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Structure theorems for vector valued Siegel modular forms of degree 2 and weight $\det^k \otimes \mathrm{Sym}(10)$

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We prove the explicit structure theorems of modules $\bigoplus_k M_{\det^k \otimes \mathrm{Sym}(10)}(\mathrm{Sp}_2(\mathbb{Z}_2))$ of vector valued Siegel modular forms of degree 2, where k runs over the set of even integers or odd integers. We also check the conjecture given by Ibukiyama [Vector valued Siegel modular forms of symmetric tensor weight of small degrees, *Comment. Math. Univ. St. Pauli* **61** (2012) 51–75.] for modules of vector valued Siegel modular forms of degree 2 of weights $\det^* \otimes \mathrm{Sym}(8)$ and $\det^* \otimes \mathrm{Sym}(10)$.

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1. Introduction

For $k, j \in \mathbb{Z}_{\geq 0}$, we denote by $M_{k,j}(\Gamma^{(2)})$ the space of vector valued Siegel modular forms of weight $\det^k \otimes \mathrm{Sym}(j)$. Here, $\mathrm{Sym}(j)$ is the symmetric tensor representation of degree j of $\mathrm{GL}_2(\mathbb{C})$. Let A_{ev} be the ring of scalar valued Siegel modular forms of degree2, level 1 and even weights:

$$A_{\mathrm{ev}} = \bigoplus_{k \in 2\mathbb{Z}} M_k(\Gamma^{(2)}),$$

where $M_k(\Gamma^{(2)}) = M_{k,0}(\Gamma^{(2)})$. For $j \in \mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, we define the graded A_{ev} module of vector valued Siegel modular forms as

$$M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)}) = \bigoplus_{k \equiv \epsilon \bmod 2} M_{k,j}(\Gamma^{(2)}).$$

For a graded module $M = \sum_{k \in \mathbb{Z}} M_k$ and an integer a define $M(a)$ by $M(a)_k = M_{a+k}$. Roughly speaking, our first main theorem is stated as follows.

Theorem 1.1. *As a grated $A_{\rm ev}$ module, $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ has the following free resolution.*

$$\begin{aligned}0 &\to A_{\rm ev}(-24)\oplus A_{\rm ev}(-26)\\ &\to A_{\rm ev}(-6)\oplus A_{\rm ev}(-8)\oplus A_{\rm ev}(-10)^2\oplus A_{\rm ev}(-12)^2\oplus A_{\rm ev}(-14)^3\\ &\quad \oplus A_{\rm ev}(-16)^2\oplus A_{\rm ev}(-18)\oplus A_{\rm ev}(-20)\to M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})\to 0. \qquad (1.1)\end{aligned}$$

As a grated $A_{\rm ev}$ module, $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ has the following free resolution.

$$\begin{aligned}0 &\to A_{\rm ev}(-27)\oplus A_{\rm ev}(-29)\\ &\to A_{\rm ev}(-9)\oplus A_{\rm ev}(-11)\oplus A_{\rm ev}(-13)\oplus A_{\rm ev}(-15)^3\oplus A_{\rm ev}(-17)^3\\ &\quad \oplus A_{\rm ev}(-19)^2\oplus A_{\rm ev}(-21)\oplus A_{\rm ev}(-23)\to M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})\to 0. \qquad (1.2)\end{aligned}$$

Moreover, maps in the free resolutions can be given explicitly. In other words, $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ is generated by 13 *modular forms of determinant weights* $6, 8, 10, 10, 12, 12, 14, 14, 14, 16, 16, 18, 20$ *and they satisfy two fundamental relations. $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ is generated by* 13 *modular forms of determinant weights* $9, 11, 13, 15, 15, 15, 17, 17, 17, 19, 19, 21, 23$ *and they satisfy two fundamental relations.*

In Sec. 6, we give relations and generators explicitly.

By Igusa [10], there exists a cusp form $\chi_{35}\in M_{35}(\Gamma^{(2)})$ of weight 35 and we have

$$M^1_{\mathrm{Sym}(0)}(\Gamma^{(2)}) = A_{\rm ev}\chi_{35}.$$

And by Igusa [9], $A_{\rm ev}$ is isomorphic to the polynomial ring of four variables over $\mathbb{C}$ and is generated by $\phi_4, \phi_6, \chi_{10}$ and χ_{12}. Here, ϕ_4 (respectively ϕ_6) is the Siegel–Eisenstein series of weight 4 (respectively 6). And χ_{10} (respectively χ_{12}) is the cusp form of weight 10 (respectively 12). We put $K_{\rm ev} = \mathrm{Frac}(A_{\rm ev})$. Let $j\in 2\mathbb{Z}_{\geq 0}$ and $\epsilon = 0$ or 1. By [20, Proposition 3.1], we have

$$\dim_{K_{\rm ev}} M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})\otimes_{A_{\rm ev}} K_{\rm ev} = j+1. \qquad (1.3)$$

We put

$$\det(M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})) = \wedge^{j+1} M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$$

then $\det(M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)}))\neq 0$. We fix an isomorphism $\wedge^{j+1}\mathrm{Sym}(j)\cong \det^{j(j+1)/2}$. Then for $f_i\in M_{k_i,j}(\Gamma^{(2)})$ $(1\leq i\leq j+1)$, we have $f_1\wedge\cdots\wedge f_{j+1}\in M_k(\Gamma^{(2)})$, where $k=\sum_i k_i + j(j+1)/2$. Since j is even, we have

$$\det(M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)}))\subset \begin{cases} A_{\rm ev}, & \text{if } j/2+\epsilon \text{ is even},\\ A_{\rm ev}\chi_{35}, & \text{if } j/2+\epsilon \text{ is odd}.\end{cases}$$

Thus, there exists a non-zero ideal $I_{j,\epsilon}$ of A_{ev} such that

$$\det(M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})) = \begin{cases} I_{j,\epsilon}, & \text{if } j/2+\epsilon \text{ is even,} \\ I_{j,\epsilon}\chi_{35}, & \text{if } j/2+\epsilon \text{ is odd.} \end{cases}$$

Definition 1.1. For $j \in 2\mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, we denote by $\gcd(I_{j,\epsilon})$ the greatest common divisor of generators of the ideal $I_{j,\epsilon}$ above. We define $f_{j,\epsilon} \in A_{\mathrm{ev}} \oplus A_{\mathrm{ev}}\chi_{35}$ by

$$f_{j,\epsilon} = \begin{cases} \gcd(I_{j,\epsilon}), & \text{if } j/2+\epsilon \text{ is even,} \\ \gcd(I_{j,\epsilon})\chi_{35}, & \text{if } j/2+\epsilon \text{ is odd.} \end{cases}$$

The element $f_{j,\epsilon}$ is defined up to a non-zero constant.

Remark 1.1.

(1) In Sec. 4, we construct $f_{j,\epsilon}$ by the alternating product of minors of matrices appearing in a free resolution of $M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$.
(2) If $M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$ is a free A_{ev} module and $\{f_1, \dots, f_{j+1}\}$ is its basis, we can take $f_{j,\epsilon}$ as $f_1 \wedge \cdots \wedge f_{j+1}$.
(3) By structure theorems of $M^{\epsilon}_{\mathrm{Sym}(2)}(\Gamma^{(2)})$ (cf. [17, 7]) and the result on a Wronskian of scalar valued Siegel modular forms (cf. [1]), we have

$$\det\left(M^{0}_{\mathrm{Sym}(2)}(\Gamma^{(2)})\right) = (\phi_4, \phi_6, \chi_{10}, \chi_{12})^2\chi_{35},$$

$$\det\left(M^{1}_{\mathrm{Sym}(2)}(\Gamma^{(2)})\right) = (\phi_4, \phi_6, \chi_{10}, \chi_{12})\chi_{35}^2.$$

Thus, we have $f_{2,0} = \chi_{35}$ and $f_{2,1} = \chi_{35}^2$ up to non-zero constants.

In Sec. 4, we shall prove

Proposition 1.1. *For $j \in 2\mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, we have*

$$f_{j,\epsilon} \in M_{35(j/2+\epsilon)}(\Gamma^{(2)}).$$

The second main result of this paper is the following theorem.

Theorem 1.2. *Let j be an even number and assume $0 \leq j \leq 10$. For $\epsilon = 0, 1$, we have*

$$f_{j,\epsilon} = \chi_{35}^{j/2+\epsilon}$$

up to a non-zero constant.

Remark 1.2. The statement of the theorem was conjectured by Ibukiyama [7] if $j = 4, 6, 8$.

We recall preceding results on the structure theorems of the graded A_{ev} module $M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$. First, Igusa [9, 10] gave generators of $A_{\mathrm{ev}} = M^{0}_{\mathrm{Sym}(0)}(\Gamma^{(2)})$ and proved that $M^{1}_{\mathrm{Sym}(0)}(\Gamma^{(2)})$ is a free A_{ev} module of rank 1. Satoh [17] proved the structure theorem of $M^{0}_{\mathrm{Sym}(2)}(\Gamma^{(2)})$. The module $M^{0}_{\mathrm{Sym}(2)}(\Gamma^{(2)})$ is generated by six modular forms, they satisfy four relations and the four relations satisfy one relation. Ibukiyama [7] proved structure theorems of $M^1_{\mathrm{Sym}(2)}(\Gamma^{(2)})$, $M^0_{\mathrm{Sym}(4)}(\Gamma^{(2)})$, $M^1_{\mathrm{Sym}(4)}(\Gamma^{(2)})$ and $M^0_{\mathrm{Sym}(6)}(\Gamma^{(2)})$. The module $M^1_{\mathrm{Sym}2}(\Gamma^{(2)})$ is generated by four modular forms and they satisfy one relation. $M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$ is free of rank $j+1$ for $(j,\epsilon) = (4,0),(4,1),(6,0)$. Van Dorp [22] proved $M^1_{\mathrm{Sym}(6)}(\Gamma^{(2)})$ is free of rank 7. Kiyuna [13] proved $M^{\epsilon}_{\mathrm{Sym}(8)}(\Gamma^{(2)})$ is free of rank 9 for $\epsilon = 0,1$. And they all gave generators and relations explicitly.

By Theorem 1.1 and Sec. 6, we can compute a basis of $M_{k,10}(\Gamma^{(2)})$ explicitly. This is one of motivations of this paper. In Sec. 8, we give some examples of Hecke eigenforms. A program for computing $M_{k,10}(\Gamma^{(2)})$ can be found at [19].

2. Definition and Notation

In this section, we define Siegel modular forms of degree two and introduce notation used throughout in this paper. We define Siegel upper half space $\mathbb{H}_2$ of degree two by

$$\mathbb{H}_2 = \{x + iy \,|\, x, y \in \mathrm{Sym}_2(\mathbb{R}), y \text{ is positive definite}\}.$$

Here, $\mathrm{Sym}_2(\mathbb{R})$ is the space of symmetric matrices with entries in $\mathbb{R}$. For a commutative ring R, we define the symplectic group of degree 2 by

$$\mathrm{Sp}_2(R) = \{g \in \mathrm{GL}_4(R) \;\big|\; {}^t g w_2 g = w_2\},$$

where $w_2 = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$. Then $\mathrm{Sp}_2(\mathbb{R})$ acts on $\mathbb{H}_2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az+b)(cz+d)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R}), \quad z \in \mathbb{H}_2.$$

Here $a,b,c,d \in M_2(\mathbb{R})$. We put $\Gamma^{(2)} = \mathrm{Sp}_2(\mathbb{Z})$.

Let $\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}}(V)$ be an irreducible polynomial representation of $\mathrm{GL}_2(\mathbb{C})$ and $\chi : \mathrm{Sp}_2(\mathbb{Z}) \to \mathbb{C}^{\times}$ a character. A V-valued holomorphic function $f : \mathbb{H}_2 \to V$ is said to be a (holomorphic) Siegel modular form of degree two, weight ρ and character χ if

$$f(g \cdot z) = \rho(cz+d)\chi(g)f(z), \quad \text{for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}).$$

We denote the space of Siegel modular forms of degree two, weight ρ and character χ by $M_\rho(\Gamma^{(2)},\chi)$. If χ is the trivial character, we simply denote $M_\rho(\Gamma^{(2)},\chi)$ by $M_\rho(\Gamma^{(2)})$.

For a non-negative integer j, we denote by V_j the space of homogeneous polynomials of u_1 and u_2 of degree j, where u_1 and u_2 are variables. We define the symmetric representation $\mathrm{Sym}(j)$ of degree j by

$$(\mathrm{Sym}(j)(\alpha)P)(u_1\ u_2) = P((u_1 u_2)\alpha)$$

for $P \in V_j$ and $\alpha \in \mathrm{GL}_2(\mathbb{C})$. We denote by det the determinant representation of $\mathrm{GL}_2(\mathbb{C})$. When $\rho = \det^k \otimes \mathrm{Sym}(j)$, we put $M_{k,j}(\Gamma^{(2)}) = M_\rho(\Gamma^{(2)})$. For

$F \in M_{k,j}(\Gamma^{(2)})$, we define the Siegel operator by

$$\Phi(F)(z) = \lim_{t\to+\infty} F\left(\begin{pmatrix} z & 0 \\ 0 & it \end{pmatrix}\right).$$

Then by Arakawa [2], we have

$$\Phi(F) \in S_{k+j}(\mathrm{SL}_2(\mathbb{Z}))u_1^j$$

if $j > 0$. Here $S_{k+j}(\mathrm{SL}_2(\mathbb{Z}))$ is the space of elliptic cusp forms of weight $k+j$, level 1. For $F \in M_{k,j}(\Gamma^{(2)})$, we call F a cusp form if $\Phi(F) = 0$ and denote by $S_{k,j}(\Gamma^{(2)})$ the space of cusp forms.

