^\omega \wedge \langle y, k \rangle \in P \wedge k \subseteq^* h \wedge \forall y', k' \in \omega (\langle y, k \rangle \not\leq_\varphi^* \langle y', k' \rangle \Rightarrow k' \not\subseteq^* h).$$

This is a  $\Pi_1^1$  definition. If  $R(h, n)$  holds, then  $n$  is the unique such integer. Thus, for  $h \in B$ ,  $\neg R(h, n) \Leftrightarrow \exists m(R(h, m) \wedge n \neq m)$ , which is  $\Pi_1^1$ . Hence,  $R$  is  $\Delta_1^1$  on  $B$ .  $\square$

Similarly, given a  $\Pi_1^1$  set  $A \subseteq P$  we define the next bad pair in  $A$  relative to  $h$  to be the  $<_\varphi$ -least  $\langle y, k \rangle \in A$  such that  $k \subseteq^* h$ . The corresponding version of Lemma 6 still holds.

Initial segments of the recursive construction can be coded by reals, as follows. Recall that  $\text{WO}_\alpha = \{w \in \text{WO}: |<_w| = \alpha\}$  and for  $\alpha < \omega_1^{\text{CK}}$ , we have  $\text{WO}_\alpha \in \Delta_1^1$ .

**Definition.** Let  $\alpha < \omega_1^{\text{CK}}$ . A real  $z \in \omega^\omega$  is  $\alpha$ -adequate if  $z = \langle w, v, h \rangle$ , where  $w \in 2^\omega$ ,  $v \in \omega^\omega$ , and  $h \in \omega^\omega$ , and the following conditions are satisfied:

- (1)  $w \in \text{WO}_\alpha$ ,
- (2) if  $n \notin \text{dom}(<_w)$ , then  $v(n) = h(n) = 0$ ,
- (3) the  $<_w$ -least element is the  $<_\varphi$ -least element,
- (4) if  $n \in \text{dom}(<_w)$  is a  $<_w$ -successor (say  $n = m_{<_w}^+$ ), then the following holds:
  - (a)  $n = \langle y, k \rangle$  is the next bad pair relative to  $h(m)$  such that  $\langle y, k \rangle \notin \text{dom}(<_w) \upharpoonright n$ ,
  - (b)  $v(n)$  is a canonical code for  $D_{v(m)}^{\omega^\omega} \setminus D_y^{\omega^\omega}$ ,
  - (c)  $h(n) = k$ .
- (5) if  $n \in \text{dom}(<_w)$  is a  $<_w$ -limit, then with  $v'$  the canonical code for

$$\bigcap_{j \in \omega} D_{v(\text{Cof}(w, n, j))}^{\omega^\omega}$$

and  $h'$  the canonical code for  $\Delta_{j \in \omega} D_{h(\text{Cof}(w, n, j))}^{\omega^\omega}$ , the following holds:

- (a)  $n = \langle y, k \rangle$  is the next pair relative to  $h'$  such that  $\langle y, k \rangle \notin \text{dom}(\langle \cdot \rangle_w) \upharpoonright n$ ,
- (b)  $v(n)$  is the canonical code for  $D_{v'}^{\omega^\omega} \setminus D_y^{\omega^\omega}$ , and
- (c)  $h(n) = k$ .

Some comments on these conditions: (1) says that  $z$  represents the construction up to stage  $\alpha$ , (2) is needed only to ensure that there can be at most one  $\alpha$ -adequate real for every  $\alpha < \omega_1^{\text{CK}}$ , (3), (4a), and (5a) state that  $\langle \cdot \rangle_w$  represents the order in which the bad pairs are picked in our construction and that we pick a new bad pair at each stage, and conditions (4b,c) and (5b,c) require  $v(n)$  and  $h(n)$  to be codes for the correct sets whenever  $n \in \text{dom}(\langle \cdot \rangle_w)$ .

We call a real *adequate* if it is  $\alpha$ -adequate for some  $\alpha < \omega_1^{\text{CK}}$ .

