International Journal of Mathematics
Vol. 26, No. 5 (2015) 1550034 (51 pages)

DOI: 10.1142/S0129167X15500342



# Siegel modular forms of genus 2 and level 2

Fabien Cléry
*Department Mathematik, Universität Siegen*
*Emmy-Noether-Campus, Walter-Flex-Strasse 3*
*57068 Siegen, Germany*
*cleryfabien@gmail.com*

Gerard van der Geer
*Korteweg-de Vries Instituut*
*Universiteit van Amsterdam, Postbus 94248*
*1090 GE Amsterdam, The Netherlands*
*G.B.M.vanderGeer@uva.nl*

Samuel Grushevsky
*Mathematics Department, Stony Brook University*
*Stony Brook, NY 11790-3651, USA*
*sam@math.sunysb.edu*

Received 7 January 2014
Accepted 16 February 2015
Published 18 March 2015

(with an appendix by Shigeru Mukai)

We study vector-valued Siegel modular forms of genus 2 on the three level 2 groups $\Gamma[2] \lhd \Gamma_1[2] \lhd \Gamma_0[2] \subset \mathrm{Sp}(4,\mathbb{Z})$. We give generating functions for the dimension of spaces of vector-valued modular forms, construct various vector-valued modular forms by using theta functions and describe the structure of certain modules of vector-valued modular forms over rings of scalar-valued Siegel modular forms.

*Keywords*: Siegel modular form; vector-valued modular form; modular forms of level 2; theta series.

Mathematics Subject Classification 2010: 14J15

## 1. Introduction

Vector-valued Siegel modular forms are the natural generalization of elliptic modular forms and in recent years there has been an increasing interest in these modular forms. One of the attractive aspects of the theory of elliptic modular forms is the presence of easily accessible examples. By contrast easily accessible examples in the theory of vector-valued Siegel modular forms have been very few. Vector-valued Siegel modular forms of genus 2 and level 1 have been considered by Satoh, Ibukiyama and others, cf. [2, 29, 17, 18, 32].

The study of local systems and point counting of curves over finite fields has made it possible to calculate Hecke eigenvalues for eigenforms of the Hecke algebra, first for vector-valued forms of genus 2 and level 1, later under some assumptions also for genus 2 and level 2 and even for genus 3 and level 1, see [8, 3–5, 7]. These methods do not require nor provide an explicit description of these modular forms. Describing explicitly these modular forms and the generators for the modules of such modular forms is thus a natural question.

The focus of this paper is genus 2 and level 2: more precisely, we will study vector-valued modular forms on the full congruence subgroup $\Gamma[2]$ of $\mathrm{Sp}(2,\mathbb{Z})$ of level 2 together with the action of $\mathfrak{S}_6 \cong \mathrm{Sp}(2,\mathbb{Z}/2\mathbb{Z})$ on these. This will lead to a wealth of results on modular forms on the congruence subgroups $\Gamma_0[2]$ and $\Gamma_1[2]$ too. We will construct many such modular forms by taking Rankin–Cohen brackets of polynomials in theta constants with even characteristics, and by using gradients of theta functions with odd characteristics. We will furthermore describe some modules of vector-valued modular forms. One major tool is studying the representations of $\mathfrak{S}_6$, the Galois group of the level 2 cover of the moduli space of principally polarized abelian surfaces, on the spaces of modular forms. The methods of [3] allow one to compute these actions assuming the conjectures made in [3] — and these give a heuristic tool to detect where one has to search for modular forms or relations among them. We apply these to get bounds on the weights of generators and relations of the modules of vector-valued forms — but note that our final results on the module structure are not conditional on the conjectures of [3].

More precisely, our results are as follows. In Theorems 9.1 and 9.2 we compute the rings of scalar-valued modular forms on $\Gamma_1[2]$ and $\Gamma_0[2]$. This computation uses Igusa's determination of the ring of scalar-valued modular forms on $\Gamma[2]$, and the result for $\Gamma_0[2]$ was already known by Ibukiyama [1]. By analyzing the action of $\mathfrak{S}_6$ on the spaces of vector-valued modular forms on $\Gamma[2]$, in Theorem 14.1 we give the generating functions for the dimensions of the spaces $M_{j,k}(\Gamma_1[2])$ of modular forms on $\Gamma_1[2]$. These results are based on Wakatsuki's [33] computation of the generating functions for $M_{j,k}(\Gamma[2])$, and their derivation uses the conjectures made in [3] — but the result fits all available data, e.g. Tsushima's calculations (cf. references in [3]). In Secs. 15–18 we construct vector-valued modular forms in two ways: using a variant of the Rankin–Cohen bracket applied to even theta constants and by using gradients of odd theta functions multiplied by suitable even theta constants in order to get modular forms of the desired level. Using these results and suitable Castelnuovo–Mumford regularity established in Sec. 19, in Theorem 20.1 we determine the generators for the module $\Sigma_2 = \bigoplus_{k,\,\mathrm{odd}} S_{2,k}(\Gamma[2])$ of cusp forms of "weight" $\mathrm{Sym}^2 \otimes \det^k$. In Theorems 21.1 and 23.1 we determine the generators for the modules $\mathcal{M}_j^\epsilon = \bigoplus_{k,k\equiv\epsilon \mod 2} M_{j,k}(\Gamma[2])$ for $\epsilon = 0, 1$ and $j = 2, 4$. In some cases we also determine the submodule of relations.

We conclude the paper by constructing an explicit generator for many cases where the space of cusp forms $S_{j,k}(\mathrm{Sp}(4,\mathbb{Z}))$ is 1-dimensional and by giving the Fourier coefficients of the module generators for $\Sigma_2^1 = \bigoplus_{k,\,\mathrm{odd}} S_{2,k}(\Gamma[2])$ and of certain generators of a module of modular forms of weight $(4,*)$.

The fact that we have two different ways of constructing vector-valued modular forms naturally leads to many identities between modular forms, some of them quite pretty. We have restricted ourselves to just giving a few samples, inviting the reader to find many more.

**Remark 1.1.** One intriguing feature of the situation is as follows. Mukai [25] recently showed that the Satake compactification of the moduli space of principally polarized abelian surfaces with a $\Gamma_1[2]$-level structure is given by the Igusa quartic — which by the results of Igusa is the Satake compactification of the moduli space of principally polarized abelian surfaces with a full level 2 structure. We will see how this remarkable fact is reflected in the structure of rings and modules of scalar-valued and vector-valued modular forms on $\Gamma[2]$ and $\Gamma_1[2]$. In an Appendix to this paper Mukai makes a very minor correction to a statement about the Fricke involution in [25] to guarantee the peaceful coexistence of his paper with the present one.

**Remark 1.2.** Another interesting feature is that the modules of vector-valued modular forms that we consider are not of finite presentation over the ring of scalar-valued modular forms. Indeed, recall that the ring of even weight scalar-valued modular forms on $\Gamma[2]$ is a quotient of a polynomial ring in five variables by a principal ideal — and the modules of vector-valued modular forms like $\bigoplus_k M_{j,k}(\Gamma[2])$ are of finite presentation only over this polynomial ring.

## 2. Preliminaries

Let $\Gamma = \mathrm{Sp}(4,\mathbb{Z})$ be the Siegel modular group. The following level 2 congruence subgroups $\Gamma[2] \lhd \Gamma_1[2] \lhd \Gamma_0[2] \subset \Gamma$ defined by

$$\Gamma[2] = \{M \in \Gamma : M \equiv 1_4 \bmod 2\},$$

$$\Gamma_1[2] = \left\{ M \in \Gamma : M \equiv \begin{pmatrix} 1_2 & * \\ 0 & 1_2 \end{pmatrix} \bmod 2 \right\}$$

and

$$\Gamma_0[2] = \left\{ M \in \Gamma : M \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 2 \right\}$$

will play a central role here.

The successive quotients can be identified as follows

$$\Gamma_1[2]/\Gamma[2] \simeq (\mathbb{Z}/2\mathbb{Z})^3, \quad \Gamma_0[2]/\Gamma[2] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4,$$

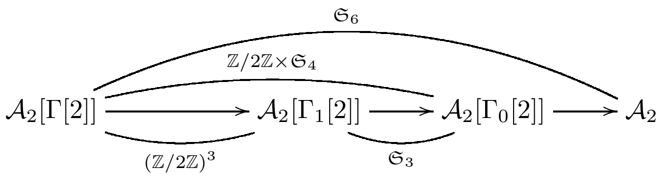
$$\Gamma_0[2]/\Gamma_1[2] \simeq \mathfrak{S}_3, \quad \Gamma/\Gamma[2] \simeq \mathfrak{S}_6,$$

with $\mathfrak{S}_n$ the symmetric group on $n$ letters; see Sec. 3 for an explicit identification.

These groups act on the Siegel upper half space

$$\mathfrak{H}_2 = \left\{ \tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbb{C}) : \tau^t = \tau, \mathrm{Im}(\tau) > 0 \right\}$$

in the usual way ($\tau \mapsto M\langle\tau\rangle = (a\tau + b)(c\tau + d)^{-1}$) and the quotient orbifolds of the action of $\Gamma, \Gamma_0[2], \Gamma_1[2]$ and $\Gamma[2]$ will be denoted by $\mathcal{A}_2$, $\mathcal{A}_2[\Gamma_0[2]]$, $\mathcal{A}_2[\Gamma_1[2]]$ and $\mathcal{A}_2[\Gamma[2]]$. We have a diagram of coverings




Recall that we have a so-called Fricke involution induced by the element

$$\begin{pmatrix} 0 & 1_2/\sqrt{2} \\ -\sqrt{2}1_2 & 0 \end{pmatrix} \tag{2.1}$$

of $\mathrm{Sp}(4, \mathbb{R})$ that normalizes $\Gamma_1[2]$ and $\Gamma_0[2]$ and thus induces an involution $W_2$ on $\mathcal{A}_2[\Gamma_1[2]]$ and $\mathcal{A}_2[\Gamma_0[2]]$.

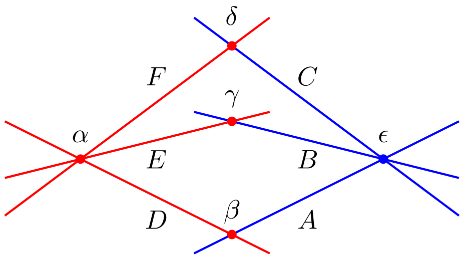
These quotients admit a Satake (or Baily–Borel) compactification obtained by adding 1-dimensional and 0-dimensional boundary components.

The Satake compactification $\mathcal{A}_2[\Gamma[2]]^*$ of $\mathcal{A}_2[\Gamma[2]]$ is obtained by adding fifteen 1-dimensional boundary components each isomorphic to $\mathcal{A}_1[2] = \Gamma(2)\backslash\mathfrak{H}_1$, where $\Gamma(2)$ denotes the principal congruence subgroup[a] of level 2 of $\mathrm{SL}(2, \mathbb{Z})$ and 15 points forming a $(15_3, 15_3)$-configuration. The group $\mathfrak{S}_6 = \Gamma/\Gamma[2]$ acts on it. One can assign to each 1-dimensional boundary component a pair $\{i, j\} \subset \{1, 2, \ldots, 6\}$ with $i \neq j$ such that any $\sigma \in \mathfrak{S}_6$ sends the component $B_{ij}$ corresponding to $\{i, j\}$ to $B_{\sigma(i)\sigma(j)}$; similarly one can assign to each 0-dimensional cusp a partition $(ij)(kl)(mn)$ of $\{i, j, k, l, m, n\} = \{1, 2, \ldots, 6\}$ into three pairs on which $\mathfrak{S}_6$ acts in the natural way such that the cusp given by $(ij)(kl)(mn)$ is a cusp of the boundary components $B_{ij}, B_{kl}$ and $B_{mn}$, cf. Lemma 3.1 and Remark 16.2. Note that $\Gamma_0[2]$ is the inverse image of a Siegel parabolic group (fixing a 0-dimensional boundary component) under the reduction mod2 map $\mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$ and $\Gamma_1[2]$ is the subgroup fixing each of the three 1-dimensional boundary components passing through this 0-dimensional cusp.

The Satake compactification of $\mathcal{A}_2[\Gamma_1[2]]$ is obtained by adding six 1-dimensional boundary components (each isomorphic to $\Gamma_0(2)\backslash\mathfrak{H}_1$ and denoted $A, \ldots, F$) and five 0-dimensional boundary components (denoted $\alpha, \ldots, \epsilon$) as in the

[a]We denote the congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$ by round brackets, those of $\mathrm{Sp}(4, \mathbb{Z})$ by square brackets.

following configuration:




The normal subgroup $\Gamma_1[2]/\Gamma[2]$ of $\Gamma_0[2]/\Gamma[2]$ acts trivially on this configuration and the induced action of the quotient $\mathfrak{S}_3$ permutes the 1-dimensional boundary cusps $A, B, C$ and $D, E, F$ and permutes the three 0-dimensional cusps $\beta, \gamma, \delta$ and fixes $\alpha$ and $\epsilon$. The Fricke involution $W_2$ interchanges $\alpha$ and $\epsilon$, fixes $\gamma$ and interchanges $\beta$ and $\delta$ as we shall see later (Corollary 10.1).

The Satake compactification of $\mathcal{A}_2[\Gamma_0[2]]$ is obtained by adding to $\mathcal{A}_2[\Gamma_0[2]]$ two 1-dimensional boundary components (the images of $D$ and $A$) each isomorphic to $\Gamma_0(2)\backslash\mathfrak{H}_1$ and three 0-dimensional cusps (the images of $\alpha, \beta$ and $\epsilon$).

We let $V$ be the standard 2-dimensional representation space of $\mathrm{GL}(2, \mathbb{C})$ and let $\rho_{j,k} : \mathrm{GL}(2, \mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}^j(V) \otimes \det(V)^{\otimes k})$ be the irreducible representation of highest weight $(j + k, k)$. By a Siegel modular form of weight $(j, k)$ on $\Gamma$ (respectively, $\Gamma_0[2], \Gamma_1[2], \Gamma[2]$) we mean a holomorphic map $f : \mathfrak{H}_2 \to \mathrm{Sym}^j(V) \otimes \det(V)^{\otimes k}$ such that

$$f(M\langle\tau\rangle) = \rho_{j,k}(c\tau + d)f(\tau) \quad \text{for all}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad (\text{respectively, } \Gamma_0[2], \Gamma_1[2], \Gamma[2]).$$

We refer to [6] and the references given there for background on Siegel modular forms. Let $\mathbb{E}$ be the Hodge bundle on $\mathcal{A}_2$ (or its pull back to $\mathcal{A}_2[\Gamma']$ for $\Gamma'$ a finite index subgroup of $\Gamma$). It corresponds to the standard representation of $\mathrm{GL}(2, \mathbb{C})$. The bundle $\mathbb{E}$ extends to “good” toroidal compactifications of $\mathcal{A}_2[\Gamma']$. Then scalar-valued modular forms of weight $k$ on $\Gamma'$ can be interpreted as sections of $L^{\otimes k}$ with $L = \det(\mathbb{E})$ on $\mathcal{A}_2[\Gamma']$. By the well-known Koecher principle such sections extend automatically to these toroidal compactifications. Similarly, if $\mathbb{E}_\rho = \mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^{\otimes k}$ is the vector bundle on $\mathcal{A}_2[\Gamma']$ corresponding to the irreducible representation $\rho = \rho_{j,k}$ then modular forms of weight $(j, k)$ on $\Gamma'$ are the sections of this vector bundle and by the Koecher principle these extend to sections over “good” toroidal compactifications. Again, we refer to [6] and the references given there for more details.

We close this section by explaining our notation for the irreducible representations of $\mathfrak{S}_6$. The irreducible representations of $\mathfrak{S}_6$ correspond bijectively to the partitions of 6. The representation corresponding to the partition $P$ will be denoted by $s[P]$, with $s[6]$ the trivial one and $s[1, 1, 1, 1, 1, 1] = s[1^6]$ the alternating representation. Their dimensions are recalled for convenience:

| $P$ | $[6]$ | $[5,1]$ | $[4,2]$ | $[4,1^2]$ | $[3^2]$ | $[3,2,1]$ | $[3,1^3]$ | $[2^3]$ | $[2^2,1^2]$ | $[2,1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| dim | 1 | 5 | 9 | 10 | 5 | 16 | 10 | 5 | 9 | 5 | 1 |

## 3. Theta Characteristics

In this paper a theta characteristic is an element of $\{0,1\}^4$ written as a row vector $(\mu_1,\mu_2,\nu_1,\nu_2)$ or as a $2\times 2$ matrix $\left[\begin{smallmatrix}\mu_1 & \mu_2\\ \nu_1 & \nu_2\end{smallmatrix}\right]$. It is called even or odd depending on the parity of $\mu_1\nu_1+\mu_2\nu_2$.

We order the six odd theta characteristics $m_1,\ldots,m_6$ lexicographically:

$$m_1=\begin{bmatrix}0&1\\0&1\end{bmatrix},\quad m_2=\begin{bmatrix}0&1\\1&1\end{bmatrix},\quad m_3=\begin{bmatrix}1&0\\1&0\end{bmatrix},$$
$$m_4=\begin{bmatrix}1&0\\1&1\end{bmatrix},\quad m_5=\begin{bmatrix}1&1\\0&1\end{bmatrix},\quad m_6=\begin{bmatrix}1&1\\1&0\end{bmatrix}.$$

Note that the sum $\sum_{i=1}^6 m_i$ is zero mod 2 and each of the 10 even theta characteristics is a sum of three different odd theta characteristics in two ways; e.g.,

$$n_1=\begin{bmatrix}0&0\\0&0\end{bmatrix}=m_1+m_4+m_6=m_2+m_3+m_5.$$

In this way each even theta characteristic is associated to a partition of $\{1,2,3,4,5,6\}$ in two triples. We use the following (lexicographic) ordering for the 10 even theta characteristics

$$n_1=\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad n_2=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad n_3=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad n_4=\begin{bmatrix}0&0\\1&1\end{bmatrix},\quad n_5=\begin{bmatrix}0&1\\0&0\end{bmatrix},$$
$$n_6=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad n_7=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad n_8=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad n_9=\begin{bmatrix}1&1\\0&0\end{bmatrix},\quad n_{10}=\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

For the ease of the reader we give the correspondence between the even $n_i$ and triples of odd ones:

$$\begin{array}{ll} n_1(146)(235) & n_6(156)(234)\\ n_2(136)(245) & n_7(123)(456)\\ n_3(135)(246) & n_8(124)(356)\\ n_4(145)(236) & n_9(126)(345)\\ n_5(134)(256) & n_{10}(125)(346)\end{array}$$

**Lemma 3.1.** (i) *An unordered pair $\{m_i,m_j\}$ of different odd theta characteristics determines uniquely an unordered quadruple of even theta characteristics, namely the $n_k$ corresponding to the four ways of writing $n_k=m_i+m_j+a=b+c+d$ with $\{m_1,\ldots,m_6\}=\{m_i,m_j,a,b,c,d\}$.* (ii) *A partition of the set of odd theta characteristics $\{m_{i_1},m_{i_2}\}\sqcup\{m_{i_3},m_{i_4}\}\sqcup\{m_{i_5},m_{i_6}\}$ in three pairs determines uniquely a quadruple of even theta characteristics such that $n=a+b+c$ with $a\in\{m_{i_1},m_{i_2}\}$, $b\in\{m_{i_3},m_{i_3}\}$ and $c\in\{m_{i_5},m_{i_6}\}$.*

For example $\{m_1, m_2\}$ corresponds to $\{n_7, n_8, n_9, n_{10}\}$ and $\{m_1, m_2\} \sqcup \{m_3, m_4\} \sqcup \{m_5, m_6\}$ corresponds to $\{n_1, n_2, n_3, n_4\}$.

An element $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of $\Gamma$ acts on $\mathbb{Z}^4$ by

$$M \cdot \begin{bmatrix} \mu_1 \\ \mu_2 \\ \nu_1 \\ \nu_2 \end{bmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \nu_1 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} (CD^t)_0 \\ (AB^t)_0 \end{pmatrix}, \tag{3.1}$$

where for a matrix $X$ the symbol $X_0$ denotes the diagonal vector (in its natural order). The quotient group $\Gamma/\Gamma[2] \cong \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$ is identified with the symmetric group $\mathfrak{S}_6$ via its action on the six odd theta characteristics. Recall that the group $\mathfrak{S}_6$ is generated by the two elements (12) and (123456) represented by elements of $\Gamma$

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

The partition of the six odd theta characteristics into three pairs defines a conjugacy class of $\Gamma_0[2]$: let $C \cong \mathfrak{S}_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$ be the subgroup of $\mathfrak{S}_6$ that stabilizes the partition $\{m_1, m_2\} \sqcup \{m_3, m_4\} \sqcup \{m_5, m_6\}$. Then the inverse image of $C$ under the quotient map $\Gamma \to \Gamma/\Gamma[2]$ equals $\Gamma_0[2]$.

Since by Lemma 3.1 this partition of the six odd theta characteristics in three disjoint pairs defines a quadruple of even ones, the group $\Gamma_0[2]/\Gamma[2] \cong C$ acts on this set $\{n_1, n_2, n_3, n_4\}$ and this defines a surjective map $C \to \mathfrak{S}_4$ with kernel generated by (12)(34)(56) that gives an isomorphism $C \cong \mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$.

Representatives of the generators of $\Gamma_1[2]/\Gamma[2] = (\mathbb{Z}/2\mathbb{Z})^3$ are given by the transformations $\tau_{11} \mapsto \tau_{11} + 1$, $\tau_{22} \mapsto \tau_{22} + 1$ and $\tau \mapsto \tau + 1_2$ corresponding to $(12), (34)$ and $(56)$. Generators of $\mathfrak{S}_3 = \Gamma_0[2]/\Gamma_1[2]$ are given by

$$X' = \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix}, \quad Y' = \begin{pmatrix} B & 0 \\ 0 & B^{-t} \end{pmatrix} \tag{3.3}$$

with $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

## 4. Theta Series

For $(\tau, z) \in \mathfrak{H}_2 \times \mathbb{C}^2$ and $\left[\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{smallmatrix}\right]$ with $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ in $\mathbb{Z}^2$ we consider the standard theta series with characteristics

$$\vartheta_{\left[\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right]}(\tau, z) = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} e^{\pi i((n+\mu/2)(\tau(n+\mu/2)^t + 2(z+\nu/2)^t))}.$$

Usually the $\mu_i, \nu_i$ will be equal to 0 or 1; in fact we will be mainly interested in the theta constants and the formula

$$\vartheta_{\left[\begin{smallmatrix} \mu+2m \\ \nu+2n \end{smallmatrix}\right]}(\tau, 0) = (-1)^{\mu \cdot n^t} \vartheta_{\left[\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right]}(\tau, 0)$$

allows us to reduce the characteristic modulo 2. The transformation behavior of the theta series under $\Gamma$ is known, cf. [21].

