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Jacobi forms that characterize paramodular forms

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Abstract The Fourier Jacobi expansions of paramodular forms are characterized from among all sequences of Jacobi forms by two conditions on the Fourier coefficients of the Jacobi forms: a growth condition and a set of linear relations. Examples, both theoretical and computational, indicate that the growth condition may be superfluous.

Keywords Jacobi forms · Paramodular forms

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1 Introduction

For theoretical purposes it would be nice to characterize the Fourier Jacobi expansions of Siegel paramodular forms of degree two from among all formal power series with Jacobi forms as coefficients. For computational purposes it would be nice if the characterization were in terms of linear relations among the Fourier coefficients of the various Jacobi forms. We achieve this goal only in a few cases.

The linear relations we study arise from a symmetry possessed by the Fourier Jacobi expansions of paramodular forms. Let $J_{k,m}$ denote the complex vector space of Jacobi forms

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of weight k and index m. Let Γ be a group commensurable with $\mathrm{Sp}_2(\mathbb{Z})$ and denote by $M_k(\Gamma)$ the complex vector space of Siegel modular forms of weight k automorphic with respect to Γ. One commensurable family is given by the paramodular groups $K(N)$:

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \quad \text{where } * \in \mathbb{Z}.$$

Each paramodular form $f \in M_k(K(N))$ has a Fourier Jacobi expansion $f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m\geq 0: N|m} \phi_m(\tau, z)\, e(m\omega)$ where $\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right)$ is in the Siegel upper half space and $\phi_m \in J_{k,m}$. These Jacobi forms ϕ_m are not independent and possess a symmetry that is best expressed by using a normalizer μ_N of the paramodular group $K(N)$ satisfying $\mu_N^2 = -I_4$ and given by $\mu_N = \left(\begin{smallmatrix} -F_N' & 0 \\ 0 & F_N \end{smallmatrix}\right)$, where the Fricke involution $F_N = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right)$ is the usual normalizer of $\Gamma_0(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : N|c\right\}$.

For $\epsilon = \pm 1$, let $M_k(K(N))^\epsilon = \{f \in M_k(K(N)) : f|_k\mu_N = \epsilon f\}$ be the plus and minus eigenspaces of μ_N. Let the Fourier Jacobi expansion map, $\mathrm{FJ} : M_k(K(N))^\epsilon \to \prod_{m\in\mathbb{Z}: m\geq 0, N|m} J_{k,m}$, be defined by $\mathrm{FJ}(f) = \sum_{m:N|m} \phi_m \xi^m$ and write, for $(\tau, z) \in \mathcal{H}_1 \times \mathbb{C}$,

$$\phi_m(\tau, z) = \sum_{n,r\in\mathbb{Z}:\, 4mn\geq r^2,\ n\geq 0} c(n, r; \phi_m)\, e(n\tau + rz).$$

These coefficients possess the symmetry

$$c(n, r; \phi_m) = \epsilon c(m/N, -r; \phi_{nN}). \tag{1}$$

We mention that $f \in M_k(K(N))^\epsilon$ is a cusp form if and only if $\mathrm{FJ}(f) \in \prod_{m\in\mathbb{Z}; m\geq 0, N|m} J_{k,m}^{\mathrm{cusp}}$. This nontrivial assertion follows from the representation of the one-dimensional cusps by matrices of the shape $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$. In fact, the one-dimensional cusps correspond to divisors t of N via $D^* = A = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, see Reefschläger [19] or compare [17].

In Theorem 2.2 we show that certain *convergent* series of Jacobi forms satisfying the symmetry (1) are in fact the Fourier Jacobi expansion of some Siegel paramodular form. However, the real question motivating this article is: Are *formal* series of Jacobi forms satisfying the symmetry (1) the Fourier Jacobi expansions of Siegel paramodular forms? Work of H. Aoki [1] essentially answers this question affirmatively for $N = 1$ and we prove this for $N \in \{2, 3, 4\}$ as well by following his method. Let us give a more definite formulation.

Definition 1.1 Let $\mathcal{X}_2^{\mathrm{semi}}(N) = \{\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \geq 0 : a, 2b, c \in \mathbb{Z} \text{ and } N|c\}$ for $N \in \mathbb{N}$. For $k \in \mathbb{Z}$, let $\Phi = \sum_{m:N|m} \phi_m \xi^m \in \prod_{m\geq 0:N|m} J_{k,m}$ be a formal power series whose coefficients are Jacobi forms. For $\epsilon \in \{-1, 1\}$, we say that Φ satisfies the *Involution*(ϵ) *condition* if

$$\forall \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \mathcal{X}_2^{\mathrm{semi}}(N), \quad c(n, r; \phi_m) = \epsilon\, c\left(\frac{m}{N}, -r; \phi_{nN}\right).$$

We say that Φ satisfies the *growth condition* if

$$\forall \rho > 1,\ \exists A > 0 : \forall \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \mathcal{X}_2^{\mathrm{semi}}(N), \quad \left|c(n, r; \phi_m)\right| \leq A\rho^{n+m}.$$

Set $\mathbb{M}_k(N)^\epsilon = \{\Phi \in \prod_{m\geq 0:N|m} J_{k,m} : \Phi \text{ satisfies Involution}(\epsilon)\}$.

We would like to know when the map $\mathrm{FJ}: M_k(K(N))^{\epsilon} \to \mathbb{M}_k(N)^{\epsilon}$ is surjective. In Theorem 2.2 we show that this map surjects onto the subspace of $\mathbb{M}_k(N)^{\epsilon}$ that satisfies the growth condition, thereby giving at least one theoretical characterization of the Fourier Jacobi expansions of Siegel paramodular forms. Details aside, this amounts to the fact that the paramodular groups are generated by the Jacobi group and an involution. By following Aoki's method however, we do prove the surjectivity of FJ onto $\mathbb{M}_k(N)^{\epsilon}$ for $N \le 4$.

Following a suggestion of the referee, a *formal Fourier Jacobi expansion* should always mean an element of $\mathbb{M}_k(N)^{\epsilon}$, a formal series of Jacobi forms that satisfies the Involution(ϵ) condition. By this terminology, a formal Fourier Jacobi expansion automatically satisfies the symmetry condition inherent in the Fourier Jacobi expansion of a paramodular μ_N-eigenform. In these terms, the following theorem proves that certain formal Fourier Jacobi expansions are in fact convergent Fourier Jacobi expansions of paramodular forms.

Theorem 1.2 *Let $N \in \{1,2,3,4\}$ and $\epsilon \in \{-1,1\}$. For all weights $k \in \mathbb{Z}$, the Fourier Jacobi expansion map* FJ *from paramodular forms to formal series of Jacobi forms that satisfy the Involution(ϵ) condition*, $\mathrm{FJ}: M_k(K(N))^{\epsilon} \to \mathbb{M}_k(N)^{\epsilon}$, *is an isomorphism.*

As a corollary we obtain new results for the generating functions of the plus and minus eigenspaces. For any prime p, $\dim S_k(K(p))$ is known in [11] for $k > 4$, in [13] for $k = 3$, 4, and for $p < 349$ and $k = 2$ in [16]. We can easily show that the generalized Siegel Φ operator, the projection from $M_k(K(N))$ to the boundary of the Satake compactification, is always surjective for any k for squarefree N. Indeed, this is due to Satake [20] when $k > 4$, and, again for squarefree N, the image is zero dimensional for $k = 2$ and at most one dimensional for $k = 4$ due to the known cusp configuration in [12] for prime level and in [17] for general N; furthermore, the lift of the Jacobi Eisenstein series of $J_{4,N}$ surjects to the image of Φ when $k = 4$. So the generating function for $\dim M_k(K(p))$ can be easily given for any p as long as we know $\dim S_2(K(p))$. In fact, the full generating functions are known for $N = 2$ by T. Ibukiyama and F. Onodera [14], the plus and minus eigenspaces being given there also, and for $N = 3$ by T. Dern [4]. Our proofs use their results. The generating function $\sum \dim M_k(K(4))t^k$ is given here for the first time by relying on the definitive results of Igusa [15] for subgroups of Γ_2 that contain the principal subgroup $\Gamma_2(2)$. These results, new for $N = 4$, are:

