Vector-valued Siegel modular forms of genus 2

Explicit constructions of Siegel modular forms using differential operators of Rankin-Cohen type

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MSc Thesis

under supervision of

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Abstract

In this thesis we will give generators for $\bigoplus_{k\equiv 1(2)} M_{\mathrm{Sym}^6\otimes\det^k}(\mathsf{Sp}_2(\mathbb{Z}))$ as a module over the ring of classical Siegel modular forms of even weight and genus 2. We first determine some explicit pluri-harmonic homogeneous polynomials and use them to define differential operators of Rankin-Cohen type, thereby generalizing Rankin-Cohen brackets that were considered by Ibukiyama, Satoh and others. We prove non-vanishing and linear independence of forms by explicit calculations of Fourier coefficients. This also allows us to determine some eigenvalues for the Hecke operators.

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Preface

Siegel modular forms were introduced by Carl Ludwig Siegel in the 1930's. They form a generalisation of elliptic modular forms and although some results can be lifted from the elliptic case to the general case (e.g. the theory of Hecke operators and bases of common eigenvectors, integrality of eigenvalues of the Hecke operators), much is yet unknown. For instance, the order of vanishing of the Eisenstein series E_4 and E_6 at some special points of the upper half-plane can be used to determine the structure of the ring of all modular forms on $\mathsf{SL}_2(\mathbb{Z})$:

$$\bigoplus_{k\in\mathbb{Z}} M_k(\mathsf{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6].$$

Such a method does not exist in general. For some instances of rings or modules of Siegel modular forms similar structure theorems exist. For most instances however, explicit generators are unknown. In this thesis I will restrict mostly to Siegel modular forms of 'genus 2' and in this case some structure theorems are known. Igusa treated the 'classical' Siegel modular forms of genus 2, and Satoh and Ibukiyama gave generators for the 'first' few modules of vector-valued Siegel modular forms of genus 2. Satoh and Ibukiyama used generalisations of Rankin-Cohen brackets (RC-brackets). These brackets are differential operators on rings of modular forms and correspond to a special class of polynomials, which I will call 'RC-polynomials'. All RC-polynomials were characterized by Ibukiyama and I will repeat some of his results and proofs in this thesis. Ibukiyama's characterisation does not give the RC-polynomials explicitly, but his student Miyawaki was able to determine a large class of RC-polynomials in terms of hypergeometric functions.

The weight of a Siegel modular form of genus 2 (an irreducible, finite dimensional representation of $\mathsf{GL}_2(\mathbb{C})$) is determined by a pair $(m,k)\in\mathbb{Z}^2$. Satoh and Ibukiyama treated the cases (m,k) with $m = 2,4$ and $k \in \mathbb{Z}$ and (m,k) with $m = 6$ and $k \in 2\mathbb{Z}$. Miyawaki's description of RC-polynomials only includes those polynomials that lead to modular forms of weight (m,k) with even k. Ibukiyama solved this problem by defining RC-brackets on triples of classical Siegel modular forms. These RC-brackets on triples are (especially in the case $m = 4$) rather complicated.

I will 'complete' Ibukiyama's structure theorem by finding all modular forms of weight $(6,k)$ with odd k. In order to do this, I will give a generalisation of Ibukiyama's RC-brackets on triples of modular forms. Unfortunately, the Siegel modular forms in the range of my generalized Rankin-Cohen brackets do not include vector-valued modular forms of 'low' weight. This problem also exists for modular forms of weight (m,k) with $m \geq 6$ and k even and Ibukiyama solved this by constructing vector-valued theta series with harmonic coefficients and by using Arakawa's vector-valued generalisation of Klingen-Eisenstein series. I will use Ibukiyama's theta series and the Klingen-Eisenstein series and a 'trick' to construct the 'missing' modular forms of weight $(6,k)$ with small k.

Outline of this thesis

Chapter 1 consists of a very short introduction to Siegel modular forms and serves as a convenient context for stating some results, definitions and notations that I will use in the proceeding chapters. I will also make my above described goals (i.e. finding all modular forms of weight $(6,k)$ with k odd) more precise and I will finish with an explicit example of a vector-valued modular form.

Chapter 2 deals with Ibukiyama's characterization of RC-polynomials. The formulation of this characterisation is rather tedious and therefore deserves some attention. After stating the characterisation it is almost a shame not to give a (partial) proof of Ibukiyama's theorem and therefore I will finish the chapter with a (partial) proof. This proof is almost identical to Ibukiyama's proof, but perhaps a little more detailed.

In Chapter 3 I will give some explicit examples of RC-polynomials and RC-brackets that can be found in the literature. These polynomials will be used in Chapter 4. Furthermore, I will generalize Ibukiyama's RC-brackets on triples of classical modular forms and give the main ingredient of the 'trick' I mentioned above: I will define an RC-bracket that acts on vector-valued modular forms.

Chapter 4 consists of two parts. In the first part I will give a one-and-a-half-page proof of the structure theorem for modular forms of weight $(2,k)$. This proof is based on the proofs due to Satoh and Ibukiyama, but uses a result by Aoki and Ibukiyama that eliminates some tedious computations that can be found in the original proofs. In the second part of the chapter I will state and prove the structure theorem for modular forms of weight $(6,k)$ with k odd. Unfortunately, I was not able to eliminate any tedious computations here. Readers that are unwilling to check my calculations are then treated with a small table of eigenvalues of the Hecke operators $T(2)$ and $T(3)$ and an argument for their correctness.

Acknowledgements

Writing this thesis would have been nearly impossible without quite a number of people. I would like to thank anyone who has been involved in one way or the other. To be a little bit more specific, I will mention some of them. In the first place I thank my supervisor Gerard van der Geer. Especially his enthusiasm during our last few meetings has finally made me appreciate the 'struggle' I sometimes went trough. I regret that we did not correspond on a more frequent basis, which is something I have myself to blame for. Secondly, Fabien Cléry, who, as the second reader, had to read trough many persistent typo's and was still able to point out numerous useful corrections. My gratitude also goes out to some people who were less formally involved. First of all my mother Yvonne, who checked my spelling and grammar. Secondly, my friends and fellow students Jan Nauta, Sam van Gool and Gesche Nord for moral and sometimes also technical support. Further—rather essential—moral support came from my sister Suzanne and Harry Rours. Without them I would probably never have started in the first place. Next, I should thank Joeri Jörg, Corine Laman, Wessel Haanstra and Tim Cardol for listening to my story about theta series and the many coffee breaks and for the latter I would also like to thank Mathijs van Zaal. Finally, I thank Rob de Boer and Michiel van Boven for the time they have given me to finish this thesis.

Christiaan van Dorp,
Amsterdam, March 2, 2012

Contents

Chapter 1

Introduction

Siegel modular forms generalize elliptic modular forms. The modular group $\mathsf{SL}_2(\mathbb{Z})$, the upper half-plane $\mathscr{H}$ and the weight of a modular form k are replaced with the "symplectic group" $\mathsf{Sp}_g(\mathbb{Z})$, the "Siegel upper half-space" $\mathscr{H}_g$ and a representation $\rho : \mathsf{GL}_g(\mathbb{C}) \to V$ on some finite dimensional vector space V respectively. Siegel modular forms are functions $f : \mathscr{H}_g \to V$ that behave like elliptic modular forms with respect to the new objects $\mathsf{Sp}_g(\mathbb{Z})$, $\mathscr{H}_g$ and ρ. These objects depend on an integer g called the "genus". Elliptic modular forms are Siegel modular forms of genus 1, which justifies the first sentence of this paragraph. If the representation ρ is not a character, then the corresponding Siegel modular forms are vector-valued functions. We will first make these notions more precise and then state some important properties of Siegel modular forms that can be found in the literature.

1.1 The symplectic group and the Siegel upper half-space

Let V be a real vector space of dimension $2g$ and choose a basis $\{e_1, \dots, e_g, f_1, \dots, f_g\}$. Define the *symplectic form* $(\cdot,\cdot)$ on V by

$$(e_i, e_j) = (f_i, f_j) = 0 \qquad \text{and} \qquad (e_i, f_j) = \delta_{ij} = -(f_j, e_i) \qquad \text{for } 1 \le i, j \le g.$$

Let $\mathsf{Sp}_g(\mathbb{R})$ be the group of linear operators on V which preserve the symplectic form. Let W be the $\mathbb{Z}$-lattice spanned by $\{e_1, \dots, e_g, f_1, \dots, f_g\}$. We denote by $\mathsf{Sp}_g(\mathbb{Z})$ the subgroup of $\mathsf{Sp}_g(\mathbb{R})$ of operators that send W to W.

We can describe an element of $\mathsf{Sp}_g(\mathbb{R})$ as a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $a, b, c, d \in \mathrm{Mat}_g(\mathbb{R})$, the space of real $g \times g$ matrices, such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & c' \\ b' & d' \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{smallmatrix}\right)$. Here $\mathbf{1}_g$ is the $g \times g$ identity matrix and we denote by x' the transpose of a matrix x. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{Sp}_g(\mathbb{R})$, then we must have $ad' - bc' = \mathbf{1}_g$, $ab' = ba'$ and $cd' = dc'$.

Let $K_g := \left\{ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \,\middle|\, \begin{smallmatrix} aa' + bb' = \mathbf{1}_g \\ ab' = ba' \end{smallmatrix} \right\}$. The group K_g is a maximal compact subgroup of $\mathsf{Sp}_g(\mathbb{R})$ and the space of cosets $\mathsf{Sp}_g(\mathbb{R})/K_g$ is in bijection with the *Siegel upper half-space* which is defined by

$$\mathscr{H}_g := \{\tau \in \mathrm{Mat}_g(\mathbb{C}) \mid \tau = \tau', \mathrm{Im}(\tau) \succ 0\}.$$

The bijection $\mathsf{Sp}_g(\mathbb{R})/K_g \longleftrightarrow \mathscr{H}_g$ can be derived from the decomposition $\mathsf{Sp}_g(\mathbb{R}) = N_g A_g K_g$ where

$$A_g := \left\{ \left(\begin{smallmatrix} u' & 0 \\ 0 & u^{-1} \end{smallmatrix} \right) \middle| u \in \mathsf{GL}_g(\mathbb{R}) \right\}, \qquad \text{and} \qquad N_g := \left\{ \left(\begin{smallmatrix} 1_g & x \\ 0 & 1_g \end{smallmatrix} \right) \middle| x = x' \in \mathrm{Mat}_g(\mathbb{R}) \right\}$$

and therefore we can send $x + iy \in \mathscr{H}_g$ to $\left(\begin{smallmatrix} 1_g & x \\ 0 & 1_g \end{smallmatrix} \right) \left(\begin{smallmatrix} u' & 0 \\ 0 & u^{-1} \end{smallmatrix} \right) \cdot K_g$ with $0 \prec y = u'u$. The groups $\mathsf{Sp}_g(\mathbb{R})$ and $\mathsf{Sp}_g(\mathbb{Z})$ act on $\mathsf{Sp}_g(\mathbb{R})/K_g$ by means of left multiplication and through the map $\mathsf{Sp}_g(\mathbb{R})/K_g \to \mathscr{H}_g$, the groups $\mathsf{Sp}_g(\mathbb{R})$ and $\mathsf{Sp}_g(\mathbb{Z})$ act also on $\mathscr{H}_g$. This last action is given by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot \tau = (a\tau + b)(c\tau + d)^{-1}.$$

Details about these facts are given in e.g. [13, 23, 21].

Let γ be given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathsf{Sp}_g(\mathbb{R})$. The derivative of the map $\tau \mapsto \gamma \cdot \tau$ at the point τ is given by

$$w \mapsto (c\tau + d)'^{-1} w (c\tau + d)^{-1} : \mathrm{Mat}_g(\mathbb{C}) \to \mathrm{Mat}_g(\mathbb{C}).$$

This implies that the *factor of automorphy* for arbitrary genus should be defined as $j(\gamma, \tau) := c\tau + d \in \mathsf{GL}_g(\mathbb{C})$.[1] Note that for any $\gamma_1, \gamma_2 \in \mathsf{Sp}_g(\mathbb{R})$ we have the relation

$$j(\gamma_1\gamma_2, \tau) = j(\gamma_1, \gamma_2 \cdot \tau) j(\gamma_2, \tau). \tag{1.1}$$

1.2 Siegel modular forms and some of their properties

Let ρ be a representation of $\mathsf{GL}_g(\mathbb{C})$ onto a complex vector space V. From now on, we will use the symbol Γ_g to denote $\mathsf{Sp}_g(\mathbb{Z})$.

Definition 1.2.1. Let f be a V-valued function on $\mathscr{H}_g$ and let $\gamma \in \mathsf{Sp}_g(\mathbb{R})$. Define the *slash operator* $|_\rho \gamma$ as follows:

$$f|_\rho \gamma(\tau) := \rho(j(\gamma, \tau))^{-1} f(\gamma \cdot \tau)$$

Definition 1.2.2. A *Siegel modular form* of *genus* g and *weight* ρ is a holomorphic function $f : \mathscr{H}_g \to V$ such that for all $\gamma \in \Gamma_g$, we have $f = f|_\rho \gamma$. If $g = 1$, then f must also be 'holomorphic at infinity'[2].

We denote by $M_\rho(\Gamma_g)$ the $\mathbb{C}$-vector space of all Siegel modular forms of weight ρ and genus g.

Let f be a Siegel modular form of weight ρ and genus g. If $s \in \mathrm{Mat}_g(\mathbb{Z})$ is an integral symmetric matrix, then $f(\tau + s) = f(\tau)$. Therefore, the function f has a Fourier expansion $f(\tau) = \sum_n a(n) e^{2\pi i \operatorname{Tr}(n\tau)}$, where the sum is taken over all half-integral symmetric matrices, i.e. matrices of the form $n = n' = (n_{ij})$ with $n_{ii}, 2n_{ij} \in \mathbb{Z}$ for all i, j. If all Fourier coefficients $a(n)$ vanish at $n \not\succeq 0$, then f is called *holomorphic at infinity*. The *Koecher Principle* tells us that $f \in M_\rho(\Gamma_g)$ is automatically holomorphic at infinity if $g > 1$. For $g = 1$ we must add holomorphicity at infinity to the definition of a modular form. We will sometimes abuse notation and write $q^n := e^{2\pi i \operatorname{Tr}(n\tau)}$, such that $f(\tau) = \sum_{n \succeq 0} a(n) q^n$.

Holomorphicity at infinity for $f \in M_\rho(\Gamma_g)$ ensures the limit

$$\lim_{t \to \infty} f\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & it \end{smallmatrix} \right) \right) =: (\Phi f)(\tau) \qquad \tau \in \mathscr{H}_{g-1}$$

[1] Note that Freitag [13] gives a slightly different definition for j then we do.

[2] explained in the paragraph below.

to exist and this limit is again a Siegel modular form. Therefore, we have a map $\Phi : M_\rho(\Gamma_g) \to M_\rho(\Gamma_{g-1})$, at which we can view $\rho : \mathsf{GL}_g(\mathbb{C}) \to \mathsf{GL}(V)$ as a representation of $\mathsf{GL}_{g-1}(\mathbb{C})$, if we use the embedding

$$z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right) : \mathsf{GL}_{g-1}(\mathbb{C}) \hookrightarrow \mathsf{GL}_g(\mathbb{C}).$$

The map Φ is called the *Siegel operator* and a Siegel modular form f in the kernel of the Siegel operator is called a *cusp form*. The space of cusp forms in $M_\rho(\Gamma_g)$ is denoted by $S_\rho(\Gamma_g)$. If $f = \sum_n a(n)q^n$ and $f \in S_\rho(\Gamma_g)$, then $a(n) \neq 0 \implies n \succ 0$.

Remark 1.3. The group $\mathsf{GL}_g(\mathbb{Z})$ of unimodular matrices can be embedded in Γ_g by means of the homomorphism $u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & u'^{-1} \end{smallmatrix}\right) : \mathsf{GL}_g(\mathbb{Z}) \to \mathsf{Sp}_g(\mathbb{Z})$. Therefore, we have an action of $\mathsf{GL}_g(\mathbb{Z})$ on the ring of Siegel modular forms of weight ρ and in particular on $M_{\det^k}(\Gamma_g)$:

$$f(u\tau u') = \det(u)^k f(\tau) = \pm f(\tau), \qquad f \in M_{\det^k}(\Gamma_g).$$

This means in particular that if $gk \equiv 1(2)$, then $f \equiv 0$ for f of weight $\det^k$. For example, all non-zero modular forms on Γ_1 are of even weight. However, modular forms of odd weight do exist on Γ_2 (see Theorem 1.4.1 below).