**Lemma 4.1.** *For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, we have the transformation behavior*

$$\begin{aligned} &\vartheta_{M\cdot\left[\mu \atop \nu\right]}(M\langle\tau\rangle, (C\tau+D)^{-t}z) \\ &\quad = \kappa(M) e^{2\pi i\phi\left(\left[\mu \atop \nu\right], M\right)} \cdot \det(C\tau+D)^{\frac{1}{2}} e^{\pi i z(C\tau+D)^{-1}Cz^t} \vartheta_{\left[\mu \atop \nu\right]}(\tau, z), \end{aligned}$$

*where $\phi(\left[\mu \atop \nu\right], M)$ is given by*

$$(2\mu B^t C\nu^t + 2(AB^t)_0(D\mu^t - C\nu^t) - \mu B^t D\mu^t - \nu A^t C\nu^t)/8,$$

*and with the action on the characteristics given by* (3.1). *Moreover, $\kappa(M)$ is an eighth root of unity* (*depending only on $M$ and not on $\mu, \nu$*).

For the theta constants $\vartheta(\tau) = \vartheta(\tau, 0)$ the transformation under $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ reduces to

$$\vartheta_{M\cdot\left[\mu \atop \nu\right]}(M\langle\tau\rangle) = \kappa(M) e^{2\pi i\phi\left(\left[\mu \atop \nu\right], M\right)} \det(C\tau+D)^{\frac{1}{2}} \vartheta_{\left[\mu \atop \nu\right]}(\tau).$$

It is convenient to introduce the *slash operator*. For $M \in \Gamma$, $k$ half-integral and a function $F$ on $\mathfrak{H}_2$ we put

$$(F|_{0,k}M)(\tau) = \det(C\tau+D)^{-k} F(M\langle\tau\rangle).$$

(Here $\sqrt{\det(C\tau+D)}$ is chosen to have positive imaginary part.) Invariance of $F$ under the slash operator expresses the fact that a function transforms like a scalar-valued modular form of weight $k$.

The action of the matrices $M = X$ and $M = Y$ (defined in 3.2) on the (column) vector of the 10 even theta constants by the slash operator $\vartheta_{n_i} \mapsto \vartheta_{n_i}|_{0,\frac{1}{2}}M$ is given by the unitary matrices

$$\rho(X) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta \end{pmatrix} \quad \text{and}$$

$$\rho(Y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^7 \\ 0 & 0 & 0 & \zeta^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta^7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \zeta^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta^7 & 0 & 0 & 0 & 0 \\ \zeta^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.1}$$

with $\zeta = e^{\pi i/4}$.

By formula (3.1) for the action of $M$ on the set of characteristics it follows that $M$ acts trivially on the set $\{0,1\}^4$ of characteristics if and only if $M \in \Gamma[2]$. Recall the (Igusa) theta groups

$$\Gamma[n, 2n] := \{M \in \Gamma : M \equiv 1_4 \bmod n, (AB^t)_0 \equiv (CD^t)_0 \equiv 0 \bmod 2n\}.$$

It turns out [21] that theta constants are scalar-valued modular forms (with a multiplier) only on the subgroup $\Gamma[4, 8]$. The transformation formula for theta constants implies that the squares of theta constants are scalar-valued modular forms of weight 1 on $\Gamma[2, 4]$, while the fourth powers of theta constants are modular forms of weight 2 on $\Gamma[2]$. In fact, it is known, see [21], that the ring of scalar-valued modular forms of integral weight on $\Gamma[2, 4]$ is generated by squares of theta constants, while the ring of scalar-valued modular forms of even weight on $\Gamma[2]$ is generated by the fourth powers of theta constants. The squares of the theta constants and fourth powers of theta constants satisfy many polynomial relations, which we will describe explicitly below for genus 2. All these polynomials identities follow from Riemann's bilinear relation, which we now recall.

We define the theta functions of the second-order to be

$$\Theta[\mu](\tau, z) = \vartheta_{\left[\begin{smallmatrix}\mu\\0\end{smallmatrix}\right]}(2\tau, 2z),$$

and call their evaluations at $z = 0$ theta constants of the second-order. These are modular forms of weight $1/2$ on $\Gamma[2, 4]$, and generate the ring of scalar-valued modular forms of half-integral weight on $\Gamma[2, 4]$. In particular, the squares of theta constants (with characteristics) are expressible in terms of theta constants of the second-order by using Riemann's bilinear relation

$$\vartheta^2_{\left[\begin{smallmatrix}\mu\\\nu\end{smallmatrix}\right]}(\tau, z) = \sum_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^2} (-1)^{\sigma \cdot \nu} \Theta[\sigma](\tau)\Theta[\sigma + \mu](\tau, z), \tag{4.2}$$

evaluated at $z = 0$. Moreover, it is known that (in genus 2) theta constants of the second-order are algebraically independent, and determine a birational morphism of the Satake compactification of $\Gamma[2,4]\backslash\mathfrak{H}_2$ onto $\mathbb{P}^3$. Thus the squares of theta constants of the second-order are simply the coordinates on $\mathbb{P}^9$ restricted to the Veronese image of $\mathbb{P}^3 \to \mathbb{P}^9$ given by the Riemann bilinear relations, and as such satisfy polynomial relations given by Igusa [20, pp. 393, 396], which will be described explicitly in the next section, where we also explicitly write down the action of $\Gamma[2]/\Gamma[2,4] = (\mathbb{Z}/2\mathbb{Z})^4$ on the squares of theta constants.

## 5. The Squares of the Theta Constants

To construct modular forms we shall use the squares and the fourth powers of the 10 even theta constants. Therefore we summarize the behavior of the squares of the theta constants under $\Gamma[2]$, cf. [21, 27]. From the transformation formula of Lemma 4.1 we obtain

$$\begin{aligned}(\vartheta_{n_j}^2|_{0,1}M)(\tau) &= \det(C\tau+D)^{-1}\vartheta_{n_j}^2(M\langle\tau\rangle)\\ &= (-1)^{\mathrm{Tr}(D-I_2)/2}e^{4\pi i\phi(n_j,M)}\vartheta_{n_j}^2(\tau).\end{aligned}$$

Here we have to compute the expression $4\phi(n_j,M)$ modulo 2 in order to get the transformation formula. Letting $M = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \Gamma[2]$ and thus $B = 2\left(\begin{smallmatrix}b_1 & b_2\\ b_3 & b_4\end{smallmatrix}\right)$ and $C = 2\left(\begin{smallmatrix}c_1 & c_2\\ c_3 & c_4\end{smallmatrix}\right)$, we get for $4\phi(\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right], M)$ the expression

$$\mu_1 b_1 + \mu_2 b_4 + \nu_1 c_1 + \nu_2 c_4 + \mu_1\mu_2(b_2+b_3) + \nu_1\nu_2(c_2+c_3) \mod 2$$

and the fact that $M \in \Gamma[2] \subset \Gamma$ implies $c_2 + c_3 \equiv 0 \bmod 2$ and $b_2 + b_3 \equiv 0 \bmod 2$, so

$$4\phi\left(\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right], M\right) \equiv \mu_1 b_1 + \mu_2 b_4 + \nu_1 c_1 + \nu_2 c_4 \bmod 2,$$

and writing

$$e^{4\pi i\phi(n_j,M)} = (-1)^{\alpha(n_j,M)},$$

we see that the $\alpha(n_j, M)$ are given for $j = 1,\ldots,10$ by the following table:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0 | $c_4$ | $c_1$ | $c_1+c_4$ | $b_4$ | $b_4+c_1$ | $b_1$ | $b_1+c_4$ | $b_1+b_4$ | $b_1+b_4+c_1+c_4$ |

The squares of the theta constants satisfy many quadratic relations. A pair of odd theta characteristics $\{m_{j_1}, m_{j_2}\}$ determines six even theta characteristics $n_i$, namely the six complementary to the four given by Lemma 3.1. These come in pairs such that the sum of $\alpha$ is the same for each pair, see the table above. For example, $m_1$ and $m_2$ determine the three pairs $(n_1,n_3)$, $(n_2,n_4)$ and $(n_5,n_6)$ (that give $c_1$ mod 2). This gives the relation

$$\vartheta_1^2\vartheta_3^2 - \vartheta_2^2\vartheta_4^2 - \vartheta_5^2\vartheta_6^2 = 0, \tag{5.1}$$

where we write $\vartheta_i$ for $\vartheta_{n_i}$. These relations form an orbit under the action of $\mathfrak{S}_6$.

## 6. The Ring of Scalar-Valued Modular Forms on $\Gamma[2]$

We review the structure of the ring $\bigoplus_k M_{0,k}(\Gamma[2])$ of scalar-valued modular forms on $\Gamma[2]$. We have graded rings

$$R = \bigoplus_k M_{0,k}(\Gamma[2]) \quad \text{and} \quad R^{\text{ev}} = \bigoplus_k M_{0,2k}(\Gamma[2]).$$

The group $\mathfrak{S}_6 = \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$ acts on $R$ and $R^{\text{ev}}$. The structure of these rings was determined by Igusa, cf. [20, 22]. The ring $R^{\text{ev}}$ is generated by the fourth powers of the 10 even theta characteristics. We shall use the following notation.

**Notation 6.1.** We denote $\vartheta_{n_i}$ by $\vartheta_i$ and $\vartheta_{n_i}^4$ by $x_i$ for $i = 1, \ldots, 10$.

Each $x_i$ is a modular form of weight 2 on $\Gamma[2]$. Formally the 10 elements $x_i$ span a 10-dimensional representation of $\mathfrak{S}_6$. The matrices $\rho(X)$ and $\rho(Y)$ given in (4.1) imply that the $\mathfrak{S}_6$-representation is $s[2^3] + s[2, 1^4]$. However, the forms $x_i$ are not linearly independent, but generate the vector space $M_{0,2}(\Gamma[2])$ of dimension 5; in fact these satisfy relations like

$$\vartheta_1^4 - \vartheta_4^4 - \vartheta_6^4 - \vartheta_7^4 = 0$$

and these relations form a representation $s[2, 1^4]$ of $\mathfrak{S}_6$. The four $n_j$ occurring in such a relation correspond to a pair of odd theta characteristics; this gives 15 such relations, see Lemma 3.1. So $M_{0,2}(\Gamma[2])$ equals $s[2^3]$ as a representation space and $x_i$ for $i = 1, \ldots, 5$ form a basis. The $x_i$ define a morphism

$$\varphi : \mathcal{A}_2[\Gamma[2]] \to \mathbb{P}^4 \subset \mathbb{P}^9$$

that extends to an embedding of the Satake compactification $\mathcal{A}_2[\Gamma[2]]^*$ into projective space $\mathbb{P}^4 \subset \mathbb{P}^9$. The $\mathbb{P}^4 \subset \mathbb{P}^9$ is given by the linear relations satisfied by the $x_i$, a basis of which can be given by

$$\begin{aligned} &x_6 = x_1 - x_2 + x_3 - x_4 - x_5, \quad x_7 = x_2 - x_3 + x_5, \\ &x_8 = x_1 - x_4 - x_5, \quad x_9 = -x_3 + x_4 + x_5, \quad x_{10} = x_1 - x_2 - x_5. \end{aligned} \tag{6.1}$$

The closure of the image of $\varphi$ is then the quartic threefold (the Igusa quartic) within this linear subspace given by the equation

$$\left(\sum_{i=1}^{10} x_i^2\right)^2 - 4\sum_{i=1}^{10} x_i^4 = 0. \tag{6.2}$$

It follows that $R^{\text{ev}}$ is generated by five fourth powers of the theta constants $u_0 = x_1, u_1 = x_2, \ldots, u_4 = x_5$ and that

$$R^{\text{ev}} \cong \mathbb{C}[u_0, \ldots, u_4]/(f)$$

with $f$ a homogeneous polynomial of degree 4 in the $u_i$. The full ring $R$ is a degree 2 extension $R^{\text{ev}}[\chi_5]/(\chi_5^2 + 2^{14}\chi_{10})$ generated by the modular form $\chi_5$ of weight 5

$$\chi_5 = \prod_{i=1}^{10} \vartheta_i.$$

This form is anti-invariant under $\mathfrak{S}_6$ (i.e. it generates the sign representation $s[1^6]$ of $\mathfrak{S}_6$) and so its square is a form of level 1 and satisfies the equation $\chi_5^2 = -2^{14}\chi_{10}$, where $\chi_{10}$ is Igusa's cusp form of weight 10 and level 1, cf. [22].

As a virtual representation of $\mathfrak{S}_6$ we thus have for even $k \geq 0$

$$M_{0,k}(\Gamma[2]) = \mathrm{Sym}^{k/2} s[2^3] - \begin{cases} 0 & 0 \leq k \leq 6, \\ \mathrm{Sym}^{k/2-4} s[2^3] & k \geq 8 \end{cases}$$

and $M_{0,k+5}(\Gamma[2]) = s[1^6] \otimes M_{0,k}(\Gamma[2])$ for even $k \geq 0$. Igusa calculated the character of $\mathfrak{S}_6$ on the spaces $M_{0,k}(\Gamma[2])$, see [20, pp. 399–402]. From his results we can deduce generating functions $\sum_{k\geq 0} m_{s[P],k} t^k$ for the multiplicities $m_{s[P],k}$ of the irreducible representations $s[P]$ (with $P$ a partition of 6) of $\mathfrak{S}_6$ in $M_{0,k}(\Gamma[2])$. We give the result in the following table:

| | | | |
|---|---|---|---|
| $s[6]$ | $\frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$ | $s[1^6]$ | $\frac{t^5(1+t^{25})}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$ |
| $s[5,1]$ | $\frac{t^{11}(1+t)}{((1-t^4)(1-t^6))^2}$ | $s[2,1^4]$ | $\frac{t^6(1+t^{11})}{((1-t^4)(1-t^6))^2}$ |
| $s[4,2]$ | $\frac{t^4(1+t^{15})}{(1-t^2)(1-t^4)^2(1-t^{10})}$ | $s[2,1^2]$ | $\frac{t^9}{(1-t^2)(1-t^4)^2(1-t^5)}$ |
| $s[4,1^2]$ | $\frac{t^{11}(1+t^4)}{(1-t)(1-t^4)(1-t^6)(1-t^{12})}$ | $s[3,1^3]$ | $\frac{t^6(1+t^4+t^{11}+t^{15})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$ |
| $s[3,3]$ | $\frac{t^7(1+t^{13})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$ | $s[2^3]$ | $\frac{t^2(1+t^{23})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$ |
| $s[3,2,1]$ | $\frac{t^8(1-t^8)}{(1-t^2)^2(1-t^5)(1-t^6)^2}$ | | |

For the convenience of the reader we give the representation type of $M_{0,k}(\Gamma[2])$ for even $k$ with $2 \leq k \leq 12$:

| $k\backslash P$ | $[6]$ | $[5,1]$ | $[4,2]$ | $[4,1^2]$ | $[3^2]$ | $[3,2,1]$ | $[3,1^3]$ | $[2^3]$ | $[2^2,1^2]$ | $[2,1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 4 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 6 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 2 | 0 | 1 | 0 |
| 8 | 1 | 0 | 3 | 0 | 0 | 1 | 1 | 3 | 0 | 0 | 0 |
| 10 | 2 | 0 | 3 | 0 | 0 | 2 | 3 | 4 | 0 | 2 | 0 |
| 12 | 3 | 1 | 6 | 1 | 0 | 3 | 4 | 5 | 0 | 2 | 0 |

## 7. The Igusa Quartic

In this section we give three models of the Igusa quartic. The first is the one given above as the image of the Satake compactification $\mathrm{Proj}(\bigoplus_k M_{0,2k}(\Gamma[2]))$ under the morphism $\varphi$ above which is the variety in $\mathbb{P}^4 \subset \mathbb{P}^9$ given by the linear equations (representing an irreducible representation $s[2,1^4]$ of $\mathfrak{S}_6$)

$$\begin{aligned} &x_6 = x_1 - x_2 + x_3 - x_4 - x_5, \quad x_7 = x_2 - x_3 + x_5, \\ &x_8 = x_1 - x_4 - x_5, x_9 = -x_3 + x_4 + x_5, x_{10} = x_1 - x_2 - x_5 \end{aligned} \tag{7.1}$$

and the quartic equation

$$\left(\sum_{i=1}^{10} x_i^2\right)^2 - 4\sum_{i=1}^{10} x_i^4 = 0. \tag{7.2}$$

This variety admits an action of $\mathfrak{S}_6$ induced by the action on the $x_i$ given by the irreducible 5-dimensional representation $s[2^3]$. It has exactly 15 singular lines given as the $\mathfrak{S}_6$-orbit of $\{(a : a-b : a : a-b : b : b : 0 : 0 : 0 : 0) : (a : b) \in \mathbb{P}^1\}$. The intersection points of such lines form the $\mathfrak{S}_6$-orbit of $(1 : 1 : 1 : 1 : 0 : 0 : 0 : 0 : 0 : 0)$ of length 15. Together these form a $(15_3, 15_3)$ configuration and are the images of the boundary components. Using Lemma 3.1 we get the following.

**Lemma 7.1.** *The fifteen* 1*-dimensional boundary components of* $\mathcal{A}_2[\Gamma[2]]^*$ *correspond one-to-one to the fifteen pairs of distinct odd theta characteristics. The* 15 0*-dimensional boundary components correspond one-to-one to the* 15 *partitions of* $\{m_1, \ldots, m_6\}$ *into three pairs.*

There is another model of the Igusa quartic given in $\mathbb{P}^4 \subset \mathbb{P}^5$ by the equations (cf. [30])

$$\sigma_1 = 0, \quad \sigma_2^2 - 4\sigma_4 = 0 \tag{7.3}$$

with $\sigma_i$ the $i$th elementary symmetric function in the six coordinates $y_1, \ldots, y_6$. We let the group $\mathfrak{S}_6$ act by $y_i \mapsto y_{\pi(i)}$ for $\pi \in \mathfrak{S}_6$. The representation on the space of the $y_i$ is $s[6] + s[5,1]$ with $\sigma_1$ representing the $s[6]$-part. We can connect the two models by using the outer automorphism

$$\psi : \mathfrak{S}_6 \to \mathfrak{S}_6 \quad \text{with } \psi(12) = (16)(34)(25) \text{ and } \psi(123456) = (134)(26)(5)$$

and the coordinate change $x_i = y_{a(i)} + y_{b(i)} + y_{c(i)}$ with $(a(i), b(i), c(i))$ given by

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| (125) | (245) | (256) | (235) | (156) | (126) | (145) | (124) | (135) | (123) |

or conversely $y_1 = (2x_1 - x_2 - x_3 - x_4)/3$, etc. (use the $\mathfrak{S}_6$-actions). In the model given by (7.3) the 1-dimensional boundary components of $\mathcal{A}_2[\Gamma[2]]^*$ form the orbit of $\{(x : x : y : y : -(x+y) : -(x+y)) : (x : y) \in \mathbb{P}^1\}$. Under our conventions the boundary component $B_{ij}$ is given by $y_a = y_b$, $y_c = y_d$, $y_e = y_f$ if $\psi(ij) = (ab)(cd)(ef)$.

Yet another way to describe the Igusa quartic as a hypersurface in $\mathbb{P}^4$ that we shall also use later is by taking $x_1, \ldots, x_4, x_5 - x_6$ as the generators of $M_{0,2}(\Gamma[2])$. Then Eq. (6.2) reads

$$(s_1^2 - 4s_2 - (x_5 - x_6)^2)^2 - 64s_4 = 0 \tag{7.4}$$

where $s_i$ is the $i$th elementary symmetric function of $x_1, x_2, x_3, x_4$. The involution $\iota = (12)(34)(56) \in \mathfrak{S}_6$ acts by sending $x_5 - x_6$ to its negative and the fixed point locus is the Steiner surface $(s_1 - 4s_2)^2 = 64s_4$ in $\mathbb{P}^3$ and it displays the quotient by $\iota$ as a double cover of $\mathbb{P}^3$ branched along the four planes given by $x_i = 0$, $i = 1, \ldots, 4$, cf. Mukai [25].

## 8. Humbert Surfaces

A Humbert surface in $\mathcal{A}_2$ (or $\mathcal{A}_2[G]$ for $G = \Gamma[2], \Gamma_1[2]$ or $\Gamma_0[2]$) is a divisor parametrizing principally polarized abelian surfaces with multiplication by an order in a real quadratic field, or abelian surfaces that are isogenous to a product of elliptic curves. Some of these Humbert surfaces play a role in the story of our modular forms.

The Humbert surface of invariant $\Delta$ in $\mathcal{A}_2[G]$ is defined in $\mathfrak{H}_2$ by all equations of the form

$$a\tau_{11} + b\tau_{12} + c\tau_{22} + d(\tau_{12}^2 - \tau_{11}\tau_{22}) + e = 0,$$

with primitive vector $(a, b, c, d, e) \in \mathbb{Z}^5$ satisfying $\Delta = b^2 - 4ac - 4de$, cf. [30]. We can take their closures in the Satake compactifications $\mathcal{A}_2[G]^*$. A Humbert surface of invariant $\Delta$ with $\Delta$ not a square intersects the boundary only in the 0-dimensional boundary components, while those with $\Delta$ a square contain 1-dimensional components.

In this paper the Humbert surfaces of invariant 1, 4 and 8 will play a role. The Humbert surface of invariant 1 is the locus of principally polarized abelian surfaces in $\mathcal{A}_2$ (respectively, in $\mathcal{A}_2[\Gamma[2]]$ etc.) that are products of elliptic curves. In $\mathcal{A}_2[\Gamma[2]]$ this locus consists of 10 irreducible components, each isomorphic to $\Gamma(2)\backslash\mathfrak{H}_1 \times \Gamma(2)\backslash\mathfrak{H}_1$ and corresponding to the vanishing of one even theta characteristic. In $\mathcal{A}_2[\Gamma_1[2]]$ this locus consists of four irreducible components, three of which are isomorphic to $\Gamma_0(2)\backslash\mathfrak{H}_1 \times \Gamma_0(2)\backslash\mathfrak{H}_1$ and one is isomorphic to $\mathrm{Sym}^2(\Gamma(2)\backslash\mathfrak{H}_1)$.

In $\mathcal{A}_2[\Gamma_0[2]]$ the Humbert surface of invariant 1 has two irreducible components. One is isomorphic to $(\Gamma_0(2)\backslash\mathfrak{H}_1)^2$ and the other one to $\mathrm{Sym}^2(\Gamma_0(2)\backslash\mathfrak{H}_1)$.

The Humbert surface of invariant 4 in $\mathcal{A}_2[\Gamma[2]]^*$ consists of 15 components. In the model of the Igusa quartic given by (7.3) these components are given by $y_i = y_j$ with $1 \leq i, j \leq 6$. The product $\prod(y_i - y_j)$ defines the $\mathfrak{S}_6$-anti-invariant modular form

$$\chi_{30} = (x_2 - x_3)(x_2 - x_4)(x_3 - x_4)(x_3 - x_5)(x_3 - x_6)(x_5 - x_6)\prod_{i=2}^{10}(x_1 - x_i)$$

of weight 30. The zero locus of $\chi_{35} = \chi_{30}\chi_5$ is supported on $H_1 + H_4$.