**Lemma 7.** *The set of all adequate reals is  $\Pi_1^1$ .*

*Proof.* Replace condition (1) above with condition (1)'  $w \in \text{WO}_{<\omega_1^{\text{CK}}}$ , which is  $\Pi_1^1$ . Conditions (2) and (3) are arithmetical. For (4),  $n$  is a  $\langle \cdot \rangle_w$ -successor,  $n = (m)_{\langle \cdot \rangle_w}^+$ , and (4b, c) are arithmetical predicates, while (4a) is  $\Pi_1^1$ . Thus, (4) is  $\Pi_1^1$ . Similarly, (5) is  $\Pi_1^1$ .  $\square$

It is immediate from the definition of  $\alpha$ -adequate that for each  $\alpha < \omega_1^{\text{CK}}$ , if there is an  $\alpha$ -adequate real, then this real is unique; denote it by  $z_\alpha$ .

**Lemma 8.** *Every adequate real is  $\Delta_1^1$ .*

*Proof.* Let  $z_\alpha$  be  $\alpha$ -adequate for some fixed  $\alpha < \omega_1^{\text{CK}}$ . Then  $z = \langle w, v, h \rangle$  equals  $z_\alpha$  iff  $z$  satisfies conditions (1) through (5). The first 3 conditions are  $\Delta_1^1$ . Conditions (4) and (5) are  $\Pi_1^1$ , because (4a) and (5a) contain a predicate  $R(n, h)$ , i.e.  $n$  is the next bad pair relative to  $h$  (where  $h = h(m)$  in 4a and  $h = h'$  in 5a). However, since  $z$  is given, we know that this  $h$  is an element of  $B = \{h \in \omega : h \in C_\infty^\omega \wedge \exists n(R(h, n))\}$ . By Lemma 6,  $R$  is  $\Delta_1^1$  on  $B$ . Thus, conditions (4) and (5) are  $\Delta_1^1$  in this case.  $\square$

Finally, we define  $V \subseteq \omega^\omega$  and  $H \in [\omega]^\omega$  as follows. Let  $x \in V$  iff

$$\forall z \in \Delta_1^1(z = \langle w, v, h \rangle \text{ adequate} \Rightarrow \forall n(n \in \text{dom}(\langle \cdot \rangle_w) \Rightarrow x \in D_{v(n)}^{\omega^\omega}))$$

and  $n \in H$  iff

$$\begin{aligned} \exists z \in \Delta_1^1(z = \langle w, v, h \rangle \text{ is adequate} \wedge \\ \forall j \leq n(\text{Cof}(j) \in \text{dom}(\langle \cdot \rangle_w) \Rightarrow n \in \Delta_{j \leq n} h(\text{Cof}(j))). \end{aligned}$$

Equivalently by Lemma 8,  $n \in H$  iff

$$\begin{aligned} \forall z \in \Delta_1^1(z = \langle w, v, h \rangle \text{ is adequate} \wedge \\ \forall j \leq n(\text{Cof}(j) \in \text{dom}(\langle \cdot \rangle_w) \Rightarrow n \in \Delta_{j \leq n} h(\text{Cof}(j))). \end{aligned}$$

**Lemma 9.**  *$V \in \Sigma_1^1$  and  $H \in \Delta_1^1$ . Moreover,  $V \neq \emptyset$  and  $H \in [\omega]^\omega$ .*

*Proof.* By Kleene's restricted quantification theorem (see for example [8, Theorem 4D.3]),  $V \in \Sigma_1^1$ . (Note: if the construction stops below  $\omega_1^{\text{CK}}$ , then  $V$  is actually  $\Delta_1^1$  but we will not need that fact.) Similarly, the first definition of  $H$  is  $\Pi_1^1$  and the second definition is  $\Sigma_1^1$ . Therefore,  $H \in [\omega]^\omega$  is  $\Delta_1^1$ . We show that  $V \neq \emptyset$ .