The action of $\mathsf{GL}_g(\mathbb{Z})$ on $M_\rho(\Gamma_g)$ has a nice consequence for the Fourier coefficients of a modular form. If $f \in M_\rho(\Gamma_g)$, $\gamma = \left(\begin{smallmatrix} u'^{-1} & 0 \\ 0 & u \end{smallmatrix}\right)$ and $f = \sum_n a(n)q^n$, then

$$f(\tau) = f|_\rho \gamma(\tau) = \rho(u)^{-1} \sum_n a(n) q^{u^{-1}nu'^{-1}}$$

and if we change summation variables, we get $f(\tau) = \sum_n \rho(u)^{-1} a(unu')q^n$. Since Fourier coefficients are unique, we see that

$$a(unu') = \rho(u)a(n).$$

1.4 The ring of classical Siegel modular forms

The 1-dimensional representations of $\mathsf{GL}_g(\mathbb{C})$ are given by powers of the determinant. A Siegel modular form of weight $\det^k$ is called a *classical Siegel modular form*. Examples of classical Siegel modular forms are *Eisenstein series* $E_{g,k}$ defined by

$$E_{g,k}(\tau) := \sum_\gamma \det j(\gamma,\tau)^{-k},$$

where the sum is taken over a complete set of representatives for the cosets $(N_g A_g \cap \Gamma_g)\backslash\Gamma_g$. We abbreviate $M_{\det^k}(\Gamma_g)$ to $M_k(\Gamma_g)$ and write simply 'k' for the weight '$\det^k$'. The Siegel operator Φ sends Eisenstein series to Eisenstein series:

$$\Phi E_{g,k} = E_{g-1,k}.$$

1.4.1 The case $g = 2$

We will write $\tau = \left(\begin{smallmatrix} \tau_1 & z \\ z & \tau_2 \end{smallmatrix}\right)$ for an element $\tau \in \mathscr{H}_2$. If $f \in M_k(\Gamma_2)$, then we can develop f as a Taylor series around $z = 0$:

$$f = \sum_{n \geq 0} f_n(\tau_1, \tau_2) z^n. \tag{1.2}$$

If $\gamma^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z})$, then

$$\gamma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathsf{Sp}_2(\mathbb{Z}).$$

For this particular γ, we have $\gamma \cdot \tau = \begin{pmatrix} \gamma^* \cdot \tau_1 & z/(c\tau_1+d) \\ z/(c\tau_1+d) & \tau_2 - cz^2/(c\tau_1+d) \end{pmatrix}$ and $\det(j(\gamma,\tau)) = j(\gamma^*,\tau_1)$. Therefore

$$f(\tau) = f|_k\gamma(\tau) = j(\gamma^*,\tau_1)^{-k} \sum_{n\geq 0} f_n(\gamma^*\cdot\tau_1, \tau_2 - cz^2/j(\gamma^*,\tau_1)) z^n j(\gamma^*,\tau_1)^{-n}. \tag{1.3}$$

If n_0 is the order of vanishing of f with respect to z around $z=0$, then by comparing (1.2) and (1.3), we get $f_{n_0}(\gamma^*\tau_1,\tau_2) = j(\gamma^*,\tau_1)^{k+n_0} f_{n_0}(\tau_1,\tau_2)$. This shows that $f_{n_0}(\tau_1,\tau_2) \in M_{k+n_0}(\Gamma_1)$ as a function of τ_1. Similarly, we get $f_{n_0}(\tau_1,\tau_2) \in M_{k+n_0}(\Gamma_1)$ as a function of τ_2. If f is a cusp form, then $\tau_i \mapsto f_{n_0}(\tau_1,\tau_2)$ for $i=1,2$ is a cusp form too. In particular, if $M_{k+n_0}(\Gamma_1)$ or, if f is a cusp form, $S_{k+n_0}(\Gamma_1)$ has dimension one, then we can write $f_{n_0}(\tau_1,\tau_2) = h(\tau_1)h(\tau_2)$ for an $h \in M_{k+n_0}(\Gamma_2)$.

When $g=2$, we can explicitly give generators for the graded ring $\bigoplus_k M_k(\Gamma_2)$. This is made clear in the following theorem due to Igusa. We write $\varphi_k = E_{2,k}$. The functions $\varphi_{10} - \varphi_4\varphi_6$ and $\varphi_{12} - \varphi_6^2$ are non-zero cusp forms of weight 10 and 12 respectively and we denote by $\chi_{10} = -4\pi^2\Delta(\tau_1)\Delta(\tau_2)z^2 + O(z^4)$ and $\chi_{12} = 12\Delta(\tau_1)\Delta(\tau_2) + O(z^2)$ their normalizations[3]. Using theta series [20] or a differential operator [2], one can construct a non-zero cusp form χ_{35} of weight 35.

Theorem 1.4.1 (Igusa)**.** *The graded ring $M := \bigoplus_k M_k(\Gamma_2)$ is generated by φ_4, φ_6, χ_{10}, χ_{12} and χ_{35} and there is a isobaric polynomial R in $\varphi_4, \varphi_6, \chi_{10}$ and χ_{12} such that*

$$\bigoplus_k M_k(\Gamma_2) \cong \mathbb{C}[\varphi_4,\varphi_6,\chi_{10},\chi_{12},\chi_{35}]/(R - \chi_{35}^2).$$

Igusa's proof, which is based on the so called *Igusa invariants* (a genus 2 analogue of the Weierstrass j-invariant), can be found in [19, 20]. The polynomial R is given explicitly in [20]. A more 'elementary' proof can be found in [12] and yet another proof, which is based on comparing dimensions and development around $z=0$, is given in [14].

The Hilbert-Poincaré series of $\bigoplus_k M_k(\Gamma_2)$ is given by the following generating function:

$$\sum_k \dim M_k(\Gamma_2) t^k = \frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

which is explained in e.g. [28]. Some non-zero dimensions of $M_k(\Gamma_2)$ are given in the following table.

k	0	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	35	36
$\dim M_k$	1	1	1	1	2	3	2	4	4	5	6	8	7	10	11	12	14	1	17

[3]The functions χ_{10} and χ_{12} are chosen such that their Fourier coefficient at $n = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ equals 1. The function $\Delta = q - 24q^2 + 252q^3 + \cdots$ is the elliptic cusp form of weight 12.

1.5 Vector-valued Siegel modular forms of genus 2

All irreducible representations ρ of $\mathsf{GL}_2(\mathbb{C})$ are up to isomorphism determined uniquely by their highest weight $(\lambda_1 \geq \lambda_2) \in \mathbb{Z}^2$. Let $m = \lambda_1 - \lambda_2 \geq 0$ and $k = \lambda_2$, then

$$\rho \cong \mathrm{Sym}^m \otimes \det^k,$$

where Sym^m and $\det^k$ denote the m-fold symmetric product and the k-th power of the determinant of the standard representation of $\mathsf{GL}_2(\mathbb{C})$. If $k < 0$, then $M_\rho(\Gamma_2) = (0)$ as a consequence of the Koecher Principle. Therefore, we only have to consider 'polynomial' representations of $\mathsf{GL}_2(\mathbb{C})$, i.e. representations with highest weight (λ_1, λ_2) where $\lambda_2 \geq 0$.

We will now introduce some notation.

We identify $m = 0$ with the classical case, i.e. $\mathrm{Sym}^0 \otimes \det^k \equiv \det^k$. We abbreviate $\mathrm{Sym}^m \otimes \det^k$ to (m, k) and write $M_{(m,k)}(\Gamma_2)$ for $M_{\mathrm{Sym}^m \otimes \det^k}(\Gamma_2)$. For fixed m, the direct sum of vector spaces

$$M_{(m,*)}(\Gamma_2) := \bigoplus_{k \geq 0} M_{(m,k)}(\Gamma_2) \qquad \text{is a module over} \qquad M_*(\Gamma_2) := \bigoplus_{k \geq 0} M_k(\Gamma_2).$$

Also, if we restrict the summation over even and odd k, then

$$M^0_{(m,*)}(\Gamma_2) := \bigoplus_{k \equiv 0(2)} M_{(m,k)}(\Gamma_2) \qquad \text{and} \qquad M^1_{(m,*)}(\Gamma_2) := \bigoplus_{k \equiv 1(2)} M_{(m,k)}(\Gamma_2)$$

are modules over $M^0_*(\Gamma_2) := \bigoplus_{k \equiv 0(2)} M_k(\Gamma_2)$.

The structure of the ring of classical Siegel modular forms of genus 2 is known in the sense of Theorem 1.4.1. Little is known about the non-classical case, but Satoh and Ibukiyama were able to give generators for $M^i_{(m,*)}(\Gamma_2)$ for small m:

$m = 2,$	$i = 0$	Satoh	[29],
$m = 2,$	$i = 1$	Ibukiyama	[17],
$m = 4,$	$i = 0, 1$	Ibukiyama	[18],
$m = 6,$	$i = 0$	idem.	

A dimension formula due to Tsushima [32] allows us to calculate the dimension of $S_{(m,k)}(\Gamma_2)$ for all m and $k > 4$.

In this thesis, we will give generators for $M^1_{(6,*)}(\Gamma_2)$ and we will give a proof of the structure theorem for $M_{(2,*)}(\Gamma_2)$ based on Igusa's structure theorem (Theorem 1.4.1). Most generators Satoh and Ibukiyama used are constructed via certain differential operators. We will study these differential operators in Chapters 2 and 3.

Perhaps more straightforward examples of vector-valued Siegel modular forms are given by vector-valued Eisenstein series. We will need them only in Chapter 4, but since we would like to give at least one example of a vector-valued Siegel modular form in this chapter, we will give a short introduction to vector-valued Eisenstein series in the following section.

1.6 Vector-valued Klingen-Eisenstein series of genus 2

Let $\rho : \mathsf{GL}_g(\mathbb{C}) \to \mathsf{GL}(V)$ be a finite dimensional representation on a vector space V. We can define a Hermitian form, the Petersson product, on $M_\rho(\Gamma_g)$ similar to the Petersson product on elliptic modular forms.

On V we have an inner product $(\cdot,\cdot)$ such that $(\rho(u)v,w) = (v,\rho(\bar{u}')w)$ for all $v,w \in V$ and $u \in \mathsf{GL}_g(\mathbb{C})$. Let $d\tau$ be the Euclidean measure on $\mathscr{H}_g$, that is:

$$d\tau = dx_{11}dy_{11}dx_{12}dy_{12}\cdots dx_{gg}dy_{gg}, \qquad \tau = x+iy, \qquad x=(x_{ij}),\ y=(y_{ij}),$$

then $\det(\mathrm{Im}(\tau))^{-g-1}d\tau$ is invariant under the action of Γ_g. We define the Petersson product on $M_\rho(\Gamma_g)$ by

$$\langle f,g\rangle = \int_{F_g} (\rho(\mathrm{Im}(\tau))f(\tau),g(\tau))\det(\mathrm{Im}(\tau))^{-g-1}d\tau, \qquad f,g\in M_{k,m}(\Gamma_2). \tag{1.4}$$

Here F_g is a fundamental domain for the action of Γ_g on $\mathscr{H}_g$. The integrand of (1.4) is invariant under the action of Γ_g, and therefore it does not matter what fundamental domain we choose. If at least one of the f and g is a cusp form, then $\langle f,g\rangle$ is well defined.

We will denote by $N_\rho(\Gamma_g)$ the orthogonal complement of $S_\rho(\Gamma_g)$ with respect to the Petersson product.

In the case of classical Siegel modular forms, the Siegel operator is surjective for even $k>2g$. This is not true for vector-valued Siegel modular forms, but when $g=2$ we can get a similar result due to Arakawa [3]:

Proposition 1.6.1. *For $k>4$ and $m>0$ we have*

$$\Phi : N_{(m,k)}(\Gamma_2) \xrightarrow{\sim} S_{k+m}(\Gamma_2). \tag{1.5}$$

The inverse of Φ in (1.5) is constructed using 'Klingen-Eisenstein series' which we will define below. Before we can do this, we need to consider a subgroup $C_{2,1}$ of Γ_2.

If $a = \left(\begin{smallmatrix} a_1 & * \\ * & * \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{C})$, then we define $a^* = a_1$. Let

$$C_{2,1} := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_2 \,\middle|\, a = \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right), c = \left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right), d = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \right\}.$$

For an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_2$, we define $\gamma^* = \left(\begin{smallmatrix} a^* & b^* \\ c^* & d^* \end{smallmatrix}\right)$. If $\gamma \in C_{2,1}$, then $\gamma^* \in \mathsf{SL}_2(\mathbb{Z})$, which can easily be seen if one computes $ad'-bc'$. The same computation shows that $d = \left(\begin{smallmatrix} * & * \\ 0 & \pm 1 \end{smallmatrix}\right)$. Moreover, if $\tau \in \mathscr{H}_2$ and $\gamma \in C_{2,1}$ then

$$(\gamma\tau)^* = \gamma^*\tau^* \qquad \text{and} \qquad j(\gamma^*,\tau^*) = \pm\det(j(\gamma,\tau)). \tag{1.6}$$

Let V be the representation space of $\rho = \mathrm{Sym}^m \otimes \det^k$ and let v_0 be a non-zero vector that satisfies $\mathrm{Sym}^m(d)v_0 = (d^*)^m v_0$ for all $d = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \mathsf{GL}_2(\mathbb{C})$. Such a vector v_0 exists and is unique up to a scalar multiple. If, for instance, we take $V = \mathbb{C}[x,y]^{(m)} := \{p \in \mathbb{C}[x,y] \mid \forall \lambda \in \mathbb{C} : p(\lambda x,\lambda y) = \lambda^m p(x,y)\}$, then every $v_0 \in \mathbb{C}x^m$ satisfies the above property: For all $u = (u_{ij}) \in \mathsf{GL}_2(\mathbb{C})$, we have $\rho(u)p(x,y) = p((x,y)u) = p(u_{11}x+u_{21}y, u_{12}x+u_{22}y)$ and therefore $\rho(d)x^m = (d^*x)^m = (d^*)^m x^m$.

Definition 1.6.2. We first choose a fixed non-zero v_0 in V. Let χ be a cusp form of weight $k+m$ on Γ_1, and write $\rho = \mathrm{Sym}^m \otimes \det^k$. We define the *Klingen-Eisenstein series $E_{m,k}(\chi)$* as follows:

$$E_{m,k}(\chi) = \sum_\gamma \chi((\gamma\cdot\tau)^*)\rho(j(\gamma,\tau)^{-1})v_0, \tag{1.7}$$

where the sum is taken over a full set of representatives of the cosets $C_{2,1}\backslash\Gamma_2$.

n	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(1,1,1)$	$(1,1,-1)$
	0	1	0	$-2\cdot3\cdot5\cdot7$	-2^27	-2^27
	0	0	0	0	-2^27	2^27
	0	0	1	$-2\cdot3\cdot5\cdot7$	-2^27	-2^27

n	$(1,1,2)$	$(2,0,0)$	$(2,1,0)$	$(2,1,1)$	$(2,2,0)$	$(2,2,1)$
	1	-2^33	$2^33^27\cdot23$	$-2^53\cdot23$	$2^43\cdot5^27\cdot167$	$2^53\cdot7\cdot19\cdot59$
	2	0	0	$-2^63\cdot137$	0	$2^73\cdot7\cdot337$
	1	0	$-2^23^37\cdot109$	$-2^63\cdot137$	$2^43\cdot5^27\cdot167$	$2^53\cdot7\cdot19\cdot59$

n	$(2,2,2)$	$(2,2,3)$	$(3,0,0)$	$(3,3,0)$	$(4,0,0)$	$(5,0,0)$
	$2^53\cdot7\cdot257$	$-2^53\cdot23$	2^23^27	$-2^43^35\cdot7^217\cdot193$	-2^623	$2\cdot3\cdot5\cdot7\cdot23$
	$2^53\cdot7\cdot257$	2^73^219	0	0	0	0
	$2^53\cdot7\cdot257$	$-2^53\cdot23$	0	$-2^43^35\cdot7^217\cdot193$	0	0

Table 1.1: Fourier coefficients of $E_{2,10}(\Delta)$. The frequencies $n=\begin{pmatrix} n_1 & r/2 \\ r/2 & n_2 \end{pmatrix}$ are written as (n_1,n_2,r) and a Fourier coefficient $a_0x^2+a_1xy+a_2y^2$ is written as a column vector $(a_0,a_1,a_2)'$.

Because of the relations in (1.6), this sum is well-defined and the choice of v_0 together with (1.1) ensure that $E_{m,k}(\chi)$ is invariant under $|_\rho\gamma$ for all $\gamma\in\Gamma_2$. If $k>4$ and $m>0$, then (1.7) converges absolutely to a modular form of weight (m,k) on Γ_2. Moreover:

$$\Phi E_{m,k}(\chi)=\chi$$

and the inverse of Φ in (1.5) is given by

$$\chi\mapsto E_{m,k}(\chi).$$

Example 1.6.3. Let $\rho=\mathrm{Sym}^2\otimes\det^{10}:\mathsf{GL}_2(\mathbb{C})\to\mathsf{GL}(\mathbb{C}[x,y]^{(2)})$ and let $\Delta=q-24q^2+252q^3+\cdots\in S_{12}(\Gamma_1)$. We will choose $v_0=x^2\in\mathbb{C}[x,y]^{(2)}$. The function $E_{2,10}(\Delta)$ is a non-zero vector-valued Siegel modular form in $M_{(2,10)}(\Gamma_2)$. Later we will see that $\dim M_{(2,10)}(\Gamma_2)=1$ and hence, $E_{(2,10)}(\Delta)$ spans $M_{(2,10)}(\Gamma_2)$. We will also be able to compute Fourier coefficients of $E_{2,10}(\Delta)$. We listed a few of them in Table 1.1. The Fourier coefficients at $n=\begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix}$ show that we indeed have $\Phi E_{2,10}(\Delta)=\Delta$.

Arakawa also studied the action of the Hecke operators on vector-valued Eisenstein series. We mention some of his results in Section 4.3.1.

Chapter 2

Differential operators on modular forms

Vector-valued Siegel modular forms can be constructed using classical Siegel modular forms by means of certain differential operators. Suppose that the representation ρ is of the form $(2,k)$. This representation ρ can be realized by

$$\rho(G): S_g \to S_g : A \mapsto \det(G)^k GAG', \qquad S_g := \{A \in \mathrm{Mat}_g(\mathbb{C}) \mid A = A'\}. \tag{2.1}$$

In [30], Shimura defined an operator $\mathscr{D}_k$ sending a classical Siegel modular form f of weight k to a vector-valued smooth function $\mathscr{D}_k f : \mathscr{H}_g \to S_g$ that behaves like a modular form meaning that $\mathscr{D}_k f|_\rho \gamma = \mathscr{D}_k f$ for all $\gamma \in \Gamma_g$. Satoh [29] then combines two functions $f \in M_k(\Gamma_2)$ and $h \in M_\ell(\Gamma_2)$ to get a vector-valued Siegel modular form

$$[f,h] := kf\mathscr{D}_\ell g - \ell g \mathscr{D}_k f \in M_{(2,k+\ell)}(\Gamma_2).$$

It turns out that the functions $[f,h]$ generate the space $\bigoplus_{k\equiv 0(2)} M_{(k,2)}(\Gamma_2)$ as a module over the ring $\bigoplus_{k\equiv 0(2)} M_k(\Gamma_2)$.