$$\sum_{k\in\mathbb{Z}} \dim M_k\big(K(2)\big)^{+} t^k = \frac{1+t^{10}+t^{23}+t^{33}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},$$

$$\sum_{k\in\mathbb{Z}} \dim M_k\big(K(2)\big)^{-} t^k = \frac{t^{11}+t^{12}+t^{21}+t^{22}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},$$

$$\sum_{k\in\mathbb{Z}} \dim M_k\big(K(3)\big)^{+} t^k = \frac{1+t^{8}+t^{10}+t^{21}+t^{23}+t^{31}}{(1-t^4)(1-t^6)^2(1-t^{12})},$$

$$\sum_{k\in\mathbb{Z}} \dim M_k\big(K(3)\big)^{-} t^k = \frac{t^{9}+t^{11}+t^{12}+t^{19}+t^{20}+t^{22}}{(1-t^4)(1-t^6)^2(1-t^{12})},$$

$$\sum_{k\in\mathbb{Z}} \dim M_k\big(K(4)\big)^{+} t^k = \frac{1+t^{6}+t^{8}+t^{10}+t^{19}+t^{21}+t^{23}+t^{29}}{(1-t^4)^2(1-t^6)(1-t^{12})},$$

$$\sum_{k\in\mathbb{Z}} \dim M_k\big(K(4)\big)^{-} t^k = \frac{t^{7}+t^{9}+t^{11}+t^{12}+t^{17}+t^{18}+t^{20}+t^{22}}{(1-t^4)^2(1-t^6)(1-t^{12})}.$$

The question of the surjectivity of $\mathrm{FJ}: M_k(K(N))^\epsilon \to \mathbb{M}_k(N)^\epsilon$ is not idle and has applications to the computation of paramodular forms. To illustrate this, in Sect. 4 we use the symmetry condition to compute $S_4(K(31))^\pm$. These computations at least make it plausible that the growth condition is superfluous. Here one may also find a lemma showing that, for prime p, initial Fourier Jacobi expansions

$$\pi_{pJ}\circ \mathrm{FJ}: S_k\big(K(p)\big)^\epsilon \to \prod_{j=1}^{J} J_{k,pj}^{\mathrm{cusp}} \quad \text{inject for } J \ge \llcorner \frac{k}{10}\left(\frac{p^2+1}{p+1}\right)\lrcorner.$$

2 A characterization of Fourier Jacobi expansions

For a ring R, let $\mathrm{Sp}_n(R) = \{\sigma \in \mathrm{GL}_{2n}(R) : \sigma' J\sigma = J\}$ define the symplectic group over R, where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0\end{smallmatrix}\right)$ and σ' is the transpose of σ. The paramodular group $K(N)$, defined in the Introduction, is generated by the translations $\left(\begin{smallmatrix} I & S \\ 0 & I\end{smallmatrix}\right)$ with $S = \left(\begin{smallmatrix} \alpha & \beta \\ \beta & \gamma/N\end{smallmatrix}\right)$ for $\alpha, \beta, \gamma \in \mathbb{Z}$, and the element $J(N)$, see [3], Theorem 9,

$$J(N) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/N \\ -1 & 0 & 0 & 0 \\ 0 & -N & 0 & 0 \end{pmatrix}.$$

Let $\mathcal{H}_n$ denote the Siegel upper half space. For $k \in \mathbb{Z}$, the paramodular forms of weight k, denoted by $M_k(K(N))$, are the $\mathbb{C}$-vector space of holomorphic $f : \mathcal{H}_2 \to \mathbb{C}$ with the property that $f|_k\sigma = f$ for all $\sigma \in K(N)$. The subspace of cusp forms is given by $S_k(K(N)) = \{f \in M_k(K(N)) : \forall\sigma \in \mathrm{Sp}_2(\mathbb{Z}), \Phi(f|_k\sigma) = 0\}$. Here the slash action, $\big(f|_k\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\big)(\Omega) = \det(C\Omega + D)^{-k} f((A\Omega + B)(C\Omega + D)^{-1})$ and the Φ operator, $(\Phi f)(\tau) = \lim_{\lambda\to+\infty} f\left(\begin{smallmatrix} i\lambda & 0 \\ 0 & \tau\end{smallmatrix}\right)$, are the usual ones, see [6]. Since μ_N^2 acts trivially on modular forms, we may decompose paramodular forms into plus and minus forms: $M_k(K(N)) = M_k(K(N))^+ \oplus M_k(K(N))^-$ where $M_k(K(N))^\epsilon = \{f \in M_k(K(N)) : f|\mu = \epsilon f\}$ for $\epsilon \in \{-1, 1\}$.

Every paramodular form $f \in M_k(K(N))$ has a Fourier expansion

$$f(\Omega) = \sum_{T\in\mathcal{X}_2^{\mathrm{semi}}(N)} a(T; f)\, e\big(\langle\Omega, T\rangle\big)$$

supported on $\mathcal{X}_2^{\mathrm{semi}}(N) = \{\left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right) \ge 0 : a, 2b, c \in \mathbb{Z} \text{ and } N|c\}$; here $e(z) = e^{2\pi i z}$ and $\langle A, B\rangle = \mathrm{tr}(AB)$. Setting $T[\sigma] = \sigma' T\sigma$, we additionally have $a(T[\sigma]; f) = \det(\sigma)^k a(T; f)$ for all $\sigma \in \hat{\Gamma}^0(N) = \big\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) : N|b\big\}$. Note that the action of $\hat{\Gamma}^0(N)$ stabilizes $\mathcal{X}_2^{\mathrm{semi}}(N)$. If we write $\Omega = \left(\begin{smallmatrix} \tau & z \\ z & \omega\end{smallmatrix}\right) \in \mathcal{H}_2$ and collect the Fourier expansion of f in powers of $\xi = e(\omega)$, then we obtain the Fourier Jacobi expansion of f: $f\left(\begin{smallmatrix} \tau & z \\ z & \omega\end{smallmatrix}\right) = \sum_{m\ge 0: N|m} \phi_m(\tau, z)\, \xi^m$ where the

$$\phi_m(\tau, z) = \sum_{n,r\in\mathbb{Z}: \left(\begin{smallmatrix} n & r/2 \\ r/2 & m\end{smallmatrix}\right)\ge 0,\, n\ge 0} a\left(\begin{pmatrix} n & r/2 \\ r/2 & m\end{pmatrix}; f\right) e(n\tau)e(rz) \tag{2}$$

are Jacobi forms of weight k and index m. This Fourier Jacobi expansion is term by term invariant under the group,

$$\Gamma_\infty(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}),$$

and this is one motivation for the definition of Jacobi forms.

Definition 2.1 Let $k, m \in \mathbb{Z}_{\geq 0}$. The $\mathbb{C}$-vector space $J_{k,m}$ of Jacobi forms of weight k and index m is the set of holomorphic $\phi : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$ satisfying:

(1) $\forall \sigma \in \Gamma_\infty(\mathbb{Z}), \tilde{\phi}|_k\sigma = \tilde{\phi}$, where $\tilde{\phi} : \mathcal{H}_2 \to \mathbb{C}$ is defined by $\tilde{\phi}\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \phi(\tau, z)e(m\omega)$.
(2) Setting $q = e(\tau)$ and $\zeta = e(z)$, the Fourier series of ϕ has the form: $\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}: n \geq 0,\, 4mn \geq r^2} c(n, r; \phi) q^n \zeta^r$.

The vector space of Jacobi cusp forms $J_{k,m}^{\text{cusp}}$ is defined by replacing $4mn \geq r^2$ by $4mn > r^2$ in item 2. If we identify a sequence $(\phi_m) \in \prod_{m \in \mathbb{Z}: m \geq 0, N|m} J_{k,m}$ with the formal power series $\sum_{m:N|m} \phi_m \xi^m$, then developing the Fourier Jacobi expansion of a paramodular form as in (2) defines a map $\mathrm{FJ} : M_k(K(N)) \to \prod_{m \geq 0:\, N|m} J_{k,m}$. Now we can state a characterization.

