

ON EXTENSIONS OF PARTIAL ISOMETRIES

MAHMOOD ETEDADIALIABADI AND SU GAO

ABSTRACT. In this paper we define a notion of S-extension for a metric space and study minimality and coherence of S-extensions. We show that every S-extension can be identified with an algebraic object. We use this algebraic representation to give a complete characterization of all finite minimal S-extensions of a given finite metric space and a complete characterization of all minimal coherent S-extensions. We also define a notion of ultraextensive metric spaces and show that every countable metric space can be extended to a countable ultraextensive metric space. We also show that the isometry group of an infinite ultraextensive metric space has a dense locally finite subgroup, generalizing results in [8][10][12][14]. We also study compact ultrametric spaces and show that every compact ultrametric space can be extended to a compact ultraextensive ultrametric space.

1. INTRODUCTION

The study in this paper was motivated by the following theorem of Solecki [12]:

Theorem 1.1 (Solecki [12]). *Every finite metric space X can be extended to a finite metric space Y such that every partial isometry of X extends to an isometry of Y .*

The main objective of this paper is to identify such extensions and related concepts with algebraic objects. Inspired by Solecki's theorem, we define the following notions. Given a metric space X , a distinguished point $a_0 \in X$ and a collection P of partial isometries of X such that $P = P^{-1}$ and $X \setminus \{a_0\} \subseteq P(a_0)$, a P -type S-extension is a pair (Y, ϕ) , where Y is a metric space extending the metric space X and ϕ is a map from P into the set of all isometries of Y such that $\phi(p)$ extends p for all $p \in P$. When P is the set of all partial isometries of X , we call (Y, ϕ) an S-extension of X . Solecki's theorem can be restated as: Every finite metric space has a finite S-extension. We should also mention that similar notions in a more general context of finite relational structures have been studied by the authors in [2].

If X is a metric space and (Y, ϕ) is a P -type S-extension of X , then we say (Y, ϕ) is *minimal* (sometimes called *irreducible* in the literature) if for all $y \in Y$ there are

2010 *Mathematics Subject Classification.* Primary 05B25, 05C12; Secondary 03C13, 51F99.

Key words and phrases. Hrushovski property, extension property for partial automorphisms (EPPA), partial isometry, S-extension, S-map, coherent, ultraextensive, ultrahomogeneous, locally finite, ultrametric.

The second author's research was partially supported by the NSF grant DMS-1800323.

partial isometries $p_1, \dots, p_n \in P$ and $x \in X$ such that

$$y = \phi(p_1) \cdots \phi(p_n)(x).$$

This allows us to associate the point $y \in Y$ with $\phi(p_1) \cdots \phi(p_n) \in \phi(\mathbb{F}(P))$, where $\mathbb{F}(P)$ is the free group generated by elements of P , and therefore to view Y as an algebraic object.

Our first main result of the paper is to give an algebraic characterization of all finite minimal P -type S-extensions of a given finite metric space. Before even stating the result, we will give a direct constructive proof of Theorem 1.1. Recall that Solecki's proof in [12] uses a result of Herwig–Lascar [5], which is in turn a generalization of a celebrated result of Hrushovski [6] on extending partial isomorphisms of finite graphs. Our proof of Theorem 1.1 follows the ideas of Herwig–Lascar's proof and is essentially the same as the approaches in Rosendal's in [10] and Pestov's in [8]; but since we have a focus of characterizing all minimal S-extensions, our proof is somewhat different from them. Specifically, our proof is not as general as [10], and unlike [8], it does not need the full generality of Herwig–Lascar's result. Here we should mention that Hubička–Konečný–Nešetřil [7] provided a combinatorial proof of Theorem 1.1.

In a sense, our proof of Theorem 1.1 gives a “canonical” algebraic construction of a P -type S-extension from a parameter we call a *feasible prekernel*, which is a suitable normal subgroup of $\mathbb{F}(P)$. Given a feasible prekernel $N \trianglelefteq \mathbb{F}(P)$, we construct a canonical algebraic S-extension (Γ_N, Φ_N) , and introduce a weight function w_N . The minimal S-extensions can then be characterized as follows.

Theorem 1.2. *Let (Y, ϕ) be a finite minimal P -type S-extension of X . Let $N = \ker(\phi)$ and $G = \Phi_N(\mathbb{F}(P))$. Then there is a G -invariant pseudometric ρ on Γ_N which is consistent with w_N such that (Y, ϕ) is isomorphic to $(\overline{\Gamma}_N^\rho, \overline{\Phi}_N^\rho)$.*

The next notion we study is that of coherence between P -type S-extensions. Here we introduce a notion of coherence between feasible prekernels, and our second main result demonstrates a correspondence between the two coherence notions.

Theorem 1.3. *Let $X_1 \subseteq X_2$ be finite metric spaces, (Y_1, ϕ_1) be a minimal P_1 -type S-extension of X_1 , and $P_1 \subseteq P_2$ where $P_2 = P_2^{-1}$ and $X_2 \setminus \{a_0\} \subseteq P_2(a_0)$. Then the following hold.*

- (i) *Let (Y_2, ϕ_2) be a P_2 -type S-extension of X_2 that is coherent with (Y_1, ϕ_1) . Then $N_2 = \ker(\phi_2)$ is a coherent extension of $N_1 = \ker(\phi_1)$.*
- (ii) *Let $N_2 \trianglelefteq \mathbb{F}(P_2)$ be a coherent extension of $N_1 = \ker(\phi_1)$. Then letting $G_2 = \Phi_{N_2}(\mathbb{F}(P_2))$, there exists a G_2 -invariant pseudometric ρ_2 on Γ_{N_2} which is consistent with w_{N_2} , such that $(Y_2, \phi_2) = (\overline{\Gamma}_{N_2}^{\rho_2}, \overline{\Phi}_{N_2}^{\rho_2})$ is coherent with (Y_1, ϕ_1) .*

Iterative coherent S-extensions lead to infinite metric spaces with striking properties. We introduce the notion of ultraextensive metric spaces and obtain some general results about them. Specifically, we call a metric space U *ultraextensive* if U is ultrahomogeneous, every finite $X \subseteq U$ has a finite S-extension (Y, ϕ) where

$Y \subseteq U$, and if $X_1 \subseteq X_2 \subseteq U$ are finite and (Y_1, ϕ_1) is a finite minimal S-extension of X_1 with $Y_1 \subseteq U$, then there is a finite minimal S-extension (Y_2, ϕ_2) of X_2 such that $Y_2 \subseteq U$ and (Y_1, ϕ_1) and (Y_2, ϕ_2) are coherent.

Recall ultrahomogeneity means that any partial isometry can be extended to a full isometry of the entire space. Thus ultraextensiveness is a strengthening of ultrahomogeneity. It follows from our results that the universal Urysohn metric space \mathbb{U} and the rational universal Urysohn space $\mathbb{Q}\mathbb{U}$ are ultraextensive. Another example of ultraextensive metric space is the countable random graph equipped with the path metric. Moreover, we will also establish the following results.

Theorem 1.4. *Every countable metric space can be extended to a countable ultraextensive metric space.*

Theorem 1.5. *If U is an ultraextensive metric space, then every countable subset $X \subseteq U$ can be extended to a countable ultraextensive $Y \subseteq U$.*

Theorem 1.6. *For any separable ultraextensive metric space U , $\text{Iso}(U)$ contains a dense locally finite subgroup.*

For compact ultrametric spaces we prove the following.

Theorem 1.7. *Every compact ultrametric space can be extended to a compact ultraextensive ultrametric space. In particular, every compact ultrametric space has a compact ultrametric S-extension.*

The rest of the paper is organized as follows. In Section 2 we give some preliminaries of S-extensions, metrics on weighted graphs, and the profinite topology. In Section 3 we give the canonical algebraic construction of finite S-extensions in the style of Herwig–Lascar. In Section 4 we study finite minimal S-extensions and give a complete characterization of them. In Section 5 we study the notion of coherent S-extensions and give a complete characterization of coherent S-extensions along with several constructions and applications. In Section 6 we study ultraextensive spaces and establish the main results mentioned above. In Section 7 we study compact ultrametric spaces and show that they admit compact ultrametric S-extensions. In Section 8 we mention some open problems.

2. PRELIMINARIES

2.1. S-extensions. We fix some notations to be used in the rest of the paper. Let (X, d_X) and (Y, d_Y) be metric spaces. When there is no danger of confusion, we simply write X for (X, d_X) and Y for (Y, d_Y) .

We say that Y is an *extension* of X if $X \subseteq Y$ and for all $x_1, x_2 \in X$, $d_Y(x_1, x_2) = d_X(x_1, x_2)$. Interchangeably, we use the same terminology when Y contains an isometric copy of X .

An *isometry* from X to Y is a bijection $\pi : X \rightarrow Y$ such that

$$d_Y(\pi(x_1), \pi(x_2)) = d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. An isometry from X to X is also called an *isometry of X* . The set of all isometries of X is denoted as $\text{Iso}(X)$. Under composition of maps, $\text{Iso}(X)$ becomes a group.

A *partial isometry* of X is an isometry between two finite subspaces of X . The set of all partial isometries of X is denoted as $\mathcal{P}(X)$. $\mathcal{P}(X)$ is not necessarily a group, but it is a groupoid. In particular, for each $p \in \mathcal{P}(X)$ we can speak of the inverse map p^{-1} , which is still a partial isometry.

If Y is an extension of X , then every partial isometry of X is also a partial isometry of Y . In symbols, we have $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$ if $X \subseteq Y$.

If $p, q \in \mathcal{P}(X)$, we say that q *extends* p , and write $p \subseteq q$, if

$$\{(x, p(x)) : x \in \text{dom}(p)\} \subseteq \{(x, q(x)) : x \in \text{dom}(q)\}.$$

We let 1_X denote the identity isometry on X , i.e., $1_X(x) = x$ for all $x \in X$. Let \mathcal{P}_X denote the set of all $p \in \mathcal{P}(X)$ such that $p \not\subseteq 1_X$. We refer to elements of \mathcal{P}_X as *nonidentity partial isometries of X* .

The main concept we study in this paper is that of a P -type S-extension.

Definition 2.1. Let X be a finite metric space and fix a distinguished point $a_0 \in X$. Let $P \subseteq \mathcal{P}_X$ be such that $P = P^{-1}$ and $X \setminus \{a_0\} \subseteq P(a_0)$. A *P -type S-extension* of X is a pair (Y, ϕ) , where $Y \supseteq X$ is an extension of X , and $\phi : P \rightarrow \text{Iso}(Y)$ such that $\phi(p)$ extends p for all $p \in P$. The map ϕ is called a *P -type S-map* for X .

When $P = \mathcal{P}_X$, we call (Y, ϕ) an *S-extension* of X and ϕ an *S-map* for X .

Note that an equivalent restatement of Solecki's theorem (Theorem 1.1) is that every finite metric space has a finite S-extension. It is well-known that the universal Urysohn space \mathbb{U} is both universal (for all separable metric spaces) and ultrahomogeneous. These imply that every separable metric space has an S-extension (Y, ϕ) where Y is isometric with \mathbb{U} .

We will need the following notion of isomorphism between P -type S-extensions.

Definition 2.2. Let X be a metric space and (Y, ϕ) and (Z, ψ) be both P -type S-extensions of X . An *isomorphism* between (Y, ϕ) and (Z, ψ) is an isometry $\pi : Y \rightarrow Z$ such that $\psi(p) \circ \pi = \pi \circ \phi(p)$ for all $p \in P$. If there is an isomorphism between (Y, ϕ) and (Z, ψ) , we say that (Y, ϕ) and (Z, ψ) are *isomorphic*, and write $(Y, \phi) \cong (Z, \psi)$.

2.2. Metrics on weighted graphs. We will study metric spaces derived from weighted graphs. A *weighted graph* is a pair (Γ, w) , where $\Gamma = (V(\Gamma), E(\Gamma))$ is a (simple undirected) graph and $w : E(\Gamma) \rightarrow \mathbb{R}_+$ (\mathbb{R}_+ denotes the set of all positive real numbers). We call w the *weight function*. If w_1, w_2 are two weight functions on Γ , then we write $w_1 \leq w_2$ if $w_1(x, y) \leq w_2(x, y)$ for all $(x, y) \in E(\Gamma)$.

Given a weighted graph (Γ, w) , let $L_w = \inf\{w(x, y) : (x, y) \in E(\Gamma)\}$ and $B_w = \sup\{w(x, y) : (x, y) \in E(\Gamma)\}$. Assuming $0 < L_w \leq B_w < \infty$, one can define a *path metric* d_w on $V(\Gamma)$ as follows: for any $x, y \in V(\Gamma)$, let

$$d_w(x, y) = \min\{B_w, \delta_w(x, y)\}$$

where

$$\delta_w(x, y) = \inf \left\{ \sum_{i=1}^n w(x_i, x_{i+1}) : x_1 = x, x_{n+1} = y, \forall i \leq n (x_i, x_{i+1}) \in E(\Gamma) \right\}.$$

In particular, $\delta_w(x, y)$ is undefined when x and y are not connected by a path in Γ , in which case $d_w(x, y) = B_w$. It is easy to verify that d_w is indeed a metric. Also, if $(x, y) \in E(\Gamma)$, then $d_w(x, y) = \delta_w(x, y)$. We note the following simple fact about d_w without proof.

Lemma 2.3. The following are equivalent:

- (i) For any $(x, y) \in E(\Gamma)$, $d_w(x, y) = w(x, y)$.
- (ii) For any $(x, y) \in E(\Gamma)$ and any x_1, \dots, x_{n+1} where $x_1 = x$, $x_{n+1} = y$, and $(x_i, x_{i+1}) \in E(\Gamma)$ for all $i = 1, \dots, n$, we have

$$w(x, y) \leq \sum_{i=1}^n w(x_i, x_{i+1}).$$

We introduce some new concepts about the consistency of metrics on weighted graphs.

Definition 2.4. Let (Γ, w) be a weighted graph and d be a metric on $V(\Gamma)$. We say that d is *consistent* with w if for all $(x, y) \in E(\Gamma)$, $d(x, y) = w(x, y)$. We say that w is *reduced* if d_w is consistent with w .

Lemma 2.5. Let (Γ, w) be a connected weighted graph. Then there is a maximal reduced weight function w^* on Γ with $w^* \leq w$.