Differential operators like $[\cdot,\cdot]$ are called *Rankin-Cohen differential operators* and these operators are very useful for finding generators for modules $\bigoplus_k M_{(m,k)}(\Gamma_2)$. We will study these operators in some detail in this chapter.

2.1 Differential operators and pluri-harmonic polynomials

The 'original' Rankin-Cohen differential operators as studied by Rankin and Cohen [6] act on pairs of modular forms of genus 1. If $f \in M_k(\Gamma_1)$ and $g \in M_\ell(\Gamma_1)$, then

$$F_n(f,g)(\tau) = \sum_{i=1}^{n} (-1)^i \binom{n}{i} (k+n-1)_i (\ell+n-1)_{n-i} \tfrac{d^{n-i}}{d\tau^{n-i}} f(\tau) \tfrac{d^i}{d\tau^i} g(\tau)$$

is a form in $M_{k+\ell+2n}(\Gamma_1)$. For $n > 0$, the function $F_n(f,g)$ will be a cusp form. The reason for $F_n(f,g)$ to be a modular form is that the operator F_n commutes with the slash operators, i.e. $F_n(f,g)|_{k+\ell+2n}\gamma = F_n(f|_k\gamma, g|_\ell\gamma)$ for any pair of holomorphic functions f and g.

Example 2.1.1. As a simple example we take $n=1$. Let f and g be modular forms of weight k and ℓ respectively, then $h := F_1(f,g) = \ell f'g - kfg'$. The derivative of f is not a modular form (unless f is constant), but since

$$\tfrac{d}{d\tau} f(\tfrac{-1}{\tau}) = \tau^{-2} f'(\tfrac{-1}{\tau}) \quad \text{and} \quad \tfrac{d}{d\tau}\tau^k f(\tau) = k\tau^{k-1} f(\tau) + \tau^k f'(\tau),$$

we have $f'(\frac{-1}{\tau}) = \tau^{k+2} f'(\tau) + k\tau^{k+1} f(\tau)$. This shows that

$$\begin{aligned} h(\tfrac{-1}{\tau}) &= \ell \left(\tau^{k+2} f'(\tau) + k\tau^{k+1} f(\tau)\right) \tau^{\ell} g(\tau) - k\tau^k f(\tau) \left(\tau^{\ell+2} g'(\tau) + \ell \tau^{\ell+1} g(\tau)\right) \\ &= \ell \tau^{\ell+k+2} f'(\tau) g(\tau) - k\tau^{k+\ell+2} f(\tau) g'(\tau) \quad = \quad \tau^{k+\ell+2} h(\tau). \end{aligned}$$

Of course $h(\tau+1) = h(\tau)$ and therefore h is a modular form of weight $k+\ell+2$.

Now take $f = E_4$ and $g = E_6$, then $h = F_1(E_4, E_6) = 6E_4' E_6 - 4E_4 E_6' \in S_{12}(\Gamma_1)$. Hence, $h = c \cdot \Delta$ for some constant c. If we consider the Fourier series of E_4 and E_6, then we get

$$E_4 = 1 + 240q + \cdots, \quad E_4' = 2\pi i 240 q + \cdots, \quad E_6 = 1 - 504q + \cdots, \quad E_6' = -2\pi i 504 q + \cdots$$

and hence $h = 2\pi i(6 \cdot 240 + 4 \cdot 504) q + \cdots = 2\pi i \cdot 3456 q + \cdots$ which shows that $c = 2\pi i \cdot 3456 \neq 0$. Note that $\Delta = -c^{-1} \det \left(\begin{smallmatrix} 4E_4 & 6E_6 \\ E_4' & E_6' \end{smallmatrix}\right)$.

Let f and g denote elliptic modular forms of weight k and ℓ respectively and let n be any non-negative integer. We can write $F_n(f,g)$ as follows:

$$F_n(f,g) = \tilde{p}(\tfrac{d}{d\tau_1}, \tfrac{d}{d\tau_2}) f(\tau_1) g(\tau_2) \Big|_{\tau_1 = \tau_2 = \tau},$$

with $\tilde{p}$ a polynomial given by

$$\tilde{p}(r,s) = \sum_{i=1}^{n} (-1)^i \binom{n}{i} (k+n-1)_i (\ell+n-1)_{n-i} r^{n-i} s^i.$$

If we substitute $r = x_1^2 + \cdots + x_{2k}^2$ and $s = x_{2k+1}^2 + \cdots + x_{2k+2\ell}^2$ for formal variables $x_1, \ldots, x_{2k+2\ell}$ and define $p(x_1, \ldots, x_{2k+2\ell}) = \tilde{p}(r,s)$, then p satisfies the following three properties.

1. The fact that p is defined using the polynomial $\tilde{p}$ is equivalent to p being invariant under orthogonal transformations meaning that

$$\forall \alpha \in \mathrm{O}(2k) \times \mathrm{O}(2\ell): \quad p(x\alpha) = p(x), \quad x = (x_1, \ldots, x_{2k+2\ell}).$$

2. The polynomial $p \in \mathbb{C}[x_1, \ldots, x_{2k+2\ell}]$ is harmonic:

$$\sum_{i=1}^{2k+2\ell} \frac{\partial^2}{\partial x_i^2} p = 0.$$

3. The polynomial p is homogeneous of weight $2n$.

It can be shown that these three properties of p imply that F_n commutes with the slash operator[1].

In the remainder of this chapter, we will generalize the Rankin-Cohen operators F_n and obtain differential operators that send classical Siegel modular forms to (possibly vector-valued) Siegel modular forms. We will follow mainly Ibukiyama [16].

[1] We will show this in a more general setting below.

2.1.1 $\mathsf{O}(d)$-invariant polynomials

Let f be a polynomial with complex coefficients in the variables given by the coefficients of the $g \times d$ matrix $x = (x_{ij})$. The group $\mathsf{O}(d)$ acts on f by

$$(\alpha \cdot f)(x) = f(x\alpha), \quad \alpha \in \mathsf{O}(d).$$

We will say that f is $\mathsf{O}(d)$-invariant if $\alpha \cdot f = f$ for all $\alpha \in \mathsf{O}(d)$ and denote by $\mathbb{C}[x]^{\mathsf{O}(d)}$ the ring of such polynomials. The above described action $f \mapsto \alpha \cdot f$ defines a representation of $\mathsf{O}(d)$ on the space of homogeneous polynomials of degree n in the coefficients of x. We will denote this space by $\mathbb{C}[x]^{(n)}$.

Theorem 2.1.2. *If $f \in \mathbb{C}[x]^{\mathsf{O}(d)}$ and $g \leq d$, then there exists a polynomial $\tilde{f}$ in $\frac{1}{2}g(g+1)$ variables such that $f(x) = \tilde{f}(y_{ij} : 1 \leq i \leq j \leq g)$, where $y = xx'$.*

Remark 2.2. If $\tilde{f}$ is a polynomial in the upper triangular coefficients of a $g \times g$ symmetric matrix y, then we will simply write $\tilde{f}(y)$ instead of $\tilde{f}(y_{ij} : 1 \leq i < j \leq g)$. A proof of Theorem 2.1.2 can be found in [33] (p. 56) or in [4]. Note that the existence of a function $\tilde{f}$ that satisfies $\tilde{f}(xx') = f(x)$ for all $x \in \mathrm{Mat}_{g,d}(\mathbb{C})$ is clear. However, the fact that a *polynomial* $\tilde{f}$ exists is more difficult to prove.

Corollary 2.2.1. *Let f be a polynomial in $\mathbb{C}[x_1, \ldots, x_t]$, where $x_i \in \mathrm{Mat}_{g_i,d_i}$, $g_i \leq d_i$, $i = 1, \ldots, t$ are matrices of formal variables and suppose that f is $\mathsf{O}(d_1) \times \cdots \times \mathsf{O}(d_t)$-invariant, that is,*

$$f(x_1\alpha_1, \ldots, x_t\alpha_t) = f(x_1, \ldots, x_t) \qquad \forall (\alpha_1, \ldots, \alpha_t) \in \mathsf{O}(d_1) \times \cdots \times \mathsf{O}(d_t).$$

Let $y_i = y_i' \in \mathrm{Mat}_{d_i}$ for $i = 1, \ldots, t$. Then there is a polynomial $\tilde{f} \in \mathbb{C}[y_1, \ldots, y_t]$ such that

$$f(x_1, \ldots, x_t) = \tilde{f}(x_1x_1', \ldots, x_tx_t').$$

Remark 2.3. We will call $\tilde{f}$ the *a*ssociated polynomial of f. If $d = (d_1, \ldots, d_t) \in \mathbb{N}^t$, then we will denote by $\mathsf{O}(d)$ the product $\mathsf{O}(d_1) \times \cdots \times \mathsf{O}(d_t)$. If $x = (x_1, \ldots, x_t)$, then we will write $\mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_t]$ and again $\mathbb{C}[x]^{\mathsf{O}(d)}$ for the subring of $\mathsf{O}(d)$-invariant polynomials of $\mathbb{C}[x]$.

Ibukiyama assumes Corollary 2.2.1 implicitly. We now give the following proof.

Proof (of Corollary 2.2.1). Let $|d| = d_1 + \cdots + d_t$ and $|g| = g_1 + \cdots + g_t$. Let $\xi \in \mathrm{Mat}(|g|, |d|)$ be a matrix of indeterminates. We will consider $x_1, \ldots, x_t$ to be parts of this matrix ξ and given an element $(\alpha_1, \ldots, \alpha_t) \in \mathsf{O}(d)$, we will construct an element α_0 of $\mathsf{O}(|d|)$ as follows:

$$\xi = \begin{pmatrix} x_1 & * & \cdots & * \\ * & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \cdots & * & x_t \end{pmatrix}, \qquad \alpha_0 = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_t \end{pmatrix}.$$

The polynomial ring $\mathbb{C}[x]$ can be seen as subring $\mathbb{C}[\xi]$ and hence, if $f \in \mathbb{C}[x]$, the we can consider f to be an element of the larger ring $\mathbb{C}[\xi]$. If $f \in \mathbb{C}[x]^{\mathsf{O}(d)}$, then in general f will not be $\mathsf{O}(|d|)$-invariant. However, if we consider the polynomial

$$\hat{f}(\xi) := \int_{\mathsf{O}(|d|)} f(\xi\alpha) d\mu(\alpha),$$

where μ is the Haar measure on $\mathsf{O}(|d|)$ such that $\mu(\mathsf{O}(|d|)) = 1$, then $\hat{f} \in \mathbb{C}[\xi]^{\mathsf{O}(|d|)}$ and

$$\hat{f}(x_1,\ldots,x_t) = \int_{\mathsf{O}(|d|)} f(x_1\alpha_1,\ldots,x_t\alpha_t)d\mu(\alpha) = \int_{\mathsf{O}(|d|)} f(x_1,\ldots,x_t)d\mu(\alpha) = f(x_1,\ldots,x_t).$$

Here we consider $\hat{f}$ to be a polynomial in $\mathbb{C}[x]$ by replacing the variables ξ_{ij} that are not in the blocks $x_1,\ldots,x_t$ with zeroes, resulting in a matrix of the form

$$\xi_0 = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_t \end{pmatrix}. \tag{2.2}$$

A group element $\alpha \in \mathsf{O}(|d|)$ acts as α_0 on ξ_0. Theorem 2.1.2 tells us that we can find a polynomial $\tilde{\hat{f}}$ such that $\tilde{\hat{f}}(\xi\xi') = \hat{f}(\xi)$. This means that $\tilde{\hat{f}}(\xi_0\xi_0') = \hat{f}(\xi_0) = f(x_1,\ldots,x_t)$. The matrix $\xi_0\xi_0'$ is of the form

$$\xi_0\xi_0' = \begin{pmatrix} x_1x_1' & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_tx_t' \end{pmatrix}$$

and hence, if we write $\tilde{f}(x_1x_1',\ldots,x_tx_t') = \tilde{\hat{f}}(\xi_0\xi_0')$, then $\tilde{f}$ solves the Corollary. □

2.3.1 Harmonic and pluri-harmonic polynomials

We again consider the polynomial ring $\mathbb{C}[x]$ where $x = (x_1,\ldots,x_t)$ and $x_i \in \mathrm{Mat}(g,d_i)$, $i = 1,\ldots,t$ are matrices with formal variables as coefficients. The Laplacian differential operator Δ acts on polynomials $f \in \mathbb{C}[x]$ in the usual way:

$$\Delta f(x_1,\ldots,x_t) = \sum_{i=1}^{t} \sum_{\substack{1\le\nu\le g \\ 1\le\mu\le d_i}} \frac{\partial^2}{\partial(x_i)_{\nu\mu}^2} f(x_1,\ldots,x_t).$$

We will write $\Delta_i := \sum_{\substack{1\le\nu\le g \\ 1\le\mu\le d_i}} \frac{\partial^2}{\partial(x_i)_{\nu\mu}^2}$ such that $\Delta = \Delta_1 + \cdots + \Delta_t$.

Definition 2.3.1. A polynomial $f \in \mathbb{C}[x]$ is called *harmonic* if $\Delta f = 0$. We will denote the subspace of $\mathbb{C}[x]$ of harmonic polynomials by $\mathrm{H}_{g,d}$.

Example 2.3.2. Let $x = (x_{ij})$ be a $g \times g$ matrix of formal variables, then $\det(x) \in \mathbb{C}[x]$. Denote by S_g the symmetric group on g elements and let $\varepsilon : \mathsf{S}_g \to \{\pm 1\}$ denote the sign on S_g. We then have

$$\Delta \det(x) = \sum_{\sigma\in\mathsf{S}_g} \varepsilon(\sigma) \sum_{\nu\mu} \frac{\partial^2}{\partial x_{\nu\mu}^2} \prod_{\ell=1}^{g} x_{\ell\sigma(\ell)} = 0,$$

since $x_{\nu\mu}$ occurs at most once in a product $\prod_\ell x_{\ell\sigma(\ell)}$. This shows that $\det(x)$ is harmonic.

If ∂x_i denotes the matrix of tangent vectors $(\partial x_i)_{\nu\mu} := \partial/\partial(x_i)_{\nu\mu}$ where $1 \le \nu \le g$ and $1 \le \mu \le d_i$, then Δ_i equals the trace of the matrix $\partial x_i \partial x_i'$. We will now define new differential operators using the coefficients of $\partial x_i \partial x_i'$ and then extend the notion of an harmonic polynomial.

Definition 2.3.3. Define for all $1 \le \mu, \nu \le g$ the differential operators

$$\Delta_i^{(\mu\nu)} := (\partial x_i \partial x_i')_{\mu\nu}, \quad i = 1, \dots, t \qquad \text{and} \qquad \Delta^{(\mu\nu)} := \Delta_1^{(\mu\nu)} + \cdots + \Delta_t^{(\mu\nu)}.$$

We will call a polynomial $f \in \mathbb{C}[x]$ *pluri-harmonic* if $\Delta^{(\mu\nu)} f(x_1, \dots, x_t) = 0$ for all $1 \le \mu, \nu \le g$. The subspace of $\mathbb{C}[x]$ consisting of pluri-harmonic polynomials is denoted by $\mathrm{PH}_{g,d}$.

Remark 2.4. The differential operator $\Delta_i^{(\mu\nu)}$ is given by

$$\Delta_i^{(\mu\nu)} f(x_1, \dots, x_t) = \sum_{j=1}^{d_i} \frac{\partial^2}{\partial (x_i)_{\mu j} \partial (x_i)_{\nu j}} f(x_1, \dots, x_t).$$

In particular $\Delta_i = \sum_{\mu=1}^{g} \Delta_i^{(\mu\mu)}$ and hence $\mathrm{PH}_{g,d} \subseteq \mathrm{H}_{g,d}$.

Example 2.4.1. (*i*) Let $g = d = 2$ and $t = 1$. The polynomial $f(x) = f(x_{11}, x_{12}, x_{21}, x_{22}) = x_{11}x_{21} + x_{12}x_{22}$ is harmonic, but is is not pluri-harmonic, since

$$\Delta^{12} f = \frac{\partial^2}{\partial x_{11} \partial x_{21}} f + \frac{\partial^2}{\partial x_{12} \partial x_{22}} f = 2.$$

(*ii*) Consider again the polynomial $\det(x) \in \mathbb{C}[x]$ with $x = (x_{ij}) \in \mathrm{Mat}_{g,d}$ as in Example 2.3.2. We have

$$\Delta^{(\mu\nu)} \det(x) = \sum_{\sigma \in \mathrm{S}_g} \varepsilon(\sigma) \sum_{j=1}^{g} \tfrac{\partial^2}{\partial x_{\mu j} \partial x_{\nu j}} \prod_{\ell=1}^{g} x_{\ell \sigma(\ell)} = 0$$

since $x_{\mu j}$ and $x_{\nu j}$ do not occur in a product $\prod_\ell x_{\ell, \sigma(\ell)}$ simultaneously. Hence, $\det(x)$ is pluri-harmonic.

The group $\mathsf{GL}_g(\mathbb{C})$ acts on $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_t]$ as follows:

$$(\gamma \cdot f)(x_1, \dots, x_t) = f(\gamma' x_1, \dots, \gamma' x_t), \qquad \gamma \in \mathsf{GL}_g(\mathbb{C}), \quad f \in \mathbb{C}[x].$$

Now we can describe the pluri-harmonic polynomials in terms of the harmonic polynomials and the action of $\mathsf{GL}_g(\mathbb{C})$.