Suppose towards a contradiction that  $V = \emptyset$ . Then for every  $x \in \omega^\omega$  there is an  $\alpha < \omega_1^{\text{CK}}$  and a  $k \in \omega$  such that for  $z_\alpha = \langle w_\alpha, v_\alpha, h_\alpha \rangle$ , we have  $k \in \text{dom}(\langle \cdot \rangle_{w_\alpha})$  and

$x \notin D_{v_\alpha(k)}^{\omega^\omega}$ . For  $k \in \text{dom}(\langle w_\alpha \rangle)$ , denote by  $y_\alpha(k)$  the code for the set removed at that stage. By assumption,

$$\omega^\omega = \bigcup_{\alpha < \omega_1^{\text{CK}}} \bigcup_{k \in \text{dom}(\langle w_\alpha \rangle)} D_{y_\alpha(k)}^{\omega^\omega}.$$

Since  $H \subseteq^* D_{h_\alpha(k)}^\omega$  for every  $k \in \text{dom}(\langle w_\alpha \rangle)$ ,

$$\limsup_{n \in H} A_n \subseteq \limsup_{n \in D_{h_\alpha(k)}^\omega} A_n.$$

In particular for every  $k \in \text{dom}(\langle w_\alpha \rangle)$ ,

$$D_{y_\alpha(k)}^{\omega^\omega} \cap \limsup_{n \in H} A_n \subseteq D_{y_\alpha(k)}^{\omega^\omega} \cap \limsup_{n \in D_{h_\alpha(k)}^\omega} A_n.$$

Hence,

$$\begin{aligned} \limsup_{n \in H} A_n &= \bigcup_{\alpha < \omega_1^{\text{CK}}} \bigcup_{k \in \text{dom}(\langle w_\alpha \rangle)} D_{y_\alpha(k)}^{\omega^\omega} \cap \limsup_{n \in H} A_n \\ &\subseteq \bigcup_{\alpha < \omega_1^{\text{CK}}} \bigcup_{k \in \text{dom}(\langle w_\alpha \rangle)} D_{y_\alpha(k)}^{\omega^\omega} \cap \limsup_{n \in D_{h_\alpha(k)}^\omega} A_n \end{aligned}$$

meets only countably many  $E$ -equivalence classes, a contradiction. Thus,  $V \neq \emptyset$ .  $\square$

We now verify that the pair  $(V, H)$  is indeed completely good. In the proof of the next lemma we use the following observation. Let  $z = \langle w, v, h \rangle$  be an adequate real. If  $m <_w n$ , then  $m, n \in P$  and  $\varphi(m) < \varphi(n)$ . This is the case, because whenever  $\langle y, k \rangle$  is a bad pair such that  $k \subseteq^* h(n)$ , also  $k \subseteq^* h(m)$ , since  $h(n) \subseteq^* h(m)$ .

**Lemma 10.** *If  $X \subseteq V$  is a nonempty  $\Sigma_1^1$  set and  $K \in [H]^\omega$  a  $\Delta_1^1$  set, then  $X \cap \limsup_{n \in K} A_n$  meets uncountably many  $E$ -classes.*

*Proof.* Suppose  $X \cap \limsup_{n \in K} A_n$  meets only countably many  $E$ -equivalence classes, i.e.  $X \cap \limsup_{n \in K} A_n \cap W = \emptyset$ . By  $\Sigma_1^1$ -separation, there is a  $\Delta_1^1$  set  $Y \subseteq \omega^\omega$  such that  $X \subseteq Y$  and  $Y \cap \limsup_{n \in K} A_n \cap W = \emptyset$ . Let  $y, k$  be a code for  $Y, K$ , respectively. Clearly,  $\langle y, k \rangle$  is a bad pair.

First, suppose the construction halted at stage  $\alpha < \omega_1^{\text{CK}}$ . Let  $z = \langle w, v, h \rangle$  be the unique  $\alpha$ -adequate real. The construction stops only if there does not exist a next bad pair which we have not picked already. Since  $\langle y, k \rangle$  is a bad pair such that  $k \subseteq^* h(n)$  for every  $n \in \text{dom}(\langle w_\alpha \rangle)$ , there must be an  $n \in \text{dom}(\langle w_\alpha \rangle)$  such that  $n = \langle y, k \rangle$ , i.e. we picked  $\langle y, k \rangle$  at that stage (otherwise, we can extend the construction by picking it now). But then  $D_{v(n)}^{\omega^\omega} \cap D_y^{\omega^\omega} = \emptyset$ , which implies  $V \cap Y = \emptyset$  and so  $V \cap X = \emptyset$ .