Proof. For all $(x, y) \in E(\Gamma)$, define $w^*(x, y) = d_w(x, y) = \delta_w(x, y)$. Then $w^* \leq w$. To see that w^* is reduced we use Lemma 2.3 and consider $(x, y) \in E(\Gamma)$. Suppose $x_1, \dots, x_{n+1} \in V(\Gamma)$ with $x_1 = x$, $x_{n+1} = y$, and $(x_i, x_{i+1}) \in E(\Gamma)$ for all $i = 1, \dots, n$. Let $\epsilon > 0$. For each $i = 1, \dots, n$, let $x_i^1 = x_i, x_i^2, \dots, x_i^{k_i+1} = x_{i+1} \in V(\Gamma)$ with $(x_i^j, x_i^{j+1}) \in E(\Gamma)$ for all $j = 1, \dots, k_i$ be such that

$$w^*(x_i, x_{i+1}) = d_w(x_i, x_{i+1}) \leq \sum_{j=1}^{k_i} w(x_i^j, x_i^{j+1}) \leq w^*(x_i, x_{i+1}) + \epsilon/n.$$

Then

$$w^*(x, y) = d_w(x, y) \leq \sum_{i=1}^n \sum_{j=1}^{k_i} w(x_i^j, x_i^{j+1}) \leq \sum_{i=1}^n w^*(x_i, x_{i+1}) + \epsilon.$$

Since ϵ is arbitrary, we have that

$$w^*(x, y) \leq \sum_{i=1}^n w^*(x_i, x_{i+1}).$$

Thus w^* is reduced. For the maximality of w^* , assume $u \leq w$ is a reduced weighted function. Then for all $(x, y) \in E(\Gamma)$, $u(x, y) = d_u(x, y) \leq d_w(x, y) = w^*(x, y)$. \square

We can always turn a metric space into a weighted graph. If (X, d) is a metric space, for any $x, y \in X$ with $x \neq y$, we add an edge between x and y with weight $w_d(x, y) = d(x, y)$. Then (X, w_d) is a connected weighted graph and w_d is a reduced weight function.

We will also consider pseudometrics on weighted graphs.

Definition 2.6. Let (Γ, w) be a weighted graph and ρ be a pseudometric on $V(\Gamma)$. We say that ρ is *consistent* with w if for all $(x, y) \in E(\Gamma)$, $\rho(x, y) = w(x, y)$.

When a weight function w satisfies $B_w < \infty$ and $L_w = 0$, one can similarly define a *path pseudometric* d_w and the distance function δ_w the same way as above. The resulting path pseudometric is consistent with w .

Definition 2.7. Let (M, ρ) be a pseudometric space. An *isometry* of (M, ρ) is a map $\varphi : M \rightarrow M$ such that for all $x, y \in M$, $\rho(\varphi(x), \varphi(y)) = \rho(x, y)$. If G is a set of isometries of (M, ρ) , we say that ρ is *G-invariant*.

Any pseudometric space (M, ρ) has a metric identification defined as follows. Let \sim be an equivalence relation defined on M by $x \sim y$ iff $\rho(x, y) = 0$. For each $x \in M$, let $[x]_\sim$ denote the \sim -equivalence class of x . Then we can define $\overline{M} = \overline{M}^\rho = M / \sim$ and a metric $\overline{\rho}$ on \overline{M} by $\overline{\rho}([x]_\sim, [y]_\sim) = \rho(x, y)$ for all $x, y \in M$. $(\overline{M}, \overline{\rho})$ is called the *metric identification* of (M, ρ) . If φ is an isometry of (M, ρ) , then we can define $\overline{\varphi} : \overline{M} \rightarrow \overline{M}$ by $\overline{\varphi}([x]_\sim) = [\varphi(x)]_\sim$ for all $x \in M$. Then $\overline{\varphi}$ is an isometry of $(\overline{M}, \overline{\rho})$. Suppose G is a set of isometries of (M, ρ) , then $\overline{G} = \overline{G}^\rho = \{\overline{\varphi} : \varphi \in G\}$ is a set of isometries of $(\overline{M}, \overline{\rho})$. We note that if G is a group, then \overline{G} is also a group.

Let (Γ, w) be a weighted graph and ρ be a pseudometric on $V(\Gamma)$ consistent with w . Let $(\overline{\Gamma}, \overline{\rho})$ denote the metric identification of the pseudometric space (Γ, ρ) . For $(x, y) \in E(\Gamma)$, define $\overline{w}([x]_\sim, [y]_\sim) = w(x, y)$. Then $(\overline{\Gamma}, \overline{w})$ is a weighted graph and $\overline{\rho}$ is consistent with \overline{w} . We will need this construction in the subsequent sections.

2.3. The profinite topology. One of the main tools we will be using is Ribes–Zalesskii theorem [9] on the profinite topology on an abstract group. Recall that if G is an abstract group, the *profinite topology* on G is the topology generated by all cosets of normal subgroups of finite index, that is, it has as a basis of open subsets all cosets of normal subgroups of finite index.

Theorem 2.8 (Ribes–Zalesskii [9]). *Let \mathbb{F} be an abstract free group and H_1, \dots, H_n be finitely generated subgroups of \mathbb{F} . Then $H_1 \cdots H_n$ is closed in the profinite topology.*

A group G is said to have *property RZ* if for any finitely generated subgroups H_1, \dots, H_n of G , $H_1 \cdots H_n$ is closed in the profinite topology of G . All groups with property RZ are residually finite. We will also use the following theorem of Coulbois [1].

Theorem 2.9 (Coulbois [1]). *If G_1 and G_2 have property RZ, then so does the free product $G_1 * G_2$.*

Herwig–Lascar [5] used the Ribes–Zalesskii theorem in their study of the extension problems. This approach was explored further by Rosendal [10] [11] to study extension problems for isometries, who showed that the Ribes–Zalesskii property for a group G is equivalent to an extension property for actions of G by isometries.

Definition 2.10. Let G be a group acting by isometries on a metric space (X, d_X) . We say that the action is *finitely approximable* if for any finite $A \subseteq X$ and finite $F \subseteq G$ there is a finite metric space (Y, d_Y) , on which G acts by isometries, and an isometry $\pi : A \rightarrow Y$ such that whenever $g \in F$ and $x, gx \in A$, then $\pi(gx) = g\pi(x)$.

Theorem 2.11 (Rosendal [10]). *The following are equivalent for a countable discrete group G :*

- (1) G has property RZ;
- (2) Any action of G by isometries on a metric space is finitely approximable.

3. FINITE S-EXTENSIONS

In this section we give a direct constructive proof of Solecki’s theorem (Theorem 1.1) following the ideas of Herwig–Lascar [5]. We will see in the following sections that the construction we present here is in some sense canonical.

For the rest of this section we fix a finite metric space X , a distinguished point $a_0 \in X$ and $P \subseteq \mathcal{P}_X$ where $P = P^{-1}$ and $X \setminus \{a_0\} \subseteq P(a_0)$. Recall that \mathcal{P}_X is the set of all nonidentity partial isometries of X . Let $\mathbb{F}(P)$ be the free group generated by elements of P . For each $p \in P$, we identify the partial isometry $p^{-1} \in P$ with the formal inverse of p in $\mathbb{F}(P)$. Thus any nonidentity element of $\mathbb{F}(P)$ is a finite word of the form $p_1 \dots p_n$ with $p_1, \dots, p_n \in P$. We use 1 to denote the identity element of $\mathbb{F}(P)$. Of course, 1 can be identified with the identity isometry 1_X .

Let H be the set of all finite words $p_1 \dots p_n$ with $p_1, \dots, p_n \in P$ such that $p_1 \dots p_n(a_0) = p_1(p_2(\dots p_n(a_0) \dots))$ is defined and $p_1 \dots p_n(a_0) = a_0$. Since X is finite, H is a finitely generated subgroup of $\mathbb{F}(P)$.

Define $\Gamma = \mathbb{F}(P)/H$. We construct a weighted graph (Γ, w) as follows:

- (1) for every $p, q \in P \cup \{1\}$ such that $p(a_0)$ and $q(a_0)$ are defined, there is an edge between pH and qH with $w(pH, qH) = d_X(p(a_0), q(a_0))$, and
- (2) for every $g, g_1, g_2 \in \mathbb{F}(P)$, if there is an edge between g_1H and g_2H , then there is an edge between gg_1H and gg_2H with $w(gg_1H, gg_2H) = w(g_1H, g_2H)$.

To see that w is well-defined, first note that if $w(g_1H, g_2H)$ is defined then there are $p, q \in P$ with $p(a_0)$ and $q(a_0)$ defined, and $g \in \mathbb{F}(P)$ such that $g_1 = gp$ and $g_2 = gq$. In this case, $w(g_1H, g_2H) = w(pH, qH) = d_X(p(a_0), q(a_0))$. Thus, to verify that w is well-defined, it suffices to make sure that if $p, q, r, s \in P$ and $d_X(p(a_0), q(a_0)) \neq d_X(r(a_0), s(a_0))$, then there does not exist $g \in \mathbb{F}(P)$ such that $gpH = rH$ and $gqH = sH$. Assume there is such a $g = p_1 \dots p_n$. Then $r^{-1}gp, s^{-1}gq \in H$, and thus $r^{-1}p_1 \dots p_n p(a_0) = a_0$ and $s^{-1}p_1 \dots p_n q(a_0) = a_0$. It follows that

$$p_1 \dots p_n(p(a_0)) = r(a_0) \text{ and } p_1 \dots p_n(q(a_0)) = s(a_0).$$

Since all p_1, \dots, p_n are partial isometries, we have $d_X(p(a_0), q(a_0)) = d_X(r(a_0), s(a_0))$.

From the finiteness of P and the definition of w , it is clear that $L_w = \inf\{w(x, y) : (x, y) \in E(\Gamma)\} > 0$ and $B_w = \sup\{w(x, y) : (x, y) \in E(\Gamma)\} < \infty$. It follows that, equipped with the path metric d_w , Γ becomes a metric space. When there is no danger of confusion, we use Γ to denote the metric space (Γ, d_w) .

We claim that Γ is essentially an extension of X . To see this, let $e : X \rightarrow \Gamma$ be defined by

$$e(a) = \begin{cases} H, & \text{if } a = a_0, \\ pH, & \text{where } p \in P \text{ and } p(a_0) = a, \text{ if } a \neq a_0. \end{cases}$$

To see that e is well-defined, first note that for any $a \neq a_0$ there is $p \in P$ with $p(a_0) = a$. If $p, q \in P$ with $p(a_0) = q(a_0)$, then $p^{-1}q \in H$ and therefore $pH = qH$. Thus e is well-defined. Furthermore, if $e(a) = pH = qH = e(b)$ and $p(a_0) = a$ and $q(a_0) = b$, then $p^{-1}q \in H$ and therefore $a = p(a_0) = q(a_0) = b$. This means that e is one-to-one. To see that e is an isometric embedding, we use the following lemmas.

Lemma 3.1. Let (Y, ϕ) be a P -type S-extension of X . Then there is $\pi : \Gamma \rightarrow Y$ such that $d_Y(\pi(g_1H), \pi(g_2H)) = w(g_1H, g_2H)$ whenever $(g_1H, g_2H) \in E(\Gamma)$.

Proof. We first expand $\phi : P \rightarrow \text{Iso}(Y)$ to a map $\psi : \mathbb{F}(P) \rightarrow \text{Iso}(Y)$ by

$$\psi(p_1 \dots p_n) = \phi(p_1) \circ \dots \circ \phi(p_n).$$

Define $\pi : \Gamma \rightarrow Y$ by $\pi(gH) = \psi(g)(a_0)$. It is easy to see that π is well-defined. Now, if $(g_1H, g_2H) \in E(\Gamma)$, then there exist $p, q \in P \cup \{1\}$ and $g \in \mathbb{F}(P)$ such that $g_1 = gp$, $g_2 = gq$, and $p(a_0), q(a_0)$ are defined. We have

$$\begin{aligned} d_Y(\pi(g_1H), \pi(g_2H)) &= d_Y(\pi(gpH), \pi(gqH)) = d_Y(\psi(g) \circ \phi(p)(a_0), \psi(g) \circ \phi(q)(a_0)) \\ &= d_Y(\phi(p)(a_0), \phi(q)(a_0)) = d_X(p(a_0), q(a_0)) \\ &= w(pH, qH) = w(gpH, gqH). \end{aligned}$$

□

Lemma 3.2. w is reduced.

Proof. We verify using Lemma 2.3 that for any $(g_1H, g_2H) \in E(\Gamma)$, $d_w(g_1H, g_2H) = w(g_1H, g_2H)$. Let $\gamma_1 = g_1, \gamma_2, \dots, \gamma_{n+1} = g_2$ be elements of $\mathbb{F}(P)$ such that for all $i = 1, \dots, n$, $(\gamma_iH, \gamma_{i+1}H) \in E(\Gamma)$. Let $Y = \mathbb{U}$ and (Y, ϕ) be a P -type S-extension of X . Let $\pi : \Gamma \rightarrow Y$ be given by Lemma 3.1. Then

$$\begin{aligned} w(g_1H, g_2H) &= d_Y(\pi(g_1H), \pi(g_2H)) \\ &\leq \sum_{i=1}^n d_Y(\pi(\gamma_iH), \pi(\gamma_{i+1}H)) = \sum_{i=1}^n w(\gamma_iH, \gamma_{i+1}H). \end{aligned}$$

□

Lemma 3.3. e is an isometric embedding.

Proof. Let $a, b \in X$ and $p, q \in P \cup \{1\}$ with $p(a_0) = a$ and $q(a_0) = b$. Then $(pH, qH) \in E(\Gamma)$. Since w is reduced, we have

$$d_w(e(a), e(b)) = d_w(pH, qH) = w(pH, qH) = d_X(p(a_0), q(a_0)) = d_X(a, b).$$

□

We identify X with

$$e(X) = \{pH : p \in P \cup \{1\} \text{ and } p(a_0) \text{ is defined}\} \subseteq \Gamma$$

and consider Γ an extension of X . For each $q \in P$, consider the partial map $\hat{q} : e(X) \rightarrow e(X)$ defined by $\hat{q}(pH) = qpH$ for all $pH, qpH \in e(X)$, i.e., whenever $p(a_0)$ and $q(p(a_0))$ are defined; note that in this case there exists $r \in P$ such that $r(a_0) = q(p(a_0))$ and $rH = qpH$. Then it is straightforward to verify that for all $a, b \in X$, $q(a) = b$ iff $\hat{q}(e(a)) = e(b)$. Thus we may identify q with \hat{q} on the domain $e(\text{dom}(q))$.

Define $\Phi : P \rightarrow \text{Iso}(\Gamma)$ by letting, for any $q \in P$,

$$\Phi(q)(gH) = qgH$$

for all $g \in \mathbb{F}(P)$. To see that $\Phi(q)$ is indeed an isometry of Γ , let $g_1, g_2 \in \mathbb{F}(P)$. From the definitions of w and δ_w , we get $\delta_w(g_1H, g_2H) = \delta_w(qg_1H, qg_2H)$ (including the case when one of these quantities is ∞). It follows that $d_w(g_1H, g_2H) = d_w(qg_1H, qg_2H)$.