Theorem 2.2 *Let $k \in \mathbb{Z}_{\geq 0}$, $N \in \mathbb{N}$ and $\epsilon \in \{-1, 1\}$. Let $\Phi = \sum_{m:N|m} \phi_m \xi^m \in \prod_{m \geq 0:\, N|m} J_{k,m}$ be a formal power series whose coefficients are Jacobi forms. There is an $f \in M_k(K(N))^\epsilon$ such that $\Phi = \mathrm{FJ}(f)$ if and only if Φ satisfies the* Involution(ϵ) *condition and the growth condition of Definition 1.1.*

Proof We first assume that $\Phi = \mathrm{FJ}(f)$ for $f \in M_k(K(N))^\epsilon$ and write each $T \in \mathcal{X}_2^{\text{semi}}(N)$ as $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$. For any $\rho > 1$, take $\lambda > 0$ with $\rho = e^{2\pi\lambda}$. By the Koecher principle there is an $A > 0$ such that $|f(\Omega)| \leq A$ on $\{\Omega = x + iY \in \mathcal{H}_2 : Y > \frac{\lambda}{2} I_2\}$. For $\Omega = X + i\lambda I_2$ we have the growth condition:

$$\begin{aligned} \left|c(n, r; \phi_m)\right| = \left|a(T; f)\right| &= \left|\int_{X \in [0,1]^3} f(\Omega) e\big(-\langle \Omega, T\rangle\big) dX\right| \\ &\leq \int_{X \in [0,1]^3} \left|f(\Omega)\right| e^{2\pi\langle \lambda I_2, T\rangle} dX \leq A\rho^{\mathrm{tr}(T)} = A\rho^{m+n}. \end{aligned}$$

For the *Involution*(ϵ) condition, we need to know the action of the involution μ_N on the Fourier expansion of f:

$$\begin{aligned} (f|\mu_N)(\Omega) &= \det(F_n)^{-k} \sum a(T; f) e\big(\langle F_N' \Omega F_N, T\rangle\big) \\ &= \sum a\big(F_N T F_N'; f\big) e\big(\langle \Omega, T\rangle\big). \end{aligned}$$

Now $F_N T F_N' = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -N \\ 1 & 0 \end{smallmatrix}\right)\frac{1}{\sqrt{N}} = \left(\begin{smallmatrix} m/N & -r/2 \\ -r/2 & Nn \end{smallmatrix}\right)$, so that we have the *Involution*(ϵ) condition:

$$c(n,r;\phi_m)=a\left(\begin{pmatrix} n & r/2\\ r/2 & m\end{pmatrix}; f\right)=\epsilon\, a\left(\begin{pmatrix} n & r/2\\ r/2 & m\end{pmatrix}; f|\mu_N\right)$$
$$=\epsilon\, a\left(\begin{pmatrix} m/N & -r/2\\ -r/2 & Nn\end{pmatrix}; f\right)=\epsilon\, c\left(\frac{m}{N},-r;\phi_{Nn}\right).$$

Now assume that $\Phi=\sum\phi_m\xi^m$ satisfies the growth and *Involution*(ϵ) conditions. For any $T=\left(\begin{smallmatrix} n & r/2\\ r/2 & m\end{smallmatrix}\right)\in\mathcal{X}_2^{\mathrm{semi}}(N)$, define $a(T)$ by $a(T)=c(n,r;\phi_m)$. On the set $\{\Omega=x+iY\in\mathcal{H}_2: Y\geq\lambda I_2\}$ the series $\sum_{T\in\mathcal{X}_2^{\mathrm{semi}}(N)}a(T)e(\langle\Omega,T\rangle)$ is majorized by a convergent series of constants. To see this, choose ρ with $1<\rho<e^{2\pi\lambda}$ so that by the growth condition there is an $A>0$ with $|a(T)|=|c(n,r;\phi_m)|\leq A\rho^{n+m}$ and so

$$\sum\bigl|a(T)\bigr|e^{-2\pi\langle Y,T\rangle}\leq\sum A\rho^{m+n}e^{-2\pi\langle Y,T\rangle}\leq A\sum_T\left(\frac{\rho}{e^{2\pi\lambda}}\right)^{m+n}$$
$$\leq A\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2n+2m+1)\left(\frac{\rho}{e^{2\pi\lambda}}\right)^{m+n}.$$

Since the convergence is uniform on compact sets, we may define a holomorphic function $f:\mathcal{H}_2\to\mathbb{C}$ via $f(\Omega)=\sum_{T\in\mathcal{X}_2^{\mathrm{semi}}(N)}a(T)e(\langle\Omega,T\rangle)$.

The absolute convergence of this series shows that $f\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right)$ is equal to the rearrangement $\sum_{m\in\mathbb{Z}_{\geq 0}:N|m}\phi_m(\tau,z)e(m\omega)$, or $f=\sum_{m\in\mathbb{Z}_{\geq 0}:N|m}\tilde{\phi}_m$. The invariance of f under the action of the group $\Gamma_\infty(\mathbb{Z})$ now follows from the invariance of the $\tilde{\phi}_m$. In particular, we have $f|E_1=f$ for

$$E_1=\begin{pmatrix}0&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}\in\Gamma_\infty(\mathbb{Z}).$$

Furthermore, the *Involution*(ϵ) condition gives us

$$a\bigl(F_NTF_N'\bigr)=c\left(\frac{m}{N},-r;\phi_{nN}\right)=\epsilon c(n,r;\phi_m)=\epsilon a(T),$$

so that

$$(f|_k\mu_N)(\Omega)=\det(F_n)^{-k}\sum_{T\in\mathcal{X}_2^{\mathrm{semi}}(N)}a(T)e\bigl(\langle F_N'\Omega F_N,T\rangle\bigr)$$
$$=\sum_T a\bigl(F_NTF_N'\bigr)e\bigl(\langle\Omega,T\rangle\bigr)=\sum_T\epsilon a(T)e\bigl(\langle\Omega,T\rangle\bigr)=\epsilon f(\Omega).$$

Following Gritsenko [8], we have $f|E_1\mu_N=f|\mu_N=\epsilon f$ and therefore that $f|(E_1\mu_N)^2=f$. The group $K(N)$ is generated by translations and the element $(E_1\mu_N)^2=-J(N)$ so that $f\in M_k(K(N))^\epsilon$. □

3 Aoki's method for $N = 2, 3$ and 4

Does Theorem 2.2 remain true without the growth condition? A method of H. Aoki [1] shows that it does for $N = 1$. We successfully use Aoki's method to show the same for $N \le 4$.

Definition 3.1 Let $j, k, m \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\epsilon \in \{-1, 1\}$. Set

$$\mathbb{M}_k^{(j)}(N)^\epsilon = \Big\{ \Phi = \sum_{m \in \mathbb{Z}:\, m \ge Nj:\, N|m} \phi_m \xi^m \in \mathbb{M}_k(N)^\epsilon \Big\},$$

$$\operatorname{ord}\phi = \min\big\{ n \in \mathbb{Z}_{\ge 0} : \exists r \in \mathbb{Z} : c(n, r; \phi) \ne 0 \big\}, \quad \text{for } \phi \in J_{k,m},$$

$$J_{k,m}(j) = \{\phi \in J_{k,m} : \operatorname{ord}\phi \ge j\}.$$

Here, as in Aoki [1, 2], precise dimensions in specific cases follow from inequalities that are in general too generous. Most dramatically, the final terms in the following Estimate diverge for $N > 5$ and large weights.