Theorem 2.4.2. *Let $g \in \mathbb{N}$, $d = (d_1, \dots, d_t) \in \mathbb{N}^t$. A polynomial $f \in \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_t]$ is pluri-harmonic if and only if $\gamma \cdot f$ is harmonic for all $\gamma \in \mathsf{GL}_g(\mathbb{C})$.*

Proof. The following identity holds for all $\gamma \in \mathsf{GL}_g(\mathbb{C})$ and all $f \in \mathbb{C}[x]$:

$$\partial x_i \partial x_i' f(\gamma' x_1, \dots, \gamma' x_t) = \gamma (\gamma \cdot \partial x_i \partial x_i' f)(x) \gamma'. \tag{2.3}$$

We will start with the "if"-part of the Theorem, so suppose that $\gamma \cdot f$ is harmonic for all $\gamma \in \mathsf{GL}_g(\mathbb{C})$. This means that the sum over i of the trace of (2.3) vanishes for all γ, that is

$$\sum_{i=1}^{t} \mathrm{Tr}\, \partial x_i \partial x_i' f(\gamma' x_1, \dots, \gamma' x_t) = 0.$$

If we replace x_j by $\gamma'^{-1} x_j$ in the argument of $\partial x_i \partial x_i' f$, we find that

$$\sum_{i=1}^{t} \mathrm{Tr}\, \gamma' \gamma \partial x_i \partial x_i' f(x) = 0$$

for all γ. If we now choose $\beta^{(\mu\nu)} := \mathbf{1}_g + e_{\mu\nu} + e_{\nu\mu}$ where $(e_{\mu\nu})_{ij} = \delta_{\mu i}\delta_{\nu j}$, then we can find a $\gamma \in \mathsf{GL}_g(\mathbb{C})$ such that $\gamma'\gamma = \beta^{(\mu\nu)}$ and

$$0 = \sum_{i=1}^{t} \operatorname{Tr} \beta^{(\mu\nu)} \partial x_i \partial x_i' f(x) = \sum_{i=1}^{t} 2\Delta_i^{(\mu\nu)} f(x) + \Delta_i f(x).$$

We already know that $\sum_{i=1}^{t} \Delta_i f(x) = 0$ and hence $\sum_{i=1}^{t} \Delta_i^{(\mu\nu)} f(x) = 0$. This implies that f is pluri-harmonic.

Now we will prove the "and only if"-part. Note that the matrix $\partial x_i \partial x_i' f(x)$ is a zero $g \times g$ matrix. Hence by equation (2.3), the matrix

$$\sum_{i=1}^{t} \partial x_i \partial x_i' f(\gamma_1' x_1, \ldots, \gamma_t' x_t)$$

and in particular its trace vanishes. This proves that $\gamma \cdot f$ is harmonic (and in fact pluri-harmonic) for all $\gamma \in \mathsf{GL}_g(\mathbb{C})$. □

2.4.1 Homogeneous polynomials

Let $d = (d_1, \ldots, d_t)$ and suppose that $\rho : \mathsf{GL}_g(\mathbb{C}) \to \mathsf{GL}(V)$ is a finite dimensional representation of $\mathsf{GL}_g(\mathbb{C})$ on some vector space V of dimension $m+1$ over $\mathbb{C}$ (the reason for writing the dimension of V in the form "$m+1$" will be made clear below).

Definition 2.4.3. A map $f : \mathbb{C}^n \to V$ is called a *polynomial map* or a *V-valued polynomial* if for any projection $\pi \in V^*$ the function $\pi \circ f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial function.

Remark 2.5. If we choose a basis $\{e_0, \ldots, e_m\}$ for V and define $\pi_i : V \to \mathbb{C}e_i$ to be the projection on the subspace of V spanned by the i-th basis vector, then $(\pi_i \circ f)_i$ is a vector of polynomial functions. On the other hand, if $f : \mathbb{C}^n \to V$ is some map and if we are able to find a basis $\{e_0, \ldots, e_m\}$ for V such that the vector $(\pi_i \circ f)_i$ is a vector of polynomial functions, then f is a V-valued polynomial.

We can identify polynomials f in $\mathbb{C}[x_1, \ldots, x_t]$ with polynomial functions $z \mapsto f(z) : \mathbb{C}^{g \times |d|} \to \mathbb{C}$ and vice versa. If $f : \mathbb{C}^{g \times |d|} \to V$ is a polynomial map, then we can find for any projection $\pi \in V^*$ a polynomial f_π in $\mathbb{C}[x]$ such that $f_\pi(z) = \pi \circ f(z)$. Recall the action of $\mathsf{GL}_g(\mathbb{C})$ on $\mathbb{C}[x]$ given by

$$(\gamma \cdot f)(x) = f(\gamma' x_1, \ldots, \gamma' x_t).$$

For every $\gamma \in \mathsf{GL}_g(\mathbb{C})$ and $f \in \mathbb{C}[x]$ the polynomial $\gamma \cdot f$ defines a polynomial function $\gamma \cdot f : \mathbb{C}^{g \times |d|} \to \mathbb{C}$. Also, if $\gamma \in \mathsf{GL}_g(\mathbb{C})$, then $\rho(\gamma) f$ is again a V-valued polynomial. Hence we can identify $\pi\rho(\gamma)f$ with a polynomial in $\mathbb{C}[x]$.

Definition 2.5.1. A V-valued polynomial f is called *homogeneous of weight ρ* if for every $\gamma \in \mathsf{GL}_g(\mathbb{C})$ and every projection $\pi \in V^*$ we have the identity

$$\gamma \cdot f_\pi = (\rho(\gamma') f)_\pi.$$

Let us keep the notation as above. We will call a V-valued polynomial f (pluri-)harmonic or $\mathsf{O}(d)$-invariant if for every projection $\pi \in V^*$ the polynomial f_π is (pluri-)harmonic or $\mathsf{O}(d)$-invariant respectively in $\mathbb{C}[x]$. Using this terminology we can state the following Lemma.

Lemma 2.5.2. *If $f : \mathbb{C}^{g\times|d|} \to V$ be a V-valued polynomial that is homogeneous of weight ρ, then f is harmonic if and only if f is pluri-harmonic.*

Proof. Choose a basis $\{e_0, \ldots, e_m\}$ for V with projections $\pi_i : V \to \mathbb{C}e_i$ and write $f_i := f_{\pi_i}$. Let $\gamma \in \mathsf{GL}_g(\mathbb{C})$. In the basis $\{e_0, \ldots, e_m\}$ the linear map $\rho(\gamma')$ is represented by a matrix $r(\gamma')$. According to Theorem 2.4.2, we have to prove that $(\gamma \cdot f_i)(x)$ is harmonic for all i. The function f is homogeneous of weight ρ and hence

$$(\gamma \cdot f_i)(x) = \sum_{j=0}^{m} r(\gamma')_{ij} f_j(x)$$

The assumption that f is a harmonic V-valued polynomial, ensures all the polynomials f_i to be harmonic in $\mathbb{C}[x]$. This means that $\Delta(\gamma \cdot f_i)(x) = \sum_{j=0}^{m} r(\gamma')_{ij} \Delta f_j(x) = 0$. This proves the Lemma. □

Example 2.5.3. Let m be an even integer, $g = 2$ and take for $V = \mathbb{C}[v_1, v_2]^{(m)}$ the space of homogeneous polynomials of degree m in the variables v_1 and v_2. If f is a V-valued polynomial in the variables x_{ij} with $1 \le i \le 2$ and $1 \le j \le d$, then f defines to a polynomial in both the variables x_{ij} and v_1 and v_2 which we also denote by f. The representation $\rho = \mathrm{Sym}^m$ acts as follows on f:

$$\rho(\gamma) f(x, v) = f(x, v\gamma), \qquad \gamma \in \mathsf{GL}_2(\mathbb{C}), \quad v = (v_1, v_2)$$

and f is homogeneous of weight ρ if for all $\gamma \in \mathsf{GL}_2(\mathbb{C})$

$$f(x, v\gamma) = f(\gamma x, v).$$

We can easily construct homogeneous polynomials f of weight ρ and type (d_1, d_2) by first taking a polynomial $h(r, s)$ in variables r and s that is homogeneous of weight $m/2$. Then we define $f(x, v) = h(v x_1 x_1' v', v x_2 x_2' v')$ with $x_i \in \mathrm{Mat}_{2,d_i}$ and $v = (v_1, v_2)$. Note that the representation space of ρ is $m + 1$ dimensional.

2.6 Rankin-Cohen differential operators

Let $\rho : \mathsf{GL}_g(\mathbb{C}) \to \mathsf{GL}(V)$ be a finite dimensional representation on the $m + 1$ dimensional $\mathbb{C}$-vector space V. Fix $g \in \mathbb{N}$ and $d = (d_1, \ldots, d_t) \in \mathbb{N}^t$ such that $g \le d_i$ for $i = 1, \ldots, t$. Let $p : \mathbb{C}^{g\times|d|} \to V$ be a V-valued polynomial that is

1. $\mathsf{O}(d)$-invariant,
2. harmonic,
3. homogeneous of weight ρ.

The polynomial map p is automatically pluri-harmonic by Lemma 2.5.2.

Definition 2.6.1. We will call a polynomial map $p : \mathbb{C}^{g\times|d|} \to V$ that satisfies the three properties above a *Rankin-Cohen polynomial map* (RC-polynomial) with respect to the representation ρ. The vector $(d_1, \ldots, d_t)/2$ is called the *type* of p and we say that p is of *even type* if d_i is an even integer for all i.

If we choose a basis $\{e_0,\dots,e_m\}$ for V and $p=(p_0,\dots,p_m)'\in\mathbb{C}[x]^{m+1}$, then the polynomials p_i are $\mathsf{O}(d)$-invariant and hence, we can find associated polynomials $\tilde{p}_i\in\mathbb{C}[y_1,\dots y_t]$ with $y_i=y_i'\in\mathrm{Mat}_g$ matrices of formal variables such that $\tilde{p}_i(xx')=p_i(x)$. Write $\tilde{p}=(\tilde{p}_0,\dots,\tilde{p}_m)'$. We will call $\tilde{p}$ the *associated polynomial* of p.

Write $\mathscr{H}_g^t:=\mathscr{H}_g\times\cdots\times\mathscr{H}_g$ for the t-fold product of the Siegel upper half-space and let $\mathrm{Hol}(\mathscr{H}_g^t,V)$ be the ring of holomorphic maps on $\mathscr{H}_g^t$ that have values in the $\mathbb{C}$-vector space V. In particular, write $\mathrm{Hol}(\mathscr{H}_g^t)=\mathrm{Hol}(\mathscr{H}_g^t,\mathbb{C})$.

Let $\tau=(\tau_{ij})\in\mathscr{H}_g$. We define $(d/d\tau)_{ij}:=\frac{1}{2}(1+\delta_{ij})\partial/\partial\tau_{ij}$. If $f\in\mathrm{Hol}(\mathscr{H}_g)$, then $\frac{d}{d\tau}f$ is a matrix consisting of partial derivatives of f. We will use $\tilde{p}$ to construct differential operators $D_{\tilde{p}}$ on $\mathrm{Hol}(\mathscr{H}_g^t)$ sending $f\in\mathrm{Hol}(\mathscr{H}_g^t)$ to $D_{\tilde{p}}f\in\mathrm{Hol}(\mathscr{H}_g,V)$.

Example 2.6.2 (Satoh [29])**.** Define the representation $\rho:\mathsf{GL}_2(\mathbb{C})\to\mathsf{GL}(S_2)$ with S_2 the space of complex symmetric 2×2 matrices as follows:

$$\rho(G)A=GAG',\qquad G\in\mathsf{GL}_2(\mathbb{C}),\quad A\in S_2.$$

Let $d=(d_1,d_2)\in(2\mathbb{N})^2$, let $x_1\in\mathrm{Mat}_{2,d_1}$ and $x_2\in\mathrm{Mat}_{2,d_2}$ and define the symmetric 2×2 matrix p with polynomials as coefficients as follows:

$$p(x_1,x_2)=d_1x_2x_2'-d_2x_1x_1'.$$

The matrix p of polynomials defines a polynomial map $p:\mathbb{C}^{g\times|d|}\to S_2$. This polynomial map is $\mathsf{O}(d)$-invariant and harmonic since

$$\Delta p=d_1\cdot 2d_2\begin{pmatrix}1&0\\0&1\end{pmatrix}-d_2\cdot 2d_1\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

If $G\in\mathsf{GL}_2(\mathbb{C})$, then $p(Gx_1,Gx_2)=Gp(x_1,x_2)G'$, which shows that p is homogeneous of weight ρ. The associated polynomial $\tilde{p}$ of p equals $\tilde{p}(y_1,y_2)=d_1y_2-d_2y_1$. If $f\in M_{d_1/2}(\Gamma_2)$ and $g\in M_{d_2/2}(\Gamma_2)$, then

$$\tilde{p}\left(\tfrac{d}{d\tau_1},\tfrac{d}{d\tau_2}\right)f(\tau_1)g(\tau_2)\Big|_{\tau_1=\tau_2=\tau}=d_1f(\tau)\tfrac{d}{d\tau}g(\tau)-d_2g(\tau)\tfrac{d}{d\tau}f(\tau)$$

defines a S_2-valued holomorphic function which is in fact a Siegel modular form of weight $\rho\otimes\det^{|d|/2}$.

The construction used in Example 2.6.2 can be generalized as follows.

Definition 2.6.3. Let p be a RC-polynomial with respect to an irreducible representation $\rho:\mathsf{GL}_g(\mathbb{C})\to V$ with associated polynomial $\tilde{p}$ (cf. Definition 2.6.1) and suppose that $f\in\mathrm{Hol}(\mathscr{H}_g^t)$. We define the differential operator $D_{\tilde{p}}$ as follows.

$$(D_{\tilde{p}}f)(\tau):=\frac{1}{(2\pi i)^n}\,\tilde{p}\left(\tfrac{d}{d\tau_1},\dots,\tfrac{d}{d\tau_t}\right)f(\tau_1,\dots,\tau_t)\Big|_{\tau_1=\cdots=\tau_t=\tau},\tag{2.4}$$

where the integer n is chosen such that $\rho(\lambda\mathbf{1}_g)p=\lambda^{2n}p$ for any scalar λ. We will refer to differential operators defined in this way as *Rankin-Cohen* differential operators (RC-operators).

Remark 2.7. The integer n in Definition 2.6.3 is well defined. Let v_0 be a vector in V such that $\rho(\lambda\mathbf{1}_g)v_0=\lambda^n v_0$ with $n_0=\lambda_1+\cdots+\lambda_g$ the sum of the coefficients of the highest weight of ρ. If $z\in\mathbb{C}^{g\times|d|}$, then we have by irreducibility of ρ:

$$\exists\gamma_z\in\mathsf{GL}_g(\mathbb{C}):\rho(\gamma_z)p(z)\in\mathbb{C}\cdot v_0.$$

Then $p(\lambda z) = \rho(\gamma_z^{-1})\rho(\lambda \mathbf{1}_g)p(\gamma_z z) = \lambda^{n_0} p(z)$. If p is non-zero, then n_0 is even, since $(-1)^{n_0} p(x) = \rho(-\mathbf{1}_g)p(x) = p(-x) = \tilde{p}(-x \cdot (-x)') = \tilde{p}(xx') = p(x)$. Now take $n = n_0/2$. We choose the factor $1/(2\pi i)^n$ in such a way that $D_{\tilde{p}}$ preserves rationality of Fourier coefficients.

The application of RC-operators to modular forms becomes clear in Theorem 2.7.1 below. First recall that we defined the slash operator $|_\rho \gamma$ for $\gamma \in \mathsf{Sp}_g(\mathbb{R})$ and a finite dimensional representation $\rho : \mathsf{GL}_g(\mathbb{C}) \to \mathsf{GL}(V)$ on a vector space V as follows:

$$f|_\rho \gamma(\tau) = \rho(j(\gamma,\tau))^{-1} f(\gamma \cdot \tau), \qquad f : \mathscr{H}_g \to V, \quad \tau \in \mathscr{H}_g,$$

where for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the factor of automorphy $j(\gamma,\tau) = c\tau + d$ and $\gamma \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$ are defined as usual. We will also define for $k = (k_1, \ldots, k_t)$ the slash operator $|_k \gamma$:

$$f|_k \gamma(\tau_1, \ldots, \tau_t) := \det(j(\gamma,\tau_1))^{-k_1} \cdots \det(j(\gamma,\tau_t))^{-k_t} f(\gamma \cdot \tau_1, \ldots, \gamma \cdot \tau_t), \qquad f \in \mathrm{Hol}(\mathscr{H}_g^t).$$

Theorem 2.7.1 (Ibukiyama)**.** *Let $f \in \mathrm{Hol}(\mathscr{H}_g^t)$ and let p be an RC-polynomial of even type $k = d/2$ with respect to a finite dimensional irreducible representation ρ. We have the following commutation relation for all $\gamma \in \mathsf{Sp}_g(\mathbb{R})$.*

$$((D_{\tilde{p}} f)|_{\rho \otimes \det^{|k|}} \gamma)(\tau) = (D_{\tilde{p}} f|_k \gamma)(\tau). \tag{2.5}$$

Theorem 2.7.1 provides us with a method to construct vector-valued Siegel modular forms from classical Siegel modular forms.

Corollary 2.7.2. *Let $f_1, \ldots, f_t$ be classical Siegel modular forms on Γ_g of weights $k_1, \ldots, k_t$. Suppose that p is an RC-polynomial with respect to a finite dimensional representation $\rho : \mathsf{GL}_g(\mathbb{C}) \to V$ that is of type $(2k_1, \ldots, 2k_t)$ and define $f(\tau_1, \ldots, \tau_t) = f_1(\tau_1) \cdots f_t(\tau_t) \in \mathrm{Hol}(\mathscr{H}_g^t)$, then $D_{\tilde{p}} f$ is a vector-valued modular form of weight $\rho \otimes \det^{k_1 + \cdots + k_t}$.*

Ibukiyama also states and proves the backward implication, which ensures that we can get all 'Rankin-Cohen type' differential operators (i.e. differential operators that send modular forms to modular forms) using RC-polynomials. To illustrate Theorem 2.7.1 and its Corollary, we will first give a non-trivial example of a vector-valued Siegel modular form.