A 0-dimensional boundary component of $\mathcal{A}_2[\Gamma[2]]^*$ has as its stabilizer a (non-normal) subgroup $\Gamma_0[2]$ in $\Gamma[2]$, hence determines a subgroup $\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$ in $\mathfrak{S}_6$. The central involution of this group fixes a component of the Humbert surface $H_4$. For our choice of $\Gamma_0[2]$ this is the surface given in the Igusa quartic by

$$x_5 - x_6 = 0, \quad \text{equivalently given by } x_7 - x_8 = 0 \text{ or } x_9 - x_{10} = 0$$

or in the model with the $y$-coordinates by $y_2 = y_5$.

The fixed point set of the Fricke involution on the Igusa quartic consists of two curves and two isolated points as we shall see in Sec. 11.

## 9. The Ring of Scalar-Valued Modular Forms on $\Gamma_1[2]$ and $\Gamma_0[2]$

We now consider modular forms on $\Gamma_1[2]$ and $\Gamma_0[2]$. Note that $M_{0,k}(\Gamma_1[2])$ is the invariant subspace of $M_{0,k}(\Gamma[2])$ under the action of $(\mathbb{Z}/2\mathbb{Z})^3 = \Gamma_1[2]/\Gamma[2]$. The space $M_{0,k}(\Gamma_1[2])$ is a representation space for $\mathfrak{S}_3 = \Gamma_0[2]/\Gamma_1[2]$. Representation theory tells us that a virtual $\mathfrak{S}_6$-representation $a_{s[6]}s[6] + a_{s[5,1]}s[5,1] + \cdots + a_{s[1^6]}s[1^6]$ in $M_{j,k}(\Gamma[2])$ contributes a virtual $\mathfrak{S}_3$-representation

$$(a_{s[6]} + a_{s[4,2]} + a_{s[2^3]})s[3] \\ + (a_{s[5,1]} + a_{s[4,2]} + a_{s[3,2,1]})s[2,1] + (a_{s[4,1^2]} + a_{s[3^2]})s[1^3] \qquad (9.1)$$

to $M_{j,k}(\Gamma_1[2])$, and hence a contribution $a_{s[6]} + a_{s[4,2]} + a_{s[2^3]}$ to the dimension of $M_{j,k}(\Gamma_0[2])$.

**Proposition 9.1.** *The generating function* $\sum_{k\geq 0} m_{s[P],k}t^k$ *of the irreducible* $\mathfrak{S}_3$ *representations in* $M_{0,k}(\Gamma_1[2])$ *is given by*

$$\sum_{k\geq 0} m_{s[P],k}t^k = \frac{N_{s[P]}}{(1-t^2)(1-t^4)^2(1-t^6)}$$

*with* $N_{s[3]} = 1 + t^{19}$, $N_{s[2,1]} = t^4 + t^8 + t^{11} + t^{15}$ *and* $N_{s[1^3]} = t^7 + t^{12}$.

We thus find the following table of representations for $M_{0,k}(\Gamma_1[2])$ for even $k \leq 12$:

| $k\backslash P$ | $[3]$ | $[2,1]$ | $[1^3]$ |
|---|---|---|---|
| 2 | 1 | 0 | 0 |
| 4 | 3 | 1 | 0 |
| 6 | 4 | 1 | 0 |
| 8 | 7 | 4 | 0 |
| 10 | 9 | 5 | 0 |
| 12 | 14 | 10 | 1 |

The structure of these rings of modular forms is as follows.

**Theorem 9.1.** *The ring of scalar-valued modular forms on* $\Gamma_0[2]$ *is generated by forms* $s_1, s_2, \alpha, s_3$ *of weight* 2, 4, 4, 6 *and a form* $\chi_{19}$ *of weight* 19 *with the ideal of relations generated by the relation* (9.6).

**Theorem 9.2.** *The ring of scalar-valued modular forms on* $\Gamma_1[2]$ *is generated by forms* $s_1, s_2, \alpha, D_1, D_2, s_3$ *of weight* 2, 4, 4, 4, 4 *and* 6 *and by a form* $\chi_7$ *in weight* 7. *The ideal of relations is generated by the relation* (9.3) *in weight* 8, *the relation* (9.4) *in weight* 12 *and the relation* (9.5) *in weight* 14.

The relations are given explicitly below. Theorem 9.1 is due to Ibukiyama, see [2, 1], but we give here an independent proof.

**Proof.** The group $\Gamma_0[2]/\Gamma[2] \simeq \mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$ acts on the ring $R^{\mathrm{ev}}$, generated by $x_1, \ldots, x_5$, but it will now be convenient to choose $x_5 - x_6 = 2x_5 - x_1 + x_2 - x_3 + x_4$

as the last generator. Then $\mathfrak{S}_4$ acts on $x_1, \ldots, x_4$ by $x_i \mapsto x_{\sigma(i)}$ and $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $x_1, \ldots, x_4$ and by $-1$ on $x_5 - x_6$. The ring of invariants is the ring $M_*^{\mathrm{ev}}(\Gamma_0[2])$, while the ring of invariants under the subgroup $(\mathbb{Z}/2\mathbb{Z})^3 = \Gamma_1[2]/\Gamma[2]$ is the ring $M_*^{\mathrm{ev}}(\Gamma_1[2])$.

The ring of invariants of the subring generated by $x_1, \ldots, x_4$ is generated by the $\mathfrak{S}_4$ elementary symmetric functions $s_1, s_2, s_3$ and $s_4$ in these $x_i$. A further invariant is $\alpha = (x_5 - x_6)^2$. We now find eight forms of weight 8, namely $s_4, s_3s_1, s_2^2, s_2s_1^2, s_1^4, \alpha s_2, \alpha s_1^2, \alpha^2$ and as we know that $\dim M_{0,8}(\Gamma_0[2]) = 7$ we find one linear relation. Since all these forms live in $M_{0,8}(\Gamma[2])$ this must be (a multiple of) the Igusa quartic relation expressing $s_4$ in the other forms. To make this explicit, note that $\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4^2$ is in $M_{0,4}(\Gamma_1[2])$, and it equals $(-s_1^2 + 4s_2 + \alpha)/8$ as one checks. We thus see that

$$64s_4 = (-s_1^2 + 4s_2 + \alpha)^2. \tag{9.2}$$

There can be no further relations because the ideal of relations among the $x_1, \ldots, x_4, x_5 - x_6$ is generated by the Igusa quartic. So $M_*^{\mathrm{ev}}(\Gamma_0[2])$ contains a subring generated by $s_1, s_2, \alpha$ and $s_3$ with Hilbert function $(1 - t^8)/(1 - t^2)(1 - t^4)^2(1 - t^6)$, and this is the Hilbert function of $M_*^{\mathrm{ev}}(\Gamma_0[2])$, see Proposition 9.1. Therefore there can be no further relations and we found the ring $M_*^{\mathrm{ev}}(\Gamma_0[2])$.

For $M_*^{\mathrm{ev}}(\Gamma_1[2])$ we look at the invariants under $(\mathbb{Z}/2\mathbb{Z})^3$. The $s[2,1]$-subspace of $M_{0,4}(\Gamma_1[2])$ has a basis $D_1, D_2$ with

$$D_1 = (x_1 - x_2)(x_3 - x_4) \quad \text{and} \quad D_2 = (x_1 - x_3)(x_2 - x_4).$$

Since the form $D_1^2 - D_1D_2 + D_2^2$ is $\mathfrak{S}_3$-invariant (and equal to $s_2^2 - 3s_1s_3 + 12s_4$) we have using (9.2) a relation in weight 8

$$16(D_1^2 - D_1D_2 + D_2^2) = 3\alpha^2 - 6(s_1^2 - 4s_2)\alpha + 3s_1^4 - 24s_1^2s_2 - 48s_1s_3 + 64s_2^2. \tag{9.3}$$

The expression

$$C = -\vartheta_5^2\vartheta_6^2 \cdots \vartheta_{10}^2 = \frac{1}{2}((x_1x_3 - x_2x_4)(x_5 + x_6) + s_1x_5x_6)$$

defines an element of $M_{0,6}(\Gamma_1[2])$ (but with a nontrivial character on $\Gamma_0[2]$) and thus can be expressed polynomially in $s_1^3, s_1s_2, s_3, \alpha s_1, D_1s_1$ and $D_2s_1$. It satisfies the relation

$$C^2 = x_5 \cdots x_{10}, \tag{9.4}$$

where $x_5 \cdots x_{10}$ is $\mathfrak{S}_3$-invariant. We thus find a subring of $M_*^{\mathrm{ev}}(\Gamma_1([2])$ generated over $M_*^{\mathrm{ev}}(\Gamma_0[2])$ by $D_1$ and $D_2$ and we have two algebraic relations, one of degree 8 and one of degree 12 given by (9.3) and (9.4). We can have no third independent algebraic relation because there are no algebraic relations among $s_1, s_2, \alpha$ and $s_3$. The Hilbert function of this subring is $(1-t^8)(1-t^{12})/(1-t^2)(1-t^4)^4(1-t^6)$ and coincides with the Hilbert function of $M_*^{\mathrm{ev}}(\Gamma_1[2])$. This shows that we found the ring $M_*^{\mathrm{ev}}(\Gamma_1[2])$.

We can construct a cusp form of weight 7 in the $s[1^3]$-subspace of $M_{0,7}(\Gamma_1[2])$, namely

$$\chi_7 = \chi_5(x_6 - x_5).$$

Since we have $\chi_5^2 = -2^{14}\chi_{10}$ we find a relation in weight 14:

$$\chi_7^2 = -2^{14}\chi_{10}\alpha. \tag{9.5}$$

Furthermore, we have the square root of the discriminant

$$\delta = (x_1 - x_2)\cdots(x_3 - x_4)$$

which is a modular form in the $s[1^3]$-subspace of $M_{0,12}(\Gamma_1[2])$. We thus find a cusp form $\chi_{19} = \chi_7\delta$ in $S_{0,19}(\Gamma_0[2])$. It satisfies the relation

$$\chi_{19}^2 = -2^{14}(x_1 - x_2)^2\cdots(x_3 - x_4)^2\chi_{10}(x_5 - x_6)^2. \tag{9.6}$$

We now show that each modular form of odd weight on $\Gamma_1[2]$ is divisible by $\chi_7$. In fact, such a form $f$ is also a modular form on $\Gamma[2]$, hence is divisible by $\chi_5$ as a modular form on $\Gamma[2]$. Next, we show that $f$ also vanishes on the component of the Humbert surface defined by $x_5 - x_6 = 0$. For this we look at the action of a representative of the element $\iota = (12)(34)(56)$ and observe that $f/\chi_5$ changes sign under this action, hence vanishes on the locus where $x_5 = x_6$. So $M_{0,k}(\Gamma_1[2]) = \chi_7 M_{0,k-7}(\Gamma_1[2])$ for odd $k$. But by a similar argument any odd weight modular form on $\Gamma_0[2]$ also vanishes on the other components of the Humbert surface $H_4$, and hence is divisible by $\delta$. □

**Remark 9.1.** Note that we have $M_{0,2k}(\Gamma_1[2])^{s[1^3]} = \delta M_{0,2k-12}(\Gamma_0[2])$.

**Remark 9.2.** Ibukiyama constructed $\chi_{19}$ as a Wronskian, see [1].

## 10. The Action of the Fricke Involution

We start by computing the action of the Fricke involution on the modular forms on $\Gamma_1[2]$. Recall that the Fricke involution $W_2$ given by formula (2.1) acts on $\mathcal{A}_2[\Gamma_1[2]]$ and $\mathcal{A}_2[\Gamma_0[2]]$ and thus induces an action on modular forms via $f \mapsto W_2(f) = f|_{j,k}W_2$ (where we sometimes omit the indices $j, k$).

**Lemma 10.1.** *The transformation formula for the* $x_i = \vartheta_i^4$ $(1 \le i \le 4)$ *under* $W_2$ *is*:

$$\begin{aligned}
x_1|_{0,2}W_2 &= (\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2 + \vartheta_4^2)^2/4,\\
x_2|_{0,2}W_2 &= (\vartheta_1^2 - \vartheta_2^2 + \vartheta_3^2 - \vartheta_4^2)^2/4,\\
x_3|_{0,2}W_2 &= (\vartheta_1^2 + \vartheta_2^2 - \vartheta_3^2 - \vartheta_4^2)^2/4,\\
x_4|_{0,2}W_2 &= (\vartheta_1^2 - \vartheta_2^2 - \vartheta_3^2 + \vartheta_4^2)^2/4.
\end{aligned}$$

**Proof.** Setting $T = 2\tau$, we get by definition

$$(\vartheta_i^4|_{0,2}W_2)(T/2) = \det(-T/\sqrt{2})^{-2}\vartheta_i^4(-T^{-1}) = 4\det(-T)^{-2}\vartheta_i^4(-T^{-1}).$$

This expression is closely related to the transformation formula of the $\vartheta_i^4$ under the element $J = t\left(\begin{smallmatrix} 0 & 1_2 \\ -1_2 & 0 \end{smallmatrix}\right)$ which reads

$$\vartheta^4_{J\cdot\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(-\tau^{-1}) = \kappa(J)^4\det(-\tau)^2\vartheta^4_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau)$$

since $8\phi([\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}], J) = 2\mu\nu^t \in 2\mathbb{Z}$. We know that $\kappa(J)^4 = \pm 1$ and we can determine its value by using $\vartheta_1^4 = \vartheta^4_{\left[\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right]}$ whose characteristic is fixed by $J$ and evaluating the latter equation at $\tau = i1_2 = -\tau^{-1}$, getting $\kappa(J)^4 = 1$. Taking into account the action of $J$ on the characteristics we thus find

$$(\vartheta_i^4|_{0,2}W_2)(\tau) = 4\vartheta^4_{w(i)}(2\tau),$$

where $[w(1), \ldots, w(10)] = [1, 5, 7, 9, 2, 8, 3, 6, 4, 10]$. We now use Riemann's bilinear relations (4.2) to see

$$\begin{aligned}
\vartheta_1^2(\tau) &= \vartheta_1^2(2\tau) + \vartheta_5^2(2\tau) + \vartheta_7^2(2\tau) + \vartheta_9^2(2\tau),\\
\vartheta_2^2(\tau) &= \vartheta_1^2(2\tau) - \vartheta_5^2(2\tau) + \vartheta_7^2(2\tau) - \vartheta_9^2(2\tau),\\
\vartheta_3^2(\tau) &= \vartheta_1^2(2\tau) + \vartheta_5^2(2\tau) - \vartheta_7^2(2\tau) - \vartheta_9^2(2\tau),\\
\vartheta_4^2(\tau) &= \vartheta_1^2(2\tau) - \vartheta_5^2(2\tau) - \vartheta_7^2(2\tau) + \vartheta_9^2(2\tau)
\end{aligned}$$

from which the result follows. □

By looking at the values of the $x_i$ at the 15 cusps for $\Gamma[2]$ we derive easily the action on the 0-dimensional and 1-dimensional cusps of $\Gamma_1[2]$. We use the notation of Sec. 2.

**Corollary 10.1.** *The action of $W_2$ on the cusps of $\mathcal{A}_2[\Gamma_1[2]]^*$ is as follows*[b]

$$W_2(\gamma) = \gamma, \quad W_2(\alpha) = \epsilon, \quad W_2(\beta) = \delta,$$
$$W_2(A) = F, \quad W_2(B) = E, \quad W_2(C) = D.$$

The action of $W_2$ on the cusps of $\Gamma_0[2]$ can be deduced immediately from this.

Using Lemma 10.1 we find that $s_1|_{0,2}W_2 = s_1$ and similarly

$$s_2|_{0,4}W_2 = 3s_1^2/8 - s_2/2 - 3\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4^2.$$

Since $\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4^2 \in M_{0,4}(\Gamma_1[2])$, we can express it in our basis and get

$$\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4^2 = -s_1^2/8 + s_2/2 + \alpha/8,$$

where $\alpha$ denotes the modular form $(x_5 - x_6)^2$ introduced in Sec. 9 and thus

$$s_2|_{0,4}W_2 = 3s_1^2/4 - 2s_2 - 3\alpha/8.$$

[b]Here the letters $\alpha, \ldots, \epsilon A, \ldots, F$ refer to the figure in Sec. 2.

As $W_2$ is an involution, we get its action on $\alpha$ using the last equation:

$$\alpha|_{0,4}W_2 = -2s_1^2 + 8s_2 + 2\alpha. \tag{10.1}$$

We can refine it as follows.

**Lemma 10.2.** *We have* $(x_5 - x_6)|_{0,2}W_2 = 4\vartheta_1\vartheta_2\vartheta_3\vartheta_4$.

**Proof.** We know that $(\vartheta_1\vartheta_2\vartheta_3\vartheta_4|_{0,2}W_2)(\tau) = 4(\vartheta_1\vartheta_5\vartheta_7\vartheta_9)(2\tau)$ but we also know that

$$\vartheta_5^2(\tau) = 2(\vartheta_1\vartheta_5 + \vartheta_7\vartheta_9)(2\tau) \quad \text{and} \quad \vartheta_6^2(\tau) = 2(\vartheta_1\vartheta_5 - \vartheta_7\vartheta_9)(2\tau)$$

and this implies $\vartheta_1\vartheta_2\vartheta_3\vartheta_4|_{0,2}W_2 = (x_5 - x_6)/4$ and thus the lemma since $W_2$ is an involution. □

We summarize the results.

**Proposition 10.1.** *The action of the involution* $W_2$ *on the generators is given by* $s_1|W_2 = s_1$, $s_2|W_2 = 3s_1^2/4 - 2s_2 - 3\alpha/8$, $\alpha|W_2 = -2s_1^2 + 8s_2 + 2\alpha$ *and* $D_1|W_2 = D_2$. *Furthermore* $s_3|W_2 = s_3 + s_1^3/8 - s_1s_2/2 - s_1\alpha/16$, $\chi_7|W_2 = \chi_7$ *and* $\chi_{19}|W_2 = -\chi_{19}$.

**Remark 10.1.** The trace of the action of $W_2$ on the space $M_{0,4}(\Gamma_1[2])$ is equal to 1.

## 11. $\mathcal{A}_2[\Gamma_1[2]]^*$ and the Igusa Quartic

In his study of moduli of Enriques surfaces Mukai found that the Satake compactification of $\mathcal{A}_2[\Gamma_1[2]]^*$ is isomorphic to the Igusa quartic, see [25]. He showed this using the geometry. We give an independent proof of this using modular forms. We will show that the scalar-valued modular forms of weight divisible by 4 define an embedding of $\mathcal{A}_2[\Gamma_1[2]]$ into projective space and that the closure of the image is the Igusa quartic.

We know that the ring of modular forms on $\Gamma[2]$ is generated by the modular forms $x_1, x_2, x_3, x_4$ and $\xi = x_5 - x_6$ of weight 2. These satisfy the relation

$$(s_1^2 - 4s_2 - \xi^2)^2 = 64s_4 \tag{11.1}$$

as we know from (9.2), but as follows also from comparing Eq. (10.1) and Lemma 10.2.

We define the following modular forms in $M_{0,4}(\Gamma_1[2])$:

$$X_1 = (x_1 + x_2 + x_3 + x_4)^2, \quad X_2 = (x_1 - x_2 + x_3 - x_4)^2,$$
$$X_3 = (x_1 + x_2 - x_3 - x_4)^2, \quad X_4 = (x_1 - x_2 - x_3 + x_4)^2$$

and

$$\eta = -16\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4^2 = 2(s_1^2 - 4s_2 - \xi^2).$$

**Proposition 11.1.** *The modular forms* $X_1, X_2, X_3, X_4$ *and* $\eta$ *generate* $M_{0,4}(\Gamma_1[2])$.

**Proof.** These forms lie in $M_{0,4}(\Gamma[2])$, are invariant under $(\mathbb{Z}/2\mathbb{Z})^3$ and linearly independent as one readily sees, cf. Theorem 9.2. □

Let $\gamma_i$ be the $i$th elementary symmetric function in the $X_1,\ldots,X_4$. Then one checks that

$$\eta^4 - 2(\gamma_1 - 4\gamma_2)\eta^2 + (\gamma_1^2 - 4\gamma_2)^2 - 64\gamma_4 = 0$$

since by Eq. (11.1) we have $\eta^2 = 2^8 s_4$. This means that $X_1,\ldots,X_4,\eta$ satisfy the equation

$$(\gamma_1^2 - 4\gamma_2 - \eta^2)^2 = 64\gamma_4 \tag{11.2}$$

which is the same as (11.1) and thus defines the Igusa quartic.

It is easy to see that the ideal of relations among the $x_i$ intersected with $\bigoplus_k M_{0,4k}(\Gamma_1[2])$ is generated by relation (11.2), hence the $X_1,\ldots,X_4,\eta$ generate a subring with Hilbert function $(1-t^{16})/(1-t^4)^5$ and since this equals the Hilbert function of $\bigoplus_k M_{0,4k}(\Gamma_1[2])$ we have the structure of $\bigoplus_k M_{0,4k}(\Gamma_1[2])$.

**Corollary 11.1 (Mukai [25]).** *The Satake compactification $\mathcal{A}_2[\Gamma_1[2]]^*$ is isomorphic to the Igusa quartic.*

It follows that there is an action of $\mathfrak{S}_6$ on $\oplus_k M_{0,4k}(\Gamma_1[2])$. This action does not preserve the set of boundary components, as $\mathcal{A}_1[\Gamma_1[2]]^*$ has only six 1-dimensional boundary components and $\mathfrak{S}_6$ acts transitively on the set of 15 singular lines. Therefore a large part of the automorphism group of $\mathcal{A}_1[\Gamma_1[2]]^*$ is not modular (i.e. not induced by elements of $\mathrm{Sp}(4,\mathbb{Q})$). To see this action we now define the modular forms

$$X_5 = (\eta + X_1 - X_2 + X_3 - X_4)/2, \quad X_6 = (-\eta + X_1 - X_2 + X_3 - X_4)/2,$$
$$X_7 = (\eta + X_1 + X_2 - X_3 - X_4)/2, \quad X_8 = (-\eta + X_1 + X_2 - X_3 - X_4)/2,$$
$$X_9 = (\eta + X_1 - X_2 - X_3 + X_4)/2, \quad X_{10} = (-\eta + X_1 - X_2 - X_3 + X_4)/2.$$

We have

$$X_5 = 4x_7x_8, \quad X_7 = 4x_5x_6, \quad X_9 = 4x_9x_{10},$$

and

$$X_6 = 4(\vartheta_1^2\vartheta_2^2 + \vartheta_3^2\vartheta_4^2)^2, \quad X_8 = 4(\vartheta_1^2\vartheta_3^2 + \vartheta_2^2\vartheta_4^2)^2, \quad X_{10} = 4(\vartheta_1^2\vartheta_4^2 + \vartheta_2^2\vartheta_3^2)^2.$$

These 10 $X_i$ generate formally a representation $s[2^3] + s[2,1^4]$ and satisfy linear relations of type $s[2,1^4]$ as the $x_i$ do. They satisfy the quartic relation $(\sum X_i)^2 - 4\sum X_i^4 = 0$.