Lemma 3.2 (Estimate) *Let $N \in \mathbb{N}$, $\epsilon \in \{-1, 1\}$, $k \in \mathbb{Z}$ and set $\delta = 0$ if $(-1)^k \epsilon = 1$ and $\delta = 1$ if $(-1)^k \epsilon = -1$. We have the inequalities*

$$\begin{aligned}
\dim M_k\big(K(N)\big)^\epsilon &\le \dim \mathbb{M}_k(N)^\epsilon \\
&\le \sum_{j=0}^{\infty} \dim\big(\mathbb{M}_k^{(j)}(N)^\epsilon / \mathbb{M}_k^{(j+1)}(N)^\epsilon\big) \\
&\le \sum_{j=0}^{\infty} \dim J_{k,Nj}(j + \delta) \\
&\le \begin{cases} \sum_{j=0}^{\infty} \sum_{i=0}^{Nj} \dim M_{k+2i-12(j+\delta)}, & k \text{ even}, \\ \sum_{j=1}^{\infty} \sum_{i=1}^{Nj-1} \dim M_{k-1+2i-12(j+\delta)}, & k \text{ odd}. \end{cases}
\end{aligned}$$

(*For k odd, $N = j = 1$ gives an empty second sum.*)

Proof The first inequality follows since $\mathrm{FJ} : M_k(K(N))^\epsilon \to \mathbb{M}_k(N)^\epsilon$ is injective, the second by the filtration $\mathbb{M}_k^{(j)}(N)^\epsilon \supseteq \mathbb{M}_k^{(j+1)}(N)^\epsilon$. For the third, consider the exact sequence

$$0 \hookrightarrow \mathbb{M}_k^{(j+1)}(N)^\epsilon \hookrightarrow \mathbb{M}_k^{(j)}(N)^\epsilon \to J_{k,Nj},$$

where the final map sends $\Phi = \sum_{i=j}^{\infty} \phi_{iN} q^{iN}$ to ϕ_{jN}. The Involution(ϵ) condition shows that the image of the last map is inside $J_{k,Nj}(j + \delta)$. This is the obvious but important point. If $\Phi \in \mathbb{M}_k^{(j)}(N)^\epsilon$ then for all $\ell < j$ we have $\phi_{N\ell} = 0$, so that $c(\ell, r; \phi_{Nj}) = \epsilon c(j, -r; \phi_{N\ell}) = 0$ and $\phi_{Nj} \in J_{k,Nj}(j)$. Furthermore, if $(-1)^k \epsilon = -1$ then $c(j, r; \phi_{Nj}) = \epsilon c(j, -r; \phi_{Nj}) = (-1)^k \epsilon c(j, r; \phi_{Nj}) = -c(j, r; \phi_{Nj})$, so $c(j, r; \phi_{Nj}) = 0$ and $\phi_{Nj} \in J_{k,Nj}(j + 1)$. Thus we may uniformly write $\phi_{Nj} \in J_{k,Nj}(j + \delta)$.

The last inequality follows from Lemma 3 on page 583 in Aoki [1], a consequence of the theory of differential operators in [5]:

$$\dim J_{k,m}(j) \le \begin{cases} \sum_{i=0}^{m} \dim M_{k+2i-12j}, & \text{if } k \text{ even}, \\ \sum_{i=1}^{m-1} \dim M_{k-1+2i-12j}, & \text{if } k \text{ odd}, m \ge 2, \\ 0, & \text{if } k \text{ odd}, \ m \le 1. \end{cases} \qquad \square$$

Lemma 3.3 *For* $N \in \{1,2,3,4,5\}$ *and* $\epsilon \in \{-1,1\}$, *let*:

$$E_{N,\delta} = \sum_{k\,\text{even}} \left(\sum_{j=0}^{\infty} \sum_{i=0}^{Nj} \dim M_{k+2i-12(j+\delta)} \right) t^k,$$

$$D_{N,\delta} = \sum_{k\,\text{odd}} \left(\sum_{j=1}^{\infty} \sum_{i=1}^{Nj-1} \dim M_{k+2i-12(j+\delta)} \right) t^k.$$

We have $E_{1,0} = ((1-t^4)(1-t^6)(1-t^{10})(1-t^{12}))^{-1}$, $D_{1,1} = E_{1,1} = 0$ *and* $D_{1,0} = t^{35} E_{1,0}$. *For* $2 \le N \le 5$ *we have*

$$E_{N,\delta} = t^{12\delta} \frac{1 + t^{10} + t^8 + \cdots + t^{14-2N}}{(1-t^4)(1-t^6)(1-t^{12})(1-t^{12-2N})},$$

$$D_{N,\delta} = t^{12\delta} \frac{t^{25-2N} + t^{11} + t^9 + \cdots + t^{15-2N}}{(1-t^4)(1-t^6)(1-t^{12})(1-t^{12-2N})}.$$

Proof Since $\dim M_\nu = 0$ for $\nu < 0$, we may make the computation slightly easier by summing over all $k \in \mathbb{Z}$ and using, for all $a \in \mathbb{Z}$, the identity $\sum_{k\in\mathbb{Z}} \dim M_{k-a} t^k = t^a / ((1-t^4)(1-t^6))$.

$$\begin{aligned} E_{N,\delta} &= \sum_{k\,\text{even}} \sum_{j=0}^{\infty} \sum_{i=0}^{Nj} \dim M_{k+2i-12(j+\delta)} t^k, \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{Nj} \left(\sum_{k\,\text{even}} \dim M_{k+2i-12(j+\delta)} t^{k+2i-12(j+\delta)} \right) t^{12(j+\delta)-2i} \\ &= \frac{1}{(1-t^4)(1-t^6)} \sum_{j=0}^{\infty} \sum_{i=0}^{Nj} t^{12(j+\delta)-2i} \\ &= \frac{t^{12\delta}}{(1-t^4)(1-t^6)} \sum_{i=0}^{\infty} \sum_{j=\ulcorner i/N \urcorner}^{\infty} t^{12j-2i} \\ &= \frac{t^{12\delta}}{(1-t^4)(1-t^6)(1-t^{12})} \sum_{i=0}^{\infty} \sum_{j=\ulcorner i/N \urcorner}^{\infty} \left(t^{12j-2i} - t^{12(j+1)-2i} \right) \\ &= \frac{t^{12\delta}}{(1-t^4)(1-t^6)(1-t^{12})} \sum_{i=0}^{\infty} t^{12\ulcorner i/N \urcorner - 2i}. \end{aligned}$$

We finish by substituting $i = N\ell + \nu$ and evaluating

$$\begin{aligned}\sum_{i=0}^{\infty} t^{12\ulcorner i/N \urcorner -2i} &= \sum_{\nu=0}^{N-1}\sum_{\ell=0}^{\infty} t^{12\ulcorner \frac{N\ell+\nu}{N}\urcorner -2(N\ell+\nu)}\\ &= \sum_{\ell=0}^{\infty} t^{12\ell-2N\ell} + \sum_{\nu=1}^{N-1}\sum_{\ell=0}^{\infty} t^{12(\ell+1)-2N\ell-2\nu}\\ &= \sum_{\ell=0}^{\infty} t^{(12-2N)\ell}\left(1+\sum_{\nu=1}^{N-1} t^{12-2\nu}\right)\\ &= \frac{1+t^{10}+t^{8}+\cdots+t^{14-2N}}{1-t^{12-2N}}.\end{aligned}$$

The proof for $D_{N,\delta}$ is quite similar.

$$\begin{aligned}D_{N,\delta} &= \sum_{k\text{ odd}}\sum_{j=1}^{\infty}\sum_{i=1}^{Nj-1} \dim M_{k-1+2i-12(j+\delta)} t^{k},\\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{Nj-1}\left(\sum_{k\text{ odd}} \dim M_{k-1+2i-12(j+\delta)} t^{k-1+2i-12(j+\delta)}\right) t^{12(j+\delta)-2i+1}\\ &= \frac{1}{(1-t^4)(1-t^6)}\sum_{j=1}^{\infty}\sum_{i=1}^{Nj-1} t^{12(j+\delta)-2i+1}\\ &= \frac{t^{12\delta}}{(1-t^4)(1-t^6)}\sum_{i=1}^{\infty}\sum_{j=\ulcorner (i+1)/N\urcorner}^{\infty} t^{12j-2i+1}\\ &= \frac{t^{12\delta}}{(1-t^4)(1-t^6)(1-t^{12})}\sum_{i=1}^{\infty}\sum_{j=\ulcorner (i+1)/N\urcorner}^{\infty} \left(t^{12j-2i+1}-t^{12(j+1)-2i+1}\right)\\ &= \frac{t^{12\delta}}{(1-t^4)(1-t^6)(1-t^{12})}\sum_{i=1}^{\infty} t^{12\ulcorner (i+1)/N\urcorner -2i+1}.\end{aligned}$$