Let $F \in M_{k,j}(\Gamma^{(2)})$. Then by Köcherer principle, we have the following Fourier expansion:

$$F\left(\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}\right) = \sum_{n,r,m} a((n,r,m);F)\exp(2\pi(nz_{11} + rz_{12} + mz_{22})).$$

Here $a((n,r,m);F) \in V_j$ and the summation index runs over the following set:

$$\{(n,r,m) \mid n,r,m \in \mathbb{Z}, n, m, 4nm - r^2 \geq 0\}.$$

Let q_{11}, q_{12}, q_{22} be variables and $\mathbb{C}[q_{12}, q_{12}^{-1}][\![q_{11}, q_{22}]\!]$ a ring of formal power series. We embed $M_{k,j}(\Gamma^{(2)})$ to $\mathbb{C}[q_{12}, q_{12}^{-1}][\![q_{11}, q_{22}]\!] \otimes_{\mathbb{C}} V_j$ by

$$F \mapsto \sum_{i=0}^{j}\left(\sum_{n,r,m} a((n,r,m);F)_i q_{11}^n q_{12}^r q_{22}^m\right) u_1^{j-i}u_2^i,$$

where for $v \in V_j$, we denote by v_i the coefficient of $u_1^{j-i}u_2^i$ of v.

3. Generators of the Ring of Scalar Valued Modular Forms

Later, we construct vector valued Siegel modular forms by scalar valued Siegel modular forms. We recall well-known results on the generator of the ring of scalar valued modular forms of degree two.

For an even integer $k \geq 4$, we denote by ϕ_k the Siegel–Eisenstein series of degree two and weight k. We normalize ϕ_k, so that $a((0,0,0);\phi_k) = 1$. We put

$$\chi_{10} = \frac{2^{10}\cdot 3^5 \cdot 5^2 \cdot 7 \cdot 53}{43867}(\phi_4\phi_6 - \phi_{10}),$$

$$\chi_{12} = \frac{2^{11}\cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 337}{131 \cdot 593}(3^2\cdot 7^2\phi_4^3 + 2\cdot 5^3\phi_6^2 - 691\phi_{12}).$$

Then we have $\chi_k \in S_k(\Gamma^{(2)})$ and $a((1,1,1);\chi_k) = 1$ for $k = 10, 12$. We can also construct χ_{10} and χ_{12} by the Saito–Kurokawa lift.

Theorem 3.1 (Igusa [9, 10]). *The modular forms $\phi_4, \phi_6, \chi_{10}$ and χ_{12} are algebraically independent over $\mathbb{C}$ and we have*

$$A_{\mathrm{ev}} = \mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}], \quad M^1_{\mathrm{Sym}(0)}(\Gamma^{(2)}) = \chi_{35}A_{\mathrm{ev}},$$

where $\chi_{35} \in S_{35}(\Gamma^{(2)})$ is a cusp form of weight 35.

Let sgn be the unique non-trivial character of $\Gamma^{(2)}$ (see [14]). Then sgn is quadratic. There exists a square root $\chi_5 \in S_5(\Gamma^{(2)}, \mathrm{sgn})$ of χ_{10}. The cusp form χ_5 has the following Fourier expansion:

$$\chi_5(Z) = \sum_{\substack{n,m,4nm-r^2\geq 0\\ n,r,m\in 1/2+\mathbb{Z}}} a((n,r,m);\chi_5)\mathbf{e}(nz_{11}+rz_{12}+mz_{22}).$$

We normalize χ_5 so that $a((1/2,1/2,1/2);\chi_5)=1$.

Fourier coefficients of Siegel–Eisenstein series ϕ_k are explicitly known and the computation is not difficult. We can compute Fourier coefficients of χ_{35} by the Wronskian given in [1, Proposition 2.1]. Since χ_5 is the Saito–Kurokawa lift of a Jacobi theta series (see [15, 6]), we can easily compute Fourier coefficients of χ_5.

4. The Determinant of the Module of Vector Valued Siegel Modular Forms

In this section, we give an another interpretation of $f_{j,\epsilon}$ in Definition 1.1 and prove Proposition 1.1.

4.1. *The determinant of a based exact sequence*

We recall the definition and basic property of the determinant of a based exact sequence over a field. We follow [5, Appendix A].

Let K be a field and W a vector space of dimension $n<\infty$ over K. We put $\det(W)=\wedge^n W$. For a one-dimensional vector space V, we put $V^{-1}=\mathrm{Hom}_K(V,K)$. For a non-zero element $v\in V$, we denote by v^{-1} the element of V^{-1} such that $v^{-1}(v)=1$.

For $i\in\mathbb{Z}$, let W_i be a finite-dimensional vector space over a field K and $d_i: W_i\to W_{i+1}$ a linear map. We put $d=\{d_i \mid i\in\mathbb{Z}\}$ and $W_\bullet=\bigoplus_{i\in\mathbb{Z}} W_i$. We assume $(W_\bullet, d)$ forms a finite exact sequence, that is there exists $n\in\mathbb{Z}_{\geq 0}$ such that $W_i=0$ for $|i|\geq n$ and $\ker(d_i)=\mathrm{Im}(d_{i-1})$ for all i. We define the determinant of $W_\bullet$ by

$$\det(W_\bullet)=\bigotimes_i \det(W_i)^{(-1)^i}.$$

Then by [5, Appendix A, Proposition 3], there is a natural isomorphism called the Euler isomorphism

$$\mathrm{Eu}_d:\det(W_\bullet)\to K.$$

For a three term exact sequence

$$0\to W_0 \xrightarrow{d_0} W_1 \xrightarrow{d_1} W_2\to 0$$

the Euler isomorphism $\det(W_0)\otimes\det(W_2)\otimes(\det(W_1))^{-1}\to K$ is given as follows. Suppose $a_1\wedge\cdots\wedge a_n\in\det(W_0)$, $c_1\wedge\cdots\wedge c_m\in\det(W_2)$ and $f\in(\det(W_1))^{-1}$. For

$c_j \in W_2$, we take $\widetilde{c}_j \in W_1$ such that $d_1(\widetilde{c}_j) = c_j$. Then we define $\mathrm{Eu}_d(a_1 \wedge \cdots \wedge a_n \wedge c_1 \cdots \wedge c_m \wedge f)$ by

$$f(d_0(a_1) \wedge \cdots \wedge d_0(a_n) \wedge \widetilde{c}_1 \wedge \cdots \wedge \widetilde{c}_m).$$

It is easy to see that this is well-defined. The definition for general exact sequences is reduced to this case.

Let W be a vector space and $e = \{e_\alpha \mid 1 \leq \alpha \leq \dim_K(W)\}$ a basis of W. Denote by $\det(e)$ by the wedge product $e_1 \wedge \cdots \wedge e_{\dim_K(W)}$. Let $(W_\bullet, d)$ be a finite exact sequence and suppose that $e = \{e(i)\}$ is a system of bases in all W_i so that each $e(i)$ is a basis in W_i. We define $\det(e) \in \det(W_\bullet)$ by

$$\det(e) = \bigotimes_i \det(e(i))^{(-1)^i}.$$

Here, we put $\det(e(i)) = 1$ if $W_i = \{0\}$. We call $(W_\bullet, d, e)$ a based exact sequence.

Definition 4.1. For a based exact sequence $(W_\bullet, d, e)$, we define its determinant by

$$\det(W_\bullet, d, e) = \mathrm{Eu}_d(\det(e)) \in K^\times.$$

Next, we explain how to compute determinants of based exact sequences. Let $(W_\bullet, d, e)$ be a based exact sequence. For $i \in \mathbb{Z}$, let $B_i = \{1 \leq \alpha \leq \dim_K W_i\}$ be an ordered set of indices of basis. Assume $d_i \neq 0$. We denote by D_i the matrix representation of d_i with respect to $e(i)$ and $e(i+1)$. We put $D_i = (m^{(i)}_{\alpha,\beta})_{\alpha \in B_{i+1}, \beta \in B_i}$. For subsets $X \subset B_{i+1}$ and $Y \subset B_i$, we define $(D_i)_{X,Y}$ by the submatrix $(m^{(i)}_{\alpha,\beta})_{\alpha \in X, \beta \in Y}$.

Definition 4.2. A collection of subsets $I_i \subset B_i$ $(i \in \mathbb{Z})$ is called admissible if

$$|I_i| = \sum_{j \geq 0} (-1)^j \dim_K W_{i-1-j}$$

and $(D_i)_{I_{i+1}, B_i \setminus I_i}$ is invertible if $d_i \neq 0$.

Remark 4.1. It can be easily proved that an admissible collection exists (see [5, Appendix A, Proposition 13]).

We can compute the determinant of a based exact sequence by the alternating product of minors. For the proof of the following theorem, see [5, Appendix A, Theorem 14].

Theorem 4.1. *Let (I_i) be an admissible collection for a based exact sequence $(W_\bullet, d, e)$. We put $\Delta_i = \det((D_i)_{I_{i+1}, B_i \setminus I_i})$. Then, we have*

$$\det(W_\bullet, d, e) = \prod_{i \in \mathbb{Z}} \Delta_i^{(-1)^{i-1}}.$$

Here, we understand $\Delta_i = 1$ if $d_i = 0$.

Example 4.1. Let $n \in \mathbb{Z}_{\geq 1}$. We define a three term based exact sequence as follows.

$$0 \to W_{-2} \xrightarrow{d_{-2}} W_{-1} \xrightarrow{d_{-1}} W_0 \to 0, \tag{4.1}$$

where $W_{-2} = K$, $W_{-1} = K^{n+1}$, $W_0 = K^n$. We take a basis $e(i)$ of W_i by the standard basis of W_i. Let ${}^t(a_1, a_2, \ldots, a_{n+1})$ be the matrix representation of d_{-2} with respect to the standard bases, $e_1, e_2, \ldots, e_{n+1}$ the standard basis of W_{-1} and $f_1, f_2, \ldots, f_n$ the standard basis of W_0. Let $B_i = \{1 \leq \alpha \leq \dim_K W_i\}$ be the ordered set of indices of basis. We assume $a_{n+1} \neq 0$. We define a subset $I_i \subset B_i$ by

$$I_i = \begin{cases} B_i, & \text{if } i \geq 0, \\ \{n+1\}, & \text{if } i = -1, \\ \emptyset, & \text{if } i \leq -2. \end{cases}$$

Then I_i is an admissible collection. Let Δ_i be the determinant defined in Theorem 4.1. Then, we have

$$\Delta_{-1} f_1 \wedge f_2 \wedge \cdots \wedge f_n = d_{-1}(e_1) \wedge d_{-1}(e_2) \wedge \cdots \wedge d_{-1}(e_n),$$

$$\Delta_{-2} = a_{n+1}.$$

Thus by Theorem 4.1,

$$\det(W_\bullet, d, e) f_1 \wedge f_2 \wedge \cdots \wedge f_n = a_{n+1}^{-1} d_{-1}(e_1) \wedge d_1(e_2) \wedge \cdots \wedge d_{-1}(e_n),$$

where $\det(W_\bullet, d, e)$ is the determinant of (4.1).

4.2. *The determinant of a free resolution of* $M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$

Next, we define the determinant of a given free resolution of $M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$ for $\epsilon = 0, 1$. And we shall prove that this determinant satisfies the property given in Definition 1.1.

Definition 4.3. Let $\epsilon = 0, 1$ and

$$0 \to F_{-r} \xrightarrow{\psi_{-r}} F_{-r+1} \xrightarrow{\psi_{-r+1}} \cdots \xrightarrow{\psi_{-2}} F_{-1} \xrightarrow{\psi_{-1}} M \to 0 \tag{Rsl}$$

a free resolution of $M = M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$ as a graded A_{ev} module. Here

$$F_i = \bigoplus_{1 \leq \nu \leq n_i} A_{\mathrm{ev}}(-a_\nu^{(i)}), \quad \text{for } -r \leq i \leq -1$$

be a graded free module, where $a_\nu^{(i)} \in \mathbb{Z}$. We take a basis $e(0) = \{f_1, \ldots, f_{j+1}\}$ of M. Define a based exact sequence $(W_\bullet, d, e)$ associated with the free resolution Rsl and $e(0)$ as follows. We put $W_0 = M \otimes_{A_{\mathrm{ev}}} K_{\mathrm{ev}}$ and $W_i = F_i \otimes_{A_{\mathrm{ev}}} K_{\mathrm{ev}}$ for $-r \leq i \leq -1$. For $i > 0$ or $i < -r$, we put $W_i = 0$. We take a basis $e(i)$ of W_i

as the basis obtained by the standard basis of F_i. We define $d_i : W_i \to W_{i+1}$ by $d_i = \psi_i \otimes_{A_{\mathrm{ev}}} \mathrm{id}_{K_{\mathrm{ev}}}$. Then, we define the determinant of the free resolution Rsl by

$$\det(\mathrm{Rsl}) = \det(W_\bullet, d, e) f_1 \wedge \cdots \wedge f_{j+1}.$$

By [5, Appendix Proposition 9], det(Rsl) does not depend on the choice of the basis $f_1, \ldots, f_{j+1}$.