Lemma 3.4. (Γ, Φ) is a P -type S-extension of X .

Proof. For any $q \in P$, $\Phi(q)$ is obviously an extension of \hat{q} . □

To construct a finite P -type S-extension of X our plan is to find a suitable normal subgroup N of finite index in $\mathbb{F}(P)$, and to use $\Gamma_N = \mathbb{F}(P)/NH$ as the underlying space of the P -type S-extension. Assuming such a normal subgroup $N \trianglelefteq \mathbb{F}(P)$ is found, we first turn Γ_N into a weighted graph (Γ_N, w_N) as follows:

- (1) for every $p, q \in P \cup \{1\}$ with $p(a_0)$ and $q(a_0)$ defined, there is an edge between pNH and qNH with $w_N(pNH, qNH) = d_X(p(a_0), q(a_0))$, and
- (2) for every $g, g_1, g_2 \in \mathbb{F}(P)$, if there is an edge between g_1NH and g_2NH , then there is an edge between gg_1NH and gg_2NH with

$$w_N(gg_1NH, gg_2NH) = w_N(g_1NH, g_2NH).$$

To guarantee that w_N is well-defined, we use a similar argument as before provided that the following condition holds:

- (C1) For every $p, q, r, s \in P \cup \{1\}$ such that $d_X(p(a_0), q(a_0)) \neq d_X(r(a_0), s(a_0))$, there does not exist $g \in \mathbb{F}(P)$ such that $gpNH = rNH$ and $gqNH = sNH$, equivalently, $N \cap pHr^{-1}sHq^{-1} = \emptyset$.

To see the equivalence in the statement of (C1), suppose $gpNH = rNH$ and $gqNH = sNH$. Then by the normality of N we have $g \in rNHp^{-1} \cap sNHq^{-1}$, and thus $rNHp^{-1} \cap sNHq^{-1} \neq \emptyset$. It follows that $N \cap pHr^{-1}sHq^{-1}N \neq \emptyset$, or $N \cap pHr^{-1}sHq^{-1} \neq \emptyset$. All steps can be reversed to establish the backward implication.

Another similar argument as before shows that d_{w_N} is a metric on Γ_N . We again define

$$e_N(a) = \begin{cases} NH, & \text{if } a = a_0, \\ pNH, & \text{where } p \in P \text{ and } p(a_0) = a, \text{ if } a \neq a_0. \end{cases}$$

In order to guarantee that e_N is one-to-one, we argue similarly as before provided that the following condition holds for N :

(C2) For every $p, q \in P \cup \{1\}$, if $p(a_0)$ and $q(a_0)$ are defined and $p(a_0) \neq q(a_0)$, then $p^{-1}q \notin NH$, equivalently, $N \cap pHq^{-1} = \emptyset$.

Finally, to guarantee that e_N is an isometric embedding, we argue similarly as in the proof of Lemma 3.3 provided that w_N is reduced, which corresponds to the following condition:

(C3) For every $p, q, r_1, s_1, \dots, r_n, s_n \in P \cup \{1\}$ such that

$$d_X(p(a_0), q(a_0)) > \sum_{i=1}^n d_X(r_i(a_0), s_i(a_0)),$$

there does not exist a path in Γ_N from pNH to qNH using translates of edges $(r_1NH, s_1NH), \dots, (r_nNH, s_nNH)$ in the same order. That is, there do not exist $g_1, \dots, g_n \in \mathbb{F}(P)$ such that

$$pNH = g_1r_1NH, \quad g_1s_1NH = g_2r_2NH,$$

.....

$$g_{n-1}s_{n-1}NH = g_nr_nNH, \quad g_ns_nNH = qH.$$

Equivalently, $N \cap pHr_1^{-1}s_1H \cdots Hr_n^{-1}s_nHq^{-1} = \emptyset$.

To summarize, we need to find $N \trianglelefteq \mathbb{F}(P)$ of finite index so that (C1), (C2) and (C3) hold. Note that these correspond to finitely many conditions, and each condition is of the form $\gamma N \cap H_1 \cdots H_n = \emptyset$ where $\gamma \in \mathbb{F}(P)$, H_1, \dots, H_n are finitely generated subgroups of $\mathbb{F}(P)$, and $\gamma \notin H_1 \cdots H_n$. For example, the condition in (C1) can be rewritten as

$$(p^{-1}qs^{-1}r)N \cap H(r^{-1}sHs^{-1}r) = \emptyset.$$

Thus, by the Ribes–Zalesskii theorem, for each condition of the form $\gamma N \cap H_1 \cdots H_n = \emptyset$, where $\gamma \notin H_1 \cdots H_n$, there is a normal subgroup of finite index satisfying the condition. Taking the intersection of all these subgroups, we obtain still a normal subgroup of finite index to satisfy all conditions (C1), (C2) and (C3).

Now

$$e_N(X) = \{pNH : p \in P \text{ and } p(a_0) \text{ is defined}\}.$$

Similar to the above, for each $q \in P$ we can define the partial map $\tilde{q} : e_N(X) \rightarrow e_N(X)$ by $\tilde{q}(pNH) = qpNH$. Then X is identified with $e_N(X)$ and q is identified with \tilde{q} with domain $e(\text{dom}(q))$. Define $\Phi_N : P \rightarrow \text{Iso}(\Gamma_N)$ by

$$\Phi_N(q)(gNH) = qgNH.$$

Then it is obvious that $\Phi_N(q)$ extends \tilde{q} for all $q \in P$.

We have thus established that (Γ_N, Φ_N) is a finite P -type S-extension of X .

Definition 3.5. Let X be a metric space and $P \subseteq \mathcal{P}_X$ be such that $X \setminus \{a_0\} \subseteq P(a_0)$. We say $N \trianglelefteq \mathbb{F}(P)$ is a *feasible P -type prekernel* for X if it satisfies (C1), (C2) and (C3). When there is no danger of confusion, we call N a *feasible prekernel*.

What we have shown in this section can be summarized as follows.

Lemma 3.6. Let X be a metric space. If $N \trianglelefteq \mathbb{F}(P)$ is a feasible P -type prekernel for X , then (Γ_N, Φ_N) is a P -type S-extension of X .

Thus Theorem 1.1 follows from the fact that for any finite metric space X there is a feasible P -type prekernel $N \trianglelefteq \mathbb{F}(P)$ that is of finite index in $\mathbb{F}(P)$. This in turn follows from property RZ of the free group $\mathbb{F}(P)$.

4. MINIMAL S-EXTENSIONS

In this section we give a complete characterization of all finite minimal P -type S-extensions of a given finite metric space. This is done by showing that the P -type S-extension we constructed in the previous section is canonical in several senses. We use the same notations from the previous section.

Throughout this section we still fix a finite metric space X , a distinguished point $a_0 \in X$ and $P \subseteq \mathcal{P}_X$ where $P = P^{-1}$ and $X \setminus \{a_0\} \subseteq P(a_0)$. We have constructed P -type S-extensions (Γ, Φ) and (Γ_N, Φ_N) for suitable $N \trianglelefteq \mathbb{F}(P)$. Here we first note that, as long as P is sufficiently rich, these P -type S-extensions do not depend on the choice of the point $a_0 \in X$. More explicitly, if $a'_0 \in X$ and $p_0(a_0) = a'_0$ for $p_0 \in P$, and if $X \setminus \{a'_0\} \subseteq P(a'_0)$, then we could similarly define $\Gamma' = \mathbb{F}(P)/H'$ and Φ' . It is easy to see that $H' = p_0 H p_0^{-1}$. Thus we may define a bijection $\pi : \Gamma \rightarrow \Gamma'$ by $\pi(gH) = g p_0^{-1} H'$ for all $g \in \mathbb{F}(P)$. It is straightforward to check that π is an isometry between Γ and Γ' such that $\pi(\Phi(q)(gH)) = \Phi'(q)(\pi(gH))$ for all $q \in P$ and $g \in \mathbb{F}(P)$. Thus π is indeed an isomorphism between the two P -type S-extensions. Similarly, when P is sufficiently rich, the finite P -type S-extension (Γ_N, Φ_N) does not depend on the choice of a_0 either.

Next we note that for any P -type S-extension (Y, ϕ) of X , the P -type S-map ϕ can be trivially extended to a map from all of $\mathbb{F}(P)$ to $\text{Iso}(Y)$ by letting

$$\hat{\phi}(p_1 \dots p_n) = \phi(p_1) \circ \dots \circ \phi(p_n)$$

for all $p_1, \dots, p_n \in \mathbb{F}(P)$. $\hat{\phi}$ is a semigroup homomorphism but not necessarily a group homomorphism. To turn it into a group homomorphism, we just need to make sure that $\phi(p^{-1}) = \phi(p)^{-1}$ for all $p \in P$, which is easy to arrange. In the rest of this paper, we will use ϕ to denote the extension $\hat{\phi}$, and thus regard ϕ as a map from $\mathbb{F}(P)$ to $\text{Iso}(Y)$. We will also tacitly assume that all the extended P -type S-maps $\phi : \mathbb{F}(P) \rightarrow \text{Iso}(Y)$ are indeed group homomorphisms and therefore their ranges are subgroups of $\text{Iso}(Y)$. We note that the extended P -type S-map $\Phi : \mathbb{F}(P) \rightarrow \text{Iso}(\Gamma_N)$ is already a group homomorphism.

The following lemma is one evidence of the canonicity of the construction (Γ_N, Φ_N) .

Lemma 4.1. Let $N \trianglelefteq \mathbb{F}(P)$ be a feasible P -type prekernel for X , that is, (Γ_N, Φ_N) is a P -type S-extension of X . Let $G = \Phi_N(\mathbb{F}(P)) \leq \text{Iso}(\Gamma_N)$ and $N_G = \ker(\Phi_N)$.

Then $N_G \trianglelefteq \mathbb{F}(P)$ is a normal subgroup and $NH = N_G H$. In particular, $(\Gamma_{N_G}, \Phi_{N_G}) = (\Gamma_N, \Phi_N)$.

Note that the notion of “feasible prekernel” is justified by Lemma 4.1 which states that such a group N can be massaged into another normal subgroup N_G that produces the same S-extension of X and $N_G = \ker(\Phi_N) = \ker(\Phi_{N_G})$.

Proof. N_G is obviously a normal subgroup of $\mathbb{F}(P)$. We only need to verify $NH = N_G H$. Let $\gamma \in N_G$. Then $\Phi_N(\gamma) = 1$ and for all $g \in \mathbb{F}(P)$, $\gamma g N H = \Phi_N(\gamma)(g N H) = g N H$. In particular $\gamma N H = N H$, and so $\gamma \in N H$. This shows that $N_G \subseteq N H$ and so $N_G H \leq N H$. Conversely, suppose $\gamma \in N$. Then for all $g \in \mathbb{F}(P)$, we have $\Phi_N(\gamma)(g N H) = \gamma g N H = g(g^{-1} \gamma g) N H = g N H$. Thus $\gamma \in \ker(\Phi_N) = N_G$. This shows that $N \leq N_G$ and so $N H \leq N_G H$. \square

Next we define minimality for P -type S-extensions.

Definition 4.2. A P -type S-extension (Y, ϕ) of X is said to be *minimal* if for any $y \in Y$ there is $g \in \mathbb{F}(P)$ such that $y = \phi(g)(a_0)$.

We state the following fact without proof.

Lemma 4.3. Let (Y, ϕ) be a P -type S-extension of X . Then the following are equivalent:

- (i) (Y, ϕ) is minimal;
- (ii) For any $y \in Y$ there exist $g \in \mathbb{F}(P)$ and $x \in X$ such that $y = \phi(g)(x)$.

Of course, the notion of minimality is motivated by the observation that if (Y, ϕ) is a P -type S-extension of X and let

$$Z = \{\phi(g)(x) : g \in \mathbb{F}(P), x \in X\},$$

then $Z \subseteq Y$ and for any $p \in P$ and $z \in Z$, $\phi(p)(z) \in Z$. Thus, by defining

$$\psi(p) = \phi(p) \upharpoonright Z$$

for all $p \in P$, we get another P -type S-extension (Z, ψ) of X which is a subextension of (Y, ϕ) .

We also note that, if (Y, ϕ) is a P -type S-extension of X , then there are many ways to define proper superextensions of (Y, ϕ) by adding points to Y and defining metrics appropriately. Thus there is no hope to give a reasonable characterization of all finite P -type S-extensions of X . Below we concentrate on characterizing finite minimal P -type S-extensions of X . We will show that all finite minimal P -type S-extensions of X are derived from P -type S-extensions of the form (Γ_N, Φ_N) .

Lemma 4.4. Let (Y, ϕ) be a P -type S-extension of X . Let $N = \ker(\phi)$. Then N is a feasible P -type prekernel for X .

Proof. Define $\Psi : \Gamma_N \rightarrow Y$ by $\Psi(g N H) = \phi(g)(a_0)$ for all $g \in \mathbb{F}(P)$. To see Ψ is well-defined, note that if $g_2^{-1} g_1 \in N H$, then for some $n \in N$ and $h \in H$, we have $\phi(g_2)^{-1} \phi(g_1)(a_0) = \phi(n) \phi(h)(a_0) = \phi(n)(a_0) = a_0$. Here we note that for any $n \in N$, $\phi(n)(a_0) = a_0$ since $n \in \ker(\phi)$, and for any $h \in H$, $\phi(h)(a_0) = a_0$ since $h(a_0) = a_0$ and $\phi(h)$ extends h . Thus $\phi(g_1)(a_0) = \phi(g_2)(a_0)$.