We finish by substituting $i = N\ell + \nu$ and evaluating

$$\begin{aligned}\sum_{i=1}^{\infty} t^{12\ulcorner (i+1)/N\urcorner -2i+1} &= \sum_{\nu=1}^{N}\sum_{\ell=0}^{\infty} t^{12\ulcorner \frac{N\ell+\nu+1}{N}\urcorner -2(N\ell+\nu)+1}\\ &= \sum_{\nu=1}^{N-1}\sum_{\ell=0}^{\infty} t^{12(\ell+1)-2N\ell-2\nu+1} + \sum_{\ell=0}^{\infty} t^{12(\ell+2)-2(N\ell+N)+1}\\ &= \sum_{\ell=0}^{\infty} t^{(12-2N)\ell}\left(\sum_{\nu=1}^{N-1} t^{13-2\nu} + t^{25-2N}\right)\\ &= \frac{t^{11}+t^{9}+\cdots+t^{15-2N}+t^{25-2N}}{1-t^{12-2N}}.\end{aligned}$$

The proof for the case $N = 1$ is similar and is given in Aoki [1]. □

Corollary 3.4 *For $N \in \{2,3,4\}$ and $\epsilon \in \{-1,1\}$ or for $N=1$ and $\epsilon=1$, all the inequalities in the Estimate of Lemma 3.2 are equalities.*

$$\forall k \in \mathbb{Z}, \quad \mathrm{FJ}: M_k(K(N))^{\epsilon} \to \mathbb{M}_k(N)^{\epsilon} \text{ is an isomorphism.}$$

$$\forall k \text{ even}, \quad \dim J_{k,Nj}(j+\delta) = \sum_{i=0}^{Nj} \dim M_{k+2i-12(j+\delta)},$$

$$\forall k \text{ odd}, \quad \dim J_{k,Nj}(j+\delta) = \sum_{i=1}^{Nj-1} \dim M_{k-1+2i-12(j+\delta)},$$

$$\sum_{k=0}^{\infty} \dim M_k(K(N))^{+} t^k = E_{N,0} + D_{N,1}$$
$$= \frac{1 + t^{10} + t^8 + \cdots + t^{14-2N} + t^{12}(t^{11} + t^9 + \cdots + t^{15-2N} + t^{25-2N})}{(1-t^4)(1-t^6)(1-t^{12})(1-t^{12-2N})},$$

$$\sum_{k=0}^{\infty} \dim M_k(K(N))^{-} t^k = E_{N,1} + D_{N,0}$$
$$= \frac{t^{12}(1 + t^{10} + t^8 + \cdots + t^{14-2N}) + t^{11} + t^9 + \cdots + t^{15-2N} + t^{25-2N}}{(1-t^4)(1-t^6)(1-t^{12})(1-t^{12-2N})}.$$

Proof Rewriting the inequalities of Lemma 3.2 as $\dim M_k(K(N))^+ \le \mathrm{coeff}(E_{N,0} + D_{N,1}, t^k)$ and as $\dim M_k(K(N))^- \le \mathrm{coeff}(E_{N,1} + D_{N,0}, t^k)$, we have $\dim M_k(K(N)) = \dim M_k(K(N))^+ + \dim M_k(K(N))^- \le \mathrm{coeff}(E_{N,0} + D_{N,1} + E_{N,1} + D_{N,0}, t^k)$. If we can show equality here, we have $\dim M_k(K(N))^+ = \mathrm{coeff}(E_{N,0} + D_{N,1}, t^k)$ and $\dim M_k(K(N))^- = \mathrm{coeff}(E_{N,1} + D_{N,0}, t^k)$ and the proof is complete. However, the generating functions $\sum_{k\in\mathbb{Z}} \dim M_k(K(N))t^k$ are known for $N = 2, 3$ and 4 and one checks equality with $E_{N,0} + E_{N,1} + D_{N,0} + D_{N,1}$. □

4 The generating function of $K(4)$

For any natural number t, the paramodular group $K(t^2)$ is conjugate, by an element of $\mathrm{Sp}_2(\mathbb{Q})$, to the following group $\tilde{\Gamma}(t)$, which is a subgroup of Γ_2 containing the principal subgroup $\Gamma_2(t)$;

$$\tilde{\Gamma}(t) = \begin{pmatrix} * & t* & * & t* \\ t* & * & t* & * \\ * & t* & * & t* \\ t* & * & t* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \quad \text{where } * \in \mathbb{Z}.$$

The proof is that $\mathrm{diag}(1, t, 1, t^{-1}) K(t^2)\, \mathrm{diag}(1, t^{-1}, 1, t) = \tilde{\Gamma}(t)$. In Igusa [15], we may find the generating function for the character X_k of the representation of $\mathrm{Sp}_2(\mathbb{F}_2) \simeq \Gamma_2/\Gamma_2(2)$ acting on $M_k(\Gamma_2(2))$. Since $\tilde{\Gamma}(2)$ contains the principal subgroup $\Gamma_2(2)$, Igusa's

Table 1 S_6 cycles

$S_3 \times S_3$		$M \in S_6$		$g(M;t)$
$(1) \times (1)$	1	(1)	1	$\frac{(1+t^5)(1-t^8)}{(1-t^2)^5}$
$(12) \times (1)$	3			
$(1) \times (12)$	3	(12)	6	$\frac{(1-t^5)(1-t^8)}{(1-t^2)^2(1+t^2)^3}$
$(12) \times (12)$	9	$(12)\,(34)$	9	$\frac{(1+t^5)(1-t^8)}{(1-t^2)^3(1+t^2)^2}$
$(123) \times (1)$	2			
$(1) \times (123)$	2	(123)	4	$\frac{(1+t^5)(1-t^8)}{(1-t^2)(1+t^2+t^4)^2}$
$(123) \times (12)$	6			
$(12) \times (123)$	6	$(12)\,(345)$	12	$\frac{(1-t^5)(1-t^8)}{(1+t^2)(1-t^2+t^4)(1+t^2+t^4)}$
$(123) \times (123)$	4	$(123)\,(456)$	4	$\frac{(1+t^5)(1-t^8)}{(1+t^2)^3(1+t^2+t^4)}$

result allows us to calculate the generating function for $\tilde{\Gamma}(2)$ by the formula

$$\sum_{k=0}^{\infty} \dim M_k\big(\tilde{\Gamma}(2)\big)\, t^k = \frac{1}{|G|} \sum_{M\in G} \sum_{k=0}^{\infty} X_k(M) t^k,$$

where $G = \tilde{\Gamma}(2)/\Gamma_2(2)$ is a finite group. Now $\mathrm{Sp}_2(\mathbb{F}_2)$ is isomorphic to the symmetric group S_6 via the permutation of the six odd theta characteristics and the group $G \simeq \mathrm{SL}_2(\mathbb{F}_2) \times \mathrm{SL}_2(\mathbb{F}_2)$ corresponds to a choice of $S_3 \times S_3 \subseteq S_6$ by the action of $\mathrm{SL}_2(\mathbb{F}_2)$ on the three even theta characteristics. We separate the elements $M \in G$ into conjugacy classes, which may be given by cycle types inside S_6, and give Igusa's computation (page 401, [15]) of $g(M;t) = \sum_{k=0}^{\infty} X_k(M) t^k$ for these conjugacy classes. Table 1 lists the cycle types in both $S_3 \times S_3$ and S_6 and gives the number of elements that have that cycle type.

This gives

$$\begin{aligned}
&\sum_{k=0}^{\infty} \dim M_k\big(K(4)\big)\, t^k \\
&\quad = \sum_{k=0}^{\infty} \dim M_k\big(\tilde{\Gamma}(2)\big)\, t^k \\
&\quad = \frac{1}{36}\Big\{g\big((1);t\big) + 6g\big((12);t\big) + 9g\big((12)(34);t\big) \\
&\qquad + 4g\big((123);t\big) + 12g\big((12)(345);t\big) + 4g\big((123)(456);t\big)\Big\} \\
&\quad = \frac{(1+t^{12})(1+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{17})}{(1-t^4)^2(1-t^6)(1-t^{12})}.
\end{aligned}$$

We mention a good cross check now that we know $\dim M_k(K(4))$. We can show that $\dim J_{k,4j}^{\mathrm{cusp}}(j) = \max\{\dim J_{k,4j}(j) - 1, 0\}$ by comparing the Taylor expansion and the theta expansion of Jacobi forms as in Eichler-Zagier [5]. By this, we can also give upper bounds for $\dim S_k(K(4))$. These upper bounds coincide with the true dimension of $S_k(K(4))$ computed from the known dimension of $M_k(K(4))$ and the dimension of the image of generalized Φ-operator on the boundary.