Proposition 4.1. *For $j \in 2\mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, let* $\det(\mathrm{Rsl})$ *be the determinant given in Definition* 4.3 *and $f_{j,\epsilon}$ an element of $A_{\mathrm{ev}} \oplus A_{\mathrm{ev}}\chi_{35}$ given in Definition* 1.1. *Then, we have*

$$\det(\mathrm{Rsl}) = f_{j,\epsilon}$$

up to a non-zero constant.

Proof. By definition of $f_{j,\epsilon}$, it is enough to prove

$$A_P \det(\mathrm{Rsl}) = \det(M) \otimes_A A_P$$

for any height one prime P of A. Here $M = M^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma^{(2)})$ and $A = A_{\mathrm{ev}}$. In the proof, we use the same notation in Definition 4.3. Let P be a height one prime of A. Since A_P is a discrete valuation ring, we can take a basis $\{f_1, \ldots, f_{j+1}t\}$ of M so that $\det(M) \otimes_A A_P = A_P f_1 \wedge \cdots \wedge f_{j+1}$. Put $\kappa = A_P/PA_P$. Since M and F_i $(-r \leq i \leq -1)$ are torsion-free A modules, the following sequence $\mathrm{Rsl} \otimes_A \kappa$ is exact.

$$0 \to F_{-r} \otimes_A \kappa \to F_{-r+1} \otimes_A \kappa \cdots \to F_{-1} \otimes_A \kappa \to M \otimes_A \kappa \to 0.$$

We define a based exact sequence $(V_\bullet, \overline{d}, \overline{e})$ over κ as follows. We put $V_0 = M \otimes_A \kappa$ and $V_i = F_i \otimes_A \kappa$ for $-r \leq i \leq -1$. Otherwise, we put $V_i = 0$. Define $\overline{d}_i : V_i \to V_{i+1}$ by $\overline{d}_i = \psi_i \otimes_A \mathrm{id}_\kappa$. Let $\overline{e}(0) = \{f_1 \otimes 1_\kappa, \ldots, f_{j+1} \otimes 1_\kappa\}$ and $\overline{e}(i)$ a basis obtained by the standard basis of F_i for $-r \leq i \leq -1$. Denote by B_i, the ordered index set of $\overline{e}(i)$. We take an admissible collection $\{I_i\}$ for a based exact sequence $(V_\bullet, \overline{d}, \overline{e})$. Then $\{I_i\}$ is an admissible collection for $(W_\bullet, d, e)$ in Definition 4.3. Then by Theorem 4.1, we have $\det(W_\bullet, d, e) \in A_P^\times$. Therefore, we have $A_P \det(\mathrm{Rsl}) = A_P f_1 \wedge \cdots \wedge f_{j+1} = \det(M) \otimes_A A_P$. □

Since Rsl is a free resolution as a graded module, there exist $f \in M_k(\Gamma^{(2)})$ and $g \in M_l(\Gamma^{(2)})$ for some k and l such that det(Rsl) is equal to f/g. By Proposition 4.1, we can take $g = 1$. We can compute the weight of f.

Proposition 4.2. *Let* $\det(\mathrm{Rsl})$ *be the determinant given in Definition* 4.3. *Then the weight of* $\det(\mathrm{Rsl})$ *is equal to* $35(j/2 + \varepsilon)$.

Proof. Let $(W_\bullet, d, e)$ be a based exact sequence given in Definition 4.3, D_i the matrix representation of d_i with respect to $e(i)$ and $e(i+1)$ and B_i the ordered index set of $e(i)$. We take an admissible collection $\{I_i\}$ for $(W_\bullet, d, e)$. For $f \in M_k(\Gamma^{(2)})$, we denote by $\mathrm{wt}(f)$ the weight of f. For $-r \le i \le -2$ and $\alpha \in B_{i+1},\ \beta \in B_i$, the weight of (α, β)-entry of D_i is equal to $a_\beta^{(i)} - a_\alpha^{(i+1)}$. Therefore, we have

$$\mathrm{wt}((\Delta_i)_{I_{i+1}, B_i \setminus I_i}) = \sum_{\alpha \in B_i \setminus I_i} a_\alpha^{(i)} - \sum_{\alpha \in I_{i+1}} a_\alpha^{(i+1)}$$

for $-r \le i \le -2$. By Theorem 4.1, we have

$$\mathrm{wt}(\det((W_\bullet, d, e))) = \sum_{-r \le i \le -2} (-1)^{i-1} \sum_{\alpha \in B_i} a_\alpha^{(i)} + \sum_{\alpha \in I_{-1}} a_\alpha^{(-1)}.$$

By

$$\mathrm{wt}(f_1 \wedge \cdots \wedge f_{j+1}) = \sum_{\alpha \in B_{-1} \setminus I_{-1}} a_\alpha^{(-1)} + \frac{1}{2} j(j+1)$$

we have

$$\mathrm{wt}(\det(\mathrm{Rsl})) = \sum_{-r \le i \le -1} (-1)^{i-1} \sum_{\alpha \in B_i} a_\alpha^{(i)} + \frac{1}{2} j(j+1).$$

Therefore, we obtain the assertion of our Proposition by [20, Proposition 3.1]. □

Corollary 4.1. *Proposition* 1.1 *is true.*

Proof. This follows from Propositions 4.1 and 4.2. □

Next, we compute $f_{j,\epsilon}$ given in Definition 1.1 for $j \le 8$.

Proposition 4.3. *For $j = 0, 2, 4, 6, 8$ and $\epsilon = 0, 1$, let $f_{j,\epsilon}$ be a Siegel modular form given in Definition* 1.1. *Then, we have*

$$f_{j,\epsilon} = \chi_{35}^{j/2+\epsilon}$$

up to a non-zero constant.

Proof. If $j = 0$, the assertion follows from the structure theorem proved by Igusa [9, 10]. If $j = 4, 6$, the assertion follows from [20], since $M^\epsilon_{\mathrm{Sym}(j)}(\Gamma^{(2)})$ is free in this case. We can check the assertion in the case when $j = 8$ by a similar computation to [20]. Source code, for checking this, can be found at https://github.com/stakemori/det_vec_vald_SMFs. Suppose $j = 2$. For simplicity, we assume $\epsilon = 1$. By Ibukiyama [7, Theorem 4.1], $M = M^1_{\mathrm{Sym}(2)}(\Gamma^{(2)})$ has the following free resolution.

$$0 \to A(-33) \to A(-21) \oplus A(-23) \oplus A(-27) \oplus A(-29) \to M \to 0. \qquad \text{(Rsl-1)}$$

Here $A = A_{\mathrm{ev}}$. In the matrix form, the second map is given by

$${}^t(-12\chi_{12}, 10\chi_{10}, -6\phi_6, 4\phi_4).$$

The third map sends the standard basis to elements defined in [7, Theorem 4.1] of determinant weights 21, 23, 27, 29 respectively. We write them as g_{21}, g_{23}, g_{27} and

g_{29}. We use the same notation in Definition 4.3 and put $f_1 = g_{21},\ f_2 = g_{23},\ f_3 = g_{27}$. We define $I_i \subset B_i$ by

$$I_i = \begin{cases} B_i, & \text{if } i \geq 0, \\ \{4\}, & \text{if } i = -1, \\ \emptyset, & \text{if } i \leq -2. \end{cases}$$

Then $\{I_i\}$ is an admissible collection. By Theorem 4.1, we have

$$\det((W_\bullet, d, e)) = (4\phi_4)^{-1}.$$

By the proof of [7, Theorem 4.1], we have

$$g_{21} \wedge g_{23} \wedge g_{27} = \phi_4 \chi_{35}^2$$

up to a non-zero constant. By definition, the determinant of the free resolution is equal to $4^{-1}\chi_{35}^2$. By Proposition 4.1, we have our assertion. □

5. Rankin–Cohen–Ibukiyama Type Differential Operators

We use Rankin–Cohen–Ibukiyama type differential operators to construct generators of $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ and $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$. We recall differential operators defined by Eholzer–Ibukiyama [3] and van Dorp [22]. We use the same notation used in [11].

Let k, l be positive integers and χ, ψ characters of $\Gamma^{(2)}$. We take scalar valued Siegel modular forms $f \in M_k(\Gamma^{(2)}, \chi)$ and $g \in M_l(\Gamma^{(2)}, \psi)$. For $j \in \mathbb{Z}_{\geq 0}$, Eholzer–Ibukiyama [3] constructed vector valued Siegel modular forms

$$\{f, g\}_{\mathrm{Sym}(j)} \in M_{k+l,j}(\Gamma^{(2)}, \chi\psi),$$
$$\{f, g\}_{\det^2 \mathrm{Sym}(j)} \in M_{k+l+2,j}(\Gamma^{(2)}, \chi\psi)$$

by Rankin–Cohen–Ibukiyama type differential operators.

Next, we review the differential operator defined by van Dorp [22]. As before, let k, l be positive integers and χ, ψ characters of $\Gamma^{(2)}$. Let $F \in M_{k,j}(\Gamma^{(2)}, \chi)$ and $g \in M_l(\Gamma^{(2)}, \psi)$ be a vector valued Siegel modular form and a scalar valued Siegel modular form respectively. Then van Dorp [22, Proposition 3.6.1] constructed

$$\{F, g\}_{\det \mathrm{Sym}(j)} \in M_{k+l+1,j}(\Gamma^{(2)}, \chi\psi)$$

by a differential operator. Though he proved the proposition only when both χ and ψ are trivial characters, the same proof works for this case.

Finally, we define differential operators on three scalar valued Siegel modular forms. We use these differential operators in order to construct vector valued Siegel modular forms of odd determinant weights from scalar valued Siegel modular forms. For $i = 1, 2, 3$, let k_i be a positive integer and χ_i a character of $\Gamma^{(2)}$. For $f_i \in$

$M_{k_i}(\Gamma^{(2)}, \chi_i)$ $(i = 1, 2, 3)$, we define differential operators as follows:

$$\begin{aligned}\{f_1, f_2, f_3\}_{\det \mathrm{Sym}(j)} &= \{\{f_1, f_2\}_{\mathrm{Sym}(j)}, f_3\}_{\det \mathrm{Sym}(j)} \\ &\in M_{k_1+k_2+k_3+1,\ j}(\Gamma^{(2)}, \chi_1\chi_2\chi_3), \\ \{f_1, f_2, f_3\}_{\det^3 \mathrm{Sym}(j)} &= \{\{f_1, f_2\}_{\det^2 \mathrm{Sym}(j)}, f_3\}_{\det \mathrm{Sym}(j)} \\ &\in M_{k_1+k_2+k_3+3,\ j}(\Gamma^{(2)}, \chi_1\chi_2\chi_3).\end{aligned}$$

6. Precise Statement of the Structure Theorems

In this section, we give generators and relations of $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ and $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ explicitly.

6.1. *Generators of $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$*

In this subsection, we define thirteen generators $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$. We define ten of them as follows.

$$\begin{aligned}F_{10} &= \{\varphi_4, \varphi_6\}_{\mathrm{Sym}(10)}, & F_{12} &= \{\varphi_4, \varphi_4^2\}_{\mathrm{Sym}(10)}, \\ G_{12} &= \{\varphi_4, \varphi_6\}_{\det^2 \mathrm{Sym}(10)}, & F_{14} &= \{\varphi_4, \varphi_4\varphi_6\}_{\mathrm{Sym}(10)}, \\ G_{14} &= \{\varphi_4, \chi_{10}\}_{\mathrm{Sym}(10)}, & H_{14} &= \{\chi_5, \varphi_4\chi_5\}_{\mathrm{Sym}(10)}, \\ F_{16} &= \{\varphi_4, \varphi_6^2\}_{\mathrm{Sym}(10)}, & G_{16} &= \{\varphi_4, \chi_{12}\}_{\mathrm{Sym}(10)}, \\ F_{18} &= \{\varphi_4, \varphi_4\chi_{10}\}_{\mathrm{Sym}(10)}, & F_{20} &= \{\varphi_4, \varphi_4\chi_{12}\}_{\mathrm{Sym}(10)}.\end{aligned}$$

We define remaining three modular forms as follows.