Second, suppose the construction continued all the way up to  $\omega_1^{\text{CK}}$ . Then there exists an  $\alpha < \omega_1^{\text{CK}}$  such that  $\alpha > \varphi(\langle y, k \rangle)$ . Let  $z = \langle w, v, h \rangle$  be  $\alpha$ -adequate. By the observation above, the pair  $\langle y, k \rangle$  was considered, hence  $n \in \text{dom}(\langle w_\alpha \rangle)$  such that  $n = \langle y, k \rangle$ . Again, this implies  $V \cap X = \emptyset$ .  $\square$

This finishes the proof of Proposition 5. We now derive a further refinement. A second key element of Harrington's proof is that  $E$  is meager on  $W \times W$ , when  $W$  is given the (subspace) Gandy–Harrington topology  $\tau_{\text{GH}}$ . This is the topology on  $\omega^\omega$  generated by the  $\Sigma_1^1$  sets. Although  $\omega^\omega$  with the Gandy–Harrington topology is not metrizable, it is strong Choquet and this enables one to redo the familiar



construction of a perfect set of inequivalent elements, using a winning strategy for the second player. While this approach would also work in our case, we will use the set  $X_{\text{low}}$  of low elements instead. This makes the construction in the proof of the main theorem more transparent, at the cost of some technicalities which we now summarize.

Let  $X_{\text{low}} = \{x \in \omega^\omega : \omega_1^{\text{CK}(x)} = \omega_1^{\text{CK}}\}$ . We will use the following facts about  $W$ ,  $X_{\text{low}}$ , and  $\tau_{\text{GH}}$ :

- (1)  $W$  and  $X_{\text{low}}$  are both nonempty  $\Sigma_1^1$  sets,
- (2)  $X_{\text{low}}$  is dense in  $\tau_{\text{GH}}$  and  $(X_{\text{low}}, \tau_{\text{GH}})$  is a Polish space, and
- (3) a nonempty  $\Sigma_1^1$  set  $A \subseteq \omega^\omega$  meets uncountably many  $E$ -equivalence classes iff  $A \cap W \neq \emptyset$  iff  $A \cap W \cap X_{\text{low}} \neq \emptyset$ .

Proofs of these facts can be found in [2].

**Proposition 11.** *Let  $E$  be a  $\Pi_1^1$  equivalence relation on  $\omega^\omega$  and  $(A_n)_{n \in \omega}$  a sequence of uniformly  $\Sigma_1^1$  subsets of  $\omega^\omega$ . If  $\limsup_{n \in K} A_n$  meets uncountably many  $E$ -equivalence classes for every  $K \in [\omega]^\omega$ , then there exists a completely good pair  $(V, H)$  such that  $V$  is a Polish space in the Gandy-Harrington topology  $\tau_{\text{GH}}$  and  $E$  is meager on  $V \times V$  (with the product topology  $\tau_{\text{GH}} \times \tau_{\text{GH}}$ ).*

*Proof.* Let  $(V, H)$  be the completely good pair given by Proposition 5. Using the facts stated above, it is easy to see that  $(V \cap W \cap X_{\text{low}}, H)$  is a completely good pair with the required additional properties.  $\square$

## 5. PROOF OF THE MAIN THEOREM

We now prove an effective version of Theorem 1. By the usual relativization and transfer arguments, this implies our main result.