The action of $W_2$ on the 10 $X_i$ is given by $X_i \mapsto X_{w(i)}$ with $(w(1),\ldots,w(10))$ given by $(1,6,8,10,7,2,5,3,9,4)$. The action of $W_2$ on $\eta$ is

$$\eta|W_2 = (X_1 - X_2 - X_3 - X_4 + \eta)/2.$$

**Construction 11.1.** We view the $X_i$ as the analogues for $\Gamma_1[2]$ of the $x_i = \vartheta_i^4$ for $\Gamma[2]$. We can also define modular forms with a character on $\Gamma_1[2]$ that play a role analogous to the role that the theta squares $\vartheta_i^2$ play for $\Gamma[2]$ as follows:

$$\begin{aligned}
U_1 &= (x_1 + x_2 + x_3 + x_4), \quad U_2 = (x_1 - x_2 + x_3 - x_4) \\
U_3 &= (x_1 + x_2 - x_3 - x_4), \quad U_4 = (x_1 - x_2 - x_3 + x_4) \\
U_5 &= 2\vartheta_5^2\vartheta_6^2, \quad U_7 = 2\vartheta_7^2\vartheta_8^2, \quad U_9 = 2\vartheta_9^2\vartheta_{10}^2, \\
U_6 &= 2(\vartheta_1^2\vartheta_2^2 + \vartheta_3^2\vartheta_4^2), \quad U_8 = 2(\vartheta_1^2\vartheta_3^2 + \vartheta_2^2\vartheta_4^2), \\
U_{10} &= 2(\vartheta_1^2\vartheta_4^2 + \vartheta_2^2\vartheta_3^2).
\end{aligned} \tag{11.3}$$

The 45 modular forms $U_iU_j$ of weight 4 with character on $\Gamma_1[2]$ satisfy equations like

$$U_1U_2 - U_3U_4 = U_7U_8, \quad U_1U_3 - U_2U_4 = U_5U_6, \quad U_1U_4 - U_2U_3 = U_9U_{10}.$$

We shall use them later to construct vector-valued modular forms on $\Gamma_1[2]$.

**Remark 11.1.** The automorphism group of the Igusa quartic is $\mathfrak{S}_6$. This implies that $\mathfrak{S}_6$ acts on the ring $R_{(4)} = \bigoplus_k M_{0,4k}(\Gamma_1[2])$. But not all automorphisms preserve the boundary $\mathcal{A}_2[\Gamma_1[2]]^* - \mathcal{A}_2[\Gamma_1[2]]$, hence not all automorphisms are induced by an action on $\mathfrak{H}_2$ as we saw above.

On the other hand we have a natural action of the subgroup $\mathfrak{G}$ generated by $\mathfrak{S}_3$ and $W_2$ on $\mathcal{A}_2[\Gamma_1[2]]$, where $\mathfrak{S}_3 = \Gamma_0[2]/\Gamma_1[2]$ is a subquotient of $\mathfrak{S}_6 = \Gamma/\Gamma[2]$. The group $\mathfrak{S}_3$ is generated by the two elements $X'$ and $Y'$ given in (3.3). To express this action on $R_{(4)}$ we choose as generators the modular forms $Y_i$ defined by $X_i = Y_{a(i)} + Y_{b(i)} + Y_{c(i)}$ with $(a(i), b(i), c(i))$ given as in Sec. 7. One then calculates the induced action.

**Lemma 11.1.** *The action of* $X'$ (*respectively*, $Y'$, *respectively*, $W_2$) *on the generators* $Y_i$ $(i = 1, \ldots, 6)$ *of* $M_{0,4}(\Gamma_1[2])$ *is given by* $(Y_1, \ldots, Y_6) \mapsto (Y_1, Y_2, Y_6, Y_4, Y_5, Y_3)$ (*respectively*, $(Y_1, Y_2, Y_6, Y_3, Y_5, Y_4)$, $(Y_5, Y_2, Y_3, Y_6, Y_1, Y_4)$).

Since $\mathfrak{G}$ is a group of automorphisms of the ring $R_{(4)}$ it acts by automorphisms on the Igusa quartic and it can be viewed as the subgroup of *modular* automorphisms of $\mathcal{A}_2[\Gamma_1[2]]^*$ (i.e. induced by an action of elements of $\mathrm{Sp}(4, \overline{\mathbb{Q}})$ on $\mathfrak{H}_2$) of $\mathfrak{S}_6$. It is the subgroup of permutations that preserve the partition $\{2\} \sqcup \{1, 5\} \sqcup \{3, 4, 6\}$ of $\{1, \ldots, 6\}$.

Finally, we give the fixed point locus of the Fricke involution.

**Lemma 11.2.** *In the model of the Igusa quartic given by* (7.3) *the fixed point locus of* $W_2$ *is given by the equations* $y_1 = y_5$ *and* $y_4 = y_6$. *It consists of a singular line and a conic section and two isolated fixed points.*

**Proof.** The action is given by the permutation $(y_1y_5)$ $(y_4y_6)$. A fixed point is either of the form $(1 : 0 : 0 : 0 : \pm 1 : -1 : \mp 1)$ or $(a : b : c : d : a : d)$ with $2a + b + c + 2d = 0$ and in the latter case the Igusa quartic equation (7.3) factors as a double line and a quadric. $\square$

## 12. Dimension Formulas for Vector-Valued Modular Forms on $\Gamma[2]$

We now give formulas for the dimension of the spaces of vector-valued modular forms $M_{j,k}(\Gamma[2])$ and $S_{j,k}(\Gamma[2])$. These formulas can be proved using the Hirzebruch–Riemann–Roch formula or the Selberg trace formula. In fact, a recent paper by Wakatsuki [33] proves the formula for $S_{j,k}(\Gamma[2])$ for $k \geq 5$ using the Selberg trace formula.

Since the group $\Gamma[2]$ contains $-1_4$ it follows that $M_{j,k}(\Gamma[2]) = (0)$ for all odd $j$. Furthermore, we have $M_{j,k}(\Gamma[2]) = S_{j,k}(\Gamma[2])$ for odd $k$.

**Theorem 12.1.** *For $k \geq 3$ odd and $j \geq 2$ even (or for $k \geq 5$ odd and $j = 0$) we have*

$$\begin{aligned}\dim M_{j,k}(\Gamma[2]) = \dim S_{j,k}(\Gamma[2]) = \frac{1}{24}[&2(j+1)k^3 + 3(j^2-2j-8)k^2 \\ &+ (j^3 - 9j^2 - 42j + 118)k + (-2j^3 - 9j^2 + 152j - 216)].\end{aligned}$$

*For $k \geq 4$ even and $j \geq 2$ even we have*

$$\begin{aligned}\dim M_{j,k}(\Gamma[2]) = \frac{1}{24}[&2(j+1)k^3 + 3(j^2-2j+2)k^2 + (j^3 - 9j^2 - 12j + 28)k \\ &+ (-2j^3 - 9j^2 + 182j - 336)].\end{aligned}$$

*Furthermore, for $k \geq 0$ even we have*

$$\dim M_{0,k}(\Gamma[2]) = \frac{(k+1)(k^2+2k+12)}{12}$$

*and $\dim M_{0,k+5}(\Gamma[2]) = \dim M_{0,k}(\Gamma[2])$ for $k \geq 0$ even.*

**Remark 12.1.** As we shall see in the next section for $k \geq 4$ even and $j + k \geq 6$ we have

$$\dim M_{j,k}(\Gamma[2]) = \dim S_{j,k}(\Gamma[2]) + 15(j + k - 4)/2.$$

We can rewrite these formulas in the form of a generating series.

**Theorem 12.2.** *The generating function for the dimension of $M_{j,k}(\Gamma[2])$ for fixed even $j \geq 2$ and $k \geq 3$ is given as*

$$\sum_{k \geq 3} \dim M_{j,k}(\Gamma[2]) t^k = \frac{\sum_{i=3}^{12} a_i t^i}{(1-t^2)^5}$$

*with $a_n = a_n(j)$ given by*

| $n$ | $a_n$ | $n$ | $a_n$ |
|---|---|---|---|
| 3 | $(j-2)(j-3)(j-4)/24$ | 4 | $j\,(2\,j^2 + 3\,j + 166)/24$ |
| 5 | $(-j^3 + 33\,j^2 - 44\,j + 72)/12$ | 6 | $-(j-1)(j^2 - 4\,j + 80)/4$ |
| 7 | $(-10\,j^2 + 25\,j - 20)/2$ | 8 | $j^3/4 - 7j^2/2 + 63j/2 - 46$ |
| 9 | $(j^3 + 39\,j^2 - 172\,j + 120)/12$ | 10 | $-j^3/12 + 11j^2/4 - 71j/3 + 36$ |
| 11 | $(-j^3 - 15\,j^2 + 106\,j - 120)/24$ | 12 | $-5j^2/8 + 25j/4 - 10$ |

**Remark 12.2.** Note that we have for Theorem 12.2 the identities $a_3 + a_5 + a_7 + a_9 + a_{11} = 0$ and $a_4 + a_6 + a_8 + a_{10} + a_{12} = 0$; see Sec. 19 for an explanation.

## 13. Representations of $\mathfrak{S}_6$ on Eisenstein Spaces

As a result of [3] we can calculate the representation of the group $\mathfrak{S}_6$ on the spaces $S_{j,k}(\Gamma[2])$ algorithmically for $j + k \geq 5$ assuming the conjectures there. This yields very helpful information for determining the structure of the modules $\mathcal{M}_j = \bigoplus_k M_{j,k}(\Gamma[2])$ and $\Sigma_j = \bigoplus_k S_{j,k}(\Gamma[2])$ and agrees in all cases we considered with the dimension formulas for $M_{j,k}(\Gamma[2])$ and $S_{j,k}(\Gamma[2])$. Moreover, for small weights the $\mathfrak{S}_6$-representation can be determined by combining the dimension formula with the cohomological calculations from [3] using point counting over finite fields or by using the module structure over $R^{\mathrm{ev}}$. In view of this it will be useful to know the representation of $\mathfrak{S}_6$ on the subspaces of the spaces of modular forms for the groups $\Gamma[2], \Gamma_1[2]$ and $\Gamma_0[2]$ generated by Eisenstein series. We will denote the orthogonal complement of the space $S_{j,k}(G)$ in $M_{j,k}(G)$ with respect to the Petersson product for $G = \Gamma[2], \Gamma_1[2]$ or $\Gamma_0[2]$ by $E_{j,k}(G)$.

**Remark 13.1.** We have $E_{j,k}(\Gamma[2]) = (0)$ if $k$ is odd.

The Eisenstein subspace $E_{j,k}(\Gamma[2])$ of $M_{j,k}(\Gamma[2])$ is also a representation space of $\mathfrak{S}_6$. By using Siegel's operator for one of the 15 boundary components of $\mathcal{A}_2[\Gamma[2]]$ it maps to the space of cusp forms $S_{j+k}(\Gamma(2)) \cong S_{j+k}(\Gamma_0(4))$ where $\Gamma(2)$ and $\Gamma_0(4)$ are the usual congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$. The dimension of $S_{2r}(\Gamma(2))$ equals $r-2$ for $r \geq 3$ and is zero otherwise. The space $S_r(\Gamma(2))$ is a representation space for the symmetric group $\mathfrak{S}_3 = \mathrm{SL}(2,\mathbb{Z})/\Gamma(2)$. The stabilizer in $\mathfrak{S}_6$ of one 1-dimensional boundary component is a group $H$ of order 48 and this group acts on the 1-dimensional boundary component via its quotient $\mathfrak{S}_3$.

As a representation space of $\mathfrak{S}_3$ the vector space $S_{2r}(\Gamma(2))$ is of the form

$$\mathrm{Sym}^r(s[2,1]) - \begin{cases} s[2,1], & r = 1, \\ s[3] + s[2,1], & r > 1 \end{cases}$$

because the ring of modular forms on $\Gamma(2)$ is generated by two modular forms of weight 2 that form an irreducible representation $s[2,1]$ and the space of Eisenstein series is a representation space $s[3] + s[2,1]$ except in weight 2, where it is a $s[2,1]$. We have

$$\mathrm{Sym}^r(s[2,1]) = (1 + [r/6] + \epsilon)s[3] + [(r+2)/3]\, s[2,1] + ([(r+3)/6] + \epsilon')s[1^3]$$

with $\epsilon = -1$ if $k \equiv 1 \bmod 6$ and $\epsilon' = -1$ if $k \equiv 4 \bmod 6$ and $\epsilon = 0$ and $\epsilon' = 0$ else. The representation of $\mathfrak{S}_6$ on the Eisenstein subspace of $M_{j,k}(\Gamma[2])$ is thus

$$\mathrm{Ind}_H^{\mathfrak{S}_6}(\mathrm{Sym}^{(j+k)/2}(s[2,1]) - s[3] - s[2,1])$$

for $j+k \geq 4$. We have

$$\mathrm{Ind}_H^{\mathfrak{S}_6}(s[3]) = s[6] + s[5,1] + s[4,2],$$
$$\mathrm{Ind}_H^{\mathfrak{S}_6}(s[2,1]) = s[4,2] + s[3,2,1] + s[2^3],$$
$$\mathrm{Ind}_H^{\mathfrak{S}_6}(s[1^3]) = s[3,1^3] + s[2,1^4].$$

**Proposition 13.1.** *For $k \geq 2$ the space $E_{j,k}(\Gamma[2])$ as a representation space of $\mathfrak{S}_6$ equals*

$$E_{j,k}(\Gamma[2]) = \mathrm{Ind}_H^{\mathfrak{S}_6}(\mathrm{Sym}^k(s[2,1]) - s[3] - s[2,1]) + \begin{cases} s[6] + s[4,2] + s[2^3], & j = 0, \\ 0, & j \geq 2. \end{cases}$$

**Corollary 13.1.** *For $k \geq 2$ the space $E_{j,2k}(\Gamma_1[2])$ as a representation of $\mathfrak{S}_3$ equals*

$$\begin{cases} a_k(s[3] + s[2,1]) - 2s[2,1], & j = 0, \\ b_{j,k}(s[3] + s[2,1]), & j \geq 2, \end{cases}$$

*where $a_k = k$ if $k$ is odd and $a_k = k+1$ if $k$ is even and $b_{j,k} = j/2 + k - 3$ if $j/2 + k$ is odd and $j/2 + k - 2$ if $j/2 + k$ is even.*

**Corollary 13.2.** *For $k \geq 2$ we have $\dim E_{0,2k}(\Gamma[2]) = 15(k-1)$. Moreover,*

$$\dim E_{0,2k}(\Gamma_1[2]) = 6[k/2] - 1 \quad \text{and} \quad \dim E_{0,2k}(\Gamma_0[2]) = 2[k/2] + 1.$$

*For $j \geq 2$ and $k \geq 2$ we have $\dim E_{j,2k}(\Gamma[2]) = 15(j/2 + k)$. Moreover,*

$$\dim E_{j,2k}(\Gamma_1[2]) = 3b_{j,k} \quad \text{and} \quad \dim E_{j,2k}(\Gamma_0[2]) = b_{j,k}.$$

## 14. Dimension Formulas for Vector-Valued Modular Forms on $\mathbf{\Gamma_1[2]}$

We now give dimension formulas for the space of modular forms and cusp forms of weight $(j,k)$ on the group $\Gamma_1[2]$; that is, we give the generating functions

$$\sum_{k \geq 3, \mathrm{odd}} \dim S_{j,k}(\Gamma_1[2]) t^k \quad \text{and} \quad \sum_{k \geq 4, \mathrm{even}} \dim M_{j,k}(\Gamma_1[2]) t^k.$$

These results can be deduced from the action of $\mathfrak{S}_6$ on the spaces $S_{j,k}(\Gamma[2])$ assuming the conjectures of [3]. Alternatively, they can be obtained by applying the holomorphic Lefschetz formula and are then not conditional on the conjectures of [3].

We start with the scalar-valued ones ($j = 0$). The generating function of $R^{\mathrm{ev}}(\Gamma_1[2])$ is computed using the ring structure given in Theorem 9.2 to be

$$\frac{(1-t^8)(1-t^{12})}{(1-t^2)(1-t^4)^4(1-t^6)}.$$

**Theorem 14.1.** *For $j > 0$ we have*

$$\sum_{k\geq 3,\mathrm{odd}} \dim S_{j,k}(\Gamma_1[2])t^k = \frac{\sum_{i=1}^{12} a_{2i+1}t^{2i+1}}{(1-t^2)(1-t^4)^4(1-t^6)}$$

*with the vector $[a_3, a_5, \ldots, a_{25}]$ of coefficients $a_i = a_i(j)$ for $j \equiv 0 \pmod 4$ equal to $1/192$ times*

$$[j^3 - 18j^2 + 104j - 192, 2j^3 + 30j^2 - 104j + 192, -2j^3 + 126j^2 - 184j + 960, -7j^3 - 24j^2 + 688j - 576, -2j^3 - 252j^2 + 704j - 1344, 8j^3 - 132j^2 - 704j + 384, 8j^3 + 180j^2 - 1472j + 1344, -2j^3 + 240j^2 - 400j + 384, -7j^3 - 18j^2 + 1048j - 1536, -2j^3 - 138j^2 + 680j - 576, 2j^3 - 18j^2 - 200j + 768, j^3 + 24j^2 - 160j + 192].$$

*For $j \equiv 2 \pmod 4$ the coefficient vector $[a_3, a_5, \ldots, a_{25}]$ of the numerator is equal to $1/192$ times*

$$[j^3 - 18j^2 + 92j - 120, 2j^3 + 30j^2 - 104j + 72, -2j^3 + 126j^2 - 136j + 552, -7j^3 - 24j^2 + 700j - 288, -2j^3 - 252j^2 + 632j - 432, 8j^3 - 132j^2 - 752j + 432, 8j^3 + 180j^2 - 1424j + 336, -2j^3 + 240j^2 - 328j - 288, -7j^3 - 18j^2 + 1036j - 984, -2j^3 - 138j^2 + 632j + 72, 2j^3 - 18j^2 - 200j + 648, j^3 + 24j^2 - 148j].$$

*For even $j \geq 2$ the generating function for even $k$ has the shape*

$$\sum_{k\geq 4,\mathrm{even}} \dim M_{j,k}(\Gamma_1[2])t^k = \frac{\sum_{i=1}^{12} a_{2i+2}t^{2i+2}}{(1-t^2)(1-t^4)^4(1-t^6)}$$

*with $[a_4, a_6, \ldots, a_{26}]$ for $j \equiv 0 \pmod 4$ being equal to $1/96$ times*

$$[j^3 - 3j^2 + 140j, j^3 + 21j^2 + 68j + 96, -3j^3 + 45j^2 - 372j + 864, -4j^3 - 36j^2 - 56j, 2j^3 - 114j^2 + 592j - 2016, 6j^3 - 30j^2 + 192j - 960, 2j^3 + 102j^2 - 656j + 1440, -4j^3 + 96j^2 - 632j + 1920, -3j^3 - 27j^2 + 324j - 288, j^3 - 63j^2 + 572j - 1440, j^3 - 3j^2 - 28j, 12j^2 - 144j + 384].$$

*For even $j \geq 2$ the generating function $\sum_{k\geq 4,\mathrm{even}} \dim M_{j,k}(\Gamma_1[2])t^k$ is of the same shape with the coefficients $[a_4, a_6, \ldots, a_{26}]$ for $j \equiv 2 \pmod 4$ being equal to $1/96$ times*

$$[j^3 - 3j^2 + 116j - 228, j^3 + 21j^2 + 68j + 540, -3j^3 + 45j^2 - 276j + 1068, -4j^3 - 36j^2 - 32j - 816, 2j^3 - 114j^2 + 448j - 1992, 6j^3 - 30j^2 + 96j - 408, 2j^3 + 102j^2 - 560j + 1848, -4j^3 + 96j^2 - 488j + 1296, -3j^3 - 27j^2 + 300j - 852, j^3 - 63j^2 + 476j - 804, j^3 - 3j^2 - 28j + 156, 12j^2 - 120j + 192].$$

**Remark 14.1.** We observe the following remarkable coincidences. For $k$ even we have:

$$\dim M_{0,k}(\Gamma[2]) = \dim M_{0,2k}(\Gamma_1[2]),$$
$$\dim M_{2,k}(\Gamma[2]) = \dim M_{2,2k}(\Gamma_1[2]),$$
$$\dim S_{2,k+1}(\Gamma[2]) = \dim S_{2,2k+1}(\Gamma_1[2]).$$

To explain two of these dimensional coincidences recall that the modular forms of weight 2 embed the moduli space $\mathcal{A}_2[\Gamma[2]]$ into projective space $\mathbb{P}^4$ and that the closure of the image is the quartic given by Eqs. (7.1) and (7.2) and is isomorphic to the Satake compactification. The hyperplane bundle of the Igusa quartic is the anti-canonical bundle; in fact, for a group $\Gamma' \subset \mathrm{Sp}(4,\mathbb{Z})$ acting freely on $\mathfrak{H}_2$ the canonical bundle is given by $\lambda^3$ with $\lambda$ the line bundle corresponding to the factor of automorphy $\det(c\tau+d)$ for a matrix $(a,b;c,d) \in \mathrm{Sp}(2g,\mathbb{Z})$. But if the group does not act freely we have to correct this; in the case at hand, the map $\mathfrak{H}_2 \to \mathcal{A}_2[\Gamma[2]]$ is ramified along the 10 components of Humbert surface $H_1$ of invariant 1. The corrected formula is then

$$K_{\mathcal{A}_2[\Gamma[2]]} = 3\lambda - 5\lambda = -2\lambda$$

where the 5 comes from 10/2 with 10 being the weight of the modular form $\chi_{10}$ defining $H_1$. In the preceding section we showed that $\mathrm{Proj}(\bigoplus_k M_{0,4k}(\Gamma_1[2]))$ is the Igusa quartic. This fits because the map $\mathcal{A}_2[\Gamma[2]] \to \mathcal{A}_2[\Gamma_1[2]]$ is ramified along the component of the Humbert surface of invariant 4 given by the vanishing of $x_5 - x_6$. Namely, the action of $(\mathbb{Z}/2\mathbb{Z})^3$ on the 15 components of $H_4$ on $\mathcal{A}_2[\Gamma[2]]$ has one orbit of length 1, three orbits of length 2, and one orbit of length 8 and the orbit of length 1 is the fixed point locus. We thus find

$$K_{\mathcal{A}_2[\Gamma_1[2]]} = 3\lambda - (5+2)\lambda = -4\lambda.$$

Note that by the Koecher principle a section of $\lambda^n$ is a modular form. No holomorphicity conditions at infinity are required.