5 An example: $\pi_{12} \circ \mathrm{FJ} : S^4(K(31)) \to \prod_{j=1}^{12} J_{4,31j}^{\mathrm{cusp}}$

Although the linear relations from the Involution(ϵ) condition are practical to implement on a computer, the growth condition is not. It is natural to wonder about the effect of omitting the growth condition and we work out one example with this in mind. In light of Theorem 2.2, when we compute formal power series over Jacobi forms satisfying the involution condition, either there will only be Fourier Jacobi expansions of paramodular forms or there will also be solutions with rapidly growing coefficients. We consider the subspaces $S_4(K(31))^-$ and

$$\mathcal{S} = \left\{ f \in S_4\big(K(31)\big)^+ : \mathrm{ord}_\xi \, \mathrm{FJ}(f) \ge 62 \right\}$$

for the following reasons: The dimensions $\dim J_{k,m}^{\mathrm{cusp}}$ are known for $k \ge 2$, see [5, 21], and so we only need to generate sufficiently many linearly independent elements of $J_{k,m}^{\mathrm{cusp}}$ to compute inside this space. Especially in weight four, see [9], theta blocks are a convenient way to construct Jacobi forms. For $d \in \mathbb{N}^8$ with $d \cdot d = 2N$, we have $T(d)(\tau, z) = \prod_{i=1}^{8} \vartheta(\tau, d_i z) \in J_{4,N}$; here $\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{(2n+1)^2}{8}} \zeta^{\frac{2n+1}{2}}$. It is easy to see that $T(d)$ is a cusp form if d has both even and odd entries. We select $K(p)$ for prime level p because T. Ibukiyama [11, 13] has given $\dim S_k(K(p))$ for $k \ge 3$; this information allows us to measure our computations against a known dimension. For weight 4, we have

$$\begin{aligned} &\dim S_4\big(K(p)\big) \\ &\quad = \frac{p^2}{576} + \frac{p}{8} - \frac{143}{576} + \left(\frac{p}{96} - \frac{1}{8}\right)\left(\frac{-1}{p}\right) + \frac{1}{8}\left(\frac{2}{p}\right) + \frac{1}{12}\left(\frac{3}{p}\right) + \frac{p}{36}\left(\frac{-3}{p}\right) \\ &\dim J_{4,m}^{\mathrm{cusp}} = \sum_{j=1}^{m} \left(\{\{4+2j\}\} - \llcorner (j^2/4m) \lrcorner \right), \end{aligned}$$

where we let $\llcorner x \lrcorner = \max\{m \in \mathbb{Z} : m \le x\}$ be the greatest integer function and where $\{\{k\}\} = \dim S_k(\mathrm{SL}_2(\mathbb{Z}))$.

V. Gritsenko has a lifting $\mathrm{Grit} : J_{k,N}^{\mathrm{cusp}} \to S_k(K(N))^\epsilon$ for $\epsilon = (-1)^k$ with the property that the Fourier Jacobi expansion of $\mathrm{Grit}(\phi)$ has leading term $\phi \, \xi^N$, see [7]. In selecting a generic example, we avoid these lifts because their Fourier coefficients satisfy special linear relations. The first prime p for which the map $\mathrm{Grit} : J_{4,p}^{\mathrm{cusp}} \to S_4(K(p))$ does not surject is $p = 31$; here $\mathrm{Grit}(J_{4,31}^{\mathrm{cusp}})$ is five dimensional and $S_4(K(31))$ six. By subtracting off the Gritsenko lift of the leading Fourier Jacobi coefficient we have $S_4(K(31))^+ = \mathrm{Grit}(J_{4,31}^{\mathrm{cusp}}) \oplus \mathcal{S}$. We will compute 12 coefficients of the Fourier Jacobi expansions from $S_4(K(31))$ in accordance with the following Lemma, noting here that $\frac{k}{10} \frac{p^2+1}{p+1} = \frac{4}{10} \frac{31^2+1}{31+1} = 12.025$.

Lemma 5.1 *Let p be a prime, $J, M, k \in \mathbb{N}$ and $\epsilon \in \{-1, 1\}$. Let $\pi_M : \prod_{j=1}^{\infty} J_{k,pj} \to \prod_{j=1}^{\llcorner M/p \lrcorner} J_{k,pj}$ be projection. The map $\pi_{pJ} \circ \mathrm{FJ} : S_k(K(p))^\epsilon \to \prod_{j=1}^{J} J_{k,pj}^{\mathrm{cusp}}$ injects for $J \ge \llcorner \frac{k}{10} (\frac{p^2+1}{p+1}) \lrcorner$.*

Proof For $T \in \mathcal{X}_2(p) = \left\{\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) > 0 : a, 2b, c \in \mathbb{Z} \text{ and } p|c\right\}$, define the Minimum function m via $m(T) = \min_x T[x]$ over $x \in \mathbb{Z}^2 \setminus \{0\}$. It is known that the $T \in \mathcal{X}_2(p)$ with $m(T) \le \frac{k}{10}\frac{p^2+1}{p+1}$ are a determining set of Fourier coefficients for $S_k(K(p))^{\epsilon}$, see [16]. Consider $f \in S_k(K(p))^{\epsilon}$ such that

$$\forall T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) \in \mathcal{X}_2(p) : \frac{m}{p} \le \frac{k}{10}\frac{p^2+1}{p+1},\ a(T; f) = 0. \tag{3}$$

We need to show that such f vanish. Take any $T \in \mathcal{X}_2(p)$ satisfying $m(T) \le \frac{k}{10}\frac{p^2+1}{p+1}$. By reduction we have $T = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\left[\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\right]$ for some $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ and $0 \le 2b \le c \le a$; in this case $c = m(T)$. If $p|\beta$ then $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \hat{\Gamma}^0(p)$ and $a(T) = \pm a\left(\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\right) = 0$ by (3) since $\frac{c}{p} \le c \le \frac{k}{10}\frac{p^2+1}{p+1}$. If β is prime to p, let $r \in \mathbb{Z}$ solve $\beta r \equiv \delta \mod p$; then $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 0 & 1 \\ 1 & r \end{smallmatrix}\right) \in \hat{\Gamma}^0(p)$ and we have $T[\sigma] = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\left[\left(\begin{smallmatrix} 0 & 1 \\ 1 & r \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} c & b+rc \\ b+rc & cr^2+2br+a \end{smallmatrix}\right) \in \mathcal{X}_2(p)$ so that $p|(cr^2 + 2br + a)$. In this case

$$a(T) = \det(\sigma)^k a\big(T[\sigma]\big) = \epsilon \det(\sigma)^k a\left(\begin{pmatrix} \frac{cr^2+2br+a}{p} & -(cr+b) \\ -(cr+b) & pc \end{pmatrix}\right) = 0$$

by (3) because $\frac{pc}{p} = c \le \frac{k}{10}\frac{p^2+1}{p+1}$. Since $a(T) = 0$ for all T with $m(T) \le \frac{k}{10}\frac{p^2+1}{p+1}$, we have $f = 0$. □

For $p = 31$ and $k = 4$, the following Proposition computes the first $J = 12$ Jacobi form coefficients of any formal power series that satisfies the Involution(ϵ) condition and finds that they are all initial Fourier-Jacobi expansions of paramodular cusp forms. This makes it at least plausible that the involution condition alone characterizes the Fourier Jacobi expansions from $S_4(K(31))^{\epsilon}$ from among all formal power series over Jacobi forms. And that is the point of this computation—to show that the growth condition may be superfluous.