**Theorem 12.** *Let  $E$  be a  $\Pi_1^1$  equivalence relation on  $\omega^\omega$  and  $(A_n)_{n \in \omega}$  a sequence of uniformly  $\Sigma_1^1$  subsets of  $\omega^\omega$ . If  $\limsup_{n \in K} A_n$  meets uncountably many  $E$ -equivalence classes for every  $K \in [\omega]^\omega$ , then there exists a  $K \in [\omega]^\omega$  such that  $\bigcap_{n \in K} A_n$  contains a perfect set of pairwise  $E$ -inequivalent elements.*

*Proof.* Let  $(V, H)$  be the completely good pair given by Proposition 11. Since  $E$  is meager on  $V \times V$  in the Gandy-Harrington topology  $\tau_{\text{GH}}$ , we can fix an increasing sequence  $(F_n)_{n \in \omega}$  of  $\tau_{\text{GH}}$ -closed nowhere dense sets such that  $E \subseteq \bigcup_{n \in \omega} F_n$ . We may assume that the diagonal  $\{(x, x) : x \in V\}$  is contained in  $F_0$ . We will recursively define a strictly increasing sequence  $j_0 < j_1 < \dots$  of natural numbers and a Cantor scheme  $(X_s)_{s \in 2^{<\omega}}$  of nonempty  $\Sigma_1^1$  subsets of  $V$  such that for all  $s, t \in 2^{<\omega}$ ,

- (1)  $\overline{X_{s \smallfrown 0}}, \overline{X_{s \smallfrown 1}} \subseteq X_s$ ,  $\overline{X_{s \smallfrown 0}} \cap \overline{X_{s \smallfrown 1}} = \emptyset$ , and  $\text{diam}(X_s) \leq 2^{-\text{lh}(s)}$ ,
- (2) if  $s \neq t \in 2^{n+1}$ , then  $X_s \times X_t \cap F_n = \emptyset$ , and
- (3) if  $s \in 2^n$ , then  $X_s \subseteq A_{j_0} \cap \dots \cap A_{j_n}$ .

Note that in (1), the closures and diameter are relative to  $(V, \tau_{\text{GH}})$ . Once this construction is completed, let  $K = \{j_0, j_1, \dots\}$  and

$$P = \bigcup_{\sigma \in 2^\omega} \bigcap_{n \in \omega} X_{\sigma \upharpoonright n}.$$

It is easy to see that  $P \subseteq \bigcap_{n \in K} A_n$  is nonempty perfect set of pairwise  $E$ -inequivalent elements.

Without loss of generality we may assume that  $A_0 = \omega^\omega$ . Start the construction with  $j_0 = 0$  and  $X_\emptyset = \omega^\omega$ . Suppose we have defined natural numbers  $j_0 < \dots < j_n$

and nonempty  $\Sigma_1^1$  sets  $X_s \subseteq A_{j_0} \cap \cdots \cap A_{j_n}$  for  $s \in 2^n$  satisfying the requirements above. By intersecting with sufficiently small basic open neighborhoods, we can split each  $X_s$  into disjoint nonempty  $\Sigma_1^1$  sets  $X_{s \smallfrown 0}$  and  $X_{s \smallfrown 1}$  satisfying requirement (1). Since  $F_n$  is closed nowhere dense, given any pair  $s \neq t \in 2^{n+1}$  we can shrink  $X_s$  and  $X_t$  so that  $X_s \times X_t \cap F_n = \emptyset$ . After finitely many iterations, we have defined  $X_s$  for  $s \in 2^{n+1}$  satisfying requirements (1) and (2).

**Claim.** *There is an  $j > j_n$  such that  $X_s \cap A_j \neq \emptyset$  for all  $s \in 2^{n+1}$ .*

*Proof.* Suppose towards a contradiction that for every  $j > j_n$  there is an  $s \in 2^{n+1}$  such that  $X_s \cap A_j = \emptyset$ . Define a binary relation  $R \subseteq \omega \times 2^{n+1}$  by  $R(j, s) \Leftrightarrow X_s \cap A_j = \emptyset$ . Since  $R$  is  $\Pi_1^1$ , there is a  $\Delta_1^1$  uniformizing function  $f: \omega \rightarrow 2^{n+1}$ . By the pigeonhole principle, there is an  $s \in 2^{n+1}$  such that  $\{j \in \omega: f(j) = s\} \cap H$  is infinite. Pick such an  $s \in 2^{n+1}$ . Then  $K = \{j \in \omega: j \in H \text{ and } f(j) = s\}$  is  $\Delta_1^1$ ,  $K \in [H]^\omega$ , and  $X_s \cap \bigcup_{n \in K} A_n = \emptyset$ . This implies that  $X_s \cap \limsup_{n \in K} A_n = \emptyset$ , contradicting the fact that  $(V, H)$  is a completely good pair.  $\square$