So the anti-canonical map of $\mathcal{A}_2[\Gamma_1[2]]$ is given by the modular forms of weight 4 for $\Gamma_1[2]$. We conclude

$$M_{0,2k}(\Gamma[2]) \cong M_{0,4k}(\Gamma_1[2]).$$

A modular form of weight $(2,2k)$ on $\Gamma[2]$ defines a section of $T^\vee \otimes H^k$ with $T^\vee$ the cotangent bundle and $H$ the hyperplane bundle on the smooth locus of the Igusa quartic minus the Humbert surface $H_1$. By a local calculation one sees that such a section extends over $H_1$. We thus see that

$$M_{2,2k}(\Gamma[2]) = H^0(\mathcal{A}_2[\Gamma[2]], \mathrm{Sym}^2\mathbb{E} \otimes \det \mathbb{E}^{2k}),$$

with $\mathbb{E}$ the Hodge bundle on $\mathcal{A}_2[\Gamma[2]]$ (corresponding to the automorphy factor $c\tau + d$).

Similarly, a modular form of weight $(2,4k)$ on $\Gamma_1[2]$ defines a section of $T^\vee \otimes H^k$ with $T^\vee$ the cotangent bundle and $H$ the hyperplane bundle on the smooth locus of the Igusa quartic minus the Humbert surface $H_1$ and one component of the Humbert surface $H_4$. By a local calculation one sees that such a section extends to the smooth locus of $\mathcal{A}_2[\Gamma_1[2]]$. By the Koecher principle it defines a modular form holomorphic on all of $\mathcal{A}_2[\Gamma_1[2]]$. We thus see that we get an isomorphism

$$M_{2,2k}(\Gamma[2]) \cong M_{2,4k}(\Gamma_1[2]).$$

## 15. Constructing Vector-Valued Modular Forms Using Brackets

We now move to constructing vector-valued modular forms. One way to construct these is by using so-called Rankin–Cohen brackets. We recall the definition of the Rankin–Cohen bracket of two Siegel modular forms and its basic properties.

Let $F$ and $G$ be two modular forms of weight $(0,k)$ and $(0,l)$ on some subgroup $\Gamma'$ of $\mathrm{Sp}(4,\mathbb{Z})$. The Rankin–Cohen bracket of $F$ and $G$ is defined by the formula

$$[F,G](\tau) = \frac{1}{2\pi i}\left(kF\frac{dG}{d\tau} - lG\frac{dF}{d\tau}\right)(\tau),$$

where

$$\frac{dF}{d\tau}(\tau) = \begin{pmatrix} \partial F/\partial\tau_{11} & \frac{1}{2}\partial F/\partial\tau_{12} \\ \frac{1}{2}\partial F/\partial\tau_{12} & \partial F/\partial\tau_{22} \end{pmatrix}(\tau).$$

We refer also to [29, 17, 32]. (Note that Satoh's definition of the bracket in [29] differs from ours: $[F,G]_{(\mathrm{Satoh})} = -\frac{1}{kl}[F,G]$.) The main fact about this bracket is the following.

**Proposition 15.1.** *If $F_i \in M_{0,k_i}(\Gamma',\chi_i)$, with $\chi_i$ a character or a multiplicative system on $\Gamma'$, then $[F_1,F_2] \in M_{2,k_1+k_2}(\Gamma',\chi_1\chi_2)$.*

Thus the bracket defines a bilinear operation:

$$M_{0,k_1}(\Gamma',\chi_1) \times M_{0,k_2}(\Gamma',\chi_2) \to M_{2,k_1+k_2}(\Gamma',\chi_1\chi_2)$$

satisfying the following properties

(i) $[F,G] = -[G,F]$,
(ii) $F[G,H] + G[H,F] + H[F,G] = 0$,
(iii) $[FG,G] = G[F,G]$.

We give some examples.

**Example 15.1.** As we saw in Sec. 5 any pair $(m_i,m_j)$ of odd theta characteristics with $1 \le i < j \le 6$ determines a quadratic relation between squares of theta constants; for example, for the pair $(1,2)$ we have

$$\vartheta_1^2\vartheta_3^2 - \vartheta_2^2\vartheta_4^2 - \vartheta_5^2\vartheta_6^2 = 0.$$

This implies the following relation between brackets:

$$[\vartheta_1^2\vartheta_3^2, \vartheta_2^2\vartheta_4^2] = -[\vartheta_1^2\vartheta_3^2, \vartheta_5^2\vartheta_6^2] = -[\vartheta_2^2\vartheta_4^2, \vartheta_5^2\vartheta_6^2],$$

and by direct computation we also have

$$[\vartheta_i^2\vartheta_j^2, \vartheta_k^2\vartheta_l^2] = 8\vartheta_i^2\vartheta_j\vartheta_k\vartheta_l^2[\vartheta_j, \vartheta_k] + 8\vartheta_i\vartheta_j^2\vartheta_k^2\vartheta_l[\vartheta_i, \vartheta_l].$$

We thus can associate to a pair $(i, j)$ (of odd theta characteristics) a form $H_{ij}$ defined by, e.g.

$$H_{12} = [\vartheta_1^2\vartheta_3^2, \vartheta_2^2\vartheta_4^4] = -[\vartheta_1^2\vartheta_3^2, \vartheta_5^2\vartheta_6^2] = -[\vartheta_2^2\vartheta_4^2, \vartheta_5^2\vartheta_6^2]$$

(up to an ambiguity of signs) and in this way using the action of $\mathfrak{S}_6$ we obtain 15 forms $H_{ij}$ with $1 \leq i < j \leq 6$ in $M_{2,4}(\Gamma[2])$.

**Example 15.2.** In analogy with Example 15.1 we can use the relation (11.3) and the analogues $U_iU_j$ from Construction 11.1 to construct 15 modular forms $H'_{ij}$

$$H'_{12} = [U_1U_2, U_3U_4] = [U_1U_2, U_7U_8] = -[U_3U_4, U_7U_8] \in M_{2,8}(\Gamma_1[2]). \quad (15.1)$$

**Remark 15.1.** We might also consider the brackets $[\Theta[\mu], \Theta[\nu]]$ of the theta constants of second-order which lie in $M_{2,1}(\Gamma[2,4])$.

## 16. Gradients of Odd Theta Functions

Another way of constructing vector-valued Siegel modular forms is by using the gradients of the six odd theta functions. The (transposed) gradients

$$G_i^t = (\partial\vartheta_{m_i}/\partial z_1, \partial\vartheta_{m_i}/\partial z_2), \quad 1 \leq i \leq 6$$

define sections of the vector bundle $\mathbb{E} \otimes \det(\mathbb{E})^{1/2}$ on the group $\Gamma[4,8]$ with $\mathbb{E}$ the Hodge bundle, see Sec. 2. In other words they are vector-valued modular forms of weight $(1, 1/2)$ on the subgroup $\Gamma[4,8]$. We identify $\mathrm{Sym}^j(\mathbb{E})$ with the $\mathfrak{S}_j$-invariant subbundle of $\mathbb{E}^{\otimes j}$. We consider expressions of the form

$$\mathrm{Sym}^j(G_{i_1}, \ldots, G_{i_j})\vartheta_{r_1}\cdots\vartheta_{r_l}, \quad (16.1)$$

where $\mathrm{Sym}^j(G_{i_1}, \ldots, G_{i_j})$ is the projection of the section of $\mathbb{E}^{\otimes j} \otimes \det(\mathbb{E})^{j/2}$ onto its $\mathfrak{S}_j$-invariant subbundle. We abbreviate $\mathrm{Sym}^a(G_i, \ldots, G_i)$ with $G_i$ occurring $a$ times by $\mathrm{Sym}^a(G_i)$ and $\mathrm{Sym}^a(G_1, \ldots, G_1, \ldots, G_6, \ldots, G_6)$ with $G_i$ occurring $a_i$ times and $a = \sum_i a_i$ is abbreviated by $\mathrm{Sym}^a(G_1^{a_1}, \ldots, G_6^{a_6})$.

**Remark 16.1.** If $V \simeq \mathbb{C}^2$ is a $\mathbb{C}$-vector space with ordered basis $e_1, e_2$ then we shall use the ordered basis $e_1^{\otimes(n-i)} \otimes e_2^{\otimes i}$ for $i = 0, \ldots, n$ for $\mathrm{Sym}^n(V)$.

We ask when the expression (16.1) gives rise to a vector-valued Siegel modular form on $\Gamma[2]$ as opposed to only on $\Gamma[4,8]$.

We shall write the $j + l$ theta characteristics occurring in (16.1) as a $4 \times (j + l)$-matrix $M$ where each characteristic is written as a length 4 column. We first write the odd theta characteristics, then the even ones. A similar problem involving polynomials in the theta constants was considered by Igusa and Salvati Manni [20, Corollary of Theorem 5; 27, Eq. 20]. One finds in an analogous manner the following.

**Proposition 16.1.** *The expression* (16.1) *gives a modular form in* $M_{j,(l+j)/2}(\Gamma[2])$ *if and only if the matrix* $M$ *satisfies* $M \cdot M^t \equiv 0 \bmod 4$. *If we write each of the* $j+l$ *characteristics in* $M$ *as* $(\epsilon_1^{(i)}\epsilon_2^{(i)}\epsilon_3^{(i)}\epsilon_4^{(i)})^t$, *then these conditions can be written equivalently as*

(i) $\sum_{i=1}^{j+l} \epsilon_a^{(i)} \equiv 0 \bmod 4$ *for any* $1 \le a \le 4$,

(ii) $\sum_{i=1}^{j+l} \epsilon_a^{(i)}\epsilon_b^{(i)} \equiv 0 \bmod 2$ *for any* $1 \le a < b \le 4$.

We also want to know the action of $\mathfrak{S}_6$. For this we have the following lemma.

**Lemma 16.1.** *The action of* $X$ (*respectively*, $Y$) *on the gradients* $G_i$ *for* $i = 1,\dots,6$ *of the odd theta functions is given by*

$$\rho(X) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta \end{pmatrix}, \quad \rho(Y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \zeta \\ \zeta^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta^7 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We give examples of modular forms constructed in this way; a number of these will be used later.

**Example 16.1.** We take

$$F = \mathrm{Sym}^6(G_1,\dots,G_6).$$

This is a modular form of weight $(6,3)$ on $\Gamma[2]$; it is $\mathfrak{S}_6$-anti-invariant and necessarily a cusp form. The space $S_{6,3}(\Gamma[2])$ is 1-dimensional and generated by $F$. The product $F\chi_5$ generates the 1-dimensional space $S_{6,8}(\Gamma)$ of level 1. A form in this space was constructed by Ibukiyama in [15], cf. [8]. He used theta functions with pluri-harmonic coefficients.

**Example 16.2.** We consider

$$G_{12} = \mathrm{Sym}^2(G_1,G_2)\vartheta_1\cdots\vartheta_6 \in M_{2,4}(\Gamma[2]).$$

We can vary this construction by taking for any pair $G_i, G_j$ of different gradients of theta functions with odd characteristics the six even $n_j$ that are complementary to the four that correspond to a pair of odd ones via Lemma 3.1. The modular forms constructed in this way form a representation of $\mathfrak{S}_6$ that is $s[3,1^3] + s[2,1^4]$.

**Remark 16.2.** The restriction of $G_{12}$ to the 1-dimensional boundary components of $\mathcal{A}_2[\Gamma[2]]^*$ vanishes on 14 of those, while it is a multiple of the unique cusp form $(\vartheta_{00}\vartheta_{01}\vartheta_{10})^4$ on $\Gamma_0(2)$ times the vector $(1,0,0)$ on the remaining boundary component. (Note that $(1,0,0)$ is a highest weight vector of our representation.) This gives the correspondence between the 15 boundary components and unordered pairs of odd theta characteristics. See also Lemma 7.1.

As a variation, consider

$$G_{11} = \mathrm{Sym}^2(G_1)\vartheta_1^2\vartheta_4^2\vartheta_6^2.$$

Also this is a modular form of weight $(2,4)$ on $\Gamma[2]$ and its orbit under $\mathfrak{S}_6$ spans the representation $s[2,1^4]$, see Example 16.5.

**Example 16.3.** We have

$$\mathrm{Sym}^2(G_1,G_2)\vartheta_2^2\vartheta_4^2\vartheta_7\vartheta_8\vartheta_9\vartheta_{10} \in S_{2,5}(\Gamma[2]).$$

For fixed $(i,j)$ there are three choices for the factor $\vartheta_a^2\vartheta_b^2\vartheta_c\vartheta_d\vartheta_e\vartheta_f$ so that $\mathrm{Sym}^2(G_i,G_j)$ times this factor is a modular form of weight $(2,5)$ on $\Gamma[2]$. Formally we find a representation $s[3^2]\oplus s[3,2,1]\oplus s[2^2,1^2]$, but we know $S_{2,5}(\Gamma[2]) = s[2^2,1^2]$. We thus find relations, for example

$$\vartheta_1\vartheta_7\vartheta_{10}\mathrm{Sym}^2(G_1,G_2) - \vartheta_4\vartheta_5\vartheta_9\mathrm{Sym}^2(G_1,G_4) + \vartheta_2\vartheta_6\vartheta_8\mathrm{Sym}^2(G_1,G_6) = 0.$$

This identity shows that

$$\mathrm{Sym}^2(G_1,G_2)\wedge\mathrm{Sym}^2(G_1,G_4)\wedge\mathrm{Sym}^2(G_1,G_6) = 0$$

which gives using the $\mathfrak{S}_6$-action

$$\mathrm{Sym}^2(G_i,G_j)\wedge\mathrm{Sym}^2(G_i,G_k)\wedge\mathrm{Sym}^2(G_i,G_l) = 0. \tag{16.2}$$

**Example 16.4.** We have

$$\mathrm{Sym}^2(G_1)\vartheta_2^2\vartheta_4^2\vartheta_5^2\vartheta_9^2\vartheta_{10}^2 \in M_{2,6}(\Gamma[2]).$$

We can build 72 modular forms of this type, 12 for each $\mathrm{Sym}^2(G_i)$ and one can show that these generate a representation $s[3,2,1]+s[3,1^3]$.

Similarly, we have

$$\mathrm{Sym}^2(G_1,G_2)\vartheta_7^3\vartheta_8^3\vartheta_9^3\vartheta_{10}^3 \in S_{2,7}(\Gamma[2]).$$

Finally,

$$\mathrm{Sym}^4(G_1)\in M_{4,2}(\Gamma[2]),\quad \mathrm{Sym}^4(G_1,G_2,G_3,G_4)\vartheta_5\vartheta_6\vartheta_7\vartheta_8 \in M_{4,4}(\Gamma[2]),$$

and

$$\mathrm{Sym}^4(G_1,G_1,G_2,G_2)\vartheta_i^2\vartheta_j^2 \in M_{4,4}(\Gamma[2])\quad \text{for } (i,j) = (1,3),(2,4) \text{ and } (5,6).$$

**Example 16.5.** For a given $1\le i\le 6$ there are 10 triples $(a,b,c)$ such that

$$f[i;a,b,c] = \mathrm{Sym}^2(G_i)\vartheta_a^2\vartheta_b^2\vartheta_c^2$$

lies in $M_{2,4}(\Gamma[2])$. The relations (5.1) in Sec. 5 imply obvious relations among these forms and using these we are reduced to four different triples for each $i$; e.g. for

$i = 1$ we have the four forms $f[1;1,2,5], f[1;1,4,6], f[1;2,3,6], f[1;3,4,5]$. In total we get 24 forms that form a $\mathfrak{S}_6$-representation $s[3,1^3]+s[2^2,1^2]+s[2,1^4]$. But since $M_{2,4}(\Gamma[2]) = s[3,1^3]+s[2,1^4]$ we have a space $s[2^2,1^2]$ of relations and these relations are generated by the $\mathfrak{S}_6$-orbit of the relation

$$f[3;2,3,8] - f[1;3,4,5] + f[2;3,4,6] - f[6;3,4,10] = 0.$$

**Example 16.6.** The form $\mathrm{Sym}^6(G_1^3, G_2^3)\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}$ is a cusp form of weight $(6,5)$ on $\Gamma[2]$ and has an orbit of 15 elements, generating formally a representation $s[5,1]+s[4,1^2]$; its contribution to $S_{6,5}(\Gamma[2])$ is $s[4,1^2]$, thus giving a $s[5,1]$ of relations.

**Example 16.7.** We have in $M_{8,4}(\Gamma[2])$ the form $\mathrm{Sym}^8(G_1^4, G_2^4)\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}$. Its $\mathfrak{S}_6$-orbit generates formally the representation $s[6]+s[5,1]+s[4,2]$. We also have the six forms $\mathrm{Sym}^8(G_i^8) \in M_{8,4}(\Gamma[2])$ that generate a representation $s[6]+s[5,1]$ and $\sum_i \mathrm{Sym}^8(G_i^8) \in M_{8,4}(\Gamma)$ and it is not zero since its image under the Siegel operator is

$$2\pi^8(\vartheta_{00}^8\vartheta_{01}^8\vartheta_{11}^8)(\tau_{11})(1,0,\ldots,0)^t.$$

**Example 16.8.** The form $\mathrm{Sym}^4(G_1, G_2, G_3^2)\vartheta_7^3\vartheta_8\vartheta_9\vartheta_{10}$ is a cusp form in $S_{4,5}(\Gamma[2])$ that generates a $s[3,2,1]$ representation in this space.

## 17. Identities Between Gradients of Odd Theta Functions and Even Theta Constants

The fact that we have two ways of constructing modular forms and that we can decompose the spaces where these forms live as $\mathfrak{S}_6$-representations, easily leads to many identities. In this section we give a number of such identities, and in some sense these can be seen as generalizations of Jacobi's famous derivative formula for genus 1

$$\left.\frac{\partial\vartheta_{11}}{\partial z}\right|_{z=0} = -\pi\vartheta_{00}\vartheta_{01}\vartheta_{10}$$

to vector-valued modular forms of genus 2. For generalizations to scalar-valued modular forms we refer to [10, 23, 12, 13].

To motivate the fact that such an identity for vector-valued modular exists, we recall some of the results of [13]. Indeed, consider Riemann's bilinear addition formula (4.2) for the case when the characteristic $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]$ is odd. Differentiating this identity with respect to $z_i$ and $z_j$, and evaluating at $z = 0$, one obtains [13], Lemma 4:

$$2\left.\frac{\partial\vartheta\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right](\tau,z)}{\partial z_i}\cdot\frac{\partial\vartheta\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right](\tau,z)}{\partial z_i}\right|_{z=0} = \sum_{\sigma\in(\mathbb{Z}/2\mathbb{Z})^2}(-1)^{\sigma\cdot\nu}\Theta[\sigma](\tau)\left.\frac{\partial\Theta[\sigma+\mu](\tau,z)}{\partial z_i\partial z_j}\right|_{z=0}.$$

Using the heat equation for the theta function, the second-order $z$-derivative in the right-hand side can be rewritten as a constant factor times the $\tau$-derivative.

By summing over $\nu$ with coefficient $(-1)^{\nu\cdot\alpha}$ the Rankin–Cohen brackets on the right can be written as linear combinations of expressions on the left: the result is [13, Lemma 5], an identity between a quadratic expression in the gradients and a combination of Rankin–Cohen brackets.

We now give an identity between the two types of vector-valued modular forms that we constructed. To rule out any ambiguities of notation we fix the coordinates by putting

$$\mathrm{Sym}^2(G_i,G_j)=\begin{bmatrix} G_i^{(1)}G_j^{(1)} \\ G_i^{(1)}G_j^{(2)}+G_i^{(2)}G_j^{(1)} \\ G_i^{(2)}G_j^{(2)} \end{bmatrix} \quad \text{for } G_i=\begin{bmatrix} G_i^{(1)} \\ G_i^{(2)} \end{bmatrix};$$

we also write the bracket, which is given as $2\times 2$ matrix-valued, as vector-valued via

$$[f,g]=\begin{bmatrix} a & b \\ b & c \end{bmatrix} \mapsto \begin{bmatrix} a \\ 2b \\ c \end{bmatrix}.$$

**Lemma 17.1.** *The following identity holds for modular forms in* $M_{2,2}(\Gamma[2,4])$:

$$\mathrm{Sym}^2(G_1,G_1)\vartheta_1^2 = 2\pi^2([\vartheta_2^2,\vartheta_5^2]+[\vartheta_4^2,\vartheta_6^2]+[\vartheta_8^2,\vartheta_9^2]);$$

*it yields similar identities under the action of* $\mathfrak{S}_6$. *Moreover, for all* $1\le i<j\le 6$ *we have the identity* $G_{ij}=-\pi^2 H_{ij}$ *in* $M_{2,4}(\Gamma[2])$. (*Here the* $G_{ij}$ *are defined in Example* 16.2 *and the* $H_{ij}$ *in Example* 15.1.)

For example, for $(i,j)=(1,2)$ we have

$$\mathrm{Sym}^2(G_1,G_2)\vartheta_1\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6 = -\pi^2[\vartheta_1^2\vartheta_3^2,\vartheta_2^2\vartheta_4^2].$$

**Proof.** The space $M_{2,4}(\Gamma[2])$ is generated by the 15 forms $G_{ij}$, because we know that $\dim M_{2,4}(\Gamma[2])=15$ and the 15 $G_{ij}$ are linearly independent by Remark 16.2. By comparing Fourier coefficients we then find the relation

$$f[1;1,2,5]=G_{12}+G_{15}=-\pi^2(H_{12}+H_{15})$$

with $f[1;1,2,5]$ defined in Example 16.5, $H_{12}=-[\vartheta_2^2\vartheta_4^2,\vartheta_5^2\vartheta_6^2]$ and $H_{15}=-[\vartheta_2^2\vartheta_8^2,\vartheta_5^2\vartheta_9^2]$. Dividing by $\vartheta_2^2\vartheta_5^2$ gives the desired identity in $M_{2,2}(\Gamma[2,4])$. The second identity also follows by comparing Fourier coefficients. □

We end with the following question.

**Question 17.1.** Is the algebra $\bigoplus_{j,k}M_{j,k}(\Gamma[4,8])$ generated over the ring $\bigoplus_{k\in\mathbb{Z}} M_{0,k}(\Gamma[4,8])$ by the $[\vartheta_a,\vartheta_b]$? Is the algebra $\bigoplus_{j,k}M_{j,k}(\Gamma[2,4])$ generated over the ring $\bigoplus_{k\in\mathbb{Z}} M_{0,k}(\Gamma[2,4])$ by the brackets $[\Theta[\mu],\Theta[\nu]]$?

## 18. Wedge Products

In this section we calculate some triple wedge products of modular forms of weight $(2,4)$ that give information about the vanishing loci of these modular forms. We start by looking at a triple wedge product of the form

$$\mathrm{Sym}^2(G_{i_1}, G_{j_1}) \wedge \mathrm{Sym}^2(G_{i_2}, G_{j_2}) \wedge \mathrm{Sym}^2(G_{i_3}, G_{j_3}),$$

where we recall that the $G_i$'s are the gradients of odd theta functions. A direct computation, by writing out the summands of this wedge product, and matching the individual terms, shows that it is equal to the sum of two triple products of Jacobian determinants, for example to

$$D(i_1, i_2) \cdot D(j_1, i_3) \cdot D(j_2, j_3) + D(i_1, j_3) \cdot D(j_1, j_2) \cdot D(i_2, i_3),$$

where $D(a,b) = G_a \wedge G_b$ are the usual Jacobian nullwerte. By a generalized Jacobi's derivative formula (see [23, 13]) each such Jacobian determinant is a product of four theta constants with characteristics, and thus we obtain an expression for such a triple wedge product as an explicit degree 12 polynomial in theta constants with characteristics.