To verify (C1), let $p, q, r, s \in P \cup \{1\}$ and $g \in \mathbb{F}(P)$ be such that $p(a_0)$, $q(a_0)$, $r(a_0)$ and $s(a_0)$ are defined, $pNH = grNH$ and $qNH = gsNH$. Applying the map Ψ to these equations, we get $\phi(p)(a_0) = \phi(g)\phi(r)(a_0)$ and $\phi(q)(a_0) = \phi(g)\phi(s)(a_0)$. We need to show that $d_X(p(a_0), q(a_0)) = d_X(r(a_0), s(a_0))$. Since $\phi(g)$ is an isometry of Y , we have

$$\begin{aligned} d_X(p(a_0), q(a_0)) &= d_Y(\phi(p)(a_0), \phi(q)(a_0)) \\ &= d_Y(\phi(g)\phi(r)(a_0), \phi(g)\phi(s)(a_0)) \\ &= d_Y(\phi(r)(a_0), \phi(s)(a_0)) = d_X(r(a_0), s(a_0)). \end{aligned}$$

To verify (C2), let $p, q \in P \cup \{1\}$ be such that $p(a_0), q(a_0)$ are defined and $pNH = qNH$. We get

$$p(a_0) = \phi(p)(a_0) = \Psi(pNH) = \Psi(qNH) = \phi(q)(a_0) = q(a_0).$$

Finally, to verify (C3), let $p, q, r_1, s_1, \dots, r_n, s_n \in P \cup \{1\}$ and $g_1, \dots, g_n \in \mathbb{F}(P)$ be such that $p(a_0), q(a_0), r_1(a_0), s_1(a_0), \dots, r_n(a_0), s_n(a_0)$ are all defined, and

$$\begin{aligned} pNH &= g_1 r_1 NH, \quad g_1 s_1 NH = g_2 r_2 NH, \\ &\dots\dots \\ g_{n-1} s_{n-1} NH &= g_n r_n NH, \quad g_n s_n NH = qH. \end{aligned}$$

Applying Ψ to all these equations, we get

$$\begin{aligned} p(a_0) &= \phi(g_1)(r_1(a_0)), \quad \phi(g_1)(s_1(a_0)) = \phi(g_2)(r_2(a_0)), \\ &\dots\dots \\ \phi(g_{n-1})(s_{n-1}(a_0)) &= \phi(g_n)(r_n(a_0)), \quad \phi(g_n)(s_n(a_0)) = q(a_0). \end{aligned}$$

It follows that

$$d_X(p(a_0), q(a_0)) = d_Y(p(a_0), q(a_0)) \leq \sum_{i=1}^n d_Y(r_i(a_0), s_i(a_0)) = \sum_{i=1}^n d_X(r_i(a_0), s_i(a_0)).$$

□

Thus, for any finite P -type S -extension (Y, ϕ) , we have that $N = \ker(\phi)$ is a normal subgroup of $\mathbb{F}(P)$ of finite index and that N is a feasible prekernel. Furthermore, we will be able to carry out the construction of (Γ_N, Φ_N) as in Section 3 as a P -type S -extension of X based on the weighted graph (Γ_N, w_N) . In particular, the weight function w_N , the path metric d_{w_N} , the isometric embedding e_N , etc. are all well-defined.

Theorem 4.5. *Let (Y, ϕ) be a finite minimal P -type S -extension of X . Let $N = \ker(\phi)$ and $G = \Phi_N(\mathbb{F}(P))$. Then there is a G -invariant pseudometric ρ on Γ_N which is consistent with w_N such that (Y, ϕ) is isomorphic to $(\overline{\Gamma_N}, \overline{\Phi_N})$.*

Proof. We again define $\Psi : \Gamma_N \rightarrow Y$ by $\Psi(gNH) = \phi(g)(a_0)$ for all $g \in \mathbb{F}(P)$. As in the proof of Lemma 4.4, Ψ is well-defined. Since ϕ is minimal, Ψ is onto.

We define a pseudometric ρ on Γ_N by

$$\rho(g_1 NH, g_2 NH) = d_Y(\Psi(g_1 NH), \Psi(g_2 NH)) = d_Y(\phi(g_1)(a_0), \phi(g_2)(a_0)).$$

It is easy to verify that ρ is indeed a pseudometric on Γ_N .

Recall that $\Phi_N : P \rightarrow \text{Iso}(\Gamma_N)$ is defined by

$$\Phi_N(p)(gNH) = pgNH$$

for all $p \in P$ and $g \in \mathbb{F}(P)$. From previous section, the $\text{Iso}(\Gamma_N)$ refers to the group of isometries for the metric space (Γ_N, d_{w_N}) . Here we claim that the maps $gNH \mapsto pgNH$ are also isometries of the pseudometric space (Γ_N, ρ) . To see this, we only need to check

$$\begin{aligned} \rho(pg_1NH, pg_2NH) &= d_Y(\phi(pg_1)(a_0), \phi(pg_2)(a_0)) \\ &= d_Y(\phi(p)\phi(g_1)(a_0), \phi(p)\phi(g_2)(a_0)) = \rho(g_1NH, g_2NH). \end{aligned}$$

Extending Φ_N to a group homomorphism from $\mathbb{F}(P)$ to $\text{Iso}(\Gamma_N)$, the group of all isometries of the pseudometric space (Γ_N, ρ) , it follows that ρ is G -invariant.

To verify that ρ is consistent with w_N , we consider an edge in the weighted graph (Γ_N, w_N) , which is of the form $(gpNH, gqNH)$ where $g \in \mathbb{F}(P)$ and $p, q \in P \cup \{1\}$ are such that $p(a_0)$ and $q(a_0)$ are defined. Note that $w_N(gpNH, gqNH) = d_X(p(a_0), q(a_0))$. We have

$$\begin{aligned} \rho(gpNH, gqNH) &= d_Y(\phi(g)\phi(p)(a_0), \phi(g)\phi(q)(a_0)) \\ &= d_Y(\phi(p)(a_0), \phi(q)(a_0)) \\ &= d_X(p(a_0), q(a_0)) = w_N(gpNH, gqNH). \end{aligned}$$

We can now consider the metric identification of the pseudometric space (Γ_N, ρ) , which is denoted by $(\overline{\Gamma_N}, \overline{\rho})$. Since ρ is consistent with w_N , so is $\overline{\rho}$. Since ρ is G -invariant, for each $\varphi \in G$ we can define an isometry $\overline{\varphi} \in \overline{G}$ for $(\overline{\Gamma_N}, \overline{\rho})$. Thus it makes sense to define $\overline{\Phi_N} : P \rightarrow \text{Iso}(\overline{\Gamma_N})$ by $\overline{\Phi_N}(p) = \overline{\Phi_N}(p)$.

Finally, let $\pi : \overline{\Gamma_N} \rightarrow Y$ be defined as $\pi([gNH]_{\sim}) = \phi(g)(a_0)$. Then π is an isometry between the metric spaces $(\overline{\Gamma_N}, \overline{\rho})$ and (Y, d_Y) . To complete the proof of the theorem, we only need to verify that for any $p \in P$, $\pi \circ \overline{\Phi_N}(p) = \phi(p) \circ \pi$. We have

$$\begin{aligned} [\pi \circ \overline{\Phi_N}(p)]([gNH]_{\sim}) &= \pi[\overline{\Phi_N}(p)([gNH]_{\sim})] = \pi([pgNH]_{\sim}) \\ &= \phi(pg)(a_0) = \phi(p)\phi(g)(a_0) \\ &= \phi(p)[\pi([gNH]_{\sim})] = [\phi(p) \circ \pi]([gNH]_{\sim}). \end{aligned}$$

□

Theorem 4.6. *Let $N \trianglelefteq \mathbb{F}(P)$ be a feasible P -type prekernel that is of finite index. Let $G = \Phi_N(\mathbb{F}(P))$. Let ρ be a G -invariant pseudometric on Γ_N which is consistent with the weight function w_N . Then $(\overline{\Gamma_N}, \overline{\Phi_N})$ is a finite minimal P -type S -extension of X .*

Proof. Consider the metric identification $(\overline{\Gamma_N}, \overline{\rho})$. Since ρ is G -invariant, \overline{G} is a set of isometries for $\overline{\Gamma_N}$. Since ρ is consistent with w_N , so is $\overline{\rho}$. Define $\overline{e_N} : X \rightarrow \overline{\Gamma_N}$ by $\overline{e_N}(a) = [e_N(a)]_{\sim}$. Then $\overline{e_N}$ is an isometric embedding from X into $\overline{\Gamma_N}$. Note that $\overline{e_N}(a_0) = [NH]_{\sim}$. Thus we can identify a_0 with $[NH]_{\sim}$. It follows from similar

arguments as before that $(\overline{\Gamma_N}, \overline{\Phi_N})$ is a P -type S-extension of X . To see that it is minimal, we just note that for any $g \in \mathbb{F}(P)$, $\overline{\Phi_N}(g)([NH]_{\sim}) = [gNH]_{\sim}$. \square

We summarize the characterization of all finite minimal P -type S-extensions of X in the following theorem.

Theorem 4.7. *The following are equivalent:*

- (i) (Y, ϕ) is a finite minimal P -type S-extension of X ;
- (ii) There exists a feasible P -type prekernel $N \trianglelefteq \mathbb{F}(P)$ of finite index, and, letting $G = \Phi_N(\mathbb{F}(P))$, there exists a G -invariant pseudometric ρ on Γ_N which is consistent with w_N , such that (Y, ϕ) is isomorphic to $(\overline{\Gamma_N}, \overline{\Phi_N})$;
- (iii) For $N = \ker(\phi)$ and $G = \Phi_N(\mathbb{F}(P))$, there exists a G -invariant pseudometric ρ on Γ_N which is consistent with w_N , such that (Y, ϕ) is isomorphic to $(\overline{\Gamma_N}, \overline{\Phi_N})$.

5. COHERENT S-EXTENSIONS

5.1. Coherent S-extensions and strongly coherent S-extensions. In this section we study a notion of coherence for (P -type) S-extensions. The terminology has been used for a different notion in Siniora–Solecki [14] which refers to a slightly stronger condition. We call their notion of coherence *strongly coherent*.

Definition 5.1 (Solecki). Let X be a metric space. An S-extension (Y, ϕ) of X is *strongly coherent* if for every triple (p, q, r) of partial isometries of X such that $p \circ q = r$, we have $\phi(p) \circ \phi(q) = \phi(r)$.

The following is a strengthening of Solecki's theorem on existence of finite S-extensions.

Theorem 5.2 (Solecki [13] [10] [14]). *Let X be a finite metric space. Then X has a finite strongly coherent S-extension (Y, ϕ) .*

The following notion of coherence is slightly weaker than the notion of strongly coherent but is sufficient for our study of ultraextensive metric spaces in subsequent sections.

Definition 5.3. Let $X_1 \subseteq X_2$ be metric spaces and (Y_i, ϕ_i) be a P_i -type S-extension of X_i for $i = 1, 2$ where $P_1 \subseteq P_2$. We say that (Y_1, ϕ_1) and (Y_2, ϕ_2) are *coherent* if

- (i) Y_2 extends Y_1 ,
- (ii) $\phi_2(p)$ extends $\phi_1(p)$ for all $p \in P_1 \subseteq P_2$, and
- (iii) letting $K_i = \phi_i(\mathbb{F}(P_i)) \leq \text{Iso}(Y_i)$ for $i = 1, 2$, and letting $\kappa : K_1 \rightarrow K_2$ be such that $\kappa(\phi_1(p)) = \phi_2(p)$ for all $p \in P_1$, then κ has a unique extension to a group isomorphic embedding from K_1 into K_2 .

The following lemma makes it precise that the notion of strong coherence is a stronger notion than coherence.

Lemma 5.4. Let $X_1 \subseteq X_2$ be finite metric spaces and (Y_1, ϕ_1) be a P_1 -type S-extension of X_1 . Let $P_2 \supseteq P_1$ be such that $P_2 = P_2^{-1}$ and $X_2 \setminus \{a_0\} \subseteq P_2(a_0)$.

Suppose (Y_2, ϕ) is a strongly coherent S-extension of $X_2 \cup Y_1$. Then there is ϕ_2 such that (Y_2, ϕ_2) is a P_2 -type S-extension of X_2 which is coherent with (Y_1, ϕ_1) .

Proof. Let (Y_2, ϕ) be a strongly coherent S-extension of $X_2 \cup Y_1$. Define $\phi_2 : P_2 \rightarrow \text{Iso}(Y_2)$ by

$$\phi_2(p) = \begin{cases} \phi(\phi_1(p)), & \text{if } p \in P_1, \\ \phi(p), & \text{if } p \in P_2 \setminus P_1. \end{cases}$$

Then (Y_2, ϕ_2) is obviously a P_2 -type S-extension of X_2 . The construction also guarantees that $Y_1 \subseteq Y_2$ and that $\phi_1(p) \subseteq \phi_2(p)$ for $p \in P_1$. Since (Y_2, ϕ) is a strongly coherent S-extension of Y_1 , ϕ restricted to $\text{Iso}(Y_1)$ is a group isomorphism embedding from $\text{Iso}(Y_1)$ into $\text{Iso}(Y_2)$. When further restricted to $K_1 = \phi_1(\mathbb{F}(P_1))$, it gives a group isomorphic embedding into $K_2 = \phi_2(\mathbb{F}(P_2))$. \square

5.2. A characterization of coherent S-extensions. Although the existence of coherent S-extensions follows from results of Solecki [13] [10] [14] by Lemma 5.4, we introduce a notion of coherent extensions for groups and utilize this notion to give a characterization of all possible minimal coherent S-extensions.

Let $X_1 \subseteq X_2$ be finite metric spaces, (Y_1, ϕ_1) be a minimal P_1 -type S-extension of X_1 , and $P_1 \subseteq P_2 \subseteq \mathcal{P}_{X_2}$ where $P_2 = P_2^{-1}$ and $X_2 \setminus \{a_0\} \subseteq P_2(a_0)$. Let (Y_2, ϕ_2) be a minimal P_2 -type S-extension of X_2 that is coherent with (Y_1, ϕ_1) . Next, we characterize all such coherent S-extensions.

Definition 5.5. Let $X_1 \subseteq X_2$ be finite metric spaces, (Y_1, ϕ_1) be a P_1 -type S-extension of X_1 and $P_1 \subseteq P_2 \subseteq \mathcal{P}_{X_2}$ where $P_2 = P_2^{-1}$ and $X_2 \setminus \{a_0\} \subseteq P_2(a_0)$. Let $N_1 = \ker(\phi_1)$. We say $N_2 \trianglelefteq \mathbb{F}(P_2)$ is a *coherent extension* of N_1 if it is a feasible P_2 -type prekernel for X_2 and satisfies the following conditions:

- (D1) $N_1 = N_2 \cap \mathbb{F}(P_1)$;
- (D2) For every $g, h, k, l \in \mathbb{F}(P_1)$ such that

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{Y_1}(\phi_1(k)(a_0), \phi_1(l)(a_0)),$$

we have $N_2 \cap gH_2k^{-1}lH_2h^{-1} = \emptyset$;

- (D3) For every $g, h \in \mathbb{F}(P_1)$ and $p, q \in P_2$ with both $p(a_0)$ and $q(a_0)$ defined, if

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{X_2}(p(a_0), q(a_0)),$$

we have $N_2 \cap gH_2p^{-1}qH_2h^{-1} = \emptyset$.

Note that since N_2 is a feasible prekernel, letting $G_2 = \Phi_{N_2}(\mathbb{F}(P_2))$, for any G_2 -invariant pseudometric ρ_2 on Γ_{N_2} which is consistent with w_{N_2} , $(\overline{\Gamma_{N_2}}, \overline{\Phi_{N_2}})$ is a minimal S-extension of X_2 .

Theorem 5.6. Let $X_1 \subseteq X_2$ be finite metric spaces, (Y_1, ϕ_1) be a minimal P_1 -type S-extension of X_1 , and $P_1 \subseteq P_2 \subseteq \mathcal{P}_{X_2}$ where $P_2 = P_2^{-1}$ and $X_2 \setminus \{a_0\} \subseteq P_2(a_0)$.