$$\begin{aligned}F_6 = \frac{1}{667705262220781424640000\chi_{12}}&(299617786417098240F_{18} \\ &+ 1789501343175B_{18}^{(0)} - 269788430692085760 0B_{18}^{(1)} + 4499692750175B_{18}^{(2)} \\ &- 3530479328750B_{18}^{(3)} + 127068578893267200 0B_{18}^{(4)} \\ &+ 310661133307499520B_{18}^{(5)} + 140785854300B_{18}^{(6)} \\ &+ 142952698392B_{18}^{(7)} - 153516996000B_{18}^{(8)} + 14151466506240000B_{18}^{(9)} \\ &- 55352584175616000 0B_{18}^{(10)}).\end{aligned}$$

$$\begin{aligned}F_8 = \frac{1}{500778946665586068480000 0\chi_{10}}&(-843455334590563968 0F_{18} \\ &- 66731865311475B_{18}^{(0)} + 161806890613036032 00B_{18}^{(1)} - 75624543557850B_{18}^{(2)} \\ &+ 5670763346875B_{18}^{(3)} - 142126285976539200 00B_{18}^{(4)}\end{aligned}$$

$$+\,41041767463461273 60B_{18}^{(5)} - 27222227916975B_{18}^{(6)} - 2824331333544B_{18}^{(7)}$$
$$+\,305313705000B_{18}^{(8)} - 1559002738648320000B_{18}^{(9)}$$
$$-\,3590159128657920000B_{18}^{(10)}).$$

$$G_{10} = \frac{1}{13800125293914960000\varphi_4^2}(-15916405775169019760640F_{18}$$
$$-\,132162848896253175B_{18}^{(0)} - 36964508172629003366400B_{18}^{(1)}$$
$$-\,61881543692326050B_{18}^{(2)} + 4575977962943750B_{18}^{(3)}$$
$$-\,20639189761094963520000B_{18}^{(4)} + 516230561016664865280B_{18}^{(5)}$$
$$-\,23611666345271550B_{18}^{(6)} - 2865001595829912B_{18}^{(7)} + 215990508690000B_{18}^{(8)}$$
$$-\,2792613466587732480000B_{18}^{(9)} - 914774164413677568 0000B_{18}^{(10)}$$
$$-\,3915785552148369900000 0B_{18}^{(11)}).$$

Here $B_{18}^{(0)},\ldots,B_{18}^{(11)}$ are given as follows:

$$\begin{aligned}
B_{18}^{(0)} &= \{\varphi_4,\varphi_4^2\varphi_6\}_{\mathrm{Sym}(10)}, & B_{18}^{(1)} &= \{\chi_5,\varphi_4^2\chi_5\}_{\mathrm{Sym}(10)},\\
B_{18}^{(2)} &= \{\varphi_6,\varphi_4^3\}_{\mathrm{Sym}(10)}, & B_{18}^{(3)} &= \{\varphi_6,\varphi_6^2\}_{\mathrm{Sym}(10)},\\
B_{18}^{(4)} &= \{\varphi_6,\chi_{12}\}_{\mathrm{Sym}(10)}, & B_{18}^{(5)} &= \{\varphi_4^2,\chi_{10}\}_{\mathrm{Sym}(10)},\\
B_{18}^{(6)} &= \{\varphi_4^2,\varphi_4\varphi_6\}_{\mathrm{Sym}(10)}, & B_{18}^{(7)} &= \{\varphi_4,\varphi_4^3\}_{\det^2\mathrm{Sym}(10)},\\
B_{18}^{(8)} &= \{\varphi_4,\varphi_6^2\}_{\det^2\mathrm{Sym}(10)}, & B_{18}^{(9)} &= \{\varphi_4,\chi_{12}\}_{\det^2\mathrm{Sym}(10)},\\
B_{18}^{(10)} &= \{\chi_5,\varphi_6\chi_5\}_{\det^2\mathrm{Sym}(10)}, & B_{18}^{(11)} &= \varphi_4\varphi_6F_8.
\end{aligned}$$

We note that the set $\{F_{18},\ B_{18}^{(0)},\ldots,B_{18}^{(11)}\}$ forms a basis of the space $M_{18,\ 10}(\Gamma^{(2)})$.

In Sec. 7.3, we shall explain $F_6,\ F_8$ and F_{10} are holomorphic Siegel modular forms.

6.2. *Generators of $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$*

We define seven of thirteen generators of $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ as follows:

$$\begin{aligned}
F_{17} &= \{\varphi_4,\varphi_6,\varphi_6\}_{\det\mathrm{Sym}(10)}, & G_{17} &= \left\{\varphi_4,\varphi_4^2,\varphi_4\right\}_{\det\mathrm{Sym}(10)},\\
H_{17} &= \{\chi_5,\varphi_6,\chi_5\}_{\det\mathrm{Sym}(10)}, & F_{19} &= \{\varphi_4,\chi_5,\varphi_4\chi_5\}_{\det\mathrm{Sym}(10)},\\
G_{19} &= \left\{\varphi_4,\varphi_4^2,\varphi_6\right\}_{\det\mathrm{Sym}(10)}, & F_{21} &= \{\varphi_4,\varphi_6,\chi_{10}\}_{\det\mathrm{Sym}(10)},\\
F_{23} &= \{\varphi_4,\varphi_6,\chi_{12}\}_{\det\mathrm{Sym}(10)}.
\end{aligned}$$

We define remaining six modular forms as follows:

$$F_9 = \frac{1}{46365083972861250125168640000\chi_{10}}(-6725180154573619200F_{19} - 7089754342125G_{19} + 22117267502616B_{19}^{(0)} + 4205094643659571200B_{19}^{(1)} - 7138425156720B_{19}^{(2)} - 5140411073357414400B_{19}^{(3)} - 2847351975761018880 0B_{19}^{(4)} - 78089462464B_{19}^{(5)}).$$

$$F_{11} = \frac{1}{67079114544070095667200 0\varphi_4^2}(-29326410727288012800F_{19} - 154720948329675G_{19} + 296435594930904B_{19}^{(0)} + 47067973368230707200B_{19}^{(1)} - 112872064939920B_{19}^{(2)} - 81197866538488627200B_{19}^{(3)} - 35718418963060162560 0B_{19}^{(4)} - 20661646909696B_{19}^{(5)} - 67079114544070095667200 0B_{19}^{(6)}).$$

$$F_{13} = \frac{1}{17600933845382823936000\varphi_6}(-5870720280512102400F_{19} + 3571610126175G_{19} - 17576872544088B_{19}^{(0)} - 3898171979838259200B_{19}^{(1)} + 4328987691600B_{19}^{(2)} + 3418705164194611200B_{19}^{(3)} + 18831546114903244800B_{19}^{(4)} - 286545870208B_{19}^{(5)} - 17600933845382823936000B_{19}^{(6)}).$$

$$F_{15} = \frac{1}{13165054156800\varphi_4}(258020594380800F_{19} - 12669930B_{19}^{(2)} + 37752728832000B_{19}^{(3)} - 143718507878400B_{19}^{(4)} - 37349B_{19}^{(5)}).$$

$$G_{15} = \frac{1}{19508428800\varphi_4}B_{19}^{(0)}.$$

$$H_{15} = \frac{1}{92155379097600\varphi_4}(594965135278080B_{19}^{(1)} - 944735220B_{19}^{(2)} + 97485935339520B_{19}^{(3)} - 4267924896614400B_{19}^{(4)} + 550172207B_{19}^{(5)}).$$

Here $B_{19}^{(0)}, \ldots, B_{19}^{(6)}$ are given by

$$B_{19}^{(0)} = \{\varphi_4, \varphi_6, \varphi_4^2\}_{\det \mathrm{Sym}(10)}, \qquad B_{19}^{(1)} = \{\varphi_4, \varphi_4\chi_5, \chi_5\}_{\det \mathrm{Sym}(10)},$$
$$B_{19}^{(2)} = \{\varphi_4, \varphi_4\varphi_6, \varphi_4\}_{\det \mathrm{Sym}(10)}, \qquad B_{19}^{(3)} = \{\chi_5, \varphi_4^2, \chi_5\}_{\det \mathrm{Sym}(10)},$$
$$B_{19}^{(4)} = \{\chi_5, \varphi_4\chi_5, \varphi_4\}_{\det \mathrm{Sym}(10)}, \qquad B_{19}^{(5)} = \{\varphi_6, \varphi_4^2, \varphi_4\}_{\det \mathrm{Sym}(10)},$$
$$B_{19}^{(6)} = \varphi_4\varphi_6F_9.$$

We note that the set $\{F_{19}, G_{19}, B_{19}^{(0)}, \ldots, B_{19}^{(6)}\}$ forms a basis of the space $M_{19,10}(\Gamma^{(2)})$. In Sec. 7.3, we shall explain that $F_9, F_{11}, F_{13}, F_{15}, G_{15}$ and H_{15} are holomorphic modular forms.

6.3. *Relations of* $M^{\mathbf{0}}_{\mathbf{Sym(10)}}(\boldsymbol{\Gamma^{(2)}})$

Lemma 6.1. *Generators of even determinant weights satisfy the following two relations.*

$$\begin{aligned} &a_{18}F_6 + a_{16}F_8 + a_{14}F_{10} + b_{14}G_{10} + a_{12}F_{12} + b_{12}G_{12} + a_{10}F_{14} + b_{10}G_{14} + c_{10}H_{14} \\ &\quad + a_8F_{16} + b_8G_{16} + a_6F_{18} + a_4F_{20} = 0 \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &\alpha_{20}F_6 + \alpha_{18}F_8 + \alpha_{16}F_{10} + \beta_{16}G_{10} + \alpha_{14}F_{12} + \beta_{14}G_{12} + \alpha_{12}F_{14} + \beta_{12}G_{14} \\ &\quad + \gamma_{12}H_{14} + \alpha_{10}F_{16} + \beta_{10}G_{16} + \alpha_8F_{18} + \alpha_6F_{20} = 0. \end{aligned} \tag{6.2}$$

Here, a_{18} *and* a_{16} *are defined by*

$$\begin{aligned} a_{18} = {}& 15643660032(67331642279\varphi_4^3\varphi_6 - 6689228000\varphi_6^3 + 2370923189526528\varphi_4^2\chi_{10} \\ &+ 4612035407616000\varphi_6\chi_{12}) \end{aligned}$$

and

$$\begin{aligned} a_{16} = {}& 5105916816000(387254979\varphi_4^4 - 20240608\varphi_4\varphi_6^2 + 6135186530304\varphi_6\chi_{10} \\ &+ 4114923337728\varphi_4\chi_{12}). \end{aligned}$$

Coefficients $a_{14}, \ldots, a_4$ *are defined by*

$$\begin{aligned} a_{14} &= -11824384\varphi_4(-392419241\varphi_4\varphi_6 + 145847127883968\chi_{10}), \\ b_{14} &= -1986496512\varphi_4(-31914599\varphi_4\varphi_6 + 10054388155392\chi_{10}), \\ a_{12} &= 830289075(23924401\varphi_4^3 - 1654848\varphi_6^2 + 1331133253632\chi_{12}), \\ b_{12} &= 76257199200(23313\varphi_4^3 - 2080\varphi_6^2 + 1434101760\chi_{12}), \\ a_{10} &= -135273640320(5849\varphi_4\varphi_6 + 412304256\chi_{10}), \\ b_{10} &= 41009272012800(-909943\varphi_4\varphi_6 + 346335575040\chi_{10}), \\ c_{10} &= 2615541477773672448000(-\varphi_4\varphi_6 + 3468906\chi_{10}), \\ a_8 &= 312420856840400\varphi_4^2, \\ b_8 &= 7055267550695755776\varphi_4^2, \\ a_6 &= -8969486460731596800\varphi_6, \\ a_4 &= 1651765429768572518400\varphi_4. \end{aligned}$$

Coefficients $\alpha_{20}, \alpha_{18}, \alpha_{16}$ *and* β_{16} *are defined by*

$$\begin{aligned}\alpha_{20} &= -5214553344(425048543296\phi_4^5 + 484587670889\phi_4^2\phi_6^2 \\ &\quad + 15696098336318400\phi_4\phi_6\chi_{10} + 62924853322985472\phi_4^2\chi_{12} \\ &\quad - 890719685310873600000\chi_{10}^2), \\ \alpha_{18} &= 5105916816000(-1811050991\phi_4^3\phi_6 - 24020864\phi_6^3 - 145283115931200\phi_4^2\chi_{10} \\ &\quad + 87510477938688\phi_6\chi_{12}), \\ \alpha_{16} &= 5912192(19495410909\phi_4^4 - 23419603319\phi_4\phi_6^2 - 1950560293550400\phi_6\chi_{10} \\ &\quad + 2818879454297088\phi_4\chi_{12})\end{aligned}$$

and

$$\begin{aligned}\beta_{16} &= 1986496512(649847848\phi_4^4 - 809420843\phi_4\phi_6^2 - 86364658900800\phi_6\chi_{10} \\ &\quad + 87573208859136\phi_4\chi_{12}).\end{aligned}$$

Coefficients $\alpha_{14}, \ldots, \alpha_6$ *are defined by*

$$\begin{aligned}\alpha_{14} &= -4151445375\phi_4(22269553\phi_4\phi_6 + 1622175242688\chi_{10}), \\ \beta_{14} &= -381285996000\phi_4(21233\phi_4\phi_6 + 1927074240\chi_{10}), \\ \alpha_{12} &= -135273640320(14911\phi_4^3 - 44156\phi_6^2 + 3872074752\chi_{12}), \\ \beta_{12} &= -205046360064000(2855021\phi_4^3 - 3764964\phi_6^2 + 718771802112\chi_{12}), \\ \gamma_{12} &= -163471342360854528000(1649\phi_4^3 - 1729\phi_6^2 + 211323168\chi_{12}), \\ \alpha_{10} &= 11238160318000(-139\phi_4\phi_6 + 11793600\chi_{10}), \\ \beta_{10} &= -593178707810304000(5947\phi_4\phi_6 + 1014249600\chi_{10}), \\ \alpha_8 &= 448474323036579840000\phi_4^2, \\ \alpha_6 &= -825882714884286259200\phi_6.\end{aligned}$$

We shall prove Lemma 6.1 in Sec. 7.4.