Example 2.7.3. In this example, we take $g = 2$. Let $\mathbf{r}$ and $\mathbf{s}$ be symmetric 2 by 2 matrices of formal variables. We will write $v = (x, y)$ and $\mathbf{r}[v] := v\mathbf{r}v'$. Furthermore, let $V = \mathbb{C}[x,y]^{(4)}$ be the vector space of homogeneous polynomials of degree 4 in x and y.

Let p be the polynomial defined by $p(\mathbf{r}, \mathbf{s}, v) = 2\mathbf{r}[v]^2 - 5\mathbf{r}[v]\mathbf{s}[v] + 2\mathbf{s}[v]^2$, then p defines a V-valued polynomial if we consider p as a function of the variables $\mathbf{r}_{ij}$ and $\mathbf{s}_{ij}$ (cf. Example 2.5.3). Let $\rho : \mathsf{GL}_2(\mathbb{C}) \to V$ be representation Sym^4 defined by

$$\gamma \cdot q(v) = q(v\gamma), \qquad q \in \mathbb{C}[x,y]^{(4)}, \quad \gamma \in \mathsf{GL}_2(\mathbb{C}).$$

Suppose that f and g are modular forms of weight 4 on Γ_2 (of course, this means that f and g are both scalar multiples of φ_4) and consider the differential operator

$$D = (2\pi i)^2 D_p = p\left(\frac{d}{d\tau_1}, \frac{d}{d\tau_2}\right)\bigg|_{\tau_1 = \tau_2 = \tau} = 2\frac{d}{d\tau_1}[v]^2 - 5\frac{d}{d\tau_1}[v]\frac{d}{d\tau_2}[v] + 2\frac{d}{d\tau_2}[v]^2\bigg|_{\tau_1 = \tau_2 = \tau}.$$

Then $h(\tau) :=$

$$D(f(\tau_1)g(\tau_2)) = 2g(\tau)\frac{d}{d\tau}[v]^2 f(\tau) - 5\frac{d}{d\tau}[v]f(\tau)\frac{d}{d\tau}[v]g(\tau) + 2f(\tau)\frac{d}{d\tau}[v]^2 g(\tau)$$

is a modular form of weight $(4,8)$. In this example, we will check that h indeed satisfies the functional equation that comes from the action of $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma_2$. First we will calculate the behaviour of $\frac{d}{d\tau}[v]f$. We have $\frac{d}{d\tau}f(-\tau^{-1}) = \tau^{-1}\frac{df}{d\tau}(-\tau^{-1})\tau^{-1}$ and also $\frac{d}{d\tau}f(-\tau^{-1}) = \frac{d}{d\tau}\det(\tau)^4 f(\tau) = \det(\tau)^4\tau^{-1}f(\tau) + \det(\tau)^4\frac{df}{d\tau}$. Therefore

$$\frac{df}{d\tau}(-\tau^{-1}) = \det(\tau)^4\tau\frac{df}{d\tau}(\tau)\tau - \det(\tau)^4\tau f(\tau). \tag{2.6}$$

Now we will calculate how $\frac{d}{d\tau}[v]^2 f$ transforms under the action of γ. Since

$$\frac{d}{d\tau}[v]^2 u(-\tau^{-1}) = \mathrm{Sym}^4(\tau^{-1})\left(\frac{d}{d\tau}[v]^2 u\right)(-\tau^{-1}) - 2(\tau^{-1})[v]\cdot\mathrm{Sym}^2(\tau^{-1})\left(\frac{d}{d\tau}[v]u\right)(-\tau^{-1})$$

for any holomorphic function u on $\mathscr{H}_2$ and in our case $f(-\tau^{-1}) = \det(-\tau)^k f(\tau)$, where k is the weight of f (in our example, $k=4$) we have $\left(\frac{d}{d\tau}[v]^2 f\right)(-\tau^{-1}) =$

$$\mathrm{Sym}^4(\tau)\det(-\tau)^k\left(k(k+1)f(\tau)\tau^{-1}[v]^2 + 2(k+1)\tau^{-1}[v]\cdot\frac{d}{d\tau}[v]f(\tau) + \frac{d}{d\tau}[v]^2 f(\tau)\right).$$

Here we used (2.6). Now we can calculate how h transforms under the action of γ:

$$\begin{aligned}
h(-\tau^{-1}) \;=\;& \; 2\det(\tau)^8 g(\tau)\mathrm{Sym}_4(\tau)\left(20\tau^{-1}[v]^2 f(\tau) + 10\tau^{-1}[v]\frac{d}{d\tau}[v]f(\tau) + \frac{d}{d\tau}[v]^2 f(\tau)\right)\\
&- \det(\tau)^4\mathrm{Sym}^2(\tau)\left(4\tau^{-1}[v]f(\tau) + \frac{d}{d\tau}[v]f(\tau)\right)\cdot\\
&\qquad\qquad\cdot\det(\tau)^4\mathrm{Sym}^2(\tau)\left(4\tau^{-1}[v]g(\tau) + \frac{d}{d\tau}[v]g(\tau)\right)\\
&+ 2\det(\tau)^8 f(\tau)\mathrm{Sym}^4(\tau)\left(20\tau^{-1}[v]^2 g(\tau) + 10\tau^{-1}[v]\frac{d}{d\tau}[v]g(\tau) + \frac{d}{d\tau}[v]^2 g(\tau)\right)\\
=\;& \det(\tau)^8\mathrm{Sym}^4(\tau)h(\tau)
\end{aligned}$$

and hence, the function h is a modular form of weight $(4,8)$. The reason why this works is the particular choice of p. We will see in Chapter 3 that p is indeed a RC-polynomial.

Remark 2.8. Corollary 2.7.2 can be extended to include Siegel modular forms on finite index subgroups of Γ_g. In particular, if f is a classical Siegel modular form on Γ_g with a character v, i.e. $f(\gamma\cdot\tau) = v(\gamma)\det(j(\gamma,\tau))^k f(\tau)$, then we can use f and a RC-operator D_p to construct a vector-valued Siegel modular form with a character or, if for instance $v^2 = 1$ and f occurs twice in the argument of D_p, then the resulting modular form is a modular form on the full group Γ_g.

Example 2.8.1. The cusp form χ_{10} has a square root χ_5 in the ring of holomorphic functions on $\mathscr{H}_2$ and for all $\gamma\in\Gamma_2$ we have $\chi_5(\gamma\cdot\tau) = v(\gamma)\det(j(\gamma,\tau))^5\chi_5(\tau)$ for some character v with $v^2 = 1$. If D_p is some RC-operator of type $(5,5,k_3,k_4,\ldots,k_t)$ and $f_3,f_4,\ldots,f_t$ are Siegel modular forms of weight $k_3,k_4,\ldots,k_t$ on Γ_2, then $D_p(\chi_5\otimes\chi_5\otimes f_3\otimes\cdots\otimes f_t)$ is a modular form on the full group Γ_2.

Before we can prove Theorem 2.7.1, we need to define the Weierstrass transform and look at some of its well-known properties. To eliminate any possible confusion regarding notation, we first give the following remark.

Remark 2.9 (multi-indices and t-tuples). If x and α are matrices of the same size, then we want to make sense of the unary and binary forms

$$x^\alpha, \quad \alpha!, \quad (x)_\alpha, \quad dx \quad \text{and} \quad \frac{\partial^\alpha}{\partial x^\alpha}. \tag{2.7}$$

If B is one of the above forms, then we shall interpret

$$B(x,\alpha)=\begin{cases}\prod_{i\le j}B(x_{ij},\alpha_{ij}) & \text{if } x \text{ and } \alpha \text{ are elements of a space of } \textit{symmetric} \text{ matrices,}\\ \prod_{ij}B(x_{ij},\alpha_{ij}) & \text{otherwise.}\end{cases}$$

We will write $|\alpha|=\sum_{ij}\alpha_{ij}$ with the same convention for symmetric matrices as above.

We can extend the above convention to t-tuples as follows. If $x=(x_1,\dots,x_t)$ and $\alpha=(\alpha_1,\dots,\alpha_t)$ are t-tuples of matrices, then we first apply a diagonal injection $x\mapsto \operatorname{diag}(x_1,\dots x_t)$ and $\alpha\mapsto \operatorname{diag}(\alpha_1,\dots,\alpha_t)$. Next, we apply the above conventions where we ignore the additional zeroes. So, for instance, we have $x^\alpha=\prod_{\mu=1}^t\prod_{ij}(x_\mu)_{ij}^{(\alpha_\mu)_{ij}}$.

Using this diagonal embedding, we also get

- $xx'=(x_1x_1',\dots,x_tx_t')$ for $x=(x_1,\dots,x_t)$,
- $\tau[x]=(\tau_1[x_1],\dots,\tau_t[x_t])$ for $\tau\in\mathscr{H}_g^t$,
- $\operatorname{Tr}(x)=\operatorname{Tr}(x_1)+\cdots+\operatorname{Tr}(x_t)$ for $x=(x_1,\dots,x_t)$ a t-tuple of square matrices and
- $\det(\tau)^k=\det(\tau_1)^{k_1}\cdots\det(\tau_t)^{k_t}$ for $\tau\in\mathscr{H}_g^t$ and $k\in\mathbb{Z}^t$.

If $d\in\mathbb{N}^t$, then we write $\mathbb{R}^{g\times d}=\mathbb{R}^{g\times d_1}\oplus\cdots\oplus\mathbb{R}^{g\times d_t}$. If $\tau=(\tau_1,\dots,\tau_t)\in\mathscr{H}_g^t$, then we write $\hat\tau=(\tau_1,\tau_1,\dots,\tau_1)\in\mathscr{H}_g^t$.

Definition 2.9.1. For a function f on $\mathbb{R}^{g\times d}$ the *Weierstrass transform* of f is defined by

$$y\mapsto\mathscr{W}[f](y)=\int_{\mathbb{R}^{g\times d}}f(y-x)e^{-\pi\operatorname{Tr}(x'x)}dx:\mathbb{R}^{g\times d}\to\mathbb{C}.$$

Lemma 2.9.2. *(i) Let x be a $g\times d$ matrix of variables and let p be a polynomial in the coefficients of x. The polynomial p is harmonic if and only if $\mathscr{W}[p](y)=p(y)$.*

(ii) If p and $\mathscr{W}[p](x)$ are homogeneous of degree n, that is $\mathscr{W}[p](\lambda x)=\lambda^n\mathscr{W}[p](x)$ and $p(\lambda x)=\lambda^n p(x)$ for all scalars λ, then p is harmonic (and hence, $p=\mathscr{W}[p]$ by (i)).

Proof. (i) If $\Delta=\operatorname{Tr}\partial x\partial x'$, then $e^{\Delta/(4\pi)}$ is a well-defined operator on $\mathbb{C}[x]$, since $e^{\Delta/(4\pi)}p(x)=\sum_{n=0}^\infty\left(\frac{\Delta}{4\pi}\right)^n p(x)/n!$ is a finite sum. In fact we have the identity

$$\mathscr{W}[p](x)=e^{\Delta/(4\pi)}p(x)\qquad\forall p\in\mathbb{C}[x].\tag{2.8}$$

The Lemma follows immediately from this identity, since $\Delta p=0\iff e^{\Delta/(4\pi)}p(x)=p(x)$. We will therefore prove Equation (2.8).

We can write $p(x-y)$ as a finite Taylor series $p(x-y)=\sum_\alpha\frac{1}{\alpha!}\frac{\partial^\alpha}{\partial y^\alpha}p(y)(-x)^\alpha$ and therefore

$$\mathscr{W}[p](y)=\sum_\alpha\frac{1}{\alpha!}\frac{\partial^\alpha}{\partial y^\alpha}p(y)(-1)^{|\alpha|}\int x^\alpha e^{-\pi\operatorname{Tr}x'x}dx=\sum_\alpha\frac{1}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}p(y)\int x^{2\alpha}e^{-\pi\operatorname{Tr}x'x}dx,$$

where the last equality holds because the integral vanishes for multi-indices α with an odd coefficient. On the other hand, we have

$$e^{\operatorname{Tr}\partial y\partial y'/(4\pi)}p(y)=\sum_\alpha\frac{1}{\alpha!}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}p(y)\frac{1}{(4\pi)^{|\alpha|}}.$$

The Lemma will therefore hold if $\int x^{2\alpha} e^{-\pi \operatorname{Tr} x' x} dx = \left(\frac{1}{4\pi}\right)^{|\alpha|} \frac{(2\alpha)!}{\alpha!}$ for all multi-indices α or, equivalently, if

$$\alpha! \int (\sqrt{\pi} x)^{2\alpha} e^{-\pi \operatorname{Tr} x' x} dx = 2^{-|2\alpha|}(2\alpha)!$$

and this is true by the duplication formula for the Gamma function.

(ii) Since $\mathscr{W}[p](x) - p(x) = e^{\Delta/(4\pi)} p(x) - p(x) = \Delta/(4\pi) p(x) + \frac{1}{2}\Delta^2/(4\pi)^2 p(x) + \cdots$ is a homogeneous polynomial of degree n but also has degree $\leq n-2$, we must have $\Delta p(x) = \Delta^2 p(x) = \cdots = 0$. Hence, $\mathscr{W}[p] = p$ is harmonic. □

Lemma 2.9.3 (Kashiwara-Vergne)**.** *Let q be a pluri-harmonic polynomial. The function $f(x) = q(x) \exp(i\pi \operatorname{Tr}(\tau[x]))$ has the following Fourier transform:*

$$\int_{\operatorname{Mat}_{g,d}(\mathbb{R})} f(x) e^{-2\pi i \operatorname{Tr}(x y')} dx = \det(\tau/i)^{-k} q(-\tau^{-1} y) \exp\left(i\pi \operatorname{Tr}(-\tau^{-1}[y])\right). \qquad (2.9)$$

We will denote the Fourier transform by $\mathscr{F}$ or $\mathscr{F}_x$, where x stands for the variable of which the argument is considered to be a function. The same holds for the Weierstrass transform.

Proof. Both sides of Equation (2.9) are entire functions on $\mathscr{H}_g^t$ as a function of τ for any fixed $y \in \operatorname{Mat}_{g,d}(\mathbb{R})$. Henceforth it suffices to prove the Lemma for $\tau = i\sigma$ with σ real. If we complete the square in the exponential function, we get

$$\mathscr{F} f(y) = \int q(x) \exp(-\pi \operatorname{Tr}(t[x + i\sigma^{-1} y])) dx \cdot \exp(-\pi \operatorname{Tr}(t^{-1}[y])).$$

Now replace x with $s^{-1}x - i\sigma^{-1}y$ where $s^2 = \sigma$ and $s \succ 0$. This is allowed by Cauchy's Theorem. We get

$$\begin{aligned} \mathscr{F} f(y) &= \det(s)^{-d} \int q(s^{-1}x - i\sigma^{-1}y) \exp(\operatorname{Tr}(x x')) dx \cdot \exp(-\pi \operatorname{Tr}(\sigma^{-1}[y])) \\ &= \det(\sigma)^{-k} e^{-\pi \operatorname{Tr}(\sigma^{-1}[y])} \mathscr{W}_x[p(-s^{-1}x)](-is^{-1}y). \end{aligned}$$

Since q is pluri-harmonic, the function $x \mapsto q(s^{-1}x)$ is harmonic and this means that it is invariant under the Weierstrass transform $\mathscr{W}$ (Lemma 2.9.2). Hence

$$\mathscr{F} f(y) = \det(\sigma)^{-k} e^{-\pi \operatorname{Tr}(\sigma^{-1}[y])} q(s^{-1} i s^{-1} y) = \det(\tau/i)^{-k} e^{\pi i \operatorname{Tr}(-\tau^{-1}[y])} q(-\tau^{-1} y).$$

This proves the Lemma. □

Lemma 2.9.4. *Define for every $x \in \operatorname{Mat}_{g,d}(\mathbb{C})$ the function $g_x(\tau) = \exp(i\pi \operatorname{Tr}(\tau[x])) \in \operatorname{Hol}(\mathscr{H}_g^t)$. If p is a RC-polynomial of even type $k = d/2$, then the function g_x satisfies the commutation relation (2.5) on Γ_g for $D_{\bar{p}}$.*

Proof. We will prove the commutation relation for generators γ of Γ_g. The only non-trivial case is $\gamma = \tau \mapsto -\tau^{-1}$. By Lemma 2.9.3 with "$q = 1$", we have

$$i^{|k|g} \cdot g_x|_k \gamma(\tau) = \det(\tau_1/i)^{-k_1} \cdots \det(\tau_t/i)^{-k_t} g_x(-\tau^{-1}) = \mathscr{F}_x g_x(\tau)$$

and if we apply $D_{\tilde{p}}$, we find

$$\begin{aligned}(D_{\tilde{p}}\mathscr{F}_x g_x(\tau))(\tau_1) &= \mathscr{F}_x[\tilde{p}(x_1x_1',\ldots,x_tx_t')g_x(\tau)]\Big|_{\tau_i=\tau_1}\\ &= \mathscr{F}_x[\tilde{p}(xx')g_x(\hat{\tau})] = \mathscr{F}_x[p(x)g_x(\hat{\tau})].\end{aligned}$$

Now we apply Lemma 2.9.3 again, but this time with "$q=p$" and we find

$$\mathscr{F}_x[p(x)g_x(\hat{\tau})] = \det(\tau_1/i)^{-|k|}p(-\hat{\tau}^{-1}x)g_x(-\hat{\tau}^{-1}).$$

The polynomial p is assumed to be homogeneous of weight ρ and hence

$$\mathscr{F}_x[p(x)g_x(\hat{\tau})] = \det(\tau_1/i)^{-|k|}\rho(-\tau_1^{-1})D_{\tilde{p}}(g_x)(-\tau_1^{-1}) = D_{\tilde{p}}(g_x)|_{\det^{|k|}\otimes\rho}\gamma(\tau_1)\cdot i^{|k|g}.$$

This shows that $D_{\tilde{p}}(g_x|_k\gamma) = D_{\tilde{p}}(g_x)|_{\rho\otimes\det^{|k|}}\gamma$. □

We can now prove Theorem 2.7.1.