- (i) Let (Y_2, ϕ_2) be a P_2 -type S-extension of X_2 that is coherent with (Y_1, ϕ_1) . Then $N_2 = \ker(\phi_2)$ is a coherent extension of $N_1 = \ker(\phi_1)$.
- (ii) Let $N_2 \trianglelefteq \mathbb{F}(P_2)$ be a coherent extension of $N_1 = \ker(\phi_1)$. Then letting $G_2 = \Phi_{N_2}(\mathbb{F}(P_2))$, there exists a G_2 -invariant pseudometric ρ_2 on Γ_{N_2}

which is consistent with w_{N_2} , such that $(Y_2, \phi_2) = (\overline{\Gamma_{N_2}}, \overline{\Phi_{N_2}})$ is coherent with (Y_1, ϕ_1) .

Proof. We first prove (i). Let (Y_2, ϕ_2) be a P_2 -type S-extension of X_2 that is coherent with (Y_1, ϕ_1) . Let $N_2 = \ker(\phi_2)$. Then by Theorem 4.5, there is a pseudometric ρ_2 on Γ_{N_2} such that (Y_2, ϕ_2) is isomorphic to $(\overline{\Gamma_{N_2}}, \overline{\Phi_{N_2}})$. By Lemma 4.4, N_2 is a feasible prekernel. Next we show that N_2 is a coherent extension of N_1 .

For (D1), note that since (Y_2, ϕ_2) is coherent with (Y_1, ϕ_1) , we have $\phi_1(\mathbb{F}(P_1)) \cong \phi_2(\mathbb{F}(P_1))$ via the map $\phi_1(g) \mapsto \phi_2(g)$. Thus,

$$N_2 \cap \mathbb{F}(P_1) = \ker(\phi_2) \cap \mathbb{F}(P_1) = \ker(\phi_1) = N_1.$$

For (D2), we need to verify that if for $g, h, k, l \in \mathbb{F}(P_1)$

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{Y_1}(\phi_1(k)(a_0), \phi_1(l)(a_0)),$$

then $N_2 \cap gH_2k^{-1}lH_2h^{-1} = \emptyset$. Toward a contradiction, assume there is $n \in N_2 \cap gH_2k^{-1}lH_2h^{-1}$. Then there are $\eta, \eta' \in H_2$ with $k\eta^{-1}g^{-1}n = l\eta'h^{-1}$, which implies

$$\phi_2(k\eta^{-1}g^{-1}n) = \phi_2(l\eta'h^{-1}).$$

From the definitions of H_2 and of N_2 , if we apply the left-hand-side element to $\phi_2(g)(a_0) = \phi_1(g)(a_0)$, the resulting value is $\phi_2(k)(a_0) = \phi_1(k)(a_0)$. Similarly, if we apply the right-hand-side element to $\phi_2(h)(a_0) = \phi_1(h)(a_0)$, the resulting value is $\phi_2(l)(a_0) = \phi_1(l)(a_0)$. Thus, both sides of the equation represent the same partial isometry of Y_1 with $\phi_1(g)(a_0)$ and $\phi_1(h)(a_0)$ in its domain and with $\phi_1(k)(a_0)$ and $\phi_1(l)(a_0)$ in its range. We conclude that $d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) = d_{Y_1}(\phi_1(k)(a_0), \phi_1(l)(a_0))$, a contradiction.

The argument for (D3) is similar. This finishes the proof of (i).

For (ii), let $G_1 = \Phi_{N_1}(\mathbb{F}(P_1))$ and, by Theorem 4.5, let ρ_1 be a G_1 -invariant pseudometric that is consistent with w_{N_1} such that $(\overline{\Gamma_{N_1}}, \overline{\Phi_{N_1}}) \cong (Y_1, \phi_1)$. For notational simplicity we assume $(Y_1, \phi_1) = (\overline{\Gamma_{N_1}}, \overline{\Phi_{N_1}})$. Let $N_2 \trianglelefteq \mathbb{F}(P_2)$ be a coherent extension of $N_1 = \ker(\phi_1)$. Since N_2 is a feasible prekernel, one can define Γ_{N_2} , w_{N_2} , and Φ_{N_2} as before.

Define a map $\pi : \Gamma_{N_1} \rightarrow \Gamma_{N_2}$ by letting $\pi(gN_1H_1) = gN_2H_2$ for all $g \in \mathbb{F}(P_1)$. To see π is well-defined, note that if $gN_1H_1 = g'N_1H_1$, then $g^{-1}g' \in N_1H_1 \leq N_2H_2$, and therefore $gN_2H_2 = g'N_2H_2$.

Recall that w_{N_1} is defined for pairs (gpN_1H_1, gqN_1H_1) where $g \in \mathbb{F}(P_1)$ and $p, q \in P_1$ with $p(a_0)$ and $q(a_0)$ defined, and its value is $d_{X_1}(p(a_0), q(a_0))$. Let $\pi(w_{N_1})$ on $\pi(\Gamma_{N_1})$ be the push-forward weight function, that is,

$$\pi(w_{N_1})(gpN_2H_2, gqN_2H_2) = w_{N_1}(gpN_1H_1, gqN_1H_1).$$

Note that (D2) implies that $\pi(w_{N_1})$ is well-defined. Also note that w_{N_2} coincides with $\pi(w_{N_1})$ on $\pi(\Gamma_{N_1})$. In fact, w_{N_2} is defined in the same way on the image of such pairs under π , that is, on pairs of the form (gpN_2H_2, gqN_2H_2) for $g \in \mathbb{F}(P_1) \subseteq \mathbb{F}(P_2)$ and $p, q \in P_1 \subseteq P_2$.

Recall that Φ_{N_2} is defined by $\Phi_{N_2}(p)(gN_2H_2) = pgN_2H_2$ for all $g \in \mathbb{F}(P_2)$ and $p \in P_2$, and is extended to a group homomorphism from $\mathbb{F}(P_2)$ to the symmetric group of Γ_{N_2} . Let $G_2 = \Phi_{N_2}(\mathbb{F}(P_2))$.

We are now ready to define a G_2 -invariant pseudometric ρ_2 on Γ_{N_2} that is consistent with w_{N_2} and satisfies $\rho_2 \upharpoonright \pi(\Gamma_{N_1}) = \pi(\rho_1)$. Here $\pi(\rho_1)$ is the pseudometric on $\pi(\Gamma_{N_1})$ defined by $\pi(\rho_1)(gN_2H_2, hN_2H_2) = \rho_1(gN_1H_1, hN_1H_1)$ for $g, h \in \mathbb{F}(P_1)$. Note that (D2) implies that $\pi(\rho_1)$ is well-defined.

We define ρ_2 as follows. First, for $g, h \in \mathbb{F}(P_1)$, define

$$\begin{aligned} \rho_2(gN_2H_2, hN_2H_2) &= \pi(\rho_1)(gN_2H_2, hN_2H_2) \\ &= \rho_1(gN_1H_1, hN_1H_1) = d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)). \end{aligned}$$

Next, for $p, q \in P_2$ with $p(a_0)$ and $q(a_0)$ defined, and for $\gamma \in \mathbb{F}(P_2)$, define

$$\rho_2(\gamma pN_2H_2, \gamma qN_2H_2) = w_{N_2}(\gamma pN_2H_2, \gamma qN_2H_2) = d_{X_2}(p(a_0), q(a_0)).$$

To see that these do not conflict with each other, note that (D3) implies that for $g, h \in \mathbb{F}(P_1)$ and $p, q \in P_2$ with both $p(a_0)$ and $q(a_0)$ defined, if

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{X_2}(p(a_0), q(a_0)),$$

then there is no $\gamma \in \mathbb{F}(P_2)$ with $\gamma pN_2H_2 = gN_2H_2$ and $\gamma qN_2H_2 = hN_2H_2$. We continue to define ρ_2 so that if $g, h \in \mathbb{F}(P_2)$ and $\rho_2(gN_2H_2, hN_2H_2)$ is already defined, then we define

$$\rho_2(\gamma gN_2H_2, \gamma hN_2H_2) = \rho_2(gN_2H_2, hN_2H_2)$$

for any $\gamma \in \mathbb{F}(P_2)$. To see that this does not create a conflict among the existing definitions of ρ_2 values, note that condition (D2) implies that for any $g, h, k, l \in \mathbb{F}(P_1)$, if there is $\gamma \in \mathbb{F}(P_2)$ such that $\gamma gN_2H_2 = kN_2H_2$ and $\gamma hN_2H_2 = lN_2H_2$, then

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) = d_{Y_1}(\phi_1(k)(a_0), \phi_1(l)(a_0)).$$

To complete the definition of ρ_2 , we consider the existing values of ρ_2 as a weight function and define ρ_2 to be the path pseudometric. Since the weight function is G_2 -invariant, it follows from the definition of the path pseudometric that the resulting ρ_2 is also G_2 -invariant.

Since for every $g, h \in \mathbb{F}(P_1)$ we have

$$\rho_2(gN_2H_2, hN_2H_2) = \pi(\rho_1)(gN_2H_2, hN_2H_2) = \rho_1(gN_1H_1, hN_1H_1),$$

we also have that $\overline{\rho_2} \upharpoonright \overline{\pi(\Gamma_{N_1})} = \overline{\pi(\rho_1)}$. Note that (D2) with $l, k = 1$ implies that the induced map $\overline{\pi} : \overline{\Gamma_{N_1}} = \overline{\Gamma_{N_1}^{\rho_1}} \rightarrow \overline{\Gamma_{N_2}} = \overline{\Gamma_{N_2}^{\rho_2}}$ is an isometric embedding. Thus $\overline{\Gamma_{N_1}} \cong \overline{\pi(\Gamma_{N_1})}$ is a subspace of $\overline{\Gamma_{N_2}}$.

Letting $Y_2 = \overline{\Gamma_{N_2}}$ and $\phi_2 = \overline{\Phi_{N_2}}$. We have that (Y_2, ϕ_2) is a P_2 -type S-extension of X_2 and $Y_1 \subseteq Y_2$ via the isomorphism of (Y_1, ϕ_1) with $(\overline{\Gamma_{N_1}}, \overline{\Phi_{N_1}})$. To see the coherence of (Y_2, ϕ_2) with (Y_1, ϕ_1) , let $p \in P_1$. Then $\overline{\Phi_{N_1}}(p)(gN_1H_1) = pgN_1H_1$ for all $g \in \mathbb{F}(P_1)$ and $\overline{\Phi_{N_2}}(p)(gN_2H_2) = pgN_2H_2$ for all $g \in \mathbb{F}(P_2)$. Via the induced embedding $\overline{\pi} : \overline{\Gamma_{N_1}} \rightarrow \overline{\Gamma_{N_2}}$ and the isomorphism of (Y_1, ϕ_1) with $(\overline{\Gamma_{N_1}}, \overline{\Phi_{N_1}})$, and because $\rho_2 \upharpoonright \pi(\Gamma_{N_1}) = \pi(\rho_1)$, we have $\phi_1(p) \subseteq \phi_2(p)$. Finally, it is clear that the map $\phi_1(p) \mapsto \phi_2(p)$ for all $p \in P_1$ generates a group isomorphic embedding from G_1 to G_2 . \square

5.3. A construction of coherent prekernel extensions. The existence of finite coherent S-extensions via Lemma 5.4 and Theorem 5.6 imply the existence of coherent prekernel extensions. In this subsection we provide a direct construction of coherent prekernel extensions of finite index.

Lemma 5.7. Let $X_1 \subseteq X_2$ be finite metric spaces, (Y_1, ϕ_1) be a P_1 -type S-extension of X_1 and $P_1 \subseteq P_2 \subseteq \mathcal{P}_{X_2}$ where $P_2 = P_2^{-1}$ and $X_2 \setminus \{a_0\} \subseteq P_2(a_0)$. Then there exists a P_2 -type S-map $\phi_{\mathbb{U}} : \mathbb{F}(P_2) \rightarrow \text{Iso}(\mathbb{U})$ such that $(\mathbb{U}, \phi_{\mathbb{U}})$ is a P_2 -type S-extension of X_2 which is coherent with (Y_1, ϕ_1) .

Proof. Following Uspenskij's proof in [15], which uses the Katetov construction of \mathbb{U} to show that the isometry group of every Polish space can be embedded into $\text{Iso}(\mathbb{U})$ (see also Sections 1.2 and 2.5 of [4] for details), we obtain an isometric embedding $i : Y_1 \rightarrow \mathbb{U}$ and a group isomorphic embedding $j : \text{Iso}(Y_1) \rightarrow \text{Iso}(\mathbb{U})$ such that for every $\varphi \in \text{Iso}(Y_1)$, $j(\varphi) \supseteq \varphi$. In addition, from the ultrahomogeneity of \mathbb{U} we obtain an isometric copy of X_2 in \mathbb{U} as a superset of X_1 . Now for each $p \in P_2$, let $\phi_{\mathbb{U}}(p) \in \text{Iso}(\mathbb{U})$ be an extension of p guaranteed to exist by the ultrahomogeneity of \mathbb{U} such that if $p \in P_1$, then $\phi_{\mathbb{U}}(p) = j(\phi_1(p))$. Then $\phi_{\mathbb{U}}$ is as required. \square

Proposition 5.8. Suppose $X_1 \subseteq X_2$ are finite metric spaces and $P_1 \subseteq P_2 \subseteq \mathcal{P}_{X_2}$ where $P_2 = P_2^{-1}$ and $X_2 \setminus \{a_0\} \subseteq P_2(a_0)$. Let (Y_1, ϕ_1) be a finite P_1 -type S-extension of X_1 and $N_1 = \ker(\phi_1)$. Then, there exists a coherent extension, $N_2 \trianglelefteq \mathbb{F}(P_2)$, of N_1 of finite index.

Proof. By Lemma 4.4, $N_1 = \ker(\phi_1)$ is a feasible prekernel. We may define w_{N_1} and Φ_{N_1} and let $G_1 = \Phi_{N_1}(\mathbb{F}(P_1))$ and $\Gamma_{N_1} = \mathbb{F}(P_1)/N_1H_1$. Since (Y_1, ϕ_1) is a minimal P_1 -type S-extension of X_1 , by Theorem 4.5, there is a G_1 -invariant pseudometric ρ_1 on Γ_{N_1} such that it is consistent with w_{N_1} , Y_1 is isometric to $(\overline{\Gamma_{N_1}}, \overline{\rho_1})$ and (Y_1, ϕ_1) is isomorphic to $(\overline{\Gamma_{N_1}}, \overline{\Phi_{N_1}})$.