To complete this step in the construction, let  $j_{n+1} = j$  and intersect each  $X_s$  with  $A_{j_{n+1}}$ . This finishes the proof of Theorem 12.  $\square$

The following parametric version of the Laczkovich–Komj ath property was also considered by Balcerzak and G ab.

**Definition.** An equivalence relation  $E$  on a Polish space  $Y$  has the *parametric Laczkovich–Komj ath property* if for every uncountable Polish space  $X$  and every sequence  $(A_n)_{n \in \omega}$  of analytic subsets of  $X \times Y$ , if  $\limsup_{n \in K} A_n(x)$  meets uncountably many  $E$ -equivalence classes for every  $x \in X$  and  $K \in [\omega]^\omega$ , then there exists a  $K \in [\omega]^\omega$  and a perfect set  $P \subseteq X$  such that  $\bigcap_{n \in K} A_n(x)$  meets perfectly many  $E$ -equivalence classes for each  $x \in P$ .

**Theorem 13** (Balcerzak–G ab [1]). *If  $E$  has the Laczkovich–Komj ath property and for every analytic set  $A \subseteq X \times X$ , the set*

$$\{x \in X: A_x \text{ meets uncountably many } E\text{-equivalence classes}\}$$

*is analytic, then  $E$  has the parametric Laczkovich–Komj ath property.*  $\square$

**Proposition 14.** *Every coanalytic equivalence relation on a Polish space has the parametric Laczkovich–Komj ath property.*

*Proof.* Let  $E$  be a coanalytic equivalence relation on a Polish space  $X$  and  $A \subseteq X \times X$  an analytic subset. Without loss of generality we may assume  $E$  is a  $\Pi_1^1$  equivalence relation on  $X = \omega^\omega$  and  $A \subseteq \omega^\omega \times \omega^\omega$  is  $\Sigma_1^1$ . Since  $A$  is  $\Sigma_1^1$ , each section  $A_x$  is  $\Sigma_1^1$  as well. Hence,  $A_x$  meets uncountably many  $E$ -equivalence classes iff  $A_x \cap W \neq \emptyset$ . Thus,

$$\{x \in \omega^\omega: A_x \text{ meets uncountably many } E\text{-equivalence classes}\}$$

is  $\Sigma_1^1$ . Hence,  $E$  has the parametric Laczkovich–Komj ath property by Theorem 1 and Theorem 13.  $\square$

## REFERENCES

- [1] Marek Balcerzak and Szymon G ab. On the Laczkovich–Komj ath property of sigma-ideals. preprint.
- [2] Su Gao. *Invariant descriptive set theory*. Taylor & Francis Group, 2009.

- [3] Su Gao, Steve Jackson, Miklós Laczkovich, and R. Daniel Mauldin. On the unique representation of families of sets. *Transactions of the American Mathematical Society*, 360(2):939–958, 2008.
- [4] Leo A. Harrington, Alexander S. Kechris, and Alain Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. *Journal of the American Mathematical Society*, 3(4):903–928, 1990.
- [5] Péter Komjáth. On the limit superior of analytic sets. *Analysis Mathematica*, 10(4):283–293, 1984.
- [6] Miklós Laczkovich. On the limit superior of sequences of sets. *Analysis Mathematica*, 3(3):199–206, 1977.
- [7] Donald A. Martin and Alexander S. Kechris. Infinite games and effective descriptive set theory. In *Analytic sets*. Academic Press, 1980.
- [8] Yiannis N. Moschovakis. *Descriptive set theory*. Number 100 in Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., 1980.
- [9] Jack H. Silver. Counting the number of equivalence classes of Borel and coanalytic equivalence relations. *Annals of Mathematical Logic*, 18(1):1–28, 1980.

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