6.4. *Relations of* $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$

Lemma 6.2. *Generators of odd determinant weights satisfy the following two relations.*

$$\begin{aligned}&d_{18}F_9 + d_{16}F_{11} + d_{14}F_{13} + d_{12}F_{15} + e_{12}G_{15} + f_{12}H_{15} + d_{10}F_{17} + e_{10}G_{17} \\ &\quad + f_{10}H_{17} + d_8F_{19} + e_8G_{19} + d_6F_{21} + d_4F_{23} = 0 \qquad (6.3)\end{aligned}$$

and

$$\begin{aligned}&\delta_{20}F_9 + \delta_{18}F_{11} + \delta_{16}F_{13} + \delta_{14}F_{15} + \epsilon_{14}G_{15} + \zeta_{14}H_{15} + \delta_{12}F_{17} + \epsilon_{12}G_{17} \\ &\quad + \zeta_{12}H_{17} + \delta_{10}F_{19} + \epsilon_{10}G_{19} + \delta_8F_{21} + \delta_6F_{23} = 0. \qquad (6.4)\end{aligned}$$

Here d_{18} *and* d_{16} *are defined by*

$$\begin{aligned} d_{18} &= -195284202946560(-199029244391\phi_4^3\phi_6 + 2697950850992640\phi_4^2\chi_{10} \\ &\quad + 3549133607846400\phi_6\chi_{12}) \end{aligned}$$

and

$$\begin{aligned} d_{16} &= -341747355156480(6216500122\phi_4^4 - 25056024021\phi_4\phi_6^2 \\ &\quad - 471257790566400\phi_6\chi_{10} + 2428645469322240\phi_4\chi_{12}). \end{aligned}$$

Coefficients $d_{14}, \ldots, d_4$ *are defined by*

$$\begin{aligned} d_{14} &= -748935563257782602956680\phi_4(-433\phi_4\phi_6 + 14131200\chi_{10}), \\ d_{12} &= -102195265536(-685230911\phi_4^3 + 35309376921600\chi_{12}), \\ e_{12} &= -1093591536500736(39827745779\phi_4^3 + 468263910297600\chi_{12}), \\ f_{12} &= -255488163840(193718497\phi_4^3 + 4258403020800\chi_{12}), \\ d_{10} &= 955493351615296(-49\phi_4\phi_6 + 3686400\chi_{10}), \\ e_{10} &= 387290963292597(7\phi_4\phi_6 + 115200\chi_{10}), \\ f_{10} &= 5177826116350101946368(-7\phi_4\phi_6 + 460800\chi_{10}), \\ d_8 &= -2133719553440976076800\phi_4^2, \\ e_8 &= 0, \\ d_6 &= -396262202781895557120\phi_6, \\ d_4 &= -660437004636492595200\phi_4. \end{aligned}$$

Coefficients δ_{20}, δ_{18} *and* δ_{16} *are defined by*

$$\begin{aligned} \delta_{20} &= -27897743278080(-1439954360064\phi_4^5 + 467496493270\phi_4^2\phi_6^2 \\ &\quad + 300859457751014400\phi_4\phi_6\chi_{10} + 342250970398848000\phi_4^2\chi_{12} \\ &\quad - 754154219249642373120\chi_{10}^2), \\ \delta_{18} &= 341747355156480(16282786754\phi_4^3\phi_6 + 2556737145\phi_6^3 \\ &\quad - 5088306946498560\phi_4^2\chi_{10} + 895389802076160\phi_6\chi_{12}) \end{aligned}$$

and

$$\begin{aligned} \delta_{16} &= -1069907947511118004224 0(-3036\phi_4^4 + 5\phi_4\phi_6^2 - 49459200\phi_6\chi_{10} \\ &\quad + 130636800\phi_4\chi_{12}). \end{aligned}$$

Coefficients $\delta_{14}, \ldots, \delta_6$ *are defined by*

$$\begin{aligned} \delta_{14} &= -8073425977344\phi_4(-8673809\phi_4\phi_6 + 260218327680\chi_{10}), \\ \epsilon_{14} &= -1093591536500736\phi_4(39827745779\phi_4\phi_6 + 659445588817920\chi_{10}), \end{aligned}$$

$$\zeta_{14} = -255488163840\phi_4(193718497\phi_4\phi_6 + 11030709116160\chi_{10}),$$
$$\delta_{12} = 955493351615296(-44\phi_4^3 - 5\phi_6^2 + 4838400\chi_{12}),$$
$$\epsilon_{12} = -55327280470371(-44\phi_4^3 - 5\phi_6^2 + 967680\chi_{12}),$$
$$\zeta_{12} = 7396894451928717706624(-44\phi_4^3 - 5\phi_6^2 + 4193280\chi_{12}),$$
$$\delta_{10} = 21337195534409760768000(-\phi_4\phi_6 + 64512\chi_{10}),$$
$$\epsilon_{10} = -356927351770457395200\chi_{10},$$
$$\delta_8 = -3962622027818955571200\phi_4^2,$$
$$\delta_6 = -660437004636492595200\phi_6.$$

We shall prove Lemma 6.2 in Sec. 7.4.

7. Proof of the Main Results

In this section, we prove Theorems 1.1 and 1.2. Also, we prove generators given in Sec. 6 are holomorphic modular forms. The actual computation is done by a computer algebra system. We use SageMath [18] and a SageMath package for Siegel modular forms degree two [19].

7.1. *Hilbert series*

For $j \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in \{0, 1\}$, we define the Hilbert series $h_{j,\epsilon}(t)$ of $M_{\mathrm{Sym}(j)}(\Gamma^{(2)})$ by

$$h_{j,\epsilon}(t) = \sum_{k=0}^{\infty} \dim_{\mathbb{C}} M_{k,j}(\Gamma^{(2)}) t^k.$$

Then $h_{10,\epsilon}(t)$ is given as follows.

Lemma 7.1.

$$h_{10,0}(t) = \frac{t^6 + t^8 + 2t^{10} + 2t^{12} + 3t^{14} + 2t^{16} + t^{18} + t^{20} - t^{24} - t^{26}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

$$h_{10,1}(t) = \frac{t^9 + t^{11} + t^{13} + 3t^{15} + 3t^{17} + 2t^{19} + t^{21} + t^{23} - t^{27} - t^{29}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

In particular, expansions of these Hilbert series are given by

$$h_{10,0}(t) = t^6 + t^8 + 3t^{10} + 4t^{12} + 7t^{14} + 9t^{16} + 13t^{18} + 17t^{20} + 22t^{22} + 27t^{24} + 35t^{26} + 41t^{28} + O(t^{30})$$

and

$$h_{10,1}(t) = t^9 + t^{11} + 2t^{13} + 5t^{15} + 6t^{17} + 9t^{19} + 13t^{21} + 16t^{23} + 21t^{25} + 28t^{27} + 33t^{29} + O(t^{31}).$$

Proof. $h_{10,0}(t)$ is given in [7]. But for the sake of completeness, we give a proof. By Tsushima [21], the dimension of $S_{k,j}(\Gamma^{(2)})$ is known if $k > 4$. By Arakawa [2], we can compute $\dim_{\mathbb{C}} M_{k,j}(\Gamma^{(2)}) - \dim_{\mathbb{C}} S_{k,j}(\Gamma^{(2)})$ if $k > 4$. Therefore, we have to prove $M_{k,10}(\Gamma^{(2)}) = 0$ if $k \leq 4$. The vanishing of $M_{k,10}(\Gamma^{(2)})$ for $k \leq 0$ follows from Freitag's work [4]. We prove $M_{k,10}(\Gamma^{(2)}) = S_{k,10}(\Gamma^{(2)})$ for $1 \leq k \leq 4$. If k is odd, this follows from the definition. By Ibukiyama [7, Lemma 2], $M_{2,10}(\Gamma^{(2)}) = S_{2,10}(\Gamma^{(2)})$. Since $S_{14}(\mathrm{SL}_2(\mathbb{Z})) = 0$, we have $M_{4,10}(\Gamma^{(2)}) = S_{4,10}(\Gamma^{(2)})$. Thus, we have $M_{k,10}(\Gamma^{(2)}) = S_{k,10}(\Gamma^{(2)})$ for $1 \leq k \leq 4$. Again, by Ibukiyama [7, Lemma 2], we have $S_{k,10}(\Gamma^{(2)}) = 0$ if $0 \leq k \leq 4$. Therefore, we have $M_{k,10}(\Gamma^{(2)}) = 0$ if $0 \leq k \leq 4$. □

7.2. *Precision*

We explain to what extent we compute Fourier coefficients of the generators. For $f \in M_k(\Gamma^{(2)})$, we consider f as an element of $\mathbb{C}[q_{12}, q_{12}^{-1}][\![q_{11}, q_{22}]\!]$ as in Se. 2.

We recall Sturm type theorem for scalar valued Siegel modular forms of degree 2 for our need. We introduce some notation. For $f \in \mathbb{C}[q_{12}, q_{12}^{-1}][\![q_{11}, q_{22}]\!]$, we denote coefficients of f as follows:

$$f = \sum_{(n,r,m)} a((n,r,m); f) q_{11}^n q_{12}^r q_{22}^m.$$

For $a \in \mathbb{R}_{\geq 0}$ and a subring $B \subset \mathbb{C}$, we define a ring by

$$R_a(B) = R'(B)/(q_{11}^{[a]+1}, q_{22}^{[a]+1}). \tag{7.1}$$

Here $[\cdot]$ is the Gauss symbol and $R'(B)$ is defined by

$$R'(B) = \{f \in B[q_{12}, q_{12}^{-1}][\![q_{11}, q_{22}]\!] \mid a((n,r,m); f) = 0 \text{ if } 4nm - r^2 < 0\}.$$

Theorem 7.1 (Kikuta–Takemori [12]). *Let $k \in \mathbb{Z}_{\geq 0}$ and p a prime number. Suppose $f \in M_k(\Gamma^{(2)})$ has $\mathbb{Z}_{(p)}$ integral Fourier coefficients. We put*

$$\overline{f} = \text{ The image of } f \text{ in } \mathbb{Z}/p\mathbb{Z}[q_{12}, q_{12}^{-1}][\![q_{11}, q_{22}]\!]$$

and

$$b_k = \begin{cases} k/10, & \text{if } k \text{ is even}, \\ (k-5)/10, & \text{if } k \text{ is odd}. \end{cases}$$

If the image of $\overline{f}$ in $R_{b_k}(\mathbb{Z}/p\mathbb{Z})$ vanishes, then we have $\overline{f} = 0$.

By the theorem above and the fact that $M_k(\Gamma^{(2)})$ has a basis of modular forms with integral Fourier coefficients, we have the following lemma.

Lemma 7.2. *Let $f \in M_k(\Gamma^{(2)})$ and assume the image of f in $R_{b_k}(\mathbb{C})$ vanishes, where b_k is as in Theorem 7.1. Then we have $f = 0$.*

Remark 7.1. The statement of [20, Lemma 5.1] is true, but the proof is not sufficient.

In Lemma 7.5, we shall prove the determinant of the first 11 generators (ordered by the determinant weight) of $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ (respectively $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$) is equal to $\chi_{35}^5(\phi_4^3-\phi_6^2)$ (respectively $\chi_{35}^6(\phi_4^3-\phi_6^2)$) up to a non-zero constant. By Lemma 7.2, it is enough to compute generators of $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ (respectively $M^1_{\mathrm{Sym}(10)}(\Gamma^{(2)})$) in $R_{18}(\mathbb{C}) \otimes_{\mathbb{C}} V_{10}$ (respectively $R_{22}(\mathbb{C}) \otimes_{\mathbb{C}} V_{10}$).

7.3. *Holomorphy of generators*

In Sec. 6, we constructed generators of small determinant weights by dividing modular forms. Since dividing by a cusp form is subtle, we introduce the following lemma.

Lemma 7.3. *For $a \in \mathbb{Z}_{\geq 1}$, let $R_a(\mathbb{C})$ be the ring defined by* (7.1). *Let*

$$f = \sum_{(n,r,m)} a((n,r,m); f) q_{11}^n q_{12}^r q_{22}^m \bmod (q_{11}^{a+1}, q_{22}^{a+1}),$$

$$g = \sum_{(n,r,m)} a((n,r,m); g) q_{11}^n q_{12}^r q_{22}^m \bmod (q_{11}^{a+1}, q_{22}^{a+1})$$

be elements of $R_a(\mathbb{C})$. Assume $a((1,-1,1); g) = 1$ and g is cuspidal, that is $a((n,r,m); g) = 0$ if $4nm - r^2 = 0$ and $n, m \leq a$. Then there uniquely exists $h \in R_{a-1}(\mathbb{C})$ such that $\widetilde{f} = \widetilde{g}h$. Here $\widetilde{f}$ (respectively $\widetilde{g}$) is the image of f (respectively g) in $R_{a-1}(\mathbb{C})$. Moreover, we have the following recursive equation.

$$a((n,r,m); h) + a((1,1,1); g) a((n, r-2, m); h) = a((n+1, r-1, m+1); f) \quad (7.2)$$

for $(n,r,m) \in \mathbb{Z}^3$ with $0 \leq n, m \leq a-1$ and $4nm - r^2 \geq 0$.