Proposition 5.2 *Let $k, p, J \in \mathbb{N}$ with p prime. Define the subspaces*

$$A(J) = \left\{\Phi = \sum_{j=1}^{J} \phi_{jp}\, \xi^{jp} \in \prod_{j=1}^{J} J_{k,jp}^{\text{cusp}} :\ \Phi \text{ satisfies Involution}(-)\right\} \quad \text{and}$$

$$B(J) = \left\{\Phi = \sum_{j=2}^{J} \phi_{jp}\, \xi^{jp} \in \prod_{j=1}^{J} J_{k,jp}^{\text{cusp}} :\ \Phi \text{ satisfies Involution}(+)\right\}.$$

For $k = 4$ and $p = 31$, the subspace $A(12)$ is trivial and the subspace $B(12)$ is one dimensional and is spanned by $\Phi_0 = \psi_{62}\xi^{62} + \psi_{93}\xi^{93} + \cdots + \psi_{12\cdot 31}\xi^{12\cdot 31}$ with initial expansions

$$\begin{aligned}\psi_{62} = \mathbf{q}^2\big(&-\zeta^{22} + 7\zeta^{21} - 15\zeta^{20} - 3\zeta^{19} + 50\zeta^{18} - 37\zeta^{17} - 47\zeta^{16}\\ &+ 19\zeta^{15} + 74\zeta^{14} + 49\zeta^{13} - 163\zeta^{12} - 13\zeta^{11} + 67\zeta^{10} + 28\zeta^{9} + 108\zeta^{8}\\ &- 84\zeta^{7} - 106\zeta^{6} - 74\zeta^{5} + 114\zeta^{4} + 162\zeta^{3} - 84\zeta^{2} - 54\zeta + 6 - 54/\zeta\\ &- 84/\zeta^{2} + 162/\zeta^{3} + 114/\zeta^{4} - 74/\zeta^{5} - 106/\zeta^{6} - 84/\zeta^{7} + 108/\zeta^{8}\\ &+ 28/\zeta^{9} + 67/\zeta^{10} - 13/\zeta^{11} - 163/\zeta^{12} + 49/\zeta^{13} + 74/\zeta^{14} + 19/\zeta^{15}\end{aligned}$$

$$
\begin{aligned}
&- 47/\zeta^{16} - 37/\zeta^{17} + 50/\zeta^{18} - 3/\zeta^{19} - 15/\zeta^{20} + 7/\zeta^{21} - 1/\zeta^{22}\big) \\
&+ \mathbf{q}^3\big(\zeta^{27} - 5\zeta^{26} + 5\zeta^{25} + 11\zeta^{24} - 19\zeta^{23} - 2\zeta^{22} - 5\zeta^{21} + 21\zeta^{20} + 39\zeta^{19} \\
&- 47\zeta^{18} - 5\zeta^{17} - 64\zeta^{16} + 19\zeta^{15} + 133\zeta^{14} - 25\zeta^{13} + 17\zeta^{12} - 131\zeta^{11} \\
&- 52\zeta^{10} + 71\zeta^{9} - 3\zeta^{8} + 159\zeta^{7} - 37\zeta^{6} - 49\zeta^{5} - 38\zeta^{4} - 86\zeta^{3} + 10\zeta^{2} \\
&+ 26\zeta + 112 + 26/\zeta + 10/\zeta^{2} - 86/\zeta^{3} - 38/\zeta^{4} - 49/\zeta^{5} - 37/\zeta^{6} \\
&+ 159/\zeta^{7} - 3/\zeta^{8} + 71/\zeta^{9} - 52/\zeta^{10} - 131/\zeta^{11} + 17/\zeta^{12} - 25/\zeta^{13} \\
&+ 133/\zeta^{14} + 19/\zeta^{15} - 64/\zeta^{16} - 5/\zeta^{17} - 47/\zeta^{18} + 39/\zeta^{19} + 21/\zeta^{20} \\
&- 5/\zeta^{21} - 2/\zeta^{22} - 19/\zeta^{23} + 11/\zeta^{24} + 5/\zeta^{25} - 5/\zeta^{26} + 1/\zeta^{27}\big) + O\big(\mathbf{q}^4\big);
\end{aligned}
$$

$$\psi_{93} = \mathbf{q}^2\big(\mathrm{coeff}\big(\psi_{62}, q^3\big)\big) + O\big(\mathbf{q}^3\big).$$

Proof It is convenient to denote $J_{k,m}^{\mathrm{cusp}}(\nu) = \{\phi \in J_{k,m}^{\mathrm{cusp}} : \mathrm{ord}\,\phi \geq \nu\}$. Let $\Phi = 0 \cdot \xi^{31} + \phi_{62}\xi^{62} + \phi_{93}\xi^{93} + \cdots + \phi_{31\nu}\xi^{31\nu} \in B(\nu)$. The space $J_{4,62}^{\mathrm{cusp}}$ is spanned by the 9 theta blocks $T(d)$ for $d = [1,1,1,1,2,4,6,8]$, $[1,1,1,2,2,2,3,10]$, $[1,1,1,2,2,4,4,9]$, $[1,1,1,2,3,6,6,6]$, $[1,1,2,2,2,2,5,9]$, $[1,1,2,4,4,5,5,6]$, $[1,2,2,2,2,3,7,7]$, $[1,3,4,4,4,4,5,5]$, $[2,2,2,2,3,3,3,9]$. The Involution(+) condition tells us that for all $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) \in \mathcal{X}_2(31)$ we have $c(n,r;\phi_m) = c(\frac{m}{31}, -r; \phi_{31n})$. Setting $n = 1$ and $m = 62$ in condition Involution(+), we have

$$c(1,r;\phi_{62}) = c(2,-r;\phi_{31}) = c(2,-r;0) = 0,$$

so that the q^1-coefficients of ϕ_{62} vanish. The subspace $J_{4,62}^{\mathrm{cusp}}(2)$ is spanned by one element, ψ_{62}, which is the following linear combination of the above nine theta blocks: $\psi_{62} = (-3,-5,-1,-2,-1,0,1,0,1) \cdot (T(d_1), \ldots, T(d_9))$. The initial expansion of ψ_{62} is as given above. Thus ϕ_{62} is some multiple of ψ_{62}, say $\phi_{62} = \alpha\psi_{62}$ for $\alpha \in \mathbb{C}$, and the subspace $B(2)$ is at most one dimensional.

The space $J_{4,93}^{\mathrm{cusp}}$ is spanned by the 16 theta blocks $T(c)$ for $c = [1,1,1,1,1,1,6,12]$, $[1,1,1,1,1,6,8,9]$, $[1,1,1,1,2,3,5,12]$, $[1,1,1,1,2,4,9,9]$, $[1,1,1,1,4,6,7,9]$, $[1,1,1,3,5,6,7,8]$, $[1,1,2,2,2,6,6,10]$, $[1,1,2,3,3,3,3,12]$, $[1,1,2,3,3,4,5,11]$, $[1,1,2,6,6,6,6,6]$, $[1,2,3,3,3,3,8,9]$, $[1,3,4,4,6,6,6,6]$, $[2,2,2,2,2,2,9,9]$, $[2,2,2,2,2,3,6,11]$, $[2,3,3,3,3,3,4,11]$, $[3,4,5,5,5,5,5,6]$. For $n = 1$ and $m = 93$ the Involution(+) conditions are $c(1,r;\phi_{93}) = c(3,-r;\phi_{31}) = 0$ so that $\phi_{93} \in J_{4,93}^{\mathrm{cusp}}(2)$. The subspace $J_{4,93}^{\mathrm{cusp}}(3)$ is trivial and the subspace $J_{4,93}^{\mathrm{cusp}}(2)$ is spanned by the following four linear combinations of theta blocks:

$$
\begin{aligned}
Q_1 &= (-1,-1,-6,-6,-4,-1,0,-1,2,0,1,0,0,0,0,0) \cdot \big(T(c_1), \ldots, T(c_{16})\big), \\
Q_2 &= (-2,-1,-9,-6,-3,0,-1,-1,3,0,0,0,1,0,0,0) \cdot \big(T(c_1), \ldots, T(c_{16})\big), \\
Q_3 &= (-1,-1,-4,-2,0,0,1,1,-2,0,0,0,0,1,0,0) \cdot \big(T(c_1), \ldots, T(c_{16})\big), \\
Q_4 &= (1,0,1,3,2,-1,0,-1,-5,0,0,0,0,0,1,0) \cdot \big(T(c_1), \ldots, T(c_{16})\big).
\end{aligned}
$$