**Proposition 18.1.** *We have the following identities in* $S_{0,15}(\Gamma[2])$ :

$$G_{12} \wedge G_{34} \wedge G_{56} = \pi^6 \chi_5 \vartheta_1^4 \vartheta_2^4 \vartheta_3^4 \vartheta_4^4 (\vartheta_5^4 - \vartheta_6^4) = -\pi^6 \chi_7 x_1 x_2 x_3 x_4;$$

$$G_{12} \wedge G_{13} \wedge G_{45} = \pi^6 \frac{\chi_5^3 \vartheta_1^2 \vartheta_4^2 \vartheta_6^2}{\vartheta_2^2 \vartheta_7^2 \vartheta_9^2}$$

*and*

$$G_{12} \wedge G_{13} \wedge G_{14} = 0.$$

Since we know that the zero divisors of the $\vartheta_i^4$ are the components of $H_1$ we deduce the following.

**Corollary 18.1.** *The modular form $G_{12}$ does not vanish outside the Humbert surface $H_1$.*

A calculation using the Fourier–Jacobi expansion of the theta constants shows that $G_{12}$ vanishes on six components of $H_1$ and does not vanish identically on the other four as one sees by using the group action. On the component given by $\tau_{12} = 0$ it equals

$$\pi^2 \vartheta_{00}^4 \vartheta_{01}^4 (\tau_{11}) \otimes \vartheta_{00}^4 \vartheta_{01}^4 \vartheta_{10}^4 (\tau_{22}) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

## 19. Bounds on the Module Generators

We now turn to the module structure of the $R^{\mathrm{ev}}$-modules

$$\mathcal{M}_j^{\epsilon} = \bigoplus_{k \equiv \epsilon \bmod 2} M_{j,k}(\Gamma[2]) \quad \text{for } \epsilon = 0 \text{ and } \epsilon = 1$$

and similar ones where the $M_{j,k}$ are replaced by spaces of cusp forms $S_{j,k}$. First note that $R^{\rm ev} = \bigoplus_{k \text{ even}} H^0(\mathcal{A}_2[\Gamma[2]], L^k)$ with $L$ the determinant of the Hodge bundle. By the Koecher principle we have $H^0(\mathcal{A}_2[\Gamma[2]], L^k) = H^0(\tilde{\mathcal{A}}_2[\Gamma[2]], L^k)$ with $\tilde{\mathcal{A}}_2[\Gamma[2]]$ the standard toroidal compactification. Similarly, we have

$$M_{j,k}(\Gamma[2]) = H^0(\mathcal{A}_2[\Gamma[2]], \mathrm{Sym}^j(\mathbb{E}) \otimes L^k) = H^0(\tilde{\mathcal{A}}_2[\Gamma[2]], \mathrm{Sym}^j(\mathbb{E}) \otimes L^k).$$

Note that the Hodge bundle $\mathbb{E}$ extends to the toroidal compactification and $L$ extends to the Satake compactification. The $R^{\rm ev}$-modules $\mathcal{M}_j^\epsilon$ for $\epsilon = 0, 1$ are of the form

$$\bigoplus_k H^0(X, F \otimes L^k)$$

over $R^{\rm ev} = \bigoplus_k H^0(X, L^k)$ and as $L$ is an ample line bundle on the Satake compactification (but only nef on $\tilde{\mathcal{A}}_2[\Gamma[2]]$) they are finitely generated (cf. [24, pp. 98–100]). Recall that we have

$$R^{\rm ev} = \mathbb{C}[u_0, \ldots, u_4]/(f)$$

with $f$ a homogeneous polynomial of degree 4 in the $u_i$. We set

$$T = \mathbb{C}[u_0, \ldots, u_4].$$

The group $\mathfrak{S}_6$ acts on it; the action on the space of homogeneous polynomials of degree 1 is the irreducible representation $s[2^3]$. So we may write $T$ as the symmetric algebra $\mathrm{Sym}^* s[2^3]$ and $R^{\rm ev}$ as the (virtual) $T$-module

$$R^{\rm ev} = T - T(-4). \tag{19.1}$$

We can view $\mathcal{M}_j^\epsilon$ as a $T$-module and then a theorem of Hilbert [26, p. 56] tells us that it is of finite presentation. But because of (19.1) it is then not of finite presentation when viewed as a module over $R^{\rm ev}$ and we get (infinite) periodicity. The fact that it is also a $R^{\rm ev}$-module implies that its Euler characteristic is zero as stated in Remark 12.2.

By what was observed in Sec. 14 the situation is similar for modules of modular forms on $\Gamma_1[2]$ over the ring $\bigoplus_k M_{0,4k}(\Gamma_1[2])$. Again we can view the modules as modules over a polynomial ring in five variables (with a non-modular $\mathfrak{S}_6$-action).

In order to determine the structure of these modules it is useful to have bounds on the weight of generators and relations of these modules. Here the notion of Castelnuovo–Mumford regularity applies. We refer to [24, I, pp. 90 ff]. Let $F_{r,s}$ be the vector bundle $\mathrm{Sym}^r(\mathbb{E}) \otimes \det(\mathbb{E})^s$ and $F'_{r,s} = \mathrm{Sym}^r(\mathbb{E}) \otimes \det(\mathbb{E})^s \otimes O(-D)$, where $D$ is the divisor at infinity of the toroidal compactification $X = \tilde{\mathcal{A}}_2[\Gamma[2]]$. So the sections of $F_{j,k}$ are the modular forms of weight $(j,k)$ on $\Gamma[2]$ and those of $F'_{j,k}$ the cusp forms of weight $(j,k)$ on $\Gamma[2]$. We consider the modules

$$\mathcal{M}_j^\epsilon = \bigoplus_{k \equiv \epsilon \bmod 2} \Gamma(X, F_{j,k}) \quad \text{and} \quad \Sigma_j^\epsilon = \bigoplus_{k \equiv \epsilon \bmod 2} \Gamma(X, F'_{j,k}).$$

Here we can consider these as modules over the polynomial ring $\mathbb{C}[u_0, \ldots, u_4]$.

Recall that one calls a vector bundle $F$ $m$-regular in the sense of Castelnuovo–Mumford with respect to an ample line bundle $\mathcal{L}$ if

$$H^i(X, F \otimes \mathcal{L}^{m-i}) = 0 \quad \text{for } i > 0.$$

The relevance of this notion is that it implies

(1) $F$ is generated by its global sections,
(2) for $k \geq 0$ the natural maps

$$H^0(X, \mathcal{L}^k) \otimes H^0(X, F \otimes \mathcal{L}^m) \to H^0(X, F \otimes \mathcal{L}^{m+k})$$

are surjective.

In our case we will apply this to the case $\mathcal{L} = \det(\mathbb{E})^2$ and $F = F_{j,r}$ or $F'_{j,r}$ for some $j$ and small $r$. However, since $\mathcal{L}$ is ample only on $\mathcal{A}_2[\Gamma[2]]$ and nef on $\tilde{\mathcal{A}}_2[\Gamma[2]]$ one needs to adapt these notions slightly. The main point is that by the Koecher Principle the sections of $F_{j,r}$ on $\mathcal{A}_2[\Gamma[2]]$ automatically extend to sections over all of $\tilde{\mathcal{A}}_2[\Gamma[2]]$. The cohomological mechanism (cf. [24, Vol. I, proof of Theorem 1.8.3]) thus works the same way.

Note that we have Serre duality

$$H^i(X, F_{j,k})^\vee = H^{3-i}(X, F_{j,3-j-k} \otimes O(D)), \quad H^i(X, F'_{j,k})^\vee = H^{3-i}(X, F_{j,3-j-k}).$$

So a necessary condition for $F'_{j,0}$ (respectively, $F'_{j,1}$) being $m$-regular with respect to $\det(\mathbb{E})^2$ is that $H^3(X, F'_{j,0} \otimes \det(\mathbb{E})^{2m-6}) = 0$ and by Serre duality this gives

$$M_{j,9-2m-j}(\Gamma[2]) = (0) \quad (\text{respectively}, M_{j,8-2m-j}(\Gamma[2]) = (0)).$$

So the dimension formulas give restrictions on the regularity. A bound on the regularity gives bounds on the weights of generators, see for example [24, Vol. I, Theorem 1.8.26].

We give here two results on the regularity.

**Proposition 19.1.** *The vector bundle* $F'_{2,1} = \mathrm{Sym}^2(\mathbb{E}) \otimes \det(\mathbb{E}) \otimes O(-D)$ *is* 3*-regular with respect to* $\det(\mathbb{E})^2$.

**Proof.** We have to prove the vanishing of $H^1(X, F'_{2,5})$, $H^2(X, F'_{2,3})$ and $H^3(X, F'_{2,1})$. By Serre duality the vanishing of the $H^3$ comes down to the non-existence of modular forms of weight $(2, 0)$. The cohomology $H^1(X, F'_{2,5})$ occurs as the first step of the Hodge filtration of the compactly supported cohomology $H^4_c(\mathcal{A}_2[\Gamma[2]], \mathbb{V}_{4,2})$, see [11, 4]. Here $\mathbb{V}_{k,l}$ is a local system defined in [4]. In fact, the Hodge filtration on $H^i_c(\mathcal{A}_2[\Gamma[2]], \mathbb{V}_{k,l})$ has the steps $H^i(X, F'_{k-l,-k})$, $H^{i-1}(X, F'_{k+l+2,-k})$, $H^{i-2}(X, F'_{k+l+2,1-l})$ and $H^{i-3}(X, F'_{k-l,l+3})$. Since $\mathbb{V}_{4,2}$ is a regular local system the $H^4_c$ consists only of Eisenstein cohomology by results of Saper and Faltings, cf. [9, 28]. By Eisenstein cohomology we mean the kernel of the natural map $H^\bullet_c(\mathcal{A}_2[\Gamma[2]], \mathbb{V}_{k,l}) \to H^\bullet(\mathcal{A}_2[\Gamma[2]], \mathbb{V}_{k,l})$. This cohomology is known by results of Harder (see [14, 31]) and does not contain a contribution of this type. Harder dealt with the case of level 1, but the results can easily be extended to the case of level 2, cf. also [31, 3]. The cohomology $H^2(X, F'_{2,3})$ occurs in $H^5_c(\mathcal{A}_2[\Gamma[2]], \mathbb{V}_{2,0})$ and again in the Eisenstein cohomology. But this contribution is zero, see [14, 4]. □

**Proposition 19.2.** *The vector bundle* $F_{2,0} = \mathrm{Sym}^2(\mathbb{E})$ *is* 3*-regular with respect to* $\det(\mathbb{E})^2$.

**Proof.** Now we have to show the vanishing of $H^1(X, F_{2,4})$, $H^2(X, F_{2,2})$ and $H^3(X, F_{2,0})$. Instead of compactly supported cohomology we now look at the Hodge filtration of $H^i(\mathcal{A}_2[\Gamma[2]], \mathbb{V}_{k,l})$ with the steps $H^i(X, F_{k-l,-k})$, $H^{i-1}(X, F_{k+l+2,-k})$, $H^{i-2}(X, F_{k+l+2,1-l})$ and $H^{i-3}(X, F_{k-l,l+3})$. The space $H^1(X, F_{2,4})$ occurs in $H^4(\mathcal{A}_2[\Gamma[2]], \mathbb{V}_{3,1})$. Again this is Eisenstein cohomology, i.e. occurs in the cokernel of the natural map $H^\bullet_c \to H^\bullet$ and it vanishes. Similarly, $H^2(X, F_{2,2})$ occurs in $H^5(\mathcal{A}_2[\Gamma[2]], \mathbb{V}_{3,1})$. For $H^3(X, F_{2,0})$ we take the Serre dual $H^0(X, F_{2,1} \otimes O(D))$. But any section of this on $\mathcal{A}_2[\Gamma[2]]$ extends by the Koecher principle to a modular form of weight $(2, 1)$ and thus vanishes; indeed, it is automatically a cusp form and if $S_{2,1}(\Gamma[2]) \neq (0)$ we land by multiplying with $\psi_4 \in M_{0,4}(\Gamma)$ in $S_{2,5}(\Gamma[2])$ which is the $\mathfrak{S}_6$-representation $s[2^2, 1^2]$, and hence $S_{2,1}(\Gamma[2])$ is a $s[2^2, 1^2]$ too; then $\chi_5 S_{2,1}(\Gamma[2])$ is a $s[4, 2]$; but $S_{2,6}(\Gamma[2])$ does not contain a $s[4, 2]$. For another argument see the proof of Lemma 20.2. □

## 20. The Module $\Sigma_2(\Gamma[2])$

In this section we determine the structure of the $R^{\mathrm{ev}}$-module of cusp forms

$$\Sigma_2 = \Sigma_2(\Gamma[2]) = \bigoplus_{k=0, k\,\mathrm{odd}}^{\infty} S_{2,k}(\Gamma[2]).$$

We construct modular forms $\Phi_i$ for $i = 1, \ldots, 10$ in the first nonzero summand $S_{2,5}(\Gamma[2])$ of $\Sigma_2$ by setting

$$\Phi_i = [x_i, \chi_5]/x_i = [\vartheta_i^4, \chi_5]/\vartheta_i^4 = 4[\vartheta_i, \vartheta_1 \ldots \hat{\vartheta}_i \ldots \vartheta_{10}].$$

**Remark 20.1.** Some Fourier coefficients of $\Phi_1$ are given in Sec. 25. Eigenvalues of Hecke operators acting on the space $S_{2,5}(\Gamma[2])$ were calculated in [3].

The main result in this section is the following.

**Theorem 20.1.** *The* 10 *modular forms* $\Phi_i$ *generate the* $R^{\mathrm{ev}}$*-module* $\Sigma_2(\Gamma[2])$.

**Remark 20.2.** As a module over the polynomial ring $T$ in five variables the module $\Sigma_2(\Gamma[2])$ is generated by the $\Phi_i$ with relations of type $s[1^6]$ in weight $(2, 5)$, $s[5, 1]$ in weight $(2, 7)$ and type $s[3^2]$ in weight $(2, 9)$ and a syzygy in weight $(2, 11)$. But over the ring of modular forms of even weight, which we recall is $T - T(-4)$, it is not of finite presentation and this pattern of (virtual) generators and relations is repeated indefinitely (modulo 8).

Before giving the proof we sketch its structure. We can calculate the action of $\mathfrak{S}_6$ on the spaces $S_{2,k}(\Gamma[2])$ of modular forms (assuming the conjectures of [3]) for small $k$. This suggests that there are nine generators in weight $(2,5)$. We construct these forms and show (directly, not using the conjectures of [3]) that these forms generate $S_{2,k}(\Gamma[2])$ over the ring of even weight scalar-valued modular forms for $k \leq 13$. We also calculate the relations up to weight 13. We then use the bound on the Castelnuovo–Mumford regularity of the module $\Sigma_2$ over $T$ and this shows that there are no further relations between our purported generators and a comparison of generating functions shows that we found the whole module $\Sigma_2$. Thus the result is independent of the conjectures in [3].

We begin by giving a table for the decomposition of $S_{2,k}(\Gamma[2])$ as a $\mathfrak{S}_6$-representation for small odd $k$. At the end of this section we shall prove that $S_{2,k}(\Gamma[2]) = (0)$ for $k = 1$ and $k = 3$.

| $S_{2,k}\backslash P$ | $[6]$ | $[5,1]$ | $[4,2]$ | $[4,1^2]$ | $[3^2]$ | $[3,2,1]$ | $[3,1^3]$ | $[2^3]$ | $[2^2,1^2]$ | $[2,1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_{2,5}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $S_{2,7}$ | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| $S_{2,9}$ | 0 | 1 | 0 | 2 | 1 | 2 | 1 | 0 | 3 | 1 | 1 |
| $S_{2,11}$ | 0 | 2 | 1 | 4 | 3 | 5 | 2 | 0 | 4 | 1 | 1 |
| $S_{2,13}$ | 0 | 2 | 2 | 6 | 5 | 9 | 4 | 1 | 8 | 2 | 1 |

Besides the $\Phi_i$ we can construct the weight $(2,5)$ forms

$$\phi_{ij} = \left(\prod_{k\neq i,j} \vartheta_k\right) [\vartheta_i, \vartheta_j] \quad (1 \leq i,j \leq 10).$$

To check that the $\phi_{ij}$ are modular forms on $\Gamma[2]$ one can use (an analogue of) Proposition 16.1. Clearly $\phi_{ii} = 0$ and $\phi_{ij} = -\phi_{ji}$. Furthermore, these satisfy $\phi_{ij} + \phi_{jk} + \phi_{ki} = 0$. One sees easily that $\Phi_i = 4\sum_{j=1}^{10} \phi_{ij}$. We also have the relations $\phi_{ij} = \phi_{1j} - \phi_{1i}$ and $\phi_{1i} = (1/40)(\Phi_1 - \Phi_i)$ and one thus obtains the relation

$$\sum_{i=1}^{10} \Phi_i = 0. \tag{20.1}$$

To prove Theorem 20.1 we begin by analyzing the $\mathfrak{S}_6$ action on the $\Phi_i$.

**Lemma 20.1.** *The* 10 *forms* $\Phi_i$ *generate the* 9-*dimensional* $s[2^2,1^2]$-*isotypic subspace of* $S_{2,5}(\Gamma[2])$ *and satisfy the relation* $\sum_{i=1}^{10} \Phi_i = 0$.

**Proof.** We calculate the action of $\mathfrak{S}_6$ on the $\Phi_i$ $(i = 1,\ldots,10)$ and find that it is formally a representation $s[2^2,1^2] + s[1^6]$. The $s[1^6]$ corresponds to the relation (20.1). Since the $\Phi_i$ are nonzero these must generate an irreducible representation $s[2^2,1^2]$ in $S_{2,5}(\Gamma[2])$. Alternatively, by restricting to the components of the Humbert surface $H_1$ one can also check that $\Phi_i$ for $i = 1,\ldots,9$ are linearly independent, cf. the proof of Lemma 20.2. □

We now give the proof of Theorem 20.1. We first show that the $\Phi_i$ generate $S_{2,7}(\Gamma[2])$ and $S_{2,9}(\Gamma[2])$ and we find relations there.

We obtain a relation in weight (2,7) as follows. A linear relation between the $x_i$ like $x_1 - x_4 - x_6 - x_7 = 0$ implies by linearity of the bracket a relation $[x_1, \chi_5] - [x_4, \chi_5] - [x_6, \chi_5] - [x_7, \chi_5] = 0$ and we can rewrite it as

$$x_1\Phi_1 - x_4\Phi_4 - x_6\Phi_6 - x_7\Phi_7 = 0.$$

Since the relations among the 10 $x_i$ generate an irreducible representation $s[2, 1^4]$, we get in this way a space $s[2, 1^4] \otimes s[1^6] = s[5, 1]$ of relations between the $\Phi_i$ over $R^{\mathrm{ev}}$ in weight (2,7).

One can check that the projections of the space generated by the $x_i\Phi_j$ to the $s[4, 1^2]$, $s[3^2]$, $s[3, 2, 1]$ and $s[2^2, 1^2]$-part do give nonzero modular forms, and comparing this to the decomposition of $S_{2,7}(\Gamma[2])$ into irreducible $\mathfrak{S}_6$ representations shows that the $\Phi_i$ generate $S_{2,7}(\Gamma[2])$ over the ring of scalar-valued modular forms. Now $M_{0,2}(\Gamma[2]) = s[2^3]$ and $S_{2,5}(\Gamma[2]) = s[2^2, 1^2]$ and comparing the two representations

$$\begin{aligned} s[2^3] \otimes s[2^2, 1^2] &= s[5, 1] + s[4, 1^2] + s[3^2] + s[3, 2, 1] + s[2^2, 1^2], \\ S_{2,7}(\Gamma[2]) &= s[4, 1^2] + s[3^2] + s[3, 2, 1] + s[2^2, 1^2] \end{aligned}$$

it follows that we must have an irreducible representation $s[5, 1]$ of relations, which is just the space given above.

In a similar way we compare the representations in weight (2,9). We find that $S_{2,5}(\Gamma[2]) \otimes M_{0,4}(\Gamma[2])$ equals as an $\mathfrak{S}_6$ representation the representation of $S_{2,9}(\Gamma_2[2])$ plus $s[3^2] + s[3, 2, 1] + s[2^2, 1^2]$; the contribution $s[3, 2, 1] + s[2^2, 1^2]$ to this excess comes from $M_{0,2}(\Gamma[2]) \otimes s[5, 1]$ where $s[5, 1]$ are the relations in weight $(2, 7)$. We are thus left with a relation space $s[3^2]$ in weight $(2, 9)$. Indeed, by calculating the projections we check that $M_{0,4} \otimes S_{2,5} \to S_{2,9}$ is surjective and the explicit relations can be computed by projection on the $s[3^2]$-subspace. Since the coefficients are not simple we refrain from giving these.

We can check again by projection on the isotopic subspaces of $S_{2,11}$ that $M_{0,6} \otimes S_{2,5} \to S_{2,11}$ is surjective. We thus find a syzygy of type $s[1^6]$ in weight $(2, 11)$. By the result on the Castelnuovo–Mumford regularity of Proposition 19.1 there can be no further relations. Therefore the $\Phi_i$ generate a submodule of $\Sigma_2(\Gamma[2])$ with Hilbert function

$$\frac{9t^5 - 5t^7 - 5t^9 + t^{11}}{(1 - t^2)^5}.$$

Since this coincides with the generating series given in Sec. 12, the $\Phi_i$ must generate the whole module. This completes the proof of the theorem.

**Remark 20.3.** If we work over the function field $\mathcal{F}$ of $\mathcal{A}_2[\Gamma[2]]$ and consider the module $\Sigma_2 \otimes \mathcal{F}$ of meromorphic sections of $\mathrm{Sym}^2(\mathbb{E})$ with $\mathbb{E}$ the Hodge bundle, then the submodule $F$ generated by the $\Phi_i$ has rank at least 3 since the wedge product

$\Phi_1 \wedge \Phi_2 \wedge \Phi_3$ does not vanish identically. Using the relation $\sum_{i=1}^{10} \Phi_i = 0$ and the five relations of weight (2,7)

$$x_6\Phi_6 = x_1\Phi_1 - x_2\Phi_2 + x_3\Phi_3 - x_4\Phi_4 - x_5\Phi_5,$$
$$x_7\Phi_7 = x_2\Phi_2 - x_3\Phi_3 + x_5\Phi_5, \quad x_8\Phi_8 = x_1\Phi_1 - x_4\Phi_4 - x_5\Phi_5,$$
$$x_9\Phi_9 = -x_3\Phi_3 + x_4\Phi_4 + x_5\Phi_5, \quad x_{10}\Phi_{10} = x_1\Phi_1 - x_2\Phi_2 - x_5\Phi_5$$

we see that $F$ is generated by $\Phi_i$ with $i = 1, \ldots, 4$. Indeed, after inverting the $x_i$ we can eliminate $\Phi_6, \ldots, \Phi_9$ and then using (20.1) and $x_{10}\Phi_{10} = x_1\Phi_1 - x_2\Phi_2 - x_5\Phi_5$ we can also eliminate $\Phi_5$. Using the relations of type $s[3^2]$ in weight $(2, 9)$ we can eliminate $\Phi_4$ too and reduce the generators of $F$ to $\Phi_1, \Phi_2, \Phi_3$. We refrain from giving the explicit relation. So outside the zero divisor of the wedge $\Phi_1 \wedge \Phi_2 \wedge \Phi_3$ the forms $\Phi_1, \Phi_2$ and $\Phi_3$ generate the bundle $\mathrm{Sym}^2(\mathbb{E}) \otimes \det(\mathbb{E})^5$.