Proof. We put $S = \{(n,r,m) \in \mathbb{Z}^3 \mid n, m, 4nm - r^2 \geq 0\}$. We define an order $\leq$ of S so that $(n,r,m) \leq (n',r',m')$ if and only if one of following conditions holds.

(1) $n + m \leq n' + m'$.
(2) $n + m = n' + m'$ and $n \leq n'$.
(3) $n = n'$, $m = m'$ and $r \leq r'$.

Let $h = \sum_{(n,r,m) \geq 0} a((n,r,m); h) q_{11}^n q_{12}^r q_{22}^m + \mathrm{mod}(q_{11}^a, q_{22}^a)$ be an element of $R_{a-1}(\mathbb{C})$. By the condition $\widetilde{f} = \widetilde{g}h$, we have

$$\sum_{\substack{T_1 + T_2 = (n+1, r-1, m+1) \\ T_1, T_2 \in S}} a(T_1; g) a(T_2; h) = a((n+1, r-1, m+1); f) \quad (7.3)$$

for $(n,r,m) \in S$ with $n, m \leq a-1$. Here addition of S is defined by entry-wise. By (7.3) and the cuspidality of g, we obtain (7.2). Thus, we can recursively define $a((n,r,m); h)$ for $(n,r,m) \in S$ with $n, m \leq a-1$ by the order of S. Uniqueness follows from the recursive equation above. □

Next, we prove holomorphy of generators.

Lemma 7.4. *Meromorphic modular forms $F_6, F_8, G_{10}, F_9, F_{11}, F_{13}, F_{15}$ and H_{15} given in Secs.* 6.1 *and* 6.2 *are holomorphic.*

Proof. For simplicity, we only prove F_9 is holomorphic. We can prove other cases in a similar way. Let N_{19} be the subspace of $M_{19,10}(\Gamma^{(2)})$ spanned by $F_{19}, G_{19}, B_{19}^{(0)}, \dots, B_{19}^{(5)}$. Then N_{19} is a subspace of $M_{19,10}(\Gamma^{(2)})$ of codimension 1. And we can calculate a basis of N_{19} explicitly from Fourier coefficients of scalar valued Siegel modular forms. Put $C_{19} = F_{19}|T(2)$, where $T(2)$ is the Hecke operator. For the explicit action of $T(2)$ on $M_{k,j}(\Gamma^{(2)})$, see [2]. Then, we have $N_{19} + \mathbb{C}C_{19} = M_{19,10}(\Gamma^{(2)})$. We can check this equality by computing vector valued modular forms in $R_6(\mathbb{Q})^{11}$. By Lemma 7.1, there uniquely exists $F_9' \in M_{9,10}(\Gamma^{(2)})$ such that $F_9' \neq 0$ up to a non-zero constant. By a similar argument in [11, Sec. 7.4], we can explicitly find rational numbers $\alpha, \beta, \gamma, \delta, c^{(i)}, d^{(i)}$ for $-1 \leq i \leq 5$ such that

$$\phi_4\phi_6 F_9' = \alpha F_{19} + \beta C_{19} + c^{(-1)} G_{19} + \sum_{i=0}^{5} c^{(i)} B_{19}^{(i)},$$

$$\phi_{10} F_9' = \gamma F_{19} + \delta C_{19} + d^{(-1)} G_{19} + \sum_{i=0}^{5} d^{(i)} B_{19}^{(i)}.$$

Then, we have $\beta = \delta \neq 0$ and $\alpha \neq \gamma$. Since C_{19} is computationally expensive, we remove C_{19}. Then, we have

$$\chi_{10} F_9' = \alpha' F_{19} + \beta' G_{19} + \sum_{i=0}^{5} e^{(i)} B_{19}^{(i)}$$

for some rational numbers $\alpha', \beta', e^{(i)}$ $(0 \leq i \leq 5)$. By explicit computation of these rational numbers, we have $F_9' = F_9$ up to a non-zero constant. Thus F_9 is holomorphic. □

7.4. *Relations*

In this subsection, we prove Lemmas 6.1 and 6.2. For simplicity, we only prove relation (6.1).

Proof [Proof of relation (6.1)]. For $k \in \mathbb{Z}_{\geq 0}$, let X_k be the following set of monomial of $\phi_4, \phi_6, \chi_{10}$ and χ_{12};

$$X_k = \{\phi_4^a \phi_6^b \chi_{10}^c \chi_{12}^d \,|\, a, b, c, d \in \mathbb{Z}_{\geq 0}, 4a + 6b + 10c + 12d = k\}.$$

We define a finite set S by

$$X_{18}F_6 \cup X_{16}F_8 \cup X_{14}F_{10} \cup X_{14}G_{10} \cup X_{12}F_{12} \cup X_{12}G_{12} \cup X_{10}F_{14} \cup X_{10}G_{14}$$
$$\cup X_{10}H_{14} \cup X_8F_{16} \cup X_8G_{16} \cup X_6F_{18}.$$

Then, we have $|S| = 27$. By computing elements of S in $R_6(\mathbb{C}) \otimes_{\mathbb{C}} V_{10}$, we see that S is linearly independent over $\mathbb{C}$. By Lemma 7.1, S is a basis of $M_{24,10}(\Gamma^{(2)})$. By numerical computation, we can check that the image of the left-hand side of (6.1) in $R_6(\mathbb{C}) \otimes_{\mathbb{C}} V_{10}$ is equal to 0. Therefore the left-hand side of (6.1) is equal to 0. □

7.5. *Determinants of generators and structure theorems*

Determinants of modular forms given in Secs. 6.1 and 6.2 are given as follows.

Lemma 7.5.

$$F_6 \wedge F_8 \wedge F_{10} \wedge G_{10} \wedge F_{12} \wedge G_{12} \wedge F_{14} \wedge G_{14} \wedge H_{14} \wedge F_{16} \wedge G_{16} = (\phi_4^3 - \phi_6^2)\chi_{35}^5 \tag{7.4}$$

up to a non-zero constant and

$$F_9 \wedge F_{11} \wedge F_{13} \wedge F_{15} \wedge G_{15} \wedge H_{15} \wedge F_{17} \wedge G_{17} \wedge H_{17} \wedge F_{19} \wedge G_{19} = (\phi_4^3 - \phi_6^2)\chi_{35}^6 \tag{7.5}$$

up to a non-zero constant.

Proof. As is explained in Sec. 7.2, we have to compute both sides of (7.4) (respectively (7.5)) in $R_{18}(\mathbb{Q})$ (respectively $R_{22}(\mathbb{Q})$). Computation is done by using SageMath [18] and a package [19] for Siegel modular forms of degree 2. See https://github.com/stakemori/det_vec_vald_SMFs for a script to check this. □

Next, we prove Theorem 1.1.

Theorem 7.2. *Theorem* 1.1 *is true, that is the sequences* (1.1) *and* (1.2) *are exact. Here the second linear map of the sequence* (1.1) *is given by sending the standard basis to* $(a_{18}, a_{16}, \ldots, a_4)$ *and* $(\alpha_{20}, \alpha_{18}, \ldots, \alpha_6)$, *where* $a_{18}, a_{16}, \ldots, a_4; \alpha_{20}, \alpha_{18}, \ldots, \alpha_6$ *are give in* (6.1) *and* (6.2). *The third map of the sequence* (1.1) *is given by sending the standard basis to* $F_6, F_8, \ldots, F_{20}$. *Linear maps of the sequence* (1.2) *is given in a similar way by using* $F_9, F_{11}, \ldots, H_{15}$, (6.3) *and* (6.4).

Proof. For simplicity, we only prove that (1.1) is exact. We denote by ψ_{-2} the second map of (1.1) and by ψ_{-1} the third map of (1.1). Because

$$\det \begin{pmatrix} a_6 & a_4 \\ \alpha_8 & \alpha_6 \end{pmatrix} = \phi_4^3 - \phi_6^2 \tag{7.6}$$

up to a non-zero constant, ψ_{-2} is injective. By Lemma 6.1, we have $\operatorname{Im}(\psi_{-2}) \subset \ker(\psi_{-1})$. By (1.3) and Lemma 7.5,

$$\{F_6, F_8, F_{10}, G_{10}, F_{12}, G_{12}, F_{14}, G_{14}, H_{14}, F_{16}, G_{16}\}$$

is a basis of $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)}) \otimes_{A_{\mathrm{ev}}} K_{\mathrm{ev}}$. Therefore, we have

$$(\ker(\psi_{-1})/\operatorname{Im}(\psi_{-2})) \otimes_{A_{\mathrm{ev}}} K_{\mathrm{ev}} = \{0\}.$$

Since $\det\left(\begin{smallmatrix} a_6 & a_4 \\ \alpha_8 & \alpha_6 \end{smallmatrix}\right)$ and $\det\left(\begin{smallmatrix} b_8 & a_4 \\ \beta_{10} & \alpha_6 \end{smallmatrix}\right)$ are co-prime in A_{ev}, we have $\ker(\psi_{-1}) = \operatorname{Im}(\psi_{-2})$. Therefore the Hilbert series of $\operatorname{Im}(\psi_{-1})$ is equal to

$$\frac{t^6 + t^8 + 2t^{10} + 2t^{12} + 3t^{14} + 2t^{16} + t^{18} + t^{20} - t^{24} - t^{26}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

which is equal to that of $M^0_{\mathrm{Sym}(10)}(\Gamma^{(2)})$ by Lemma 7.1. Therefore ψ_{-1} is surjective. This completes the proof. □

Finally, we prove Theorem 1.2.

Proof [Proof of Theorem 1.2]. By Proposition 4.3, it is enough to prove the assertion when $j = 10$. We can prove the assertion of the theorem by a similar argument to the proof of Proposition 4.3 by using Theorem 4.1, Lemma 7.5 and (7.6). □

8. Examples and a Table of Fourier Coefficients

Since constructions of modular forms given in Secs. 6.1 and 6.2 are complicated, the author has tested them in several ways. By the explicit structure theorems, we can construct a basis of the space $M_{k,10}(\Gamma^{(2)})$ and compute the action of Hecke operators on the space. In the case, when Hecke eigenvalues are known, we test if eigenforms have correct eigenvalues. We also test the generalized Ramanujan conjecture for a non-CAP cuspidal eigenform.

For tests, we introduce the following notation. Let $F \in M_{k,j}(\Gamma^{(2)})$ be a Hecke eigenform and p a prime number. We denote by $Q_p(F; X)$ the polynomial of degree 4 of X so that $\prod_{p\ \mathrm{:prime}} Q_p(F; p^{-s})^{-1}$ is the spinor L-function of F and denote by $R_p(F; X)$ the polynomial of degree 5 of X so that $\prod_{p\ \mathrm{:prime}} R_p(F; p^{-s})^{-1}$ is the standard L-function of F. That is, $Q_p(F; X)$ and $R_p(F; X)$ are defined as follows.

$$Q_p(F; X) = (1 - \alpha_0 X)(1 - \alpha_0\alpha_1 X)(1 - \alpha_0\alpha_2 X)(1 - \alpha_0\alpha_1\alpha_2 X),$$
$$R_p(F; X) = (1 - X)(1 - \alpha_1 X)(1 - \alpha_1^{-1} X)(1 - \alpha_2 X)(1 - \alpha_2^{-1} X).$$

Here $\alpha_0, \alpha_1, \alpha_2$ are the p-Satake parameters of F.

8.1. *Test for Klingen–Eisenstein series and the K–R–S lift*

Since Hecke eigenvalues of Eisenstein series and lifts are known, we test them. Let $k, j \in 2\mathbb{Z}_{\geq 0}$. For a normalized eigenform $f = \sum_{n=1}^{\infty} a(n; f)q^n \in S_{k+j}(\mathrm{SL}_2(\mathbb{Z}))$ with $k > 4$, Arakawa [2] defined the Klingen–Eisenstein series $E(f) \in M_{k,j}(\Gamma^{(2)})$. The Klingen–Eisenstein series $E(f)$ satisfies the following identities:

$$\begin{aligned} \Phi(E(f)) &= f u_1^j, \\ Q_p(E(f); X) &= Q_p^{(1)}(f; X) Q_p^{(1)}(f; p^{(k-2)}X). \end{aligned} \tag{8.1}$$

Here p is a prime number and $Q_p^{(1)}(f; X) = 1 - a(p; f)X + p^{k+j-1}X^2$.

For a normalized eigenform $f = \sum_{n=1}^{\infty} a(n; f)q^n \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, Ramakrishnan and Shahidi [16] constructed a lift $\mathrm{KS}(f) \in M_{k+1,k-2}(\mathrm{SL}_2(\mathbb{Z}))$ called the Kim–Ramakrishnan–Shahidi lift (K–R–S lift in short). The K–R–S lift $\mathrm{KS}(f)$ satisfies the following identity:

$$Q_p(\mathrm{KS}(f); X) = \prod_{i=0}^{3}(1 - \alpha_p^i \beta_p^{3-i} X), \tag{8.2}$$

where p is a prime number. α_p and β_p are defined by $1 - a(p; f)X + p^{k-1}X^2 = (1 - \alpha_p X)(1 - \beta_p X)$.