Proof (of Theorem 2.7.1). Let n be as in Remark 2.7, i.e. $n=\frac{1}{2}(\lambda_1+\cdots+\lambda_g)$ with $(\lambda_1,\ldots,\lambda_g)$ the highest weight of the irreducible representation ρ of $\mathsf{GL}_g(\mathbb{C})$.

Let f be a holomorphic function on $\mathscr{H}_g^t$. We need to show that

$$D_{\tilde{p}}(f|_k\gamma) = (D_{\tilde{p}}f)|_{\rho\otimes\det^{|k|}}\gamma \qquad \forall\gamma\in\Gamma_g \tag{2.10}$$

Fix $\gamma\in\Gamma_g$. For some functions Q_α and S_α that depend on p and γ, but not on f, we have

$$D_p(f|_k\gamma)(\tau_1) = \sum_\alpha Q_\alpha(\tau_1)f^{(\alpha)}(\gamma\cdot\hat{\tau}) \quad \text{and} \quad (D_pf)|_{\rho\otimes\det^{|k|}}\gamma(\tau_1) = \sum_\alpha S_\alpha(\tau_1)f^{(\alpha)}(\gamma\cdot\hat{\tau})$$

where the sums are taken over multi-indices $\alpha=(\alpha_{ij})$ with $|\alpha|=n$ and $f^{(\alpha)}=\frac{\partial^\alpha}{\partial\tau^\alpha}f$ denotes the α-th derivative of f. This means that equation (2.10) will certainly hold for all holomorphic functions f on $\mathscr{H}_g^t$ if $Q_\alpha=S_\alpha$ for all multi-indices α. This will be the case if for some functions g_β we have $\nabla^n g\cdot Q = \nabla^n g\cdot S$ where

$$\nabla^n g := \left(g_\beta^{(\alpha)}\right)_{|\alpha|=|\beta|=n}$$

is invertible on the whole of $\mathscr{H}_g^t$.

Let $x=(x_1,\ldots,x_t)$ be t-tuple of matrices of independent variables. We have seen that the coefficients of xx' are algebraically independent over $\mathbb{C}$ (Theorem 2.1.2). This implies that the polynomials $(xx')^\alpha$ are all algebraically independent. Therefore, we can find matrices $x_\beta\in\mathrm{Mat}_{g,d}(\mathbb{C})$ such that the matrix $(x_\beta x_\beta')^\alpha$ is invertible. Now choose $g_\beta(\tau)=e^{\pi i\,\mathrm{Tr}(\tau[x_\beta])}$ and write $g=(g_\beta\delta_{\alpha,\beta})$, then $\nabla^n g=(x_\beta x_\beta')^\alpha g$ is invertible on the whole of $\mathscr{H}_g$. We can now prove the Theorem by proving the commutation relation (2.10) for the functions g_β. This was done in Lemma 2.9.4. □

Chapter 3

Explicit constructions of RC-operators

In Chapter 2 we gave a method to construct vector-valued modular forms from classical modular forms. This was done by means of RC-operators coming from RC-polynomials. In this chapter we will construct some explicit RC-polynomials. We start with a lemma that simplifies the Laplacian on $\mathrm{O}(d)$-invariant polynomials.

3.1 The Laplacian acting on $\mathrm{O}(d)$-invariant polynomials

Lemma 3.1.1 (Eholzer, Ibukiyama)**.** *Let $d \geq g$ be positive integers, $x \in \mathrm{Mat}_{g,d}$ be a matrix of variables x_{ij} and suppose that $p \in \mathbb{C}[x]^{\mathrm{O}(d)}$ is an $\mathrm{O}(d)$-invariant polynomial in the variables x_{ij}. Furthermore, let $\tilde{p} \in \mathbb{C}[\mathbf{r}]$ be the associated polynomial of p (cf. Theorem 2.1.2), where $\mathbf{r} = \mathbf{r}' = (\mathbf{r}_{ij})$ is a symmetric $g \times g$ matrix. Define $d/d\mathbf{r}$ by $(d/d\mathbf{r})_{ij} = \frac{1}{2}(1+\delta_{ij})\partial/\partial\mathbf{r}_{ij}$ and let*

$$(\mathbf{L}_d\tilde{p})(\mathbf{r}) := \mathrm{Tr}(\mathbf{r}\left(\tfrac{d}{d\mathbf{r}}\right)^2 + d/2\tfrac{d}{d\mathbf{r}})\tilde{p}.$$

Then $(\Delta p)(x) = 4(\mathbf{L}_d\tilde{p})(xx')$.

Proof. The polynomial $\tilde{p}$ satisfies $\tilde{p}(xx') = p(x)$. Note that $\partial/\partial x_{\nu\mu}(xx')_{ij} = (\delta_{\nu i}x_{j\mu} + \delta_{\nu j}x_{i\mu})$ and therefore

$$\frac{\partial}{\partial x_{\nu\mu}}\tilde{p}(xx') \quad = \quad \sum_{1\leq i,j\leq g}\frac{\partial}{\partial\mathbf{r}_{ij}}\tilde{p}(xx')(\delta_{\nu i}x_{j\mu}+\delta_{\nu j}x_{i\mu})(1+\delta_{ij})/2. \tag{3.1}$$

The factor $(1+\delta_{ij})/2$ appears in (3.1) because we take the sum not over $1 \leq i \leq j \leq g$, but over $1 \leq i \leq g$ and $1 \leq j \leq g$. If we simplify (3.1), we get

$$\frac{\partial}{\partial x_{\nu\mu}}\tilde{p}(xx') = \sum_{i=1}^{g}\frac{\partial}{\partial\mathbf{r}_{i\nu}}\tilde{p}(xx')x_{i\mu}(1+\delta_{i\nu}) = 2(x'\tfrac{d}{d\mathbf{r}}\tilde{p}(xx'))_{\mu\nu}. \tag{3.2}$$

Now we can calculate the coefficients on the diagonal of $\partial x\partial x'\tilde{p}(xx')$ as follows:

$$\begin{aligned}(\partial x\partial x'\tilde{p}(xx'))_{\nu\nu} \quad &= \quad \sum_{\mu=1}^{d}\frac{\partial}{\partial x_{\nu\mu}}\sum_{i=1}^{g}x_{i\mu}(1+\delta_{i\nu})\frac{\partial}{\partial\mathbf{r}_{i\nu}}\tilde{p}(xx') && (3.3)\\ &= \quad \sum_{\mu=1}^{d}(1+\delta_{\nu\nu})\frac{\partial}{\partial r_{\nu\nu}}\tilde{p}(xx') \quad + \quad \sum_{\mu=1}^{d}\sum_{i=1}^{g}x_{i\mu}(1+\delta_{i\nu})\frac{\partial}{\partial x_{\nu\mu}}\frac{\partial}{\partial\mathbf{r}_{i\nu}}\tilde{p}(xx'). && (3.4)\end{aligned}$$

The left sum of (3.4) equals $2d\frac{\partial}{\partial \mathbf{r}_{\nu\nu}}\tilde{p}(xx')$ and for the sum on the right we can use (3.2) again:

$$\frac{\partial}{\partial x_{\nu\mu}}\frac{\partial}{\partial r_{i\nu}}\tilde{p}(xx') = 2(x'\frac{d}{d\mathbf{r}}\frac{\partial}{\partial \mathbf{r}_{i\nu}}\tilde{p}(xx'))_{\mu\nu}$$

and hence, the right sum in (3.4) equals

$$\begin{aligned} 2\sum_{\mu=1}^{d}\sum_{i=1}^{g} x_{i\mu}(1+\delta_{i\nu})(x'\tfrac{d}{d\mathbf{r}}\tfrac{\partial}{\partial \mathbf{r}_{i\nu}}\tilde{p}(xx'))_{\mu\nu} &= 2\sum_{i=1}^{g}(xx'\tfrac{d}{d\mathbf{r}})_{i\nu}(1+\delta_{i\nu})\tfrac{\partial}{\partial \mathbf{r}_{i\nu}}\tilde{p}(xx') \\ &= 4\left(xx'\tfrac{d}{d\mathbf{r}}\tfrac{d}{d\mathbf{r}}\tilde{p}(xx')\right)_{\nu\nu}. \end{aligned}$$

The result now follows after summation over ν. □

Remark 3.2. If $d=(d_1,\ldots,d_t)\in\mathbb{N}^t$ and p is an $\mathrm{O}(d)$-invariant polynomial in $\mathbb{C}[x_1,\ldots,x_t]$ with associated polynomial $\tilde{p}(\mathbf{r}_1,\ldots,\mathbf{r}_t)$, then we define

$$\mathbf{L}_d := \mathrm{Tr}\big(\mathbf{r}_1(\tfrac{d}{d\mathbf{r}_1})^2 + d_1/2\tfrac{d}{d\mathbf{r}_1}\big)+\cdots+\mathrm{Tr}\big(\mathbf{r}_t(\tfrac{d}{d\mathbf{r}_t})^2 + d_t/2\tfrac{d}{d\mathbf{r}_t}\big)$$

and then $(\Delta p)(x_1,\ldots,x_t) = (\mathbf{L}_d\tilde{p})(x_1x_1',\ldots,x_tx_t')$. It is possible to get similar results for the operators $\Delta^{(\mu,\nu)}$ (Definition 2.3.3). See for instance Eholzer and Ibukiyama's article ([8] p.p. 6).

Example 3.2.1. If $g=2$ and $t=1$, the differential operator $\mathbf{L}_d$ has the following form:

$$\begin{aligned} \mathbf{L}_d &= \mathbf{r}_{11}\left(4\frac{\partial^2}{\partial \mathbf{r}_{11}^2}+\frac{\partial^2}{\partial \mathbf{r}_{12}^2}\right)+4\mathbf{r}_{12}\left(\frac{\partial^2}{\partial \mathbf{r}_{11}\partial \mathbf{r}_{12}}+\frac{\partial^2}{\partial \mathbf{r}_{12}\partial \mathbf{r}_{22}}\right)+\mathbf{r}_{22}\left(4\frac{\partial^2}{\partial \mathbf{r}_{22}^2}+\frac{\partial^2}{\partial \mathbf{r}_{12}^2}\right) \\ &+ 2d\left(\frac{\partial}{\partial \mathbf{r}_{11}}+\frac{\partial}{\partial \mathbf{r}_{22}}\right). \end{aligned}$$

If $g=1$, then we simply have $\mathbf{L}_d = \mathbf{r}\frac{\partial^2}{\partial \mathbf{r}^2}+d/2\frac{\partial}{\partial \mathbf{r}}$. We can now use the operator $\mathbf{L}_d$ to find polynomials $\tilde{p}$ that are the associated polynomials of RC-polynomials p directly. The fact that $\tilde{p}$ comes from a polynomial p is only needed in the proof of Theorem 2.7.1.

3.3 Some examples of RC-polynomials from the literature

We will now discuss some examples that can be found in the literature. From now on, we will restrict to the cases $g=1$ and $g=2$.

The polynomials of weight (m,k) in 2 'variables' (i.e. $t=2$) were completely described by Miyawaki [24], but the examples we give can also be found in less recent articles.

3.3.1 The symmetric tensor representation Sym^2

We will first discuss the Rankin-Cohen brackets defined by Satoh and Ibukiyama [17].

On the space of symmetric complex 2×2 matrices S_2 we have a Lie-product $u\times v = uJv - vJu$, where $J=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Suppose that $u=\left(\begin{smallmatrix}a&b\\b&c\end{smallmatrix}\right)$ and $v=\left(\begin{smallmatrix}a'&b'\\b'&c'\end{smallmatrix}\right)$, then

$$u\times v = \begin{pmatrix}2ab'-2ba' & ac'-ca'\\ ac'-ca' & 2bc'-2cb'\end{pmatrix}.$$

Note that $u \times v = -v \times u$. If $g \in \mathsf{GL}_2(\mathbb{C})$, then we have $(gug') \times (gvg') = gug'Jgvg' - gvg'Jgug' = \det(g)g(u \times v)g'$, since $g'Jg = \det(g)J$. Therefore, if $\rho : \mathsf{GL}_2(\mathbb{C}) \to \mathsf{GL}(S_2)$ is the representation discussed in Example 2.6.2, and let $\Lambda^2\rho : \mathsf{GL}_2(\mathbb{C}) \to \mathsf{GL}(\Lambda^2 S_2)$ denote the alternating square of ρ, then $\times$ defines an isomorphism $\Lambda^2(S_2) \to S_2$ such that

$$\begin{array}{rccc} (\Lambda^2\rho)(g): & \Lambda^2 S_2 & \longrightarrow & \Lambda^2 S_2 \\ & \downarrow \times & & \downarrow \times \\ (\rho \otimes \det)(g): & S_2 & \longrightarrow & S_2 \end{array}$$

is commutative. Hence, $\rho \otimes \det \cong \Lambda^2\rho$ as representations and this isomorphism is given by the map $\times$.

Definition 3.3.1. Define for $f_i \in M_{k_i}(\Gamma_2)$

$$[f_1, f_2] := \frac{1}{2\pi i}\left(k_1 f_1 \tfrac{d}{d\tau} f_2 - k_2 f_2 \tfrac{d}{d\tau} f_1\right)$$

and

$$[f_1, f_2, f_3] := \frac{1}{(2\pi i)^2}\left(k_1 f_1 \tfrac{d}{d\tau} f_2 \times \tfrac{d}{d\tau} f_3 - k_2 f_2 \tfrac{d}{d\tau} f_1 \times \tfrac{d}{d\tau} f_3 + k_3 f_3 \tfrac{d}{d\tau} f_1 \times \tfrac{d}{d\tau} f_2\right).$$

Remark 3.4. The following equality holds for every $f_i \in M_{k_i}(\Gamma_2)$:

$$k_1 f_1 [f_1, f_2, f_3] = [f_1, f_2] \times [f_1, f_3].$$

Lemma 3.4.1. *The Let $f_i \in M_{k_i}(\Gamma_2)$, then $[f_1, f_2]$ and $[f_1, f_2, f_3]$ are modular forms in $M_{(2,k_1+k_2)}(\Gamma_2)$ and $M_{(2,k_1+k_2+k_3+1)}(\Gamma_2)$ respectively.*

Proof. The fact that $[f_1, f_2] \in M_{(2,k_1+k_2)}(\Gamma_2)$ was proven in Example 2.6.2. The product $[f_1, f_2] \times [f_1, f_3]$ is then a modular form of weight $(2, 2k_1 + k_2 + k_3 + 1)$ and this modular form is visibly divisible by f_1 in the $M_*(\Gamma_2)$-module $M_{(2,*)}(\Gamma_2)$. This proves the Lemma. □

In Chapter 4 we will see that the brackets $[\cdot,\cdot]$ and $[\cdot,\cdot,\cdot]$ generate $M_{(2,*)}(\Gamma_2)$.

3.4.1 RC-polynomials of weight $(m, 0)$ and $(m, 2)$.

The following example is very important, because it allows us to compute Fourier coefficients of classical Siegel modular forms without using L-series and class numbers as described in e.g. [9].

Example 3.4.2 (Resnikoff)**.** Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be symmetric 2×2 matrices with formal variables as coefficients. Define the polynomials p_0, p_1 and p_2 in the coefficients of $\mathbf{r}_1$ and $\mathbf{r}_2$ as follows: $\det(\mathbf{r}_1 + \lambda\mathbf{r}_2) = p_0 + \lambda p_1 + \lambda^2 p_2$. Then $p_0 = \det(\mathbf{r}_1)$ and $p_2 = \det(\mathbf{r}_2)$, and

$$P_{k_1,k_2}(\mathbf{r}_1, \mathbf{r}_2) = 2k_2(2k_2 - 1)p_0 - (2k_1 - 1)(2k_2 - 1)p_1 + 2k_2(2k_2 - 1)p_2$$

is pluri-harmonic of type (k_1, k_2) [5, 8] and of weight $\det^2$. This means that if $f_i \in M_{k_i}(\Gamma_2)$ for $i = 1, 2$ then the function

$$D_P(f_1, f_2) = \frac{1}{(2\pi i)^2} P_{k_1,k_2}\left(\tfrac{d}{d\tau_1}, \tfrac{d}{d\tau_2}\right)\Big|_{\tau_1=\tau_2=\tau} f_1(\tau_1) f_2(\tau_2)$$

is a classical Siegel modular form of weight k_1+k_2+2. In particular, we can choose $f_1=f_2=f$ for some modular form f of weight w. Then by $\det(\frac{d}{d\tau}f)=2f\det(\frac{d}{d\tau})f-2\det(\frac{d}{d\tau}f^2)$, we have

$$D_P(f,f)=\frac{2w-1}{(2\pi i)^2}\left((8w-2)f\det(\tfrac{d}{d\tau})f-(2w-1)\det(\tfrac{d}{d\tau})f^2\right),$$

which is a modular form of weight $2w+2$. Define the operator D_w by $D_w f:=(8w-2)\det(\frac{d}{d\tau})f-(2w-1)\det(\frac{d}{d\tau})f^2$, then D_w is an operator that sends $M_w(\Gamma_2)$ to $M_{2w+2}(\Gamma_2)$. This can be used to compute Fourier coefficients.