We now give some wedges of the $\Phi_i$ that give information about the vanishing loci of the $\Phi_i$.

**Proposition 20.1.** *We have in $S_{0,18}(\Gamma[2])$ the identity*

$$\Phi_1 \wedge \Phi_2 \wedge \Phi_3 = 25\chi_5^2(x_6 - x_5)(3\vartheta_5^2\vartheta_6^2\vartheta_7^2\vartheta_8^2\vartheta_9^2\vartheta_{10}^2 + \vartheta_1^2\vartheta_2^2\vartheta_3^2(\vartheta_6^2\vartheta_8^2\vartheta_9^2 - \vartheta_5^2\vartheta_7^2\vartheta_{10}^2))/8.$$

The proposition is proved by brute force by computing a basis of the space of modular forms involved. We know that $\Phi_1 \wedge \Phi_2 \wedge \Phi_3$ is divisible by $\chi_5^2$ and $x_5 - x_6$, hence the quotient is a form $f_6$ of weight 6, and actually a cusp form. In $S_{0,6}(\Gamma[2])$, a representation of type $s[2^3]$, there are five linearly independent modular forms $g_i$ $(i = 1, \ldots, 5)$ that are products of squares of six theta constants with characteristics given by

$$[1, 2, 3, 5, 7, 10], [1, 2, 3, 6, 8, 9], [1, 2, 4, 5, 8, 10], [1, 3, 4, 5, 8, 9], [5, 6, 7, 8, 9, 10]$$

respectively. By computing the Fourier expansion of $f_6$ and of the latter forms $g_1, \ldots, g_5$, we get the proposition.

Using the same method, we find

$$\Phi_1 \wedge \Phi_2 \wedge \Phi_4 = 25\chi_5^2(x_6 - x_5)(g_1 + g_2 - 2g_3 - 3g_5)/8,$$
$$\Phi_1 \wedge \Phi_3 \wedge \Phi_4 = 25\chi_5^2(x_6 - x_5)(g_1 + g_2 - 2g_4 + 3g_5)/8,$$
$$\Phi_2 \wedge \Phi_3 \wedge \Phi_4 = 25\chi_5^2(x_6 - x_5)(-g_1 + g_2 + 2g_3 - 2g_4 - g_5)/8.$$

We now prove that $S_{2,1}(\Gamma[2])$ and $S_{2,3}(\Gamma[2])$ are both zero.

**Lemma 20.2.** *We have $S_{2,1}(\Gamma[2]) = (0)$ and $S_{2,3}(\Gamma[2]) = (0)$.*

**Proof.** Since multiplication by $x_1$ is injective the vanishing of $S_{2,3}(\Gamma[2])$ implies the vanishing of $S_{2,1}(\Gamma[2])$. We thus have to prove that $S_{2,3}(\Gamma[2]) = (0)$. The injectivity of multiplication by $x_1$ applied to $S_{2,3}(\Gamma[2])$ and $\dim S_{2,5}(\Gamma[2]) = 9$ implies that $\dim S_{2,3}(\Gamma[2]) \leq 9$. But since not every $\Phi_i$ is divisible by $x_1$, as follows from calculating the restriction to the components of the Humbert surface $H_1$:

$$\Phi_i\left(\begin{pmatrix}\tau_{11} & 0\\ 0 & \tau_{22}\end{pmatrix}\right) = c_i(\vartheta_{00}\vartheta_{01}\vartheta_{10})^4(\tau_{11}) \otimes (\vartheta_{00}\vartheta_{01}\vartheta_{10})^4(\tau_{22})\begin{pmatrix}0\\2\\0\end{pmatrix}$$

with $c_i = -1/4$ for $i = 1,\dots,9$ and $c_{10} = 9/4$. We see that $\dim S_{2,3}(\Gamma[2]) < 9$. By multiplication by $\psi_4$ and $\psi_6$ in $M_{0,4}(\Gamma)$ and $M_{0,6}(\Gamma)$ we land in $S_{2,7}(\Gamma[2])$ and $S_{2,9}(\Gamma[2])$ and by inspection we see that the only irreducible representations in common in $S_{2,7}(\Gamma[2])$ and $S_{2,9}(\Gamma[2])$ are of type $s[4,1^2]$, $s[3^2]$, $s[3,2,1]$ and $s[2^2,1^2]$, so for dimension reasons we must have $S_{2,3}(\Gamma[2]) = s[3^2]$ if it is nonzero. If $S_{2,3}(\Gamma[2]) = s[3^2]$ we find that $S_{2,3}(\Gamma_1[2]) = s[1^3]$ as an $\mathfrak{S}_3$-representation and since $M_{0,2}(\Gamma_1[2]) = s[3]$ we find a representation $s[1^3]$ in $S_{2,5}(\Gamma_1[2])$. But we know that $S_{2,5}(\Gamma_1[2]) = (0)$. Therefore $S_{2,3}(\Gamma[2]) = (0)$. □

## 21. The Module $\Sigma_2(\Gamma_1[2])$

Recall that the ring $R^{\mathrm{ev}}(\Gamma[2]) = \bigoplus_k M_{0,2k}(\Gamma[2])$ is abstractly isomorphic to the ring $R' = \bigoplus_k M_{0,4k}(\Gamma_1[2])$.

We therefore look at the following two $R'$-modules

$$\Sigma^1 = \Sigma^1(\Gamma_1[2]) = \bigoplus_k S_{2,4k+1}(\Gamma_1[2]) \quad \text{and}$$
$$\Sigma^3 = \Sigma^3(\Gamma_1[2]) = \bigoplus_k S_{2,4k+3}(\Gamma_1[2]).$$

As before, we can consider these as modules over a polynomial ring in five variables as well as over the ring of scalar-valued modular forms.

**Theorem 21.1.** *The module $\Sigma^1(\Gamma_1[2])$ is generated over the ring $R'$ by the nine cusp forms of weight $(2,9)$ generating a $\mathfrak{S}_3$ representation $3s[2,1]+3s[1^3]$. The module $\Sigma^3(\Gamma_1[2])$ is generated by the four modular forms of weight $(2,7)$ forming a $\mathfrak{S}_3$-representation $s[2,1]+2s[1^3]$ and the two pairs of forms of weight $(2,11)$ each forming a $\mathfrak{S}_3$-representation $s[2,1]$.*

**Remark 21.1.** The generating functions for the dimensions of the graded pieces of these modules are

$$\frac{9t^9 - 5t^{13} - 5t^{17} + t^{21}}{(1-t^4)^5} \quad \text{and} \quad \frac{4t^7 + 4t^{11} - 8t^{15}}{(1-t^4)^5}.$$

This follows now from the results on $\Sigma_2(\Gamma[2])$.

**Proof of Theorem 21.1.** In order to construct the generators explicitly, we look at the eigenspaces of the action of $\Gamma_1[2]/\Gamma[2] = (\mathbb{Z}/2\mathbb{Z})^3$; this group is generated by $(12), (34)$ and $(56) \in \mathfrak{S}_6$. So for a triple $\epsilon$ of signs we have a corresponding eigenspace $M_{j,k}^{\epsilon} \subset M_{j,k}(\Gamma[2])$. We have maps

$$M_{0,k_1}^{\epsilon} \times M_{j,k_2}^{\epsilon} \to M_{j,k_1+k_2}(\Gamma_1[2]), \quad M_{0,k_1}^{\epsilon} \times M_{0,k_2}^{\epsilon} \to M_{2,k_1+k_2}(\Gamma_1[2]),$$

given by the product $(f, g) \mapsto fg$, respectively, by the Rankin–Cohen bracket $(f, g) \mapsto [f, g]$.

For example, a form in the $s[1^6]$-part of $M_{0,k}(\Gamma[2])$ gives rise to a form in $M_{0,k}^{---}$; so as soon as this $s[1^6]$-part is not empty we get forms in $M_{2,k+5}(\Gamma_1[2])$ by taking the bracket $\psi \mapsto [\psi, \chi_5]$. Using this idea we can construct forms in the following way.

As the four generators in weight $(2, 7)$ one can take: $F_1 = (x_5 - x_6)(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4)$, and

$$F_2 = (x_5 + x_6)(\Phi_5 - \Phi_6), \quad F_3 = (x_7 + x_8)(\Phi_7 - \Phi_8),$$
$$F_4 = (x_9 + x_{10})(\Phi_9 - \Phi_{10}),$$

with $F_1$, $F_2 + F_3 + F_4$ generating $2s[1^3]$ and $F_1 - F_3$ and $F_2 - F_3$ generating a $s[2, 1]$.

To construct cusp forms of weight $(2, 9)$ we take $A_1 = s_1 F_2$, $A_2 = s_1 F_3$ and $A_3 = s_1 F_4$ and

$$\begin{aligned} A_4 &= (x_5 + x_6)(x_7 + x_8)(\Phi_9 - \Phi_{10}), & A_7 &= (x_5 + x_6)\xi(\Phi_1 - \Phi_2 + \Phi_3 - \Phi_4), \\ A_5 &= (x_7 + x_8)(x_9 + x_{10})(\Phi_5 - \Phi_6), & A_8 &= (x_9 + x_{10})\xi(\Phi_1 - \Phi_2 - \Phi_3 + \Phi_4), \\ A_6 &= (x_5 + x_6)(x_9 + x_{10})(\Phi_7 - \Phi_8), & A_9 &= (x_7 + x_8)\xi(\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4). \end{aligned}$$

Finally, to construct generators of weight $(2, 11)$ we consider

$$\begin{aligned} L_1 &= (x_1^3 + x_2^3 - x_3^3 - x_4^3)(\Phi_7 - \Phi_8), \\ M_1 &= \xi(x_7 + x_8)(x_9 + x_{10})(\Phi_1 - \Phi_2 + \Phi_3 - \Phi_4), \\ L_2 &= (x_1^3 - x_2^3 - x_3^3 + x_4^3)(\Phi_9 - \Phi_{10}), \\ M_2 &= \xi(x_5 + x_6)(x_7 + x_8)(\Phi_1 - \Phi_2 - \Phi_3 + \Phi_4), \\ L_3 &= (x_1^3 - x_2^3 + x_3^3 - x_4^3)(\Phi_5 - \Phi_6), \\ M_3 &= \xi(x_5 + x_6)(x_9 + x_{10})(\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4). \end{aligned}$$

The generators that we need are the nine forms $A_i$ of weight $(2, 9)$ generating a representation $3s[2, 1] + 3s[1^3]$, the four modular forms $F_i$ of weight $(2, 7)$ generating a representation $s[2, 1] + 2s[1^3]$ and the two pairs $L_3 - L_1$, $L_3 - L_2$ and $M_1 - M_2$, $M_1 - M_3$ of weight $(2, 11)$, each generating a representation $s[2, 1]$. One checks that these forms generate up to weight $(2, 19)$ and that the Castelnuovo–Mumford regularity is bounded by 3. This finishes the proof. □

## 22. The Module $\mathcal{M}_2(\Gamma[2])$ and its $\Gamma_1[2]$-Analogue

In this section, we determine the structure of the $R^{\mathrm{ev}}$-module

$$\mathcal{M}_2 = \mathcal{M}_2(\Gamma[2]) = \bigoplus_{k=0}^{\infty} M_{2,2k}(\Gamma[2]).$$

In Example 16.2 we constructed 15 modular forms $G_{ij}$ in $M_{2,4}(\Gamma[2])$; recall that these are proportional to the $H_{ij}$. As shown in Remark (16.2) these are linearly independent.

**Theorem 22.1.** *The $R^{\mathrm{ev}}$-module $\mathcal{M}_2$ is generated by the fifteen modular forms $G_{ij}$.*

As in the preceding section the $\mathfrak{S}_6$ action is an essential tool for proving this theorem. We list the representations involved:

| $M_{2,k}\backslash P$ | $[6]$ | $[5,1]$ | $[4,2]$ | $[4,1^2]$ | $[3^2]$ | $[3,2,1]$ | $[3,1^3]$ | $[2^3]$ | $[2^2,1^2]$ | $[2,1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $M_{2,4}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| $M_{2,6}$ | 0 | 0 | 1 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 |
| $M_{2,8}$ | 0 | 0 | 2 | 1 | 0 | 3 | 3 | 2 | 1 | 2 | 0 |
| $M_{2,10}$ | 1 | 2 | 5 | 3 | 0 | 5 | 5 | 3 | 2 | 3 | 0 |
| $M_{2,12}$ | 0 | 1 | 7 | 4 | 1 | 11 | 8 | 6 | 4 | 4 | 0 |

The generating function for $\dim M_{2,k}$ with $k \geq 4$ even is

$$\frac{15t^4 - 19t^6 + 5t^8 - t^{10}}{(1-t^2)^5}. \tag{22.1}$$

We know already that the 15 forms $G_{ij}$ generate a $\mathfrak{S}_6$-representation $s[3,1^3] + s[2,1^4]$.

**Lemma 22.1.** *The* 15 *forms $G_{ij}$ with $1 \leq i < j \leq 6$ form a basis of the space $M_{2,4}(\Gamma[2])$.*

**Remark 22.1.** By using the forms $F_{ij} = [x_i, x_j] = [\vartheta_i^4, \vartheta_j^4]$ we find in this way the $\wedge^2 s[2^3] = s[3,1^3]$-part of $M_{2,4}$ as the $x_i$ generate a $s[2^3]$. The $F_{ij}$ can be expressed in the $G_{ij}$, for example

$$F_{12} = \frac{1}{\pi^2}(-G_{12} + G_{56} - G_{15} - G_{26}).$$

Now $M_{2,6}$ decomposes as $s[4,2] + 2s[3,2,1] + s[3,1^3] + s[2^3]$ as a representation space for $\mathfrak{S}_6$ and $M_{0,2} = s[2^3]$ and since

$$\begin{aligned} s[2,1^4] \otimes s[2^3] &= s[4,2] + s[3,2,1], \\ s[3,1^3] \otimes s[2^3] &= s[4,2] + s[4,1^2] + s[3,2,1] + s[3,1^3] + s[2^3] \end{aligned} \tag{22.2}$$

we expect to find relations of type $s[4,2] + s[4,1^2]$. One checks that $M_{0,2} \otimes M_{2,4}$ generates $M_{2,6}$. We get a $s[4,1^2]$ of relations of the form

$$x_i F_{jk} - x_j F_{ik} + x_k F_{ij} = 0. \tag{22.3}$$

These relations follow immediately from the Jacobi identity for brackets. The relations of type $s[4,2]$ either come from the vanishing of a $s[4,2]$ in the right-hand sides of (22.2) or from an identification of a copy of $s[4,2]$ in these right-hand sides. The latter is the case. We give an example of such a relation:

$$x_1(2G_{23} - G_{25} + G_{35} + G_{56}) - x_2(G_{24} + G_{45}) - x_3(G_{13} - G_{15}) - x_5G_{26} \\ + x_8(G_{36} + G_{56}) - x_9(G_{34} - G_{45}) + x_{10}(G_{12} - G_{15}) = 0.$$

Denote the left-hand side of the relation (22.3) by $R_{ijk}$. Then we have the syzygy

$$x_iR_{jkl} - x_jR_{ikl} + x_kR_{ijl} - x_lR_{ijk} = 0 \tag{22.4}$$

and it generates an irreducible representation $s[3^2]$ of relations in weight $(2,8)$.

In a similar way we expect a syzygy of type $s[1^6]$ in weight $(2,10)$. Write $R_{ijkl}$ for the left-hand side of (22.4). Then we have

$$x_1R_{2345} - x_2R_{1345} + x_3R_{1245} - x_4R_{1235} + x_5R_{1234} = 0. \tag{22.5}$$

This is a $S_6$-anti-invariant syzygy in weight $(2,10)$. By using the result on the regularity Proposition 19.2 we can derive now as we did above that we cannot have more relations. The $G_{ij}$ thus generate a submodule of $\mathcal{M}_2^{\text{ev}}$ with Hilbert function given by (22.1). Since this coincides with the generating function of our module $\mathcal{M}_2^{\text{ev}}$ we have found our module. This proves the theorem.

Since the forms $H'_{ij}$ defined in Example 15.2 satisfy similar relations we can deduce in a completely analogous way the following theorem.

**Theorem 22.2.** *The* 15 *modular forms* $H'_{ij} \in M_{2,8}(\Gamma_1[2])$ *from Example* 15.2 *generate the module* $\bigoplus_k M_{2,4k}(\Gamma_1[2])$ *over the module* $\bigoplus_k M_{0,4k}(\Gamma_1[2])$.

## 23. Other Modules

### 23.1. *The module* $\mathcal{M}_4$

We treat the $R^{\text{ev}}$-module $\mathcal{M}_4 = \bigoplus_k M_{4,2k}(\Gamma[2])$.

**Theorem 23.1.** *The module* $\mathcal{M}_4$ *over* $R^{\text{ev}}$ *is generated by six modular forms of weight* $(4,2)$ *generating a representation* $s[2,1^4]$, 15 *modular forms of weight* $(4,4)$ *generating a representation* $s[2,1^4]$ *and five modular forms of weight* $(4,4)$ *generating a representation* $s[2^3]$.

The proof is similar to the cases given above. First we look where the generators should appear, we then construct these and check that these generate $M_{4,k}$ for small $k$, and then use the bound on the Castelnuovo–Mumford regularity to bound the weight of the generators and relations.

The $\mathfrak{S}_6$ representations for small $k$ are as follows:

| $M_{4,k}\backslash P$ | $[6]$ | $[5,1]$ | $[4,2]$ | $[4,1^2]$ | $[3^2]$ | $[3,2,1]$ | $[3,1^3]$ | $[2^3]$ | $[2^2,1^2]$ | $[2,1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $M_{4,2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $M_{4,4}$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $M_{4,6}$ | 0 | 0 | 2 | 0 | 0 | 3 | 2 | 2 | 1 | 2 | 0 |
| $M_{4,8}$ | 1 | 2 | 5 | 2 | 0 | 6 | 4 | 3 | 2 | 4 | 0 |
| $M_{4,10}$ | 1 | 2 | 8 | 4 | 1 | 12 | 8 | 6 | 5 | 6 | 0 |
| $M_{4,12}$ | 2 | 5 | 14 | 8 | 3 | 20 | 13 | 9 | 8 | 8 | 0 |

The generating series is

$$\sum_{k\in 2\mathbb{Z}_{>0}} \dim M_{4,k} t^k = \frac{5t^2+10t^4-10t^6-10t^8+5t^{10}}{(1-t^2)^5}.$$

We also give the cusp forms:

| $S_{4,k}\backslash P$ | $[6]$ | $[5,1]$ | $[4,2]$ | $[4,1^2]$ | $[3^2]$ | $[3,2,1]$ | $[3,1^3]$ | $[2^3]$ | $[2^2,1^2]$ | $[2,1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_{4,4}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $S_{4,6}$ | 0 | 0 | 1 | 0 | 0 | 2 | 1 | 1 | 1 | 1 | 0 |
| $S_{4,8}$ | 0 | 1 | 3 | 2 | 0 | 5 | 3 | 2 | 2 | 3 | 0 |
| $S_{4,10}$ | 1 | 2 | 6 | 4 | 1 | 10 | 7 | 4 | 5 | 5 | 0 |
| $S_{4,12}$ | 1 | 4 | 11 | 8 | 3 | 18 | 12 | 7 | 8 | 7 | 0 |

The generating function is

$$\sum_{k\in\mathbb{Z}_{\geq 2}} \dim S_{4,2k} t^{2k} = \frac{5t^4+45t^6-95t^8+55t^{10}-10t^{12}}{(1-t^2)^5}.$$

Using the map $S_{4,2}\times M_{0,2}\to S_{4,6}$ we see that $S_{4,2}=(0)$. In weight $(4,2)$ we find a space of Eisenstein series $s[2,1^4]$ of dimension 5 instead of the usual $s[2^3]+s[2,1^4]$. We now construct generators for our module. We expect generators $s[2,1^4]$ in weight $(4,2)$, of type $s[2^3]+s[2,1^4]$ in weight $(4,4)$, relations of type $s[6]+s[4,2]$ both in weight $(4,6)$ and $(4,8)$ and a syzygy of type $s[2^3]$ in weight $(4,10)$. That is what we shall find.

**Proposition 23.1.** *The forms* $E_i=\mathrm{Sym}^4(G_i)$ *for* $i=1,\ldots,6$ *are modular forms of weight* $(4,2)$ *and satisfy the* $s[1^6]$*-type relation* $E_1-E_2-E_3+E_4-E_5+E_6=0$. *They generate the space* $M_{4,2}=M_{4,2}^{s[2,1^4]}$.

Here our convention is that if $G_i=[a,b]^t$ then $\mathrm{Sym}^4(G_i)=[a^4,4a^3b,6a^2b^2,4ab^3,b^4]^t$. The following lemma is proved by a direct calculation.

**Lemma 23.1.** *We have* $E_1\wedge E_2\wedge\cdots\wedge E_5=-96\pi^6\chi_5^4$.

**Corollary 23.1.** *Every linear relation of the form* $\sum f_iE_i=0$ *with* $f_i\in R^{\mathrm{ev}}$ *is a multiple of* $E_1-E_2-E_3+E_4-E_5+E_6=0$. *The* $E_i$ *generate a submodule of* $\mathcal{M}_4$ *with generating function* $5t^2/(1-t^2)^5$.