Since $P_1 \subseteq P_2$, we have that all of N_1 , H_1 and $\mathbb{F}(P_1)$ are subgroups of $\mathbb{F}(P_2)$. We will find a coherent extension, $N_2 \trianglelefteq \mathbb{F}(P_2)$, of N_1 of finite index.

Let $\mathcal{G} = G_1 * \mathbb{F}(P_2 \setminus P_1)$ be the free product of G_1 with $\mathbb{F}(P_2 \setminus P_1)$. We define a group homomorphism $\psi : \mathbb{F}(P_2) \rightarrow \mathcal{G}$ by letting

$$\psi(p) = \begin{cases} \Phi_{N_1}(p), & \text{if } p \in P_1, \\ p, & \text{otherwise} \end{cases}$$

for all $p \in P_2$. Since H_2 is a finitely generated subgroup of $\mathbb{F}(P_2)$, $\psi(H_2)$ is a finitely generated subgroup of \mathcal{G} . We will find $M \trianglelefteq \mathcal{G}$ of finite index and set $N_2 = \psi^{-1}(M)$. To guarantee that N_2 is a coherent extension of N_1 , we need M to satisfy the following corresponding conditions:

- (R1) For every $p, q, r, s \in P_2 \cup \{1\}$ such that $d_{X_2}(p(a_0), q(a_0)) \neq d_{X_2}(r(a_0), s(a_0))$, we have $M \cap \psi(p)\psi(H_2)\psi(r)^{-1}\psi(s)\psi(H_2)\psi(q)^{-1} = \emptyset$;
- (R2) For every $p, q \in P_2 \cup \{1\}$, if $p(a_0)$ and $q(a_0)$ are defined and $p(a_0) \neq q(a_0)$, we have $M \cap \psi(p)\psi(H_2)\psi(q)^{-1} = \emptyset$;

(R3) For every $p, q, r_1, s_1, \dots, r_n, s_n \in P_2 \cup \{1\}$ such that

$$d_{X_2}(p(a_0), q(a_0)) > \sum_{i=1}^n d_{X_2}(r_i(a_0), s_i(a_0)),$$

we have

$$M \cap \psi(p)\psi(H_2)\psi(r_1)^{-1}\psi(s_1)\psi(H_2) \cdots \psi(H_2)\psi(r_n)^{-1}\psi(s_n)\psi(H_2)\psi(q)^{-1} = \emptyset;$$

(S1) $M \cap G_1 = \{1\}$;

(S2) For every $g, h, k, l \in \mathbb{F}(P_1)$ such that

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{Y_1}(\phi_1(k)(a_0), \phi_1(l)(a_0)),$$

we have $M \cap \psi(g)\psi(H_2)\psi(k)^{-1}\psi(l)\psi(H_2)\psi(h)^{-1} = \emptyset$;

(S3) For every $g, h \in \mathbb{F}(P_1)$ and $p, q \in P_2$ with both $p(a_0)$ and $q(a_0)$ defined, if

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{X_2}(p(a_0), q(a_0)),$$

we have $M \cap \psi(g)\psi(H_2)\psi(p)^{-1}\psi(q)\psi(H_2)\psi(h)^{-1} = \emptyset$.

To see that (S1) implies (D1), note that $N_2 \cap \mathbb{F}(P_1) = \psi^{-1}(M) \cap (\psi^{-1}(G_1) \cap \mathbb{F}(P_1)) = (\psi^{-1}(M) \cap \psi^{-1}(G_1)) \cap \mathbb{F}(P_1) = \psi^{-1}(M \cap G_1) \cap \mathbb{F}(P_1) = \ker(\psi) \cap \mathbb{F}(P_1) = N_1$. The other conditions for M obviously imply the corresponding conditions for N_2 . Note also that each of conditions (R1), (R2) and (R3) is a finite collection of conditions of the form $\gamma M \cap L_1 \cdots L_n = \emptyset$ for $\gamma \in \mathcal{G}$ and finitely generated subgroups L_1, \dots, L_n (in fact each L_i is a conjugate of $\psi(H_2)$) with $\gamma \notin L_1 \cdots L_n$. Since G_1 is finite, condition (S1) is also a finite collection of conditions of the form $\gamma M \cap \{1\} = \emptyset$ for nonidentity $\gamma \in G_1$. Conditions (S2) and (S3) appear to be about infinitely many elements in $\mathbb{F}(P_1)$. However, since $G_1 = \psi(\mathbb{F}(P_1))$ is finite, they all end up being about finitely many elements of G_1 , and so each of (S2) and (S3) is still a finite collection of conditions of the form $\gamma M \cap L_1 \cdots L_n = \emptyset$ for finitely generated subgroups L_1, \dots, L_n . We verify that in each case, $\gamma \notin L_1 \cdots L_n$.

Using the P_2 -type S-map $\phi_{\mathbb{U}}$ from Lemma 5.7, we note that for any $g \in \mathbb{F}(P_2)$, if $\psi(g) = 1$, then $\phi_{\mathbb{U}}(g) = 1$. This follows from the definition of ψ and of $\phi_{\mathbb{U}}$.

For (R1), we need to verify that $1 \notin \psi(p)\psi(H_2)\psi(r)^{-1}\psi(s)\psi(H_2)\psi(q)^{-1}$. Toward a contradiction, if $1 \in \psi(p)\psi(H_2)\psi(r)^{-1}\psi(s)\psi(H_2)\psi(q)^{-1}$, then there exist $\eta_1, \eta_2 \in H_2$ such that $\psi(p\eta_1 r^{-1} s \eta_2 q^{-1}) = 1$. Let $\alpha = s \eta_2 q^{-1}$ and $\beta = r \eta_1^{-1} p^{-1}$. Then $\psi(\alpha) = \psi(\beta)$ and therefore $\phi_{\mathbb{U}}(\alpha) = \phi_{\mathbb{U}}(\beta)$. Since $\eta_1, \eta_2 \in H_2$, $\phi_{\mathbb{U}}(\alpha)(q(a_0)) = s(a_0)$ and $\phi_{\mathbb{U}}(\beta)(p(a_0)) = r(a_0)$. Now since $\phi_{\mathbb{U}}(\alpha) = \phi_{\mathbb{U}}(\beta)$ is an isometry, we should have $d_{X_2}(p(a_0), q(a_0)) = d_{X_2}(r(a_0), s(a_0))$.

For (R2), similar argument shows that if $\alpha = p \eta_1 q^{-1}$, then $\phi_{\mathbb{U}}(\alpha)(q(a_0)) = p(a_0)$. Now since $\psi(\alpha) = 1$, $\phi_{\mathbb{U}}(\alpha) = 1$ and therefore $q(a_0) = p(a_0)$.

For (R3), if

$$1 \in \psi(p)\psi(H_2)\psi(r_1)^{-1}\psi(s_1)\psi(H_2) \cdots \psi(H_2)\psi(r_n)^{-1}\psi(s_n)\psi(H_2)\psi(q)^{-1}$$

then for some $h_1, \dots, h_{n+1} \in H_2$ we have

$$1 = \psi(p)\psi(h_1)\psi(r_1)^{-1}\psi(s_1)\psi(h_2) \cdots \psi(h_n)\psi(r_n)^{-1}\psi(s_n)\psi(h_{n+1})\psi(q)^{-1}.$$

Consider the sequence

$$\begin{aligned} b_0 &= \phi_{\mathbb{U}}(p)(a_0) = p(a_0), \\ b_1 &= \phi_{\mathbb{U}}(ph_1r_1^{-1}s_1)(a_0), \\ b_2 &= \phi_{\mathbb{U}}(ph_1r_1^{-1}s_1h_2r_2^{-1}s_2)(a_0), \\ &\dots\dots \\ b_n &= \phi_{\mathbb{U}}(ph_1r_1^{-1}s_1h_2\cdots r_n^{-1}s_n)(a_0) = \phi_{\mathbb{U}}(qh_{n+1}^{-1})(a_0) = q(a_0). \end{aligned}$$

We have

$$\begin{aligned} d_{\mathbb{U}}(b_0, b_1) &= d_{\mathbb{U}}(\phi_{\mathbb{U}}(r_1h_1^{-1}p^{-1})(b_0), \phi_{\mathbb{U}}(r_1h_1^{-1}p^{-1})(b_1)) \\ &= d_{\mathbb{U}}(r_1(a_0), s_1(a_0)) = d_{X_2}(r_1(a_0), s_1(a_0)), \end{aligned}$$

and similarly $d_{\mathbb{U}}(b_1, b_2) = d_{X_2}(r_2(a_0), s_2(a_0)), \dots, d_{\mathbb{U}}(b_{n-1}, b_n) = d_{X_2}(r_n(a_0), s_n(a_0))$.

Thus

$$d_{X_2}(p(a_0), q(a_0)) \leq \sum_{i=1}^n d_{X_2}(b_{i-1}, b_i) = \sum_{i=1}^n d_{X_2}(r_i(a_0), s_i(a_0)).$$

For (S2), we need to verify that if

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{Y_1}(\phi_1(k)(a_0), \phi_1(l)(a_0)),$$

then $1 \notin \psi(g)\psi(H_2)\psi(k)^{-1}\psi(l)\psi(H_2)\psi(h)^{-1}$. Toward a contradiction, assume $1 \in \psi(g)\psi(H_2)\psi(k)^{-1}\psi(l)\psi(H_2)\psi(h)^{-1}$. Then there are $\eta, \eta' \in H_2$ with

$$\psi(g)\psi(\eta)\psi(k)^{-1} = \psi(h)\psi(\eta')\psi(l)^{-1}.$$

From the definitions of H_2 and of ψ , if we apply the left-hand-side element to $\psi(k)(a_0) = \phi_1(k)(a_0)$, the resulting value is $\psi(g)(a_0) = \phi_1(g)(a_0)$. Similarly, if we apply the right-hand-side element to $\psi(l)(a_0) = \phi_1(l)(a_0)$, the resulting value is $\psi(h)(a_0) = \phi_1(h)(a_0)$. Thus, both sides of the equation represent the same partial isometry of Y_1 with $\phi_1(k)(a_0)$ and $\phi_1(l)(a_0)$ in its domain and with $\phi_1(g)(a_0)$ and $\phi_1(h)(a_0)$ in its range. We conclude that $d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) = d_{Y_1}(\phi_1(k)(a_0), \phi_1(l)(a_0))$, a contradiction.

The argument for (S3) is similar.

Now by Coulbois' theorem (Theorem 2.9), the group $\mathcal{G} = G_1 * \mathbb{F}(P_2 \setminus P_1)$ has property RZ. Thus, there exists $M \trianglelefteq \mathcal{G}$ of finite index such that all conditions (R1)–(S3) are satisfied. Consequently, $N_2 = \psi^{-1}(M) \trianglelefteq \mathbb{F}(P_2)$ is a coherent extension of N_1 of finite index. \square

5.4. Extending isometry groups. In this subsection we apply the algebraic method from the preceding subsection to obtain a construction of S-extensions with prescribed isometry groups. Since we deal with only finite isometry groups, it suffices to consider groups extended by one more generator.

Theorem 5.9. *Let X_1 be a finite metric space and (Y_1, ϕ_1) be a finite minimal P_1 -type S-extension of X_1 . Let $G_1 = \phi_1(\mathbb{F}(P_1))$ and $G_2 = \langle G_1, k \rangle$ be an overgroup of G_1 with one element $k \notin G_1$. Then there exist a finite metric space $X_2 \supseteq X_1$, $l \in \mathcal{P}_{X_2}$ and a P_2 -type S-extension (Y_2, ϕ_2) of X_2 , where $P_2 = P_1 \cup \{l, l^{-1}\}$, such that (Y_1, ϕ_1) and (Y_2, ϕ_2) are coherent and $G_2 \cong \phi_2(\mathbb{F}(P_2))$.*

Proof. By Theorem 4.5, Y_1 is isometric to $(\overline{\Gamma_{N_1}}, \overline{\Phi_{N_1}})$ where $N_1 = \ker(\phi_1)$. Let $X_2 = Y_1 \cup \{a\}$ be the one point extension of Y_1 where $d_{X_2}(a, b) = \text{diam}(Y_1)$ for every $b \in Y_1$. Let $P_2 = P_1 \cup \{l, l^{-1}\}$ where $l = \{(a_0, a)\}$ is the partial isometry that sends a_0 to a . We define a homomorphism $\phi_2 : \mathbb{F}(P_2) \rightarrow G_2$ such that $\phi_2 \upharpoonright_{\mathbb{F}(P_1)} = \phi_1$ and $\phi_2(l) = k$. Let $N_2 = \ker(\phi_2)$. We claim there exists a G_2 -invariant pseudometric ρ_2 on Γ_{N_2} which is consistent with w_{N_2} , such that $(Y_2, \phi_2) \cong (\overline{\Gamma_{N_2}}, \overline{\Phi_{N_2}})$ is as desired. Note that because of the definition of X_2 and P_2 , $H_2 = H_1$. By Theorem 5.6, it suffices to show that N_2 is a coherent extension of N_1 .

- (C1) For every $p, q, r, s \in P_2 \cup \{1\}$ such that $d_{X_2}(p(a_0), q(a_0)) \neq d_{X_2}(r(a_0), s(a_0))$, we have $N_2 \cap pH_1r^{-1}sH_1q^{-1} = \emptyset$. If p, q, r, s are different from l , then since $(\overline{\Gamma_{N_1}}, \overline{\Phi_{N_1}})$ is a P_1 -type S-extension of X_1 , by (C0) we have $N_2 \cap pH_1r^{-1}sH_1q^{-1} = N_1 \cap pH_1r^{-1}sH_1q^{-1} = \emptyset$. If one of p, q, r, s is equal to l , then we have $\phi_2(ph_1r^{-1}sh_2q^{-1}) = 1$ for some $h_1, h_2 \in H_1$. Since l appears exactly once in $ph_1r^{-1}sh_2q^{-1}$, this means $\phi_2(l) = k \in G_1$, which is a contradiction. Other cases are obvious.
- (C2) For every $p, q \in P_2 \cup \{1\}$, if $p(a_0)$ and $q(a_0)$ are defined and $p(a_0) \neq q(a_0)$, we have $N_2 \cap pH_1q^{-1} = \emptyset$. This is similar to (C1).
- (C3) For every $p, q, r_1, s_1, \dots, r_n, s_n \in P_2 \cup \{1\}$ such that

$$d_{X_2}(p(a_0), q(a_0)) > \sum_{i=1}^n d_{X_2}(r_i(a_0), s_i(a_0)),$$

we have $N_2 \cap pH_1r_1^{-1}s_1H_1 \cdots H_1r_n^{-1}s_nH_1q^{-1} = \emptyset$. This is also similar to (C1).