Since $\det(\frac{d}{d\tau})q^n=-4\pi^2\det(n)q^n$, the Fourier coefficients at $n\not>0$ of $\det(\frac{d}{d\tau})f$ vanish. Hence $D_w f$ is a cusp form. This shows that $D_4\varphi_4\in S_{10}(\Gamma_2)$ and therefore $D_4\varphi_4=c\cdot\chi_{10}$ for some constant c. Write $\varphi_4(\tau)=\sum_{n\geq 0}a_4(n)q^n$, then $\varphi_4^2(\tau)=\sum_{n\geq 0}q^n\sum_{n_1+n_2=n}a_4(n_1)a_4(n_2)$ and

$$\begin{aligned}D_4\varphi_4 &= -120\pi^2\sum_n a_4(n)q^n\sum_m a_4(m)\det(m)q^m+28\pi^2\sum_n q^n\det(n)\sum_{n_1+n_2=n}a_4(n_1)a_4(n_2)\\ &= -4\pi^2\sum_n q^n\sum_{n_1+n_2=n}(30\det(n_2)-7\det(n))a_4(n_1)a_4(n_2).\end{aligned}$$

Take $n=\left(\begin{smallmatrix}1&1/2\\1/2&1\end{smallmatrix}\right)$, then there are only two ways to write n as the sum of two non-negative matrices, namely $n=n+0$ and $n=0+n$. Hence

$$\begin{aligned}(D_4\varphi_4)\hat{}(n) &= -4\pi^2(30\det(0)-7\det(n))a_4(n)a_4(0)-4\pi^2(30\det(n)-7\det(n))a_4(0)a_4(n)\\ &= 4\pi^2 7\cdot 2^7 3\cdot 5\cdot 7-4\pi^2 23\cdot\tfrac{3}{4}2^7 3\cdot 5\cdot 7=-\pi^2 2^7 3\cdot 5\cdot 7\cdot 41.\end{aligned}$$

Therefore, $-\pi^2 2^7 3\cdot 5\cdot 7\cdot 41\chi_{10}=D_4\varphi_4$ and then Fourier coefficients c_{10} of χ_{10} (normalized such that $c_{10}\left(\begin{smallmatrix}1&1/2\\1/2&1\end{smallmatrix}\right)=1$) are given by

$$c_{10}(n)=(2^5 3\cdot 5\cdot 7\cdot 41)^{-1}\sum_{n_1+n_2=n}(30\det(n_2)-7\det(n))a_4(n_1)a_4(n_2).$$

This and similar computations were used by Resnikoff to compute Fourier coefficients and give explicit differential equations for classical Siegel modular forms of genus 2 [26, 25, 27].

We already mentioned in Example 2.5.3 that it is easy to construct homogeneous polynomials of weight $(m,0)$ and type (k_1,k_2). The same holds for RC-polynomials of weight $(m,0)$; everything reduces to the genus 1 case which was discussed in Section 2.1. We give the following example:

Example 3.4.3. Let $p(r,s)=5s^3-30s^2r+42sr^2-14r^3$, then $\mathbf{L}_{(4,6)}p=0$. Now let $\mathbf{r}$ and $\mathbf{s}$ denote symmetric 2×2 matrices of formal variables $\mathbf{r}_{ij}$ and $\mathbf{s}_{ij}$ and write $v=(x,y)$. Define $P(\mathbf{r},\mathbf{s},v)=p(\mathbf{r}[v],\mathbf{s}[v])$, then also $\mathbf{L}_{(4,6)}P(\mathbf{r},\mathbf{s},v)=0$ as a simple consequence of the chain rule.

We will also need a RC-polynomial of weight $(m,2)$. These polynomials were described by Eholzer and Ibukiyama [8].

Example 3.4.4. Again, let $\mathbf{r}$ and $\mathbf{s}$ denote symmetric 2×2 matrices. p_0, p_1 and p_2 as in Example 3.4.2. Let $(k_1,k_2)\in\mathbb{N}^2$ and suppose that P_0, P_1 and P_2 are RC-polynomials of weight $(m,0)$ and type (k_1+2,k_2), (k_1+1,k_2+1) and (k_1,k_2+2) respectively (cf. Example 3.4.3). The polynomial

$$P=2(2k_2-1)(k_2+m/2)p_0P_0-(2k_1-1)(2k_2-1)p_1P_1+2(2k_1-1)(k_1+m/2)p_2P_2$$

is then an RC-polynomial of weight $(m,2)$ and type (k_1,k_2).

3.5 RC-polynomials of weight $(m,1)$

We have now seen RC-polynomials of weight $(m,0)$ and $(m,2)$. In Chapter 4 we also need polynomials of weight $(m,1)$. Ibukiyama's brackets $[\cdot,\cdot,\cdot]$ correspond to such polynomials and Ibukiyama was also able to construct RC-polynomials of weight $(4,1)$. In this section, we will generalize Ibukiyama's polynomials. Our generalization gives a method to construct RC-polynomials of weight $(m,1)$ for all even m.

Let x and y be variables and write $v=(x,y)'$ and $\sigma=vv'$. Let $\mathbf{r}=\begin{pmatrix} r_{11} & r_{12}\\ r_{12} & r_{22}\end{pmatrix}$, $\mathbf{s}=\begin{pmatrix} s_{11} & s_{12}\\ s_{12} & s_{22}\end{pmatrix}$ and $\mathbf{t}=\begin{pmatrix} t_{11} & t_{12}\\ t_{12} & t_{22}\end{pmatrix}$ be matrices of formal variables and write $\mathbf{r}[v]=(x,y)\mathbf{r}(x,y)'$. If $f\in\mathbb{C}[r,s,t]$, then we will write $X(f)$ for $f(\mathbf{r}[v],\mathbf{s}[v],\mathbf{t}[v])\in\mathbb{C}[\mathbf{r},\mathbf{s},\mathbf{t},v]$. As usual we define $\frac{d}{d\mathbf{r}}=\frac{1}{2}\left(\frac{\partial}{\partial r_{ij}}(1+\delta_{ij})\right)$.

Definition 3.5.1. Let $k=(k_1,k_2,\dots,k_a)$. We define the following differential operators:

$$\begin{aligned}
\mathbf{L}_k &= \sum_{i=1}^{a}\mathrm{Tr}\left(\mathbf{r}_i(\tfrac{d}{d\mathbf{r}_i})^2+k_i\tfrac{d}{d\mathbf{r}_i}\right):\mathbb{C}[\mathbf{r}_1,\dots,\mathbf{r}_a,v]\to\mathbb{C}[\mathbf{r}_1,\dots,\mathbf{r}_a,v]\\
\mathbf{L}_k &= \sum_{i=1}^{a} r_i\tfrac{\partial^2}{\partial r_i^2}+k_i\tfrac{\partial}{\partial r_i}:\mathbb{C}[r_1,\dots,r_a]\to\mathbb{C}[r_1,\dots,r_a]\\
\mathbf{M}_k &= \mathbf{r}\times\mathbf{s}[v]\left(k_3X(\cdot)+\mathbf{t}[v]X(\tfrac{\partial}{\partial t}\cdot)\right)\\
&+ \mathbf{t}\times\mathbf{r}[v]\left(k_2X(\cdot)+\mathbf{s}[v]X(\tfrac{\partial}{\partial s}\cdot)\right)\\
&+ \mathbf{s}\times\mathbf{t}[v]\left(k_1X(\cdot)+\mathbf{r}[v]X(\tfrac{\partial}{\partial r}\cdot)\right):\mathbb{C}[r,s,t]\to\mathbb{C}[\mathbf{r},\mathbf{s},\mathbf{t},v].
\end{aligned}$$

The operator $\mathbf{M}_k$ is only defined if $a=3$.

Note that this definition of $\mathbf{L}_k$ is slightly different than the definition given before. The subscript k equals the 'type', which seems to be more natural now.

Proposition 3.5.2. *The following commutation relation holds for* $\mathbf{M}$ *and* $\mathbf{L}$*:*

$$\mathbf{L}_{(k_1,k_2,k_3)}\circ\mathbf{M}_{(k_1,k_2,k_3)}=v'v\mathbf{M}_{(k_1,k_2,k_3)}\circ\mathbf{L}_{(k_1+1,k_2+1,k_3+1)}.$$

The polynomial ring $\mathbb{C}[\mathbf{r},\mathbf{s},\mathbf{t},v]$ can be written as a direct sum $\bigoplus_{n\geq 0}\mathbb{C}[\mathbf{r},\mathbf{s},\mathbf{t},v]^{(n)}$ by which we mean that a polynomial in $\mathbb{C}_n[\mathbf{r},\mathbf{s},\mathbf{t},v]$ is homogeneous of degree n as a polynomial in x and y. We may therefore prove the Proposition on the homogeneous parts $\mathbb{C}[\mathbf{r},\mathbf{s},\mathbf{t},v]^{(n)}$. This is convenient since it allows us to assume that $v'v=\mathrm{Tr}(\sigma)=1$ which will simplify the notation slightly. Before we can prove this Proposition, we need the following Lemma.

Lemma 3.5.3. *Let f be a polynomial in $\mathbb{C}[r,s,t]$. If we assume that $v'v=1$, we have the following identities*

$$\begin{aligned}
\mathrm{Tr}\tfrac{d}{d\mathbf{r}}\mathbf{M}_k(f) &= \mathbf{M}_k(\tfrac{\partial}{\partial r}f)+\mathbf{s}\times\mathbf{t}[v]X(\tfrac{\partial}{\partial r}f) && (3.5)\\
&+ \left(k_3X(f)+\mathbf{t}[v]X(\tfrac{\partial}{\partial t}f)\right)\mathrm{Tr}\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]\\
&+ \left(k_2X(f)+\mathbf{s}[v]X(\tfrac{\partial}{\partial s}f)\right)\mathrm{Tr}\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v],\\
\mathbf{r}[v]\mathbf{M}_k(f) &= \mathbf{M}_k(rf)-\mathbf{s}\times\mathbf{t}[v]\mathbf{r}[v]X(f), && (3.6)\\
\mathrm{Tr}\,\mathbf{r}(\tfrac{d}{d\mathbf{r}})^2\mathbf{M}_k(f) &= \mathbf{M}_k(r\tfrac{\partial^2}{\partial r^2}f)+\mathbf{r}[v]\mathbf{s}\times\mathbf{t}[v]X(\tfrac{\partial^2}{\partial r^2}f) && (3.7)\\
&+ \left(\mathbf{r}\times\mathbf{s}[v]+\mathbf{r}[v]\,\mathrm{Tr}\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]\right)\cdot\left(k_3X(\tfrac{\partial}{\partial r}f)+\mathbf{t}[v]X(\tfrac{\partial^2}{\partial r\partial t}f)\right)\\
&+ \left(\mathbf{t}\times\mathbf{r}[v]+\mathbf{r}[v]\,\mathrm{Tr}\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]\right)\cdot\left(k_2X(\tfrac{\partial}{\partial r}f)+\mathbf{s}[v]X(\tfrac{\partial^2}{\partial r\partial s}f)\right).
\end{aligned}$$

Proof. Note that $\frac{d}{d\mathbf{r}}X(f)=\sigma X(\frac{\partial}{\partial r}f)$ and $\mathbf{r}[v]X(f)=X(rf)$. We start with identity (3.5). We have

$$\begin{aligned}\frac{d}{d\mathbf{r}}\left(\mathbf{r}\times\mathbf{s}[v](k_3X(f)+\mathbf{t}[v]X(\tfrac{\partial}{\partial t}f))\right) &= (k_3X(f)+\mathbf{t}[v]X(\tfrac{\partial}{\partial t}f))(\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]) \\ &+ \mathbf{r}\times\mathbf{s}[v](k_3X(\tfrac{\partial}{\partial r}f)\sigma+\mathbf{t}[v]X(\tfrac{\partial^2}{\partial r\partial t}f)\sigma)\end{aligned} \tag{3.8}$$

and similarly

$$\begin{aligned}\frac{d}{d\mathbf{r}}\left(\mathbf{t}\times\mathbf{r}[v](k_2X(f)+\mathbf{s}[v]X(\tfrac{\partial}{\partial s}f))\right) &= (k_2X(f)+\mathbf{s}[v]X(\tfrac{\partial}{\partial s}f))(\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]) \\ &+ \mathbf{t}\times\mathbf{r}[v](k_2X(\tfrac{\partial}{\partial r}f)\sigma+\mathbf{s}[v]X(\tfrac{\partial^2}{\partial r\partial s}f)\sigma).\end{aligned} \tag{3.9}$$

If we differentiate the third term in $\mathbf{M}_k(f)$, we get

$$\begin{aligned}\frac{d}{d\mathbf{r}}\left(\mathbf{s}\times\mathbf{t}[v](k_1X(f)+\mathbf{r}[v]X(\tfrac{\partial}{\partial r}f))\right) &= \mathbf{s}\times\mathbf{t}[v]\left(k_1X(\tfrac{\partial}{\partial r}f)\sigma+\sigma X(\tfrac{\partial}{\partial r}f)+\mathbf{r}[v]\sigma X(\tfrac{\partial}{\partial r}f)\right) \\ &= \mathbf{s}\times\mathbf{t}[v]\left(k_1X(\tfrac{\partial}{\partial r}f)\sigma+\mathbf{r}[v]\sigma X(\tfrac{\partial^2}{\partial r^2}f)\right)+\mathbf{s}\times\mathbf{t}[v]\sigma X(\tfrac{\partial}{\partial r}f).\end{aligned} \tag{3.10}$$

If we combine equations (3.8), (3.9) and (3.10) and take the trace, we get identity (3.5).

We now proceed with (3.6). If we consider the first two terms of $\mathbf{M}_k(rf)$, we see that the factor r can be replaced by a factor $\mathbf{r}[v]$. In the third term we get an extra term due to differentiation with respect to r:

$$\begin{aligned}\mathbf{s}\times\mathbf{t}[v]\left(k_1X(rf)+\mathbf{r}[v]X(\tfrac{\partial}{\partial r}rf)\right) &= \mathbf{s}\times\mathbf{t}[v]\left(k_1\mathbf{r}[v]X(f)+\mathbf{r}[v]^2X(\tfrac{\partial}{\partial r}f)+\mathbf{r}[v]X(\tfrac{\partial}{\partial r}f)\right) \\ &= \mathbf{r}[v]\mathbf{s}\times\mathbf{t}[v]\left(k_1X(f)+\mathbf{r}[v]X(\tfrac{\partial}{\partial r}f)\right)+\mathbf{r}[v]\mathbf{s}\times\mathbf{t}[v]X(\tfrac{\partial}{\partial r}f).\end{aligned}$$

Finally, we prove identity (3.7). We have by (3.8), (3.9) and (3.10) that

$$\begin{aligned}\mathbf{r}(\tfrac{d}{dr})^2\mathbf{M}_k(f) &= \mathbf{r}\tfrac{d}{d\mathbf{r}}\mathbf{M}_k(\tfrac{\partial}{\partial r}f)\sigma+\mathbf{s}\times\mathbf{t}[v]\mathbf{r}\tfrac{d}{d\mathbf{r}}\sigma X(\tfrac{\partial}{\partial r}f) \\ &+ \mathbf{r}\tfrac{d}{d\mathbf{r}}(k_3X(f)+t[v]X(\tfrac{\partial}{\partial t}f))\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v] \\ &+ \mathbf{r}\tfrac{d}{d\mathbf{r}}(k_2X(f)+s[v]X(\tfrac{\partial}{\partial s}f))\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v].\end{aligned}$$

If g is a function of $\mathbf{r}$ and A is a 2 by 2 matrix not depending on the $\mathbf{r}_{ij}$, then $\frac{d}{d\mathbf{r}}(g(\mathbf{r})A)=\frac{d}{d\mathbf{r}}g\cdot A$. Applying this, we get

$$\begin{aligned}\mathbf{r}\tfrac{d}{d\mathbf{r}}\mathbf{M}_k(\tfrac{\partial}{\partial r}f)\sigma &= \mathbf{r}\sigma^2\mathbf{M}_k(\tfrac{\partial^2}{\partial r^2}f)+\mathbf{r}\sigma^2\mathbf{r}\times\mathbf{t}[v]X(\tfrac{\partial^2}{\partial r^2}f) \\ &+ \mathbf{r}\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]\sigma\cdot(k_3X(\tfrac{\partial}{\partial r}f)+\mathbf{t}[v]X(\tfrac{\partial^2}{\partial r\partial t}f)) \\ &+ \mathbf{r}\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]\sigma\cdot(k_2X(\tfrac{\partial}{\partial r}f)+\mathbf{s}[v]X(\tfrac{\partial^2}{\partial r\partial s}f))\end{aligned}$$

and hence

$$\begin{aligned}\mathbf{r}(\tfrac{d}{dr})^2\mathbf{M}_k(f) &= \mathbf{r}\sigma^2\mathbf{M}_k(\tfrac{\partial^2}{\partial r^2}f)+2\mathbf{r}\sigma^2\mathbf{r}\times\mathbf{t}[v]X(\tfrac{\partial^2}{\partial r^2}f) \\ &+ \left(\mathbf{r}\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]\sigma+\mathbf{r}\sigma\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]\right)\cdot(k_3X(\tfrac{\partial}{\partial r}f)+\mathbf{t}[v]X(\tfrac{\partial^2}{\partial r\partial t}f)) \\ &+ \left(\mathbf{r}\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]\sigma+\mathbf{r}\sigma\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]\right)\cdot(k_2X(\tfrac{\partial}{\partial r}f)+\mathbf{s}[v]X(\tfrac{\partial^2}{\partial r\partial s}f)).\end{aligned}$$

Note that $\mathrm{Tr}\,\mathbf{r}\sigma^2=\mathbf{r}[v]$ and that $\mathrm{Tr}\left(\mathbf{r}\frac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]\sigma+\mathbf{r}\sigma\frac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]\right)=\mathbf{t}\times\mathbf{r}[v]+\mathrm{Tr}(\frac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v])$. The result now follows from (3.6). $\square$

Proof (of Proposition 3.5.2). Using Lemma 3.5.3, we see that

$$\begin{aligned}
\mathrm{Tr}(\mathbf{r}\tfrac{d}{d\mathbf{r}}\mathbf{M}_k(f)+k_1\tfrac{d}{d\mathbf{r}}\mathbf{M}_k(f)) \quad = \quad & \mathbf{M}_k(r\tfrac{\partial^2}{\partial r^2}f+k_1\tfrac{\partial}{\partial r}f)+\mathbf{M}_k(\tfrac{\partial}{\partial r}f) \\
+ \quad & k_1k_3X(f)\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]+k_1k_2X(f)\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v] \\
+ \quad & \mathbf{r}[v]\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]\cdot\Big(k_3X(\tfrac{\partial}{\partial r}f)+\mathbf{t}[v]X(\tfrac{\partial^2}{\partial r\partial t}f)\Big) \\
+ \quad & \mathbf{r}[v]\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]\cdot\Big(k_2X(\tfrac{\partial}{\partial r}f)+\mathbf{s}[v]X(\tfrac{\partial^2}{\partial r\partial s}f)\Big) \\
+ \quad & k_1\mathbf{t}[v]X(\tfrac{\partial}{\partial t}f)\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]+k_1\mathbf{s}[v]X(\tfrac{\partial}{\partial s}f)\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v].
\end{aligned}$$