Let

$$\Delta = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + O(q^6) \in S_{12}(\mathrm{SL}_2(\mathbb{Z})),$$

$$f_{16} = q + 216q^2 - 3348q^3 + 13888q^4 + 52110q^5 + O(q^6) \in S_{16}(\mathrm{SL}_2(\mathbb{Z}))$$

be normalized eigenforms of weight 12 and 16, respectively. Then, we have

$$F_6 = E(f_{16}), \quad F_{13} = \mathrm{KS}(\Delta). \tag{8.3}$$

Fourier coefficients of F_6 and F_{13} are given in Table 1. We test (8.3). By Table 1 and the explicit action of Hecke operators given in [2], we have the following identities.

$$Q_2(F_6; X) = (1 - 2^3 \cdot 3^3 X + 2^{15} X^2)(1 - 2^7 \cdot 3^3 X + 2^{23} X^2),$$

$$Q_2(F_{13}; X) = (1 - 2^9 \cdot 3^2 \cdot 29X + 2^{33} X^2)(1 + 2^{14} \cdot 3X + 2^{33} X^2).$$

Thus, we can check (8.1) and (8.2) for $p = 2$. We note that Fourier coefficients of the K–R–S lift $\mathrm{KS}(\Delta)$ and the non-lift eigenform of weight $\det^{13} \mathrm{Sym}(10)$ were given in [8].

8.2. *Test for the generalized Ramanujan conjecture*

Next, we test the generalized Ramanujan conjecture for the cusp form F_9. By Table 1, we have

$$R_2(F_9; X) = 1 + \frac{667}{512}X + \frac{845}{2048}X^2 - \frac{845}{2048}X^3 - \frac{667}{512}X^4 - X^5.$$

Roots of $R_2(F_9; X)/(X - 1)$ in the complex field with 53 bits precision are given by $\alpha, \overline{\alpha}, \beta, \overline{\beta}$. Here α and β are given as

$$\alpha = -0.966296691713208 - 0.257430968580139i,$$

$$\beta = -0.185070495786792 - 0.982725247253387i.$$

Then, we have $|\alpha| = |\beta| = 1.00000000000000$. Thus, we can check that the absolute values of 2-Satake parameters of F_9 are equal to 1.

8.3. *Table of Fourier coefficients*

For $i, j \in \mathbb{Z}_{\geq 0}$ and $v \in V_j$, we denote by v_i the coefficient $u_1^{j-i} u_2^i$. Then Fourier coefficients of F_6, F_{13} and F_9 are given as follows.

Table 1. Fourier coefficients of some of generators.

$(n,r,m),i$	$a((n,r,m);F_6)_i$	$a((n,r,m);F_{13})_i$	$a((n,r,m);F_9)_i$
(1, 0, 1), 0	−330	0	0
(1, 0, 1), 1	0	4	−4
(1, 0, 1), 2	1350	0	0
(1, 0, 1), 3	0	−16	72
(1, 0, 1), 4	−2100	0	0
(1, 0, 1), 5	0	0	0
(1, 0, 1), 6	−2100	0	0
(1, 0, 1), 7	0	16	−72
(1, 0, 1), 8	1350	0	0
(1, 0, 1), 9	0	−4	4
(1, 0, 1), 10	−330	0	0
(1, 1, 1), 0	−88	0	0
(1, 1, 1), 1	−440	−2	2
(1, 1, 1), 2	−720	−9	9
(1, 1, 1), 3	−240	−16	16
(1, 1, 1), 4	840	−14	14
(1, 1, 1), 5	1512	0	0
(1, 1, 1), 6	840	14	−14
(1, 1, 1), 7	−240	16	−16
(1, 1, 1), 8	−720	9	−9
(1, 1, 1), 9	−440	2	−2
(1, 1, 1), 10	−88	0	0
(1, 0, 2), 0	−7524	0	0
(1, 0, 2), 1	0	−72	−888
(1, 0, 2), 2	61560	0	0
(1, 0, 2), 3	0	384	9168
(1, 0, 2), 4	−120960	0	0
(1, 0, 2), 5	0	−3024	−18144
(1, 0, 2), 6	110880	0	0
(1, 0, 2), 7	0	6816	−18048
(1, 0, 2), 8	114480	0	0
(1, 0, 2), 9	0	576	3072
(1, 0, 2), 10	−49248	0	0
(1, 1, 2), 0	−4224	0	0
(1, 1, 2), 1	−21120	32	448
(1, 1, 2), 2	−17280	144	2016
(1, 1, 2), 3	57600	256	6720
(1, 1, 2), 4	94080	224	14112
(1, 1, 2), 5	−8064	1344	8064
(1, 1, 2), 6	−47040	3136	−10416
(1, 1, 2), 7	28800	1984	−16800
(1, 1, 2), 8	−37440	−144	−9072
(1, 1, 2), 9	−69120	−256	−1792
(1, 1, 2), 10	−27648	0	0
(1, 0, 3), 0	−46552	0	0
(1, 0, 3), 1	0	544	8096
(1, 0, 3), 2	571320	0	0
(1, 0, 3), 3	0	−3712	−166784

(Continued)

Table 1. (*Continued*)

$(n, r, m), i$	$a((n,r,m);F_6)_i$	$a((n,r,m);F_{13})_i$	$a((n,r,m);F_9)_i$
(1, 0, 3), 4	−1933680	0	0
(1, 0, 3), 5	0	29568	950208
(1, 0, 3), 6	1826160	0	0
(1, 0, 3), 7	0	−27648	−3021696
(1, 0, 3), 8	−1731960	0	0
(1, 0, 3), 9	0	55008	448416
(1, 0, 3), 10	658584	0	0
(1, 1, 3), 0	−30600	0	0
(1, 1, 3), 1	−153000	−198	−3162
(1, 1, 3), 2	0	−891	−14229
(1, 1, 3), 3	918000	−1824	−68064
(1, 1, 3), 4	1020600	−2226	−171822
(1, 1, 3), 5	−793800	−9576	−538776
(1, 1, 3), 6	−1827000	−20454	−950586
(1, 1, 3), 7	−954000	−49056	−1171296
(1, 1, 3), 8	529200	−53649	−882783
(1, 1, 3), 9	675000	−35286	−138762
(1, 1, 3), 10	405000	−9360	49680
(2, 0, 2), 0	−1177200	0	0
(2, 0, 2), 1	0	−512	710656
(2, 0, 2), 2	4669200	0	0
(2, 0, 2), 3	0	51200	2936832
(2, 0, 2), 4	−4754400	0	0
(2, 0, 2), 5	0	0	0
(2, 0, 2), 6	−4754400	0	0
(2, 0, 2), 7	0	−51200	−2936832
(2, 0, 2), 8	4669200	0	0
(2, 0, 2), 9	0	512	−710656
(2, 0, 2), 10	−1177200	0	0
(2, 1, 2), 0	−857088	0	0
(2, 1, 2), 1	−2142720	3840	−526080
(2, 1, 2), 2	959040	17712	−79920
(2, 1, 2), 3	5132160	123456	2292960
(2, 1, 2), 4	544320	178752	4211760
(2, 1, 2), 5	−6459264	0	0
(2, 1, 2), 6	544320	−178752	−4211760
(2, 1, 2), 7	5132160	−123456	−2292960
(2, 1, 2), 8	959040	−17712	79920
(2, 1, 2), 9	−2142720	−3840	526080
(2, 1, 2), 10	−857088	0	0
(2, 2, 2), 0	−323136	0	0
(2, 2, 2), 1	−1615680	−3840	168960
(2, 2, 2), 2	−2643840	−17280	760320
(2, 2, 2), 3	−881280	−30720	1351680
(2, 2, 2), 4	3084480	−26880	1182720
(2, 2, 2), 5	5552064	0	0
(2, 2, 2), 6	3084480	26880	−1182720
(2, 2, 2), 7	−881280	30720	−1351680

Table 1. (*Continued*)

$(n, r, m), i$	$a((n, r, m); F_6)_i$	$a((n, r, m); F_{13})_i$	$a((n, r, m); F_9)_i$
(2, 2, 2), 8	−2643840	17280	−760320
(2, 2, 2), 9	−1615680	3840	−168960
(2, 2, 2), 10	−323136	0	0
(1, 0, 4), 0	−169290	0	0
(1, 0, 4), 1	0	−2112	192
(1, 0, 4), 2	2770200	0	0
(1, 0, 4), 3	0	18432	946176
(1, 0, 4), 4	−11592000	0	0
(1, 0, 4), 5	0	−40320	−806400
(1, 0, 4), 6	15724800	0	0
(1, 0, 4), 7	0	−279552	3600384
(1, 0, 4), 8	−5961600	0	0
(1, 0, 4), 9	0	−58368	−15409152
(1, 0, 4), 10	−2396160	0	0
(1, 1, 4), 0	−130944	0	0
(1, 1, 4), 1	−654720	480	−8640
(1, 1, 4), 2	535680	2160	−38880
(1, 1, 4), 3	6071040	7680	85440
(1, 1, 4), 4	4152960	16800	480480
(1, 1, 4), 5	−11539584	−20160	1532160
(1, 1, 4), 6	−9495360	−87360	2538480
(1, 1, 4), 7	8634240	237120	12636960
(1, 1, 4), 8	7974720	450000	16683120
(1, 1, 4), 9	526080	−98880	12584640
(1, 1, 4), 10	−1595136	−127680	3583680
(2, 0, 3), 0	−6886080	0	0
(2, 0, 3), 1	0	23424	18843648
(2, 0, 3), 2	42832800	0	0
(2, 0, 3), 3	0	−1112640	−40886016
(2, 0, 3), 4	−77112000	0	0
(2, 0, 3), 5	0	2711520	213393600
(2, 0, 3), 6	48988800	0	0
(2, 0, 3), 7	0	858240	−188828064
(2, 0, 3), 8	−49021200	0	0
(2, 0, 3), 9	0	94896	1836432
(2, 0, 3), 10	14375880	0	0
(2, 1, 3), 0	−5889024	0	0
(2, 1, 3), 1	−14722560	−43776	−12594432
(2, 1, 3), 2	17876160	−180144	−41900112
(2, 1, 3), 3	57836160	−619968	−69337824
(2, 1, 3), 4	15865920	−670656	−26205648
(2, 1, 3), 5	−10668672	−608832	39412800
(2, 1, 3), 6	5201280	454944	−81808608
(2, 1, 3), 7	−3536640	66048	−204111936
(2, 1, 3), 8	−6860160	−452304	15414048
(2, 1, 3), 9	20615040	−157152	28999872
(2, 1, 3), 10	12369024	149760	11128320

(*Continued*)

Table 1. (*Continued*)

$(n, r, m), i$	$a((n, r, m); F_6)_i$	$a((n, r, m); F_{13})_i$	$a((n, r, m); F_9)_i$
(2, 2, 3), 0	−3043008	0	0
(2, 2, 3), 1	−15215040	28800	3701760
(2, 2, 3), 2	−18865440	129600	16657920
(2, 2, 3), 3	15828480	−222528	26492160
(2, 2, 3), 4	38243520	−1383648	14985600
(2, 2, 3), 5	−4572288	−1546272	−107190720
(2, 2, 3), 6	−47416320	−104160	−266572320
(2, 2, 3), 7	−38315520	1022208	−241800480
(2, 2, 3), 8	10149840	859248	−99740880
(2, 2, 3), 9	21342960	87792	−27100080
(2, 2, 3), 10	6402888	−65520	−5613840
(2, 0, 4), 0	−26840160	0	0
(2, 0, 4), 1	0	9216	−62435328
(2, 0, 4), 2	218721600	0	0
(2, 0, 4), 3	0	7716864	478814208
(2, 0, 4), 4	−472550400	0	0
(2, 0, 4), 5	0	−18192384	−2721890304
(2, 0, 4), 6	531014400	0	0
(2, 0, 4), 7	0	19378176	878641152
(2, 0, 4), 8	168220800	0	0
(2, 0, 4), 9	0	−3612672	−671612928
(2, 0, 4), 10	−57565440	0	0
(2, 1, 4), 0	−22394880	0	0
(2, 1, 4), 1	−55987200	109056	19932672
(2, 1, 4), 2	114134400	638496	74130912
(2, 1, 4), 3	312249600	1700736	−98737344
(2, 1, 4), 4	−111081600	986496	−764603616
(2, 1, 4), 5	−550851840	−451584	465768576
(2, 1, 4), 6	−299779200	−2142336	1968341088
(2, 1, 4), 7	63187200	−10740096	2322377664
(2, 1, 4), 8	2851200	−6888096	2872643616
(2, 1, 4), 9	−77299200	2110848	662090112
(2, 1, 4), 10	−41679360	309120	310270080
(2, 2, 4), 0	−14625792	0	0
(2, 2, 4), 1	−73128960	−69632	23166976
(2, 2, 4), 2	−58659840	−313344	104251392
(2, 2, 4), 3	204134400	2785280	347504640
(2, 2, 4), 4	369546240	11210752	729759744
(2, 2, 4), 5	87026688	13590528	1242759168
(2, 2, 4), 6	−94456320	5218304	1525751808
(2, 2, 4), 7	50457600	1187840	727695360
(2, 2, 4), 8	4078080	2377728	−138829824
(2, 2, 4), 9	−62607360	2719744	−147718144
(2, 2, 4), 10	−31494144	860160	−13762560
(3, 0, 3), 0	90596880	0	0
(3, 0, 3), 1	0	−7771680	293116320