**Proof.** If $\sum_i f_i E_i = 0$ is a relation not in the ideal generated by $E_1 - E_2 - E_3 + E_4 - E_5 + E_6$ then over the function field $\mathcal{F}$ of $\mathcal{A}_2[\Gamma[2]]$ we can eliminate $E_6$ and $E_5$ and then the wedge would be zero contradicting Lemma 23.1. $\square$

To construct the forms in the $s[2,1^4]$ space of $M_{4,4}(\Gamma[2])$ we consider the 15 modular forms in the $\mathfrak{S}_6$-orbit of

$$D_{1234} = \mathrm{Sym}^4(G_1, G_2, G_3, G_4)\vartheta_5\vartheta_6\vartheta_7\vartheta_8 \in M_{4,4}(\Gamma[2]).$$

The formal representation is of type $s[3,1^3] + s[2,1^4]$, but these forms satisfy an irreducible representation of type $s[3,1^3]$ of relations generated by

$$4D_{1234} - D_{1235} - D_{1236} - D_{1245} - D_{1246} - D_{1345} - D_{1346} - D_{2345} - D_{2346} = 0.$$

These forms are cusp forms and generate the space of cusp forms $S_{4,4} = S_{4,4}^{s[2,1^4]}$. A basis is given by the forms $D_{1256}, D_{1345}, D_{1346}, D_{1356}$ and $D_{3456}$ as follows from the fact that

$$D_{1256} \wedge D_{1345} \wedge D_{1346} \wedge D_{1356} \wedge D_{3456} = -\pi^{20}\chi_5^6.$$

In fact, we find many linear identities between these forms. Simplifying one of those leads to an identity like

$$\begin{aligned}&\mathrm{Sym}^4(G_1, G_2, G_3, G_4)\vartheta_6\vartheta_7\vartheta_8\\&\quad = \mathrm{Sym}^4(G_1, G_3, G_4, G_5)\vartheta_3\vartheta_4\vartheta_9 + \mathrm{Sym}^4(G_1, G_3, G_4, G_6)\vartheta_1\vartheta_2\vartheta_{10}.\end{aligned}$$

Finally we construct generators in the $s[2^3]$-part of $M_{4,4}(\Gamma[2])$. We consider expressions

$$K_{i,j,k,l} = \mathrm{Sym}^4(G_i, G_i, G_j, G_j)\vartheta_k^2\vartheta_l^2$$

for appropriate quadruples $(i,j,k,l)$. For example we take $K_{1,2,1,3} \in M_{4,4}(\Gamma[2])$. These modular forms satisfy many relations, e.g.

$$K_{1,2,1,3} - K_{1,2,2,4} - K_{1,2,5,6} = 0 \quad \text{due to } \vartheta_1^2\vartheta_3^3 - \vartheta_2^2\vartheta_4^2 - \vartheta_5^2\vartheta_6^2 = 0.$$

We find 30 such forms in the $\mathfrak{S}_6$-orbit and as it turns out these are linearly independent and generate a $s[4,2] + s[3,2,1] + s[2^3]$ subspace of $M_{4,4}$. If $p$ denotes the projection on the $s[2^3]$-subspace the five forms $R_1 = p(K_{1,2,1,3})$, $R_2 = p(K_{1,2,2,4})$, $R_3 = p(K_{1,3,1,10})$, $R_4 = p(K_{1,3,4,9})$ and $R_5 = p(K_{1,4,2,10})$ form a basis of $s[2^3]$-subspace of $M_{4,4}(\Gamma[2])$ as a calculation shows. As it turns out their wedge is zero since these satisfy a ($\mathfrak{S}_6$-anti-invariant) relation

$$x_2R_1 - (x_2 + x_5)R_2 - (x_2 - x_4)R_3 - (x_1 - x_2 - x_5)R_4 + (x_2 - x_3 + x_5)R_5 = 0.$$

We now prove the theorem. One can show that this module is 3-regular in the sense of Castelnuovo–Mumford as in Sec. 19. Then one checks that these generators generate the spaces $M_{4,2k}$ for $k \leq 3$. By [24, Theorem 1.8.26] this suffices. This finishes the proof.

### 23.2. *The module* $\Sigma_4$

Another case is $\Sigma_4^{\mathrm{odd}}(\Gamma[2])$, where the representations are as follows:

| $S_{4,k}\backslash P$ | $[6]$ | $[5,1]$ | $[4,2]$ | $[4,1^2]$ | $[3^2]$ | $[3,2,1]$ | $[3,1^3]$ | $[2^3]$ | $[2^2,1^2]$ | $[2,1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_{4,3}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_{4,5}$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| $S_{4,7}$ | 0 | 1 | 1 | 1 | 1 | 3 | 1 | 0 | 2 | 1 | 0 |
| $S_{4,9}$ | 0 | 2 | 2 | 3 | 2 | 6 | 3 | 1 | 5 | 3 | 1 |
| $S_{4,11}$ | 0 | 4 | 5 | 7 | 4 | 12 | 5 | 2 | 8 | 4 | 1 |

We expect generators in weight $(4,5)$ of type $s[3,2,1]+s[2^2,1^2]+s[2,1^4]$; relations of type $s[5,1]+s[4,2]+s[4,1^2]+s[3,2,1]$ in weight $(4,7)$ and a generator of type $s[3,1^3]$ in weight $(4,9)$ and then periodic if viewed as a module over the ring of even weight scalar-valued modular forms on $\Gamma[2]$. For the generators of the $s[3,2,1]$ part of $S_{4,5}(\Gamma[2])$ we refer to Example 16.8 and we invite the reader to construct the remaining ones; in fact, $S_{4,5}(\Gamma[2])$ is generated by 30 cusp forms of the following shape

$$F_{abcd} = \frac{\chi_5}{\vartheta_a\vartheta_b\vartheta_c\vartheta_d}\mathrm{Sym}^2([\vartheta_a,\vartheta_b],[\vartheta_c,\vartheta_d])$$

for appropriate quadruples $(a,b,c,d)$ of distinct integers between 1 and 10.

## 24. Modular Forms of Level One

We can use our constructions and results to obtain modular forms of level 1. Note that the modules of vector-valued modular forms of level 1 for $j=2,4$ and 6 were determined by Ibukiyama, Satoh and van Dorp, see [15, 16, 29, 32]. Ibukiyama used theta series with harmonic coefficients. Here is a list of all cases where $\dim S_{j,k}(\Gamma) = 1$ for $k \geq 4$:

| $j$ | $k$ | | $j$ | $k$ | | $j$ | $k$ |
|---|---|---|---|---|---|---|---|
| 0 | $10,12,14,35,39,41,43$ | | 10 | $9,11$ | | 20 | 5 |
| 2 | $14,21,23,25$ | | 12 | $6,7$ | | 24 | 4 |
| 4 | $10,12,15,17$ | | 14 | 7 | | 28 | 4 |
| 6 | $8,10,11,13$ | | 16 | $6,7$ | | 30 | 4 |
| 8 | $8,9,11$ | | 18 | $5,6$ | | 34 | 4 |

We know that $\dim S_{j,2}(\Gamma)=0$ for $j=2,\ldots,10,14$.

In all cases we can write down an explicit form generating the space. For $j=0$ we know the generators by Igusa's description of the ring of modular forms. We give a number of these generators below, but note that all can be obtained from theta series with spherical coefficients for the $E_8$ lattice.

**Example 24.1.** (1) The form $\sum_{i=1}^{10}\chi_5\vartheta_i^8\Phi_i$ generates the space $S_{2,14}(\Gamma)$. This form is a multiple of the Rankin–Cohen bracket $[E_4,\chi_{10}]$ that occurs in the work of Satoh [29].

(2) The forms $[E_4,E_6,\chi_{10}]$ and $[E_4,E_6,\chi_{12}]$ generate $S_{2,21}(\Gamma)$ and $S_{2,23}(\Gamma)$, see [16].

(3) The form $\sum_{i=1}^{10} \mathrm{Sym}^2(\Phi_i)$ generates the space $S_{4,10}(\Gamma)$.
(4) The form $A = \chi_5 \mathrm{Sym}^6(G_1, \ldots, G_6)$ generates $S_{6,8}(\Gamma)$. A candidate generator for $S_{6,13}(\Gamma)$ is $\{E_4, A\}$, where we use the notation of [32].
(5) The form $\mathrm{Sym}^{12}(G_1, G_1, \ldots, G_6, G_6)$ generates $S_{12,6}(\Gamma)$.



## 25. Some Fourier Expansions

We give in the following two tables a few Fourier coefficients of $\Phi_1 \in S_{2,5}(\Gamma[2])$ and of $D_{1234} \in S_{4,4}(\Gamma[2])^{s[2,1^4]}$. We write the Fourier series as

$$\sum_{a,b,c} A(a,b,c) e^{\pi i (a\tau_{11} + b\tau_{12} + c\tau_{22})} = \sum_{a,c} \gamma(a,c) P(a,c) q_1^a q_2^c,$$

where the first sum runs over the triples $(a,b,c)$ of integers with $b^2 - 4ac < 0$. For a fixed pair $(a,c)$ we collect the coefficients of $q_1^a q_2^c = e^{\pi i (a\tau_{11} + c\tau_{22})}$ in the form of a vector of Laurent polynomials $\gamma(a,c)P(a,c)$ in $r = \exp \pi i \tau_{12}$ with $\gamma(a,c)$ an integer. Note that we have $P(c,a)$ equals $P(a,c)$ read in retrograde order: $P(c,a)_i = P(a,c)_{n-i}$ with $n = 3$ (for $\Phi_1$) or $n = 5$ (for $D_{1234}$).

| $[a,c]$ | $\gamma(a,c)$ | $P(a,c)$ |
|---|---|---|
| $[1,1]$ | 64 | $-r + 1/r$<br>$-r - 1/r$<br>$-r + 1/r$ |
| $[2,1]$ | 1280 | $r - 1/r$<br>$0$<br>$0$ |
| $[2,2]$ | 1280 | $r^3 - 3r + 3/r - 1/r^3$<br>$2r^3 - 2r - 2/r + 2/r^3$<br>$r^3 - 3r + 3/r - 1/r^3$ |
| $[3,1]$ | 64 | $3r^3 - 13r + 13/r - 3/r^3$<br>$3r^3 + 9r + 9/r + 3/r^3$<br>$r^3 + 9r - 9/r - 1/r^3$ |
| $[3,2]$ | 1280 | $-4r^3 + 12r - 12/r + 4/r^3$<br>$-4r^3 + 4r + 4/r - 4/r^3$<br>$-3r^3 - 3r + 3/r + 3/r^3$ |
| $[3,3]$ | 64 | $-13r^5 + 121r^3 - 250r + 250/r - 121/r^3 + 13/r^5$<br>$-35r^5 + 121r^3 - 230r - 230/r + 121/r^3 - 35/r^5$<br>$-13r^5 + 121r^3 - 250r + 250/r - 121/r^3 + 13/r^5$ |
| $[4,1]$ | 1280 | $-r^3 - 5r + 5/r + 1/r^3$<br>$0$<br>$0$ |
| $[4,2]$ | 1280 | $-r^5 + 5r^3 - 10r + 10/r - 5/r^3 + 1/r^5$<br>$-2r^5 - 10r^3 + 12r + 12/r - 10/r^3 - 2/r^5$<br>$-r^5 - 3r^3 + 14r - 14/r + 3/r^3 + 1/r^5$ |
| $[4,3]$ | 1280 | $5r^5 + 19r^3 + 14r - 14/r - 19/r^3 - 5/r^5$<br>$20r^5 - 28r^3 + 8r + 8/r - 28/r^3 + 20/r^5$<br>$12r^5 - 28r^3 + 24r - 24/r + 28/r^3 - 12/r^5$ |

(*Continued*)

(*Continued*)

| $[a,c]$ | $\gamma(a,c)$ | $P(a,c)$ |
|---|---|---|
| $[4,4]$ | 1280 | $-5r^7+19r^5-25r^3+15r-15/r+25/r^3-19/r^5+5/r^7$<br>$-10r^7+66r^5-10r^3-46r-46/r-10/r^3+66/r^5-10/r^7$<br>$-5r^7+19r^5-25r^3+15r-15/r+25/r^3-19/r^5+5/r^7$ |
| $[5,1]$ | 64 | $-5r^3+145r-145/r+5/r^3$<br>$-27r^3-27r-27/r-27/r^3$<br>$-9r^3-27r+27/r+9/r^3$ |
| $[5,2]$ | 1280 | $8r^5-8r^3-16r+16/r+8/r^3-8/r^5$<br>$8r^5+16r^3-24r-24/r+16/r^3+8/r^5$<br>$3r^5+14r^3-3r+3/r-14/r^3-3/r^5$ |
| $[5,3]$ | 64 | $-5r^7-270r^5+190r^3-745r+745/r-190/r^3+270/r^5+5/r^7$<br>$17r^7-270r^5+242r^3+659r+659/r+242/r^3-270/r^5+17/r^7$<br>$13r^7-250r^5+242r^3+217r-217/r-242/r^3+250/r^5-13/r^7$ |
| $[6,1]$ | 1280 | $5r^3-3r+3/r-5/r^3$<br>$0$<br>$0$ |
| $[6,2]$ | 1280 | $-13r^5+5r^3+50r-50/r-5/r^3+13/r^5$<br>$2r^5+18r^3-20r-20/r+18/r^3+2/r^5$<br>$3r^5-3r^3-6r+6/r+3/r^3-3/r^5$ |
| $[1,1]$ | 256 | $0$<br>$0$<br>$1$<br>$0$<br>$0$ |
| $[1,3]$ | 512 | $0$<br>$0$<br>$-1/r-4-r$<br>$2/r-2r$<br>$-2/r+4-2r$ |
| $[1,5]$ | 256 | $0$<br>$0$<br>$1/r^2+16/r+20+16r+r^2$<br>$-4/r^2-32/r+32r+4r^2$<br>$4/r^2+16/r-40+16r+4r^2$ |
| $[1,7]$ | 1024 | $0$<br>$0$<br>$-2/r^2-9/r-9r-2r^2$<br>$8/r^2+18/r-18r-8r^2$<br>$-6/r^2+6/r+6r-6r^2$ |
| $[3,3]$ | 1024 | $4/r^2-4/r-4r+4r^2$<br>$-10/r^2+8/r-8r+10r^2$<br>$-11/r^2-8/r+30-8r+11r^2$<br>$-10/r^2+8/r-8r+10r^2$<br>$4/r^2-4/r-4r+4r^2$ |
| $[3,5]$ | 1024 | $-6/r^3-2/r^2+8/r+8r-2r^2-6r^3$<br>$24/r^3-18/r+18r-24r^3$<br>$-39/r^3+26/r^2-68/r-68r+26r^2-39r^3$<br>$+30/r^3-8/r^2+46/r-46r+8r^2-30r^3$<br>$-6/r^3-16/r^2+22/r+22r-16r^2-6r^3$ |

(*Continued*)

(*Continued*)

| $[a, c]$ | $\gamma(a, c)$ | $P(a, c)$ |
|---|---|---|
| $[3, 7]$ | 1024 | $4/r^4 + 8/r^3 - 16/r^2 + 16/r - 24 + 16r - 16r^2 + 8r^3 + 4r^4$<br>$-22/r^4 - 32/r^3 + 56/r^2 - 16/r + 16r - 56r^2 + 32r^3 + 22r^4$<br>$47/r^4 + 16/r^3 - 2/r^2 + 32/r + 78 + 32r - 2r^2 + 16r^3 + 47r^4$<br>$-42/r^4 + 32/r^3 - 96/r^2 - 144/r + 144r + 96r^2 - 32r^3 + 42r^4$<br>$12/r^4 - 24/r^3 + 48/r^2 + 48/r - 168 + 48r + 48r^2 - 24r^3 + 12r^4$ |
| $[5, 5]$ | 256 | $60/r^4 + 64/r^3 + 72/r^2 - 96/r - 200 - 96r + 72r^2 + 64r^3 + 60r^4$<br>$-324/r^4 - 720/r^2 + 567/r - 576r + 720r^2 + 324r^4$<br>$525/r^4 - 128/r^3 + 936/r^2 - 192/r + 634 - 192r + 936r^2 - 128r^3 + 525r^4$<br>$-324/r^4 - 720/r^2 + 576/r - 576r + 720r^2 + 324r^4$<br>$60/r^4 + 64/r^3 + 72/r^2 - 96/r - 200 - 96r + 72r^2 + 64r^3 + 60r^4$ |

## Acknowledgments

Research of the first author was supported by NWO under grant 613.000.901; research of the third author was supported in part by National Science Foundation under the grant DMS-12-01369. The first author thanks the Riemann Center and the Institute of Algebraic Geometry of the Leibniz University Hannover for their support. The second author thanks the Mathematical Sciences Center of Tsinghua University for hospitality enjoyed there. The second author also thanks Jonas Bergström for useful comments. This paper builds on the work done with Jonas Bergström and Carel Faber in [3].

## Appendix A. Correction to "Igusa Quartic and Steiner Surfaces"

by Shigeru Mukai

This correction concerns the definition of the Fricke involution in [25]. The paragraph before [25, Theorem 2] is not precise enough to determine our Fricke involution of $H_2/\Gamma_1(2)$. In fact, the two explanations, the analytic and the moduli-theoretic one, conflict with each other. It should read as follows:

"The element $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 & I_2 \\ -2I_2 & 0 \end{smallmatrix}\right) \in \mathrm{Sp}(4, \mathbb{R})$ belongs to the normalizer of $\Gamma_0(2)$, and induces an involution of the quotient $H_2/\Gamma_0(2)$, which is called the Fricke involution. Moduli-theoretically, the Fricke involution maps a pair $(A, G)$ to $(A/G, A_{(2)}/G)$. We note that a 2-dimensional vector space $V$ is *almost* isomorphic to its dual $V^\vee$, or more precisely, we have canonically $V \simeq V^\vee \otimes \det V$. Hence the quotient $A_{(2)}/G$ is canonically isomorphic to $G$ via the Weil pairing. Therefore, the Fricke involution has a canonical lift on $H_2/\Gamma_1(2)$, which we call the (*canonical*) *Fricke involution of* $H_2/\Gamma_1(2)$. Our Fricke involution is the composite of $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 & I_2 \\ -2I_2 & 0 \end{smallmatrix}\right)$ and the involution $\left(\begin{smallmatrix} J_2 & 0 \\ 0 & J_2 \end{smallmatrix}\right)$, where we put $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It commutes with each element of $\Gamma_0(2)/\Gamma_1(2) \simeq \mathfrak{S}_3$ and $H_2/\Gamma_1(2)$ has an action of the product group $C_2 \times \mathfrak{S}_3$. Two pairs $(A, G)$ and $(A/G, A_{(2)}/G)$ (in $H_2/\Gamma_1(2)$) are geometrically related to each other by Richelot's theorem. See Remark 7."

## References


[1] H. Aoki and T. Ibukiyama, Simple graded rings of Siegel modular forms, differential operators and Borcherds products, *Internat. J. Math.* **16** (2005) 249–279.

[2] H. Aoki, On vector-valued Siegel modular forms of degree 2 with small levels, *Osaka J. Math.* **49** (2012) 625–651.

[3] J. Bergström, C. Faber and G. van der Geer, Siegel modular forms of genus 2 and level 2: Cohomological computations and conjectures, *Int. Math. Res. Not.* **2008** (2008), Article ID: rnn 100, 20 pp.

[4] J. Bergström, C. Faber and G. van der Geer, Siegel modular forms of degree three and the cohomology of local systems, *Selecta Math. (N.S.)* **20** (2014) 83–124.

[5] J. Bergström and G. van der Geer, Cohomology of local systems and Picard modular forms, in preparation.

[6] J. Bruinier, G. van der Geer, G. Harder and D. Zagier, *The* 1-2-3 *of Modular Forms*, Universitext (Springer-Verlag, 2007).

[7] F. Cléry and G. van der Geer, Generators for modules of Picard modular forms, *Nagoya Math. J.* **212** (2013) 19–57.

[8] C. Faber and G. van der Geer, Sur la cohomologie des systèmes locaux sur les espaces des modules des courbes de genre 2 et des surfaces abéliennes, I, II, *C. R. Acad. Sci. Paris Sér. I* **338** (2004) 381–384, 467–470.

[9] G. Faltings, *On the Cohomology of Locally Symmetric Hermitian Spaces*, Lecture Notes in Mathematics, Vol. 1029 (Springer-Verlag, 1984), pp. 55–98.

[10] J. Fay, On the Riemann–Jacobi formula, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **5** (1979), pp. 61–73.

[11] E. Getzler, Topological recursion relations in genus 2, in *Integrable Systems and Algebraic Geometry* (World Scientific, 1998), pp. 73–106.

[12] D. Grant, A generalization of Jacobi's derivative formula to dimension two, *J. Reine Angew. Math.* **392** (1988) 125–136.

[13] S. Grushevsky and R. Salvati Manni, Two generalizations of Jacobi's derivative formula, *Math. Res. Lett.* **12** (2005) 921–932.

[14] G. Harder, The Eisenstein motive for the cohomology of $\mathrm{Sp}(4, \mathbb{Z})$, in *Geometry and Arithmetic*, eds. C. Faber, G. Farkas and R. de Jong (European Mathematical Society, 2012).

[15] T. Ibukiyama, Vector valued Siegel modular forms of symmetric tensor representation, unpublished.

[16] T. Ibukiyama, Vector valued Siegel modular forms of $\det^k \mathrm{Sym}(4)$ and $\det^k \mathrm{Sym}(6)$, A short announcement, unpublished.

[17] T. Ibukiyama, On differential operators on automorphic forms and invariant pluriharmonic polynomials, *Comm. Math. Univ. Sancti Pauli* **48** (1999) 103–117.

[18] T. Ibukiyama, Differential operators and structures of vector valued Siegel modular forms, in *Algebraic Number Theory and Related Topics* (in Japanese) (Kyoto, 2000) *Sūrikaisekikenkyūsho Kōkyūroku* **1200** (2001) 71–81.

[19] T. Ibukiyama, Vector valued Siegel modular forms of symmetric tensor weight of small degrees, *Comm. Math. Univ. Sancti Pauli* **61** (2012) 51–75.

[20] J.-I. Igusa, On Siegel modular forms of genus two (II), *Amer. J. Math.* **86** (1964) 392–412.

[21] J.-I. Igusa, *Theta Functions*, Die Grundlehren der mathematischen Wissenschaften, Vol. 194 (Springer-Verlag, 1972).

[22] J.-I. Igusa, On the ring of modular forms of degree two over $\mathbb{Z}$, *Amer. J. Math.* **101** (1979) 149–183.

[23] J.-I. Igusa, On Jacobi's derivative formula and its generalizations, *Amer. J. Math.* **102** (1980) 409–446.
[24] R. Lazarsfeld, *Positivity in Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Vol. 49 (Springer-Verlag, 2004).
[25] S. Mukai, Igusa quartic and Steiner surfaces, in *Compact Moduli Spaces and Vector Bundles*, Contemporary Mathematics, Vol. 564 (American Mathematical Society, 2012), pp. 205–210.
[26] I. Peeva, *Graded Syzygies*, Algebra and Applications, Vol. 14 (Springer-Verlag, 2011).
[27] R. Salvati Manni, Modular varieties with level 2 theta structure, *Amer. J. Math.* **116** (1994) 1489–1511.
[28] L. Saper, $\mathcal{L}$ modules and the conjecture of Rapoport and Goresky–MacPherson, *Astérisque* **298** (1995) 319–334.
[29] T. Satoh, On certain vector valued Siegel modular forms of degree two, *Math. Ann.* **274** (1986) 335–352.
[30] G. van der Geer, *Hilbert Modular Surfaces*, Ergebnisse der Mathematik, Vol. 16 (Springer-Verlag, 1988).
[31] G. van der Geer, Rank one Eisenstein cohomology of local systems on the moduli of abelian varieties, *Sci. China Math.* **54** (2011) 1621–1634.
[32] C. van Dorp, Generators for a module of vector-valued modular forms of degree 2, preprint (2013), arXiv:1301.2910.
[33] S. Wakatsuki, Dimension formulas for spaces of vector-valued Siegel cusp forms of degree two, *J. Number Theory* **132** (2012) 200–253.