- (D1) $N_1 = N_2 \cap \mathbb{F}(P_1)$. If $g \in N_2 \cap \mathbb{F}(P_1)$ then $\phi_1(g) = \phi_2(g) = 1$. Therefore, $g \in N_1$.
- (D2) For every $g, h, k, l \in \mathbb{F}(P_1)$ such that

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{Y_1}(\phi_1(k)(a_0), \phi_1(l)(a_0)),$$

we have $N_2 \cap gH_1k^{-1}lH_1h^{-1} = \emptyset$. This is a direct consequence of (D1).

- (D3) For every $g, h \in \mathbb{F}(P_1)$ and $p, q \in P_2$ with both $p(a_0)$ and $q(a_0)$ defined, if

$$d_{Y_1}(\phi_1(g)(a_0), \phi_1(h)(a_0)) \neq d_{X_2}(p(a_0), q(a_0)),$$

we have $N_2 \cap gH_1p^{-1}qH_1h^{-1} = \emptyset$. This is a direct consequence of (D1). \square

We remark that it is possible to give a combinatorial proof of Theorem 5.9. This is done in [3] Lemma 5.1, which was in turn motivated by a result of Rosendal (Lemma 16 of [11]). As in [3], Theorem 5.9 can be used to show that the Hall's universal locally finite group can be embedded as a dense subgroup of the isometry group of the Urysohn space.

6. ULTRAEXTENSIVE METRIC SPACES

In this section we study ultraextensive metric spaces.

Definition 6.1. A metric space U is *ultraextensive* if

- (i) U is ultrahomogeneous, i.e., there is a ϕ such that (U, ϕ) is an S-extension of U ;
- (ii) Every finite $X \subseteq U$ has a finite S-extension (Y, ϕ) where $Y \subseteq U$;
- (iii) If $X_1 \subseteq X_2 \subseteq U$ are finite and (Y_1, ϕ_1) is a finite minimal S-extension of X_1 with $Y_1 \subseteq U$, then there is a finite minimal S-extension (Y_2, ϕ_2) of X_2 such that $Y_2 \subseteq U$ and (Y_1, ϕ_1) and (Y_2, ϕ_2) are coherent.

Motivated by Hrushovski [6], Solecki [12] and Vershik [16], Pestov in [8] introduced a notion of *Hrushovski–Solecki–Vershik property*, which is correspondent to the first two clauses of the above definition. He used the notion to study the nonexistence of uniform and coarse embeddings from the universal Urysohn metric space into reflexive Banach spaces. He also gave a proof of Solecki’s theorem (Theorem 1.1) using Herwig–Lascar’s theorem [5].

Recall that the random graph is the Fraïssé limit of the class of all finite graphs. We equip it with the path metric and turn it into a metric space, which is denoted by \mathcal{R} .

Proposition 6.2. The Urysohn space \mathbb{U} , the rational Urysohn space \mathbb{QU} and the random graph \mathcal{R} are ultraextensive.

Proof. The ultraextensiveness for \mathbb{U} follows directly from its universality and ultrahomogeneity, and from Theorem 5.6.

The space \mathbb{QU} is also ultrahomogeneous and universal for all finite metric spaces with rational distances. From our proof of Theorem 1.1 it is clear that if X is a finite metric space with rational distances, then there is a finite S-extension (Y, ϕ) of X where the distances of Y are finite sums of the distances in X , and therefore also rational. This implies clause (ii) of the definition of ultraextensiveness for \mathbb{QU} . The same observation applies to the proof of Theorem 5.6. Namely, in every construction of the proof of Theorem 5.6 we used the path (pseudo)metric to define new distances. Thus the distances in Y_2 are finite sums of distances in $Y_1 \cup X_2$. Therefore, if distances in X_1, X_2, Y_1 are rational, then we can find Y_2 with rational distances. Together with the ultrahomogeneity and universality of \mathbb{QU} , this implies clause (iii) of the definition of ultraextensiveness for \mathbb{QU} .

Note that the random graph \mathcal{R} as a metric space has only distances 0, 1 and 2. In fact, two distinct vertices have distance 1 if and only if they are connected with an edge. If we endow every finite graph with such a metric, namely, two distinct vertices have distance 1 if they are connected with an edge, and have distance 2 otherwise, then \mathcal{R} as a metric space is ultrahomogeneous and universal for this class of finite metric spaces. Then clause (ii) of the definition of ultraextensiveness for \mathcal{R} follows from this universality of \mathcal{R} and from Hrushovski’s theorem [6]. Finally, in Theorem 5.6, if X_1, X_2, Y_1 are finite metric spaces coming from graphs, then they have distances 0, 1 and 2, and our constructions give that the distances in Y_2 are natural numbers. Now if we redefine every distance ≥ 3 to be 2 in Y_2 , then any isometry of Y_2 continues to be an isometry in this new metric, and from ultrahomogeneity and universality we again obtain clause (iii) of the definition of ultraextensiveness for \mathcal{R} . \square

Theorem 6.3. *Every countable metric space can be extended to a countable ultra-extensive metric space.*

Proof. Let X be a countable metric space. Write X as an increasing union of finite metric spaces F_n for $n = 1, 2, \dots$. For $n \geq 1$, inductively define increasing sequences of finite metric spaces X_n, Y_n and Z_n as follows. Let $X_1 = F_1$ and (Y_1, ϕ_1) be a finite minimal S-extension of $X_1 = F_1$. We define $Y_1 \subseteq Z_1$ such that for every $D \subseteq D' \subseteq Y_1$ and a minimal S-extension of D , (E, ϕ) , where $E \subseteq Y_1$, there exists a minimal S-extension of D' , (E', ϕ') , where $E' \subseteq Z_1$ and (E, ϕ) and (E', ϕ') are coherent. Note that this is possible since there are only finitely many triples (D, D', E) and for any such triple by Theorem 5.6 we can fix a coherent extension E' . Finally, to construct Z_1 , we add $E' \setminus E$ to Y_1 for all E' corresponding to the triple (D, D', E) such that the union of the new points $(E' \setminus E)$ and $E \subseteq Y_1$ is an isometric copy of E' . Then, this new set with the path metric is Z_1 . Let X_2 be the metric space that is obtained by adding $F_2 \setminus F_1$ to Z_1 such that the union of $(F_2 \setminus F_1)$ and F_1 is isometric to F_2 and the distance between points in $F_2 \setminus F_1$ and $Z_1 \setminus F_1$ comes from the path metric.

In general, assume finite $Y_{n-1} \subseteq Z_{n-1} \subseteq X_n$ has been defined. Apply Theorem 5.6 to find (Y_n, ϕ_n) a finite minimal S-extension of $X_n \supseteq X_{n-1}$ that is coherent with (Y_{n-1}, ϕ_{n-1}) . We use a similar construction to the construction of Z_1 from Y_1 to define $Z_n \supset Y_n$. Note that Z_n has the property that every minimal S-extension in Y_n (that is, $D \subseteq E \subseteq Y_n$ where (E, ϕ) is a minimal S-extension of D) has a coherent minimal S-extension in Z_n for every $D \subseteq D' \subseteq Y_n$. Let X_{n+1} be the metric space that is obtained by adding $F_{n+1} \setminus F_n$ to Z_n such that the union of $(F_{n+1} \setminus F_n)$ and F_n is isometric to F_{n+1} and the distance between points in $F_2 \setminus F_1$ and $Z_1 \setminus F_1$ comes from the path metric.

Let Y be the union of the increasing sequence Y_n . We verify that Y is ultra-extensive. To verify Definition 6.1 (i), let $p \in \mathcal{P}_Y$. Then there is $n \geq 1$ such that $p \in \mathcal{P}_{X_n}$. Let n_p be the least such n . Then for all $m \geq n_p$, $p \subseteq \phi_m(p) \subseteq \phi_{m+1}(p)$ by the coherence of (Y_m, ϕ_m) and (Y_{m+1}, ϕ_{m+1}) . Define $\phi(p) = \bigcup_{m \geq n_p} \phi_m(p)$. Then $\phi(p)$ is an isometry of Y that extends p .

For Definition 6.1 (ii), let $F \subseteq Y$ be finite. Then there is n such that $F \subseteq X_n$, and it follows that $(Y_n, \phi_n \upharpoonright \mathcal{P}_F)$ is an S-extension of F .

Finally, for Definition 6.1 (iii), let $F \subseteq F' \subseteq Y$ be finite and assume that (E, ψ) is a finite minimal S-extension of F with $E \subseteq Y$. Then, there is a natural number n such that $E \subseteq Y_n$. By the construction of Z_n , there exists a minimal S-extension of F' , (E', ϕ') (corresponding to the triple (F, F', E)), such that $E' \subseteq Z_n \subseteq Y$ and that (E', ϕ') is coherent with (E, ψ) . \square

Theorem 6.4. *Let U be an ultraextensive metric space and $X \subseteq U$ be a countable subset. Then there exists a countable ultraextensive subset $Y \subseteq U$ with $X \subseteq Y$.*

Proof. The proof is similar to that of Theorem 6.3. The differences are that in the construction Y_n and Z_n are obtained by applying clauses (ii) and (iii) of the definition of ultraextensive metric space for U ; and $X_{n+1} = F_{n+1} \cup Z_n$. \square

Pestov [8] showed that $\text{Iso}(\mathbb{U})$ contains a countable dense locally finite subgroup. Solecki strengthened this result by showing that $\text{Iso}(\mathbb{QU})$ contains a countable dense locally finite subgroup. Rosendal [10] presented a different proof of the result by Solecki. Here we note that such dense locally finite subgroups are present in the isometry group of every separable ultraextensive space.

Theorem 6.5. *For every separable ultraextensive metric space U , $\text{Iso}(U)$ contains a dense locally finite subgroup.*

Proof. Note that $\text{Iso}(U)$ has a countable dense subset D . Let $X \subseteq U$ be a countable dense subset with the property that for all $x \in X$ and $\varphi \in D$, $\varphi(x) \in X$. Apply Theorem 6.4 to obtain a countable ultraextensive $Y \subseteq U$ with $X \subseteq Y$. Then $\text{Iso}(Y)$ is dense in $\text{Iso}(U)$.

It suffices to show that $\text{Iso}(Y)$ contains a dense locally finite subgroup. As in the proof of Theorem 6.4 we can write Y as an increasing union $\bigcup_n Y_n$. We also have group isomorphic embeddings from each $\text{Iso}(Y_n)$ to $\text{Iso}(Y_{n+1})$. Let $G = \bigcup_n \text{Iso}(Y_n)$. Then it is clear that G is locally finite and G is dense in $\text{Iso}(Y)$. \square

7. COMPACT ULTRAMETRIC SPACES

In this section we show that every compact ultrametric space can be extended to a compact ultraextensive ultrametric space. We first study finite ultrametric spaces and show that the notions of homogeneity, ultrahomogeneity, and ultraextensiveness coincide on finite ultrametric spaces.

We will use the following fact about homogeneity for every minimal S-extension.

Lemma 7.1. *Let X be a metric space and (Y, ϕ) be a minimal S-extension of X . Then Y is homogeneous.*

Proof. Let $y_1, y_2 \in Y$. Since (Y, ϕ) is minimal, there are $g_1, g_2 \in \mathbb{F}(\mathcal{P}_X)$ such that $\phi(g_1)(a_0) = y_1$ and $\phi(g_2)(a_0) = y_2$. Hence, $\phi(g_1 g_2^{-1})(y_2) = y_1$. Since $\phi(g_1 g_2^{-1})$ is an isometry of Y , Y is homogeneous. \square

Theorem 7.2. *Let Y be a finite ultrametric space. Then the following are equivalent:*

- (i) Y is homogeneous;
- (ii) Y is ultrahomogeneous;
- (iii) Y is ultraextensive.

Proof. (i) \Rightarrow (ii): Let $D(Y) = \{d(x, y) : x \neq y \in Y\}$. We prove this by induction on $|D(Y)|$. If $|D(Y)| = 1$ then Y is clearly ultrahomogeneous. Suppose $|D(Y)| > 1$ and let r be the least element of $D(Y)$. For each $x \in Y$ let $B_r(x) = \{y \in Y : d(x, y) \leq r\} = \{x\} \cup \{y \in Y : d(x, y) = r\}$. Then for any $x, y \in Y$, either $B_r(x) = B_r(y)$ or $B_r(x) \cap B_r(y) = \emptyset$. In the latter case, we also have that for any $z_1 \in B_r(x)$ and $z_2 \in B_r(y)$, $d(z_1, z_2) = d(x, y)$. Let $Y_1 = \{B_r(x) : x \in Y\}$. Then Y_1 is a partition of Y . For disjoint $B_r(x)$ and $B_r(y)$, we define $d_1(B_r(x), B_r(y)) = d(x, y)$. It is easy to check that (Y_1, d_1) is again an ultrametric space, and $D(Y_1) = D(Y) \setminus \{r\}$. If $\varphi \in \text{Iso}(Y)$, then φ induces an isometry φ_1 of Y_1 , where $\varphi_1(B_r(x)) = B_r(\varphi(x))$. Since Y

is homogeneous, so is Y_1 , and by the inductive hypothesis, Y_1 is ultrahomogeneous. Now suppose $p : A \rightarrow B$ is a partial isometry of Y . It induces a partial isometry $p_1 : \{B_r(a) : a \in A\} \rightarrow \{B_r(b) : b \in B\}$ of Y_1 . Thus there is an isometry $\varphi_1 \in \text{Iso}(Y_1)$ extending p_1 . Note that for any $x, y \in Y$, $B_r(x)$ is isometric to $B_r(y)$ by the homogeneity of Y , and each $B_r(x)$ is ultrahomogeneous. Now for each $B_r(x) \in Y_1$, we define an isometry from $B_r(x)$ to $\varphi_1(B_r(x))$ as follows. If $B_r(x) \cap A = \emptyset$, then we arbitrarily fix an isometry from $B_r(x)$ to $\varphi_1(B_r(x))$. If $B_r(x) \cap A \neq \emptyset$, then $|B_r(x) \cap A| = |\varphi_1(B_r(x)) \cap B|$, and we fix an isometry from $B_r(x)$ to $\varphi_1(B_r(x))$ that sends each $a \in B_r(x) \cap A$ to $p(a) \in \varphi_1(B_r(x)) \cap B$. Putting all of these isometries together, we obtain an isometry of Y extending p . Thus Y is ultrahomogeneous.

(ii) \Rightarrow (iii): We use a similar induction as in the above proof. If $|D(Y)| = 1$ then Y is clearly ultraextensive. Assume $|D(Y)| > 1$ and let r be the least element of Y . Define Y_1 similarly as above. Then by the inductive hypothesis Y_1 is ultraextensive. For any $x \in Y$, $B_r(x)$ is also ultraextensive. Arbitrarily fix an $x \in Y$ and let $Y_2 = B_r(x)$. Consider $Y_1 \times Y_2$ and define a metric d' by

$$d'((B_r(y_1), z_1), (B_r(y_2), z_2)) = \max\{d_1(B_r(y_1), B_r(y_2)), d(z_1, z_2)\}.$$

Then (Y, d) is isometric to $(Y_1 \times Y_2, d')$. Thus we will view Y as $Y_1 \times Y_2$. Enumerate the elements of Y_1 by $b_1 = B_r(y_1), \dots, b_m = B_r(y_m)$. We show that Y is ultraextensive.