Table 1. (*Continued*)

$(n, r, m), i$	$a((n, r, m); F_6)_i$	$a((n, r, m); F_{13})_i$	$a((n, r, m); F_9)_i$
(3, 0, 3), 2	−370623600	0	0
(3, 0, 3), 3	0	31086720	−5276093760
(3, 0, 3), 4	576525600	0	0
(3, 0, 3), 5	0	0	0
(3, 0, 3), 6	576525600	0	0
(3, 0, 3), 7	0	−31086720	5276093760
(3, 0, 3), 8	−370623600	0	0
(3, 0, 3), 9	0	7771680	−293116320
(3, 0, 3), 10	90596880	0	0
(3, 1, 3), 0	76869648	982800	78246000
(3, 1, 3), 1	128116080	6088572	127495620
(3, 1, 3), 2	−215706240	6418602	−1735483050
(3, 1, 3), 3	−500027040	−10190880	−4765793760
(3, 1, 3), 4	−163462320	−12927348	−3737171340
(3, 1, 3), 5	211495536	0	0
(3, 1, 3), 6	−163462320	12927348	3737171340
(3, 1, 3), 7	−500027040	10190880	4765793760
(3, 1, 3), 8	−215706240	−6418602	1735483050
(3, 1, 3), 9	128116080	−6088572	−127495620
(3, 1, 3), 10	76869648	−982800	−78246000
(3, 2, 3), 0	52747524	−336960	−28019520
(3, 2, 3), 1	175825080	−4224384	−185186304
(3, 2, 3), 2	73083060	−7988544	−238997952
(3, 2, 3), 3	−230433120	−2283264	−681633792
(3, 2, 3), 4	−91982520	3661056	−1807327872
(3, 2, 3), 5	231544656	0	0
(3, 2, 3), 6	−91982520	−3661056	1807327872
(3, 2, 3), 7	−230433120	2283264	681633792
(3, 2, 3), 8	73083060	7988544	238997952
(3, 2, 3), 9	175825080	4224384	185186304
(3, 2, 3), 10	52747524	336960	28019520
(3, 3, 3), 0	24141672	0	0
(3, 3, 3), 1	120708360	2822958	−60464718
(3, 3, 3), 2	209213280	12703311	−272091231
(3, 3, 3), 3	112602960	22583664	−483717744
(3, 3, 3), 4	−136919160	19760706	−423253026
(3, 3, 3), 5	−297892728	0	0
(3, 3, 3), 6	−136919160	−19760706	423253026
(3, 3, 3), 7	112602960	−22583664	483717744
(3, 3, 3), 8	209213280	−12703311	272091231
(3, 3, 3), 9	120708360	−2822958	60464718
(3, 3, 3), 10	24141672	0	0
(3, 0, 4), 0	326856600	0	0
(3, 0, 4), 1	0	2776320	2263168512
(3, 0, 4), 2	−1641477600	0	0
(3, 0, 4), 3	0	−62830080	24539277312

(*Continued*)

Table 1. (*Continued*)

$(n,r,m),i$	$a((n,r,m);F_6)_i$	$a((n,r,m);F_{13})_i$	$a((n,r,m);F_9)_i$
(3, 0, 4), 4	2166662400	0	0
(3, 0, 4), 5	0	257897472	−67879440384
(3, 0, 4), 6	−5298585600	0	0
(3, 0, 4), 7	0	−56975360	−17592942592
(3, 0, 4), 8	1035417600	0	0
(3, 0, 4), 9	0	25223168	−19634782208
(3, 0, 4), 10	−337469440	0	0
(3, 1, 4), 0	307618560	−3144960	147847680
(3, 1, 4), 1	512697600	2991168	−3146836608
(3, 1, 4), 2	−1336953600	15352416	8555416128
(3, 1, 4), 3	−1717977600	2852352	30943036800
(3, 1, 4), 4	1515628800	−8116416	28559248704
(3, 1, 4), 5	1488533760	−170868096	31554907776
(3, 1, 4), 6	1663804800	−222638976	53683211232
(3, 1, 4), 7	538617600	−21808512	11210955840
(3, 1, 4), 8	667958400	24104736	28350336096
(3, 1, 4), 9	−247795200	−10392192	10016214912
(3, 1, 4), 10	−273231360	604800	−3811420800
(3, 2, 4), 0	222409800	524160	−157783680
(3, 2, 4), 1	741366000	−2020512	2059469856
(3, 2, 4), 2	27518400	−22212576	4078981728
(3, 2, 4), 3	−1971331200	−29117568	1767635712
(3, 2, 4), 4	−1429344000	11738496	22163764224
(3, 2, 4), 5	1016064000	79466688	41201572608
(3, 2, 4), 6	1429747200	105032256	−6501266688
(3, 2, 4), 7	1036108800	46791168	−31413522432
(3, 2, 4), 8	−68040000	11937024	−18115881984
(3, 2, 4), 9	−600624000	−24086016	1296961536
(3, 2, 4), 10	−228153600	−8225280	198881280
(3, 3, 4), 0	132554880	0	0
(3, 3, 4), 1	662774400	−3103776	−221061312
(3, 3, 4), 2	873763200	−13966992	−994775904
(3, 3, 4), 3	−481593600	−28571904	−6932343744
(3, 3, 4), 4	−1840003200	−34822368	−19620915552
(3, 3, 4), 5	−1050779520	4584384	−36233378496
(3, 3, 4), 6	247867200	65927232	−43852301136
(3, 3, 4), 7	359625600	78051264	−37908340896
(3, 3, 4), 8	−382708800	43049808	−22157484048
(3, 3, 4), 9	−458553600	17788992	−3375905472
(3, 3, 4), 10	−117576960	4280640	977223360
(4, 0, 4), 0	−1297280640	0	0
(4, 0, 4), 1	0	39649280	102482575360
(4, 0, 4), 2	4768675200	0	0
(4, 0, 4), 3	0	−64225280	−515930849280
(4, 0, 4), 4	1794374400	0	0
(4, 0, 4), 5	0	0	0
(4, 0, 4), 6	1794374400	0	0
(4, 0, 4), 7	0	64225280	515930849280

Table 1. (*Continued*)

$(n,r,m),i$	$a((n,r,m);F_6)_i$	$a((n,r,m);F_{13})_i$	$a((n,r,m);F_9)_i$
(4, 0, 4), 8	4768675200	0	0
(4, 0, 4), 9	0	−39649280	−102482575360
(4, 0, 4), 10	−1297280640	0	0
(4, 1, 4), 0	−1017027840	−12841920	−18394044480
(4, 1, 4), 1	−1408876800	−25269696	−66242925504
(4, 1, 4), 2	3807691200	29798928	−84305711952
(4, 1, 4), 3	4438684800	−337919040	102921684768
(4, 1, 4), 4	−5083444800	−796243392	559765292688
(4, 1, 4), 5	3433691520	0	0
(4, 1, 4), 6	−5083444800	796243392	−559765292688
(4, 1, 4), 7	4438684800	337919040	−102921684768
(4, 1, 4), 8	3807691200	−29798928	84305711952
(4, 1, 4), 9	−1408876800	25269696	66242925504
(4, 1, 4), 10	−1017027840	12841920	18394044480
(4, 2, 4), 0	−976318464	−16343040	7339376640
(4, 2, 4), 1	−2521436160	−49397760	16163143680
(4, 2, 4), 2	817067520	193628160	150300794880
(4, 2, 4), 3	5306618880	354017280	378355630080
(4, 2, 4), 4	176924160	26664960	376012922880
(4, 2, 4), 5	−11901883392	0	0
(4, 2, 4), 6	176924160	−26664960	−376012922880
(4, 2, 4), 7	5306618880	−354017280	−378355630080
(4, 2, 4), 8	817067520	−193628160	−150300794880
(4, 2, 4), 9	−2521436160	49397760	−16163143680
(4, 2, 4), 10	−976318464	16343040	−7339376640
(4, 3, 4), 0	−549910272	6619200	−371112000
(4, 3, 4), 1	−1953803520	19256000	1196382400
(4, 3, 4), 2	−1626229440	125698320	−124038853200
(4, 3, 4), 3	1428370560	305362240	−428832439200
(4, 3, 4), 4	1673474880	345011520	−443326489200
(4, 3, 4), 5	872919936	0	0
(4, 3, 4), 6	1673474880	−345011520	443326489200
(4, 3, 4), 7	1428370560	−305362240	428832439200
(4, 3, 4), 8	−1626229440	−125698320	124038853200
(4, 3, 4), 9	−1953803520	−19256000	−1196382400
(4, 3, 4), 10	−549910272	−6619200	371112000
(4, 4, 4), 0	−356713984	0	0
(4, 4, 4), 1	−1783569920	5865472	−15539372032
(4, 4, 4), 2	−2918568960	26394624	−69927174144
(4, 4, 4), 3	−972856320	46923776	−124314976256
(4, 4, 4), 4	3404997120	41058304	−108775604224
(4, 4, 4), 5	6128994816	0	0
(4, 4, 4), 6	3404997120	−41058304	108775604224
(4, 4, 4), 7	−972856320	−46923776	124314976256
(4, 4, 4), 8	−2918568960	−26394624	69927174144
(4, 4, 4), 9	−1783569920	−5865472	15539372032
(4, 4, 4), 10	−356713984	0	0

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References

[1] H. Aoki and T. Ibukiyama, Simple graded rings of Siegel modular forms, differential operators and Borcherds products, *Internat. J. Math.* **16**(3) (2005) 249–279.
[2] T. Arakawa, Vector valued Siegel's modular forms of degree two and the associated Andrianov L-functions, *Manuscripta Math.* **44**(1–3) (1983) 155–185.
[3] W. Eholzer and T. Ibukiyama, Rankin–Cohen type differential operators for Siegel modular forms, *Internat. J. Math.* **9**(4) (1998) 443–463.
[4] E. Freitag, Ein verschwindungssatz für automorphe formen zur siegelschen modulgruppe, *Math. Z.* **165**(1) (1979) 11–18.
[5] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants* (Birkhäuser, Boston, 1994).
[6] V. A. Gritsenko and V. V. Nikulin, Igusa modular forms and 'the simplest' Lorentzian Kac–Moody algebras, *Sb. Math.* **187**(11) (1996) 1601.
[7] T. Ibukiyama, Vector valued Siegel modular forms of symmetric tensor weight of small degrees, *Comment. Math. Univ. St. Pauli* **61** (2012) 51–75.
[8] T. Ibukiyama and H. Katsurada, Exact critical values of the symmetric fourth L function and vector valued Siegel modular forms, *J. Math. Soc. Japan* **66**(1) (2014) 139–160.
[9] J. Igusa, On Siegel modular forms of genus two, *Amer. J. Math.* **84** (1962) 175–200.
[10] J. Igusa, On Siegel modular forms of genus two (II), *Amer. J. Math.* **86** (1964) 392–412.
[11] H. Katsurada and S. Takemori, Congruence primes of the Kim–Ramakrishnan–Shahidi lift, *Exp. Math.* (2015), to appear.
[12] T. Kikuta and S. Takemori, Sturm bounds for Siegel modular forms of degree 2 and odd weights, Preprint, arXiv:1508.01610 (2015).
[13] T. Kiyuna, Vector-valued Siegel modular forms of weight $\det \otimes \mathrm{Sym}(8)$, *Internat. J. Math.* **26**(1) (2015) 1550004.
[14] H. Maaß, Die Multiplikatorsysteme zur Siegelschen Modulgruppe, *Nachr. Akad. Wiss. Göttingen, II, Math.-Phys. Klasse* **11** (1964) 125–135.
[15] H. Maaß, Über ein Analogon zur Vermutung von Saito–Kurokawa, *Invent. Math.* **60**(1) (1980) 85–104.
[16] D. Ramakrishnan and F. Shahidi, Siegel modular forms of genus 2 attached to elliptic curves, Technical Report 2, Mathematical Research Note 14 (2007).
[17] T. Satoh, On certain vector valued Siegel modular forms of degree two, *Math. Ann.* **274**(2) (1986) 335–352.
[18] W. A. Stein *et al.*, Sage Mathematics Software (Version 6.5), The Sage Development Team, 2015, http://www.sagemath.org.
[19] S. Takemori, A SageMath package for Siegel modular forms of degree two, https://github.com/stakemori/degree2.
[20] S. Takemori, On the computation of the determinant of vector-valued Siegel modular forms, *LMS J. Comput. Math.* **17**(A) (2014) 247–256.

[21] R. Tsushima, An explicit dimension formula for the spaces of generalized automorphic forms with respect to $\mathrm{Sp}(2, \mathbf{Z})$, *Proc. Japan Acad. Ser. A Math. Sci.* **59**(4) (1983) 139–142.

[22] C. H. van Dorp, Vector-valued Siegel modular forms of genus 2, Master's thesis, Universiteit van Amsterdam (2011).