Some Fourier coefficients for these Q_i are in Table 2. We use the Involution(+) condition for $n = 2$ and $m = 93$ to find the q^2-coefficients of ϕ_{93}.

$$c(2,r;\phi_{93}) = c(3,-r;\phi_{62}) = \alpha c(3,-r;\psi_{62}). \tag{4}$$

Table 2 Fourier coefficients of the basis Q_i for $J^{\text{cusp}}_{4,93}(2)$

r	$c(2, r; Q_1)$	$c(2, r; Q_2)$	$c(2, r; Q_3)$	$c(2, r; Q_4)$
0	114	300	6	−226
1	−38	−145	−69	12
2	14	−47	89	−24
3	−60	−1	41	146
4	−34	84	−72	72
5	40	−69	−53	9
6	65	−27	41	−28
7	15	209	74	−174
8	−42	−113	0	45
9	−49	−137	−103	−22
10	−65	−72	−49	117
11	137	303	190	−6
12	−33	−93	−55	16
13	61	−44	−42	−36
14	−79	−10	−72	−54
15	−42	0	67	23
16	67	30	101	−3
17	−40	−99	−122	45
18	73	149	19	−26
19	−57	−55	−3	18
20	3	−23	31	−24
21	7	−6	−5	−2
22	−7	9	−28	9
23	19	30	24	0
24	−18	−35	−8	7
25	7	14	1	−12
26	−1	−2	0	6
27	0	0	0	−1

The coefficients $c(3, -r; \psi_{62})$ are known and displayed in the statement of the Proposition. The unique element $\phi_{93} \in J^{\text{cusp}}_{4,93}(2)$ satisfying equation (4) is $\alpha\psi_{93}$ where $\psi_{93} = -Q_1 - Q_4$. This shows that the subspace $B(3)$ is at most one dimensional. Continuing in this way on a computer, we showed that $J^{\text{cusp}}_{4,31j}(j) = \{0\}$ for $j = 3, \ldots, 12$ and hence that $\dim B(12) \leq 1$.

We discuss the minus space. The space $J^{\text{cusp}}_{4,31}$ is spanned by 5 theta blocks $T(b)$ for $b =$ [1, 1, 1, 1, 1, 2, 2, 7], [1, 1, 1, 1, 1, 4, 4, 5], [1, 1, 1, 1, 2, 2, 5, 5], [1, 1, 2, 2, 2, 4, 4, 4], [2, 2, 3, 3, 3, 3, 3, 3]. For even weights, the Involution($-$) conditions are quite restrictive. We have $c(j, r; \phi_{31j}) = -c(j, -r; \phi_{31j})$, so that the q^j-coefficients of ϕ_{31j} must vanish. However, the q^1-coefficients of the five theta blocks $T(b_i)$ are already linearly independent, so $A(1)$ is trivial. Now that we know that the first Jacobi coefficient vanishes, by the same reasoning as for the plus space, the only possible element of $A(2)$ is a multiple of ψ_{62}; however the extra condition that the q^2-coefficients of ϕ_{62} vanish shows that $A(2)$ is trivial. The triviality of $A(12)$ now follows from $J^{\text{cusp}}_{4,31j}(j) = \{0\}$ for $j = 3, \ldots, 12$.

By Theorem 2.2 we have a map $\pi_{12\cdot 31} \circ \mathrm{FJ} : S_4(K(31))^- \to A(12)$ and, by Lemma 5.1, this map is injective; hence $S_4(K(31))^-$ is trivial. From Ibukiyama's result, $\dim S_4(K(31)) = 6$, we may conclude that $\dim S_4(K(31))^+ = 6$ and $\dim \mathcal{S} = 1$. Therefore $\Phi_0 \in \pi_{12\cdot 31}\,\mathrm{FJ}(\mathcal{S}) \subseteq B(12)$ and $\dim B(12) = 1$. □

From another point of view, the merit of the preceding computations consists in providing upper bounds for the dimension of spaces of paramodular cusp forms. In this particular case, relying on Ibukiyama's dimension formula for the existence of forms, we have shown the following Corollary.

Corollary 5.3 $\dim S_4(K(31))^+ = 6$ *and* $\dim S_4(K(31))^- = 0$.

6 Final remarks

We conclude by comparing the Involution condition with the following weaker inequality; for general N, we cannot even show that the right hand side is finite:

$$\dim S_k\big(K(N)\big)^{(-1)^k} \le \sum_{j=1}^{\infty} \dim J_{k,Nj}^{\mathrm{cusp}}(j). \tag{5}$$

For the case $N = 31$ and $k = 4$ we have demonstrated the equality $\dim S_4(K(31))^+ = \sum_{j=1}^{20} \dim J_{4,31j}^{\mathrm{cusp}}(j)$ or $6 = 5 + 1 + 0 + 0 + \cdots + 0$. However tempting it may be to replace the 20 by ∞, we cannot be sure about that equality because we have only computed $\dim J_{4,31j}^{\mathrm{cusp}}(j) = 0$ for $3 \le j \le 20$. We can, however, be certain about inequalities; for example with $N = 29$ and $k = 4$, we can show the inequality $\dim S_4(K(29))^+ < \sum_{j=1}^{\infty} \dim J_{4,29j}^{\mathrm{cusp}}(j)$ or $5 < 5 + 1 + 0 + \cdots$. The space $J_{4,58}^{\mathrm{cusp}}(2)$ is one dimensional, spanned by Ψ say, but there is no element in $J_{4,87}^{\mathrm{cusp}}(2)$ whose q^2-terms equal the q^3-terms of Ψ. Hence there does exists a $\Phi = \Psi\xi^2$ satisfying the Involution(+) conditions to second order that is not the initial Fourier Jacobi expansion of any paramodular cusp form from $S_4(K(29))^+$. All the Φ that satisfy the Involution(+) condition to third order, however, are the initial Fourier Jacobi expansions of paramodular cusp forms from $S_4(K(29))^+$. Hence, again in this example, the Involution condition continues to compute the space $S_4(K(29))^+$ correctly even when the inequality (5) is strict.

One case where the convergence of the series $\sum_{j=1}^{\infty} \dim J_{k,Nj}^{\mathrm{cusp}}(j)$ is known for all weights k is $N = 5$. Here the sum need only be taken to $j = \llcorner k/2 \lrcorner$ because $\operatorname{ord}\phi \le (k + 2m)/12$ for $\phi \in J_{k,m}$. A more refined estimate of Gritsenko and Hulek [9] shows that $j \le \llcorner (3k-6)/8 \lrcorner$ suffices when $N = 5$. Since $N = 5$ is also the first level where the inequality (5) can be strict, it is of some interest to ponder this data. Table 3 gives the values of $\dim J_{k,5j}^{\mathrm{cusp}}(j)$ for $1 \le k \le 15$ and for $j \le \llcorner (3k-6)/8 \lrcorner$. These were computed by using theta blocks to span the spaces of Jacobi forms. For weights $k \le 15$, the dimensions of $S_k(K(5))^{\pm}$ in Table 3 may be found in a manner similar to that used to prove Corollary 5.3. One sees in Table 3 that the inequality (5) is already strict for weight $k = 12$ and hence that the method that was used for $1 \le N \le 4$ to prove the surjectivity of $\mathrm{FJ} : M_k(K(N))^{\epsilon} \to \mathbb{M}_k(N)^{\epsilon}$ for all weights will not work for $N = 5$.

Table 3 Values of $\dim J_{k,5j}^{\mathrm{cusp}}(j)$

k	j						
	1	2	3	4	5	$\sum_{j=1}^{\infty} \dim J_{k,5j}^{\mathrm{cusp}}(j)$	$\dim S_k(K(5))^{(-1)^k}$
1						0	0
2						0	0
3						0	0
4						0	0
5	1					1	1
6	1					1	1
7	1					1	1
8	2	0				2	2
9	2	0				2	2
10	3	1	0			4	4
11	3	1	0			4	4
12	4	2	1			7	6
13	3	2	1	0		6	5
14	5	3	2	0		10	8
15	4	3	2	0		9	8

We mention two references that were added in revision. For the introduction to theta blocks see [10]. For results relevant to this article, see [18].

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