Since Y is finite and ultrahomogeneous, it is enough to show that for every minimal S-extension (Y_0, ϕ_0) of X where $X, Y_0 \subseteq Y$ there is a group embedding $\pi : \text{Iso}(Y_0) \rightarrow \text{Iso}(Y)$ such that $\pi(g) \upharpoonright Y_0 = g$.

Since (Y_0, ϕ_0) is a minimal S-extension, by Lemma 7.1, Y_0 is homogeneous and therefore ultrahomogeneous by the previous argument. It follows that the non-empty intersections of Y_0 with $b_i = B_r(y_i)$ are isometric. That is, if $Y_0 \cap B_r(y_i) \neq \emptyset$ and $Y_0 \cap B_r(y_j) \neq \emptyset$, then $Y_0 \cap B_r(y_i)$ and $Y_0 \cap B_r(y_j)$ are isometric. Arbitrarily fix such a non-empty intersection Y_{02} . Let $Y_{01} = \{B_r(x) : x \in Y_0\}$. Then Y_0 is isometric to $Y_{01} \times Y_{02}$ as a subset of $Y_1 \times Y_2$. Now, for every $g \in \text{Iso}(Y_0)$, g induces an isometry of Y_{01} , which we denote by $\phi_{01}(g)$. Furthermore, for every $g \in \text{Iso}(Y_0)$ and every $1 \leq i, j \leq m$ such that $\phi_{01}(g)(b_i) = b_j$, g induces an isometry of Y_{02} , which we denote by $\phi(i, j)(g)$. More precisely, if $g(b_i, z_1) = (b_j, z_2)$ for some $1 \leq i, j \leq m$ and $z_1, z_2 \in Y_{02}$, then $\phi(i, j)(g)(z_1) = z_2$. Since Y_1 and Y_2 are ultraextensive, there are group embeddings $\pi_{01} : \text{Iso}(Y_{01}) \rightarrow \text{Iso}(Y_1)$ and $\pi_{02} : \text{Iso}(Y_{02}) \rightarrow \text{Iso}(Y_2)$ such that $\pi_{01}(g) \upharpoonright Y_{01} = g$ and $\pi_{02}(g) \upharpoonright Y_{02} = g$. Let $\pi : \text{Iso}(Y_0) \rightarrow \text{Iso}(Y)$ be such that for $b_i \in Y_1$ and $z \in Y_2$ where $\phi_{01}(g)(b_i) = b_j$ we have

$$\pi(g)(b_i, z) = (\pi_{01}(\phi_{01}(g))(b_i), \pi_{02}(\phi(i, j)(g))(z)).$$

Then, π is as desired. That is, π is a group embedding and if $g(b_i, z) = (b_j, z')$, then $\pi(g)(b_i, z) = (b_j, z')$. Therefore, Y is ultraextensive.

(iii) \Rightarrow (i) is obvious. □

In view of Theorem 7.2 it is easy to construct finite ultrahomogeneous or ultraextensive ultrametric spaces.

Definition 7.3. Let (Γ, w) be a connected (undirected) weighted graph. The *maximum path metric* on Γ is the metric defined by

$$d(x, y) = \inf\{\max\{w(y_i, y_{i+1}) : i = 1, \dots, n\} : y_1 = x, y_{n+1} = y \text{ and} \\ (y_i, y_{i+1}) \text{ is an edge in } \Gamma \text{ for all } i = 1, \dots, n\}.$$

If (Γ, w) is a connected finite weighted graph, then it is easy to see that Γ with the maximum path metric is an ultrametric space.

Proposition 7.4. Let X be a finite ultrametric space. Then X can be extended to a finite ultraextensive ultrametric space Y . Furthermore, there is such Y so that the set of distances in X and Y are the same.

Proof. By Theorem 7.2 it suffices to construct an extension of X that is homogeneous. We use the same notation as in the proof of Theorem 7.2. Our proof will be by induction on $|D(X)|$. If $|D(X)| = 1$ then X is already homogeneous. Assume $|D(X)| > 1$ and let r be the least element of $D(X)$. Define $X_1 = \{B_r(x) : x \in X\}$ and d_1 on X_1 . Then $|D(X_1)| = |D(X)| - 1$. By the inductive hypothesis, X_1 can be extended to a homogeneous Y_1 with the same distances as in X_1 . Now each $B_r(x)$ is a homogeneous space with every pair of points having distance r . Let $N = \max\{|B_r(x)| : x \in X\}$ and let $x_0 \in X$ be such that $|B_r(x_0)| = N$. Then $X_2 = B_r(x_0)$ is a homogeneous extension of each of $B_r(x)$. It follows that $Y_1 \times X_2$ is a homogeneous ultrametric space extending X . \square

Lemma 7.5. Let $\epsilon > 0$. Let $X_1 \subseteq X_2$ be finite ultrametric spaces such that X_1 is an ϵ -net in X_2 . Let (Y_1, ϕ_1) be a finite minimal S-extension of X_1 such that Y_1 is an ultrametric space with the same distances as in X_1 . Then there is a minimal S-extension (Y_2, ϕ_2) of X_2 such that Y_2 is an ultrametric space, (Y_2, ϕ_2) is coherent with (Y_1, ϕ_1) , and Y_1 is an ϵ -net in Y_2 .

Proof. By Lemma 7.1 and Theorem 7.2, Y_1 is homogeneous and therefore ultraextensive. It is enough to construct an S-extension (Y_2, ϕ_2) of X_2 that satisfies the prescribed conditions, as a minimal S-extension can always be extracted from an S-extension. Since X_1 is an ϵ -net in X_2 , we have that the set of $B_{<\epsilon}(x) = \{y \in X_2 : d_{X_2}(x, y) < \epsilon\}$, when x varies over X_1 , is a partition of X_2 . Let $B_{<\epsilon} = X_2 / \sim$ where \sim identifies all points of X_1 and the metric on $B_{<\epsilon}$ is the maximum path metric. Then $B_{<\epsilon}$ extends $B_{<\epsilon}(x)$ for every $x \in X_1$ and therefore X_2 corresponds to a subset of the product $X_1 \times B_{<\epsilon}$. Now Y_1 is a homogeneous extension of X_1 with the same distances as X_1 . In particular any distance between distinct points in Y_1 is $\geq \epsilon$. We can let $Y_2 = Y_1 \times B$ where B is a homogeneous extension of $B_{<\epsilon}$ with the same set of distances as $B_{<\epsilon}$. Then Y_2 is obviously homogeneous, and therefore ultrahomogeneous. It is clear that Y_2 is an ultrametric space, and that for every $y_2 \in Y_2$ there is $y_1 \in Y_1$ with $d_{Y_2}(y_2, y_1) < \epsilon$. We define a group homomorphism $\phi_2 : \mathbb{F}(\mathcal{P}_{X_2}) \rightarrow \text{Iso}(Y_2)$ such that for every $p \in \mathcal{P}_{X_1}$

$$\phi_2(p)(y_1, y_2) = (\phi_1(p)(y_1), y_2)$$

and for every $p \in \mathcal{P}_{X_2} \setminus \mathcal{P}_{X_1}$ let $\phi_2(p)$ be an isometry of Y_2 such that $p \subseteq \phi_2(p)$. Note that since Y_2 is ultrahomogeneous, it is possible to find $\phi_2(p)$ as required for every $p \in \mathcal{P}_{X_2} \setminus \mathcal{P}_{X_1}$. It is clear that (Y_2, ϕ_2) is coherent with (Y_1, ϕ_1) . \square

Theorem 7.6. *Every compact ultrametric space can be extended to a compact ultraextensive ultrametric space. In particular, every compact ultrametric space has a compact ultrametric S-extension.*

Proof. Let $\{X_k\}_{k=1}^{\infty}$ be an increasing sequence of finite subsets of X such that for each k , X_k is a $\frac{1}{2^k}$ -net. Then by Proposition 7.4 and Lemma 7.5, there is a sequence of S-extensions $\{(Y_k, \phi_k)\}_{k=1}^{\infty}$ such that $\{Y_k\}_{k=1}^{\infty}$ is an increasing sequence of finite ultraextensive ultrametric spaces, (Y_k, ϕ_k) is an S-extension of X_k , (Y_{k+1}, ϕ_{k+1}) is coherent with (Y_k, ϕ_k) , and Y_k is a $\frac{1}{2^k}$ -net in Y_{k+1} . Let Y be the completion of $\bigcup_{k=1}^{\infty} Y_k$. Then, Y is clearly an ultrametric space; Y is compact since $\bigcup_{k=1}^{\infty} Y_k$ is totally bounded. In fact, Y_k is a $\frac{1}{2^k}$ -net in Y . Since each Y_k is ultraextensive, so is $\bigcup_{k=1}^{\infty} Y_k$. We show that Y is ultraextensive.

We first show that Y is ultrahomogeneous. For this, let $p : A \rightarrow B$ be a partial isometry of Y . Let $\frac{1}{2^k}$ be less than the smallest non-zero distance between points of A . Since Y_k is a $\frac{1}{2^k}$ -net in Y , there are $A_k, B_k \subseteq Y_k$ and $p_k : A_k \rightarrow B_k$ such that points in A are approximated by points in A_k , points in B are approximated by points in B_k , and consequently p_k is also a partial isometry. Each p_k can be extended via $\phi_n(p_k)$ for $n > k$ to $\bigcup_{n>k} \phi_n(p_k)$, an isometry of $\bigcup_{k=1}^{\infty} Y_k$, and then uniquely to an isometry P_k of Y . Since $\text{Iso}(Y)$ is compact, the collection of P_k has an accumulation point φ , which is an isometry of Y . Since each P_k approximates p with an error less than $\frac{1}{2^k}$, it follows that $\varphi \supseteq p$. This shows that any partial isometry of Y can be extended to an isometry of Y . In particular, it also shows that any partial isometry of X can be extended to an isometry of Y , thus there is a suitable ϕ such that (Y, ϕ) is an S-extension of X .

For the remaining properties of ultraextensiveness, it suffices to show that any finite subset of Y can be extended to a finite homogeneous, and therefore ultraextensive, subset of Y . For this, let $A \subseteq Y$ be finite and let $\frac{1}{2^k}$ be less than the smallest non-zero distance between points in A . Since Y_k is a $\frac{1}{2^k}$ -net in Y , there is a set $A_k \subseteq Y_k$ such that for each $a \in A$ there is a unique point $a_k \in A_k$ such that $d(a, a_k) < \frac{1}{2^k}$. Consider the set $Z_k = (Y_k \setminus A_k) \cup A$. It is easy to see that the map $\pi : Z_k \rightarrow Y_k$ defined by $\pi(a) = a_k$ for $a \in A$ and $\pi(y) = y$ otherwise is an isometry. Thus Z_k is a finite homogenous subset of Y extending A . \square

8. OPEN PROBLEMS

One general problem is to determine if a certain class of finite metric spaces admit finite S-extensions in the same class. For example, we do not know if the class of finite Euclidean metric spaces has this property.

Question 1. *Let $X \subseteq \mathbb{R}^n$ be a finite subset. Does X have a finite S-extension (Y, ϕ) with $Y \subseteq \mathbb{R}^m$ for some $m \geq n$?*

As stated in Theorem 6.5, we know that the isometry group of every ultraextensive metric space has a dense locally finite subgroup. It is of interest to know

if the isomorphism group of other well-know mathematical objects has the same property. In particular

Question 2. *Does the homeomorphism group of the Hilbert cube have a dense locally finite subgroup?*

Question 3. *Does the linear isometry group of the Gurarij space have a dense locally finite subgroup?*

REFERENCES

- [1] T. Coulbois, *Free product, profinite topology and finitely generated subgroups*, International Journal of Algebra and Computation, 11 (2001), no. 2, 171–184.
- [2] M. Etedadialiabadi, S. Gao, *On extensions of partial isomorphisms*, The Journal of Symbolic Logic, to appear. Available at arXiv:1908.02965.
- [3] M. Etedadialiabadi, S. Gao, F. Le Maître, J. Melleray, *Dense locally finite subgroups of automorphism groups of ultraextensive spaces*, preprint. Available at arXiv:2004.14138.
- [4] S. Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics (2009).
- [5] B. Herwig, D. Lascar, *Extending partial automorphisms and the profinite topology on free groups*, Transactions of the American Mathematical Society 352(1999), 1985–2021.
- [6] E. Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica 12 (1992), 411–416.
- [7] Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil, *A combinatorial proof of the extension property for partial isometries*, Commentationes Mathematicae Universitatis Carolinae 60 (2019), no. 1, 39–47.
- [8] V. Pestov, *A theorem of Hrushovski–Solecki–Vershik applied to uniform and coarse embeddings of the Urysohn metric space*, Topology and its Applications 155 (2008), 1561–1575.
- [9] L. Ribes, P. A. Zalesskii, , *On the profinite topology on a free group*, The Bulletin of the London Mathematical Society 25 (1993), 37–43.
- [10] C. Rosendal, *Finitely approximable groups and actions part I: the Ribes–Zalesskii property*, The Journal of Symbolic Logic 76 (2011), no. 4, 1297–1306.
- [11] C. Rosendal, *Finitely approximable groups and actions part II: Generic representations*, The Journal of Symbolic Logic 76 (2011), no. 4, 1307–1321.
- [12] S. Solecki, *Extending partial isometries*, Israel Journal of Mathematics 150 (2005), 315–332.
- [13] S. Solecki, *Notes on a strengthening of the Herwig–Lascar Extension Theorem*, unpublished notes, 2009. Available at <https://faculty.math.illinois.edu/~ssolecki/papers/HervLascfin.pdf>
- [14] D. Siniora, S. Solecki, *Coherent extension of partial automorphisms, free amalgamation, and automorphism groups*, The Journal of Symbolic Logic 85 (2020), no. 1, 199–223.
- [15] V. V. Uspenskij, *On the group of isometries of the Urysohn universal metric space*, Commentationes Mathematicae Universitatis Carolinae 31 (1990), no. 1, 181–182.
- [16] A. M. Vershik, *Globalization of the partial isometries of metric spaces and local approximation of the group of isometries of Urysohn space*, Topology and Its Applications 155 (2008), no. 14, 1618–1626.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, 1155 UNION CIRCLE #311430,
DENTON, TX 76203, USA

E-mail address: mahmood.etedadialiabadi@unt.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, 1155 UNION CIRCLE #311430,
DENTON, TX 76203, USA

E-mail address: sgao@unt.edu