If we now consider $\mathbf{L}_k(\mathbf{M}_k(f))$, we get $\mathbf{M}_k(\mathbf{L}_{k+(1,1,1)}(f))+U$ where the term U turns out to vanish identically. In order to see that $U=0$, first consider the coefficient of $k_1k_2X(f)$. This term equals

$$\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{t}\times\mathbf{r}[v]+\mathrm{Tr}\,\tfrac{d}{d\mathbf{s}}\mathbf{r}\times\mathbf{s}[v]=0.$$

We get similar results for the coefficients of k_1k_3 and k_2k_3.
Now consider the coefficient of $k_3X(\frac{\partial}{\partial r}f)$. This coefficient equals

$$\mathbf{r}[v]\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]+\mathbf{r}[v]\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{t}}\mathbf{s}\times\mathbf{t}[v]=0$$

and the other 5 terms involving $k_2X(\frac{\partial}{\partial r}f)$, $k_1X(\frac{\partial}{\partial t}f)$, etc. vanish too.
Finally, consider the coefficient of $X(\frac{\partial^2}{\partial r\partial t}f)$. This expression equals

$$\mathbf{r}[v]\mathbf{t}[v]\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{r}}\mathbf{r}\times\mathbf{s}[v]+\mathbf{t}[v]\mathbf{r}[v]\,\mathrm{Tr}\,\tfrac{d}{d\mathbf{t}}\mathbf{s}\times\mathbf{t}[v]=0,$$

and as before the same result holds for the coefficients of $X(\frac{\partial^2}{\partial r\partial s}f)$ and $X(\frac{\partial^2}{\partial s\partial t}f)$. This proves the Proposition. □

Example 3.5.4. The polynomial $p(r,s)=r-s$ lies in the kernel of $\mathbf{L}_{(5,5,7)}$. This means that $q(\mathbf{r},\mathbf{s},\mathbf{t},v)=\mathbf{M}_{(4,4,6)}(p)$ is pluri-harmonic and homogeneous of weight $(4,15)$. Using q, we can define a differential operator

$$D_q=q\left(\tfrac{d}{d\tau_1},\tfrac{d}{d\tau_2},\tfrac{d}{d\tau_3}\right)\Big|_{\tau_1=\tau_2=\tau_3=\tau}$$

that sends a triple of classical Siegel modular forms of weights 4, 4 and 6 respectively to a vector-valued Siegel modular form of weight $(4,15)$. Using these methods, Ibukiyama [18] finds generators for the $M^0_*(\Gamma_2)$-module $\bigoplus_{k\equiv 1(2)}M_{(4,k)}(\Gamma_2)$.

3.6 A RC-operator on vector-valued Siegel modular forms

All RC-operators that were considered in the previous sections act on t-tuples of classical Siegel modular forms. This allows us to construct vector-valued Siegel modular forms from classical Siegel modular forms. However, we will show here that it is also possible to construct vector-valued modular forms from other *vector-valued* modular forms using a differential operator. In the following Proposition we will give an example of such a differential operator. We will need this operator in Chapter 4 to find modular forms of weight $(6,11)$ and weight $(6,13)$.

Proposition 3.6.1. *Let φ and F be modular forms in $M_\ell(\Gamma_2)$ and $M_{(m,k)}(\Gamma_2)$ respectively. Define $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)'$ and $\mathbb{D} = \nabla\nabla'$. The function*

$$\{F, \varphi\} := (k + \tfrac{1}{2}m - 1)\big(\tfrac{d}{d\tau}\varphi \times \mathbb{D}F\big)[v] - \ell\varphi \cdot \big(\tfrac{d}{d\tau} \times \mathbb{D}F\big)[v]$$

is then a modular form of weight $(m, k+\ell+1)$.

The bracket $\{\cdot,\cdot\}$ can be regarded as a generalisation of the usual Rankin-Cohen brackets.

Proof. Let $\rho = \det^{k+\ell+1}\mathrm{Sym}^m$. We have to prove that $\{F,\varphi\}|_\rho\gamma = \{F,\varphi\}$ for $\gamma \in \Gamma_2$. We only have to prove this on generators for Γ_2 and the Proposition clearly holds for $\gamma = \tau \mapsto \tau + s$ where s is any integer symmetric matrix. The case $\gamma = \tau \mapsto u'\tau u$ for a $u \in \mathsf{GL}_2(\mathbb{Z})$ is also trivial.
Let $\gamma = \tau \mapsto -\tau^{-1}$. We have the following identities

$$\begin{aligned}
\mathbb{D}F(-\tau^{-1}, v) &= \mathbb{D}F(\tau, v\tau)\det(\tau)^k = \tau(\mathbb{D}F)(\tau, v\tau)\tau\det(\tau)^k \\
\left(\tfrac{d}{d\tau}\varphi\right)(-\tau^{-1}) &= \tau\left(\tfrac{d}{d\tau}\varphi\right)(\tau)\tau\det(\tau)^\ell + \ell\tau\varphi(\tau)\det(\tau)^k
\end{aligned}$$

and therefore

$$\begin{aligned}
\left(\tfrac{d}{d\tau}\varphi\right)(-\tau^{-1}) \times \mathbb{D}F(-\tau^{-1}, v) &= \det(\tau)^{k+\ell+1}\tau\left(\tfrac{d}{d\tau}\varphi(\tau) \times (\mathbb{D}F)(\tau, v\tau)\right)\tau \\
&+ \ell\det(\tau)^{k+\ell}\varphi(\tau)\tau \times (\tau(\mathbb{D}F)(\tau, v\tau)\tau).
\end{aligned}$$

Now consider

$$\left(\tfrac{d}{d\tau} \times \mathbb{D}F\right)(-\tau^{-1}, v) = -\mathbb{D} \times \left(\tfrac{d}{d\tau}F\right)(-\tau^{-1}, v).$$

Since $\frac{d}{d\tau}F(-\tau^{-1}, v) = \tau^{-1}\left(\frac{d}{d\tau}F\right)(-\tau^{-1}, v)\tau^{-1}$ and

$$\begin{aligned}
\tfrac{d}{d\tau}F(-\tau^{-1}, v) &= \tfrac{d}{d\tau}\left(F(\tau, v\tau)\det(\tau)^k\right) \\
&= \det(\tau)^k\left(\tfrac{d}{d\tau}F\right)(\tau, v\tau) + k\det(\tau)^k\tau^{-1}F(\tau, v\tau) + \det(\tau)^k\frac{dv\tau}{d\tau} \cdot (\nabla F)(\tau, v\tau),
\end{aligned}$$

the following identity holds:

$$\begin{aligned}
\left(\tfrac{d}{d\tau} \times \mathbb{D}F\right)(-\tau^{-1}, v) &= -\det(\tau)^k\mathbb{D} \times \left(\tau\tfrac{dF}{d\tau}(\tau, v\tau)\tau\right) \\
&\quad -k\det(\tau)^k\mathbb{D} \times \tau F(\tau, v\tau) \\
&\quad -\det(\tau)^k\mathbb{D} \times \left(\tau\frac{dv\tau}{d\tau} \cdot (\nabla F)(\tau, v\tau)\tau\right).
\end{aligned} \tag{3.11}$$

Note again that $\mathbb{D}F(\tau, v\tau) = \tau(\mathbb{D}F)(\tau, v\tau)\tau$ and write $\mathbb{D}_{v\tau}f(v\tau) := (\mathbb{D}f)(v\tau)$ for any function f. We then have $\mathbb{D} \times \left(\tau\frac{dF}{d\tau}(\tau, v\tau)\tau\right) = (\tau\mathbb{D}_{v\tau}\tau) \times \left(\tau\frac{dF}{d\tau}(\tau, v\tau)\tau\right) = \det(\tau)\tau\left(\mathbb{D}_{v\tau} \times \frac{dF}{d\tau}(\tau, v\tau)\right)\tau$ and hence

$$-\det(\tau)^k\mathbb{D} \times \left(\tau\tfrac{dF}{d\tau}(\tau, v\tau)\tau\right) = \det(\tau)^{k+1}\tau\left(\tfrac{d}{d\tau} \times \mathbb{D}F\right)(\tau, v\tau)\tau.$$

Similarly, we have

$$-k\det(\tau)^k\mathbb{D} \times \tau F(\tau, v\tau) = k\det(\tau)^k\tau \times (\tau(\mathbb{D}F)(\tau, v\tau)\tau).$$

We will now first assume the following identity that applies to the term in (3.11):

$$\left\{-\mathbb{D} \times \left(\tau\frac{dv\tau}{d\tau} \cdot (\nabla F)(\tau, v\tau)\tau\right)\right\}[v] = \tfrac{1}{2}(m-2)(\tau \times (\tau(\mathbb{D}F)(\tau, v\tau)\tau))[v], \tag{3.12}$$

where the v in square brackets in the left hand side of the equation is not in the range of $\mathbb{D}$. With this identity, we have

$$\begin{aligned}
\{F,\varphi\}(-\tau^{-1},v) &= (k+\tfrac{1}{2}m-1)\det(\tau)^{k+\ell+1}\left(\tfrac{d}{d\tau}\varphi(\tau)\times(\mathbb{D}F)(\tau,v\tau)\right)[v\tau] \\
&+ (k+\tfrac{1}{2}m-1)\ell\det(\tau)^{k+\ell}\varphi(\tau)\cdot\left(\tau\times(\tau(\mathbb{D}F)(\tau,v\tau)\tau)\right)[v] \\
&- \ell\det(\tau)^{k+\ell+1}\varphi(\tau)\cdot\left(\tfrac{d}{d\tau}\times\mathbb{D}F\right)(\tau,v\tau)[v\tau] \\
&- \ell k\det(\tau)^{k+\ell}\varphi(\tau)\cdot\left(\tau\times(\tau(\mathbb{D}F)(\tau,v\tau)\tau)\right)[v] \\
&- \tfrac{1}{2}(m-2)\ell\det(\tau)^{k+\ell}\varphi(\tau)\cdot\left(\tau\times(\tau(\mathbb{D}F)(\tau,v\tau)\tau)\right)[v] \\
&= \{F,\varphi\}(\tau,v\tau)\det(\tau)^{k+\ell+1},
\end{aligned}$$

which proves the Proposition modulo identity (3.12).

Now we will show that equation (3.12) is valid. Using the relation $\mathbb{D}f(v\tau)=\tau\mathbb{D}_{v\tau}\tau f(v\tau)$, and after interchanging $\mathbb{D}_{v\tau}$ and $\frac{dv\tau}{d\tau}$, we get

$$-\mathbb{D}\times\left(\tau\frac{dv\tau}{d\tau}\cdot(\nabla F)(\tau,v\tau)\tau\right)=\tau\det(\tau)\left(\frac{dv\tau}{d\tau}\times(\nabla\mathbb{D}F)(\tau,v\tau)\right)\tau.$$

The two components of $\frac{dv\tau}{d\tau}$ are $\frac{1}{2}\begin{pmatrix}2x & y\\ y & 0\end{pmatrix}$ and $\frac{1}{2}\begin{pmatrix}0 & x\\ x & 2y\end{pmatrix}$ respectively and if we write $(\alpha,\beta)=(x,y)\tau$, then

$$\det(\tau)\begin{pmatrix}2x & y\\ y & 0\end{pmatrix}=\begin{pmatrix}2\tau_2\alpha-2z\beta & -z\alpha+\tau_1\beta\\ -z\alpha+\tau_1\beta & 0\end{pmatrix}=\begin{pmatrix}\tau_2 & -z\\ -z & \tau_1\end{pmatrix}\alpha+\begin{pmatrix}\tau_2\alpha-2z\beta & \tau_1\beta\\ \tau_1\beta & -\tau_1\alpha\end{pmatrix}$$

and

$$\det(\tau)\begin{pmatrix}0 & x\\ x & 2y\end{pmatrix}=\begin{pmatrix}0 & \tau_2\alpha-z\beta\\ \tau_2\alpha-z\beta & -2z\alpha+2\tau_1\beta\end{pmatrix}=\begin{pmatrix}\tau_2 & -z\\ -z & \tau_1\end{pmatrix}\beta+\begin{pmatrix}-\tau_2\beta & \tau_2\alpha\\ \tau_2\alpha & -2z\alpha+\tau_1\beta\end{pmatrix}.$$

The coefficients of the matrix $\mathbb{D}F$ are homogeneous polynomials in α and β of degree $m-2$ and therefore $\alpha\frac{\partial}{\partial\alpha}F+\beta\frac{\partial}{\partial\beta}F=(m-2)F$. Hence

$$\det(\tau)\frac{dv\tau}{d\tau}\times(\nabla\mathbb{D}F)(\tau,v\tau)=\tfrac{1}{2}(m-2)\det(\tau)\left(\tau^{-1}\times(\mathbb{D}F)(\tau,(\alpha,\beta))\right)\quad+\quad U$$

where $U[v]=0$. The right hand side of (3.12) contains the expression $\tau\times(\tau(\mathbb{D}F)(\tau,v\tau)\tau)$ which equals $\det(\tau)\tau\left(\tau^{-1}\times(\mathbb{D}F)(\tau,v\tau)\right)\tau$. Therefore equation (3.12) holds. □

Suppose that F and φ are modular forms of weight (m,k) and ℓ respectively. If F was already constructed using an RC-operator of weight $(m,0)$ and type (k_1,k_2) with $k_1+k_2=k$, then $\{F,\varphi\}$ coincides with a modular form constructed using the differential operator $\mathbf{M}$. We will make this clear in the following example.

Example 3.6.2. Let $p(s,t)=7\cdot 6s^2-2\cdot 7\cdot 5st+5\cdot 4t^2$ and $q(s,t)=7s-5t$ and define $Q(\mathbf{r},\mathbf{s},\mathbf{t},v)=\mathbf{M}_{5,5,7}(q)$, then Q is a pluri-harmonic polynomial of weight $(4,1)$ and type $(4,4,6)$ and $P(\mathbf{s},\mathbf{t},v):=p(\mathbf{s}[v],\mathbf{t}[v])$ is a pluri-harmonic polynomial of weight $(4,0)$ and type $(*,4,6)$. Define $F_{10}(\tau)=D_P(\varphi_4,\varphi_6)$ and $G_{15}=D_Q(\varphi_4,\varphi_4,\varphi_6)$, then $F_{10}\in M_{(4,10)}(\Gamma_2)$ and $G_{15}\in M_{(4,15)}(\Gamma_2)$. According to Proposition 3.6.1, the function $\{F_{10},\varphi_4\}$ is a modular form of weight $(4,15)$. Not surprisingly[1], the form $\{F_{10},\varphi_4\}$ equals G_{15} up to a multiplicative scalar. We can verify this by looking at the defining polynomials P and Q:

$$\begin{aligned}
P(\mathbf{s},\mathbf{t},v) &= 7\cdot 6\mathbf{s}[v]^2-2\cdot 7\cdot 5\mathbf{s}[v]\mathbf{t}[v]+5\cdot 4\mathbf{t}[v]^2 \\
Q(\mathbf{r},\mathbf{s},\mathbf{t},v) &= \mathbf{r}\times\mathbf{s}[v](6(7\mathbf{s}[v]-5\mathbf{t}[v])-5\mathbf{t}[v])-\mathbf{r}\times\mathbf{t}[v](4(7\mathbf{s}[v]-5\mathbf{t}[v])+7\mathbf{s}[v]) \\
&+ \mathbf{s}\times\mathbf{t}[v](4(7\mathbf{s}[v]-5\mathbf{t}[v])).
\end{aligned}$$

[1] $\dim M_{(4,15)}(\Gamma_2)=1$

Since $\mathbb{D}\mathbf{r}[v] = 2\mathbf{r}$, we get

$$\tfrac{1}{2}\mathbb{D}P(\mathbf{s},\mathbf{t},v) = 7\cdot 6\cdot 2\mathbf{s}[v]\mathbf{s} - 2\cdot 7\cdot 5\mathbf{s}[v]\mathbf{t} - 2\cdot 7\cdot 5\mathbf{t}[v]\mathbf{s} + 5\cdot 4\cdot 2\mathbf{t}[v]\mathbf{t}.$$

The operator $\frac{d}{d\tau}\times$ acts on both φ_4 and φ_6 and therefore we should multiply $\mathbb{D}P$ with $(\mathbf{s}+\mathbf{t})$:

$$\begin{aligned}
\tfrac{1}{4}(\mathbf{s}+\mathbf{t})\times\mathbb{D}P &= -7\cdot 5\mathbf{s}[v]\mathbf{s}\times\mathbf{t} + 5\cdot 4\mathbf{t}[v]\mathbf{s}\times\mathbf{t} + 7\cdot 6\mathbf{s}[v]\mathbf{t}\times\mathbf{s} - 7\cdot 5\mathbf{t}[v]\mathbf{t}\times\mathbf{s}\\
&= 11\cdot\mathbf{s}\times\mathbf{t}(5\mathbf{t}[v] - 7\mathbf{s}[v]).
\end{aligned}$$

The factor $\frac{d}{d\tau}\varphi_4$ in $\{F_{10},\varphi_4\}$ is represented in the polynomial Q as $\mathbf{r}$, and if we multiply $\mathbb{D}P$ with $\mathbf{r}$ we get

$$\begin{aligned}
\tfrac{1}{4}\mathbf{r}\times\mathbb{D}P &= 7\cdot 6\mathbf{s}[v]\mathbf{r}\times\mathbf{s} - 7\cdot 5\mathbf{s}[v]\mathbf{r}\times\mathbf{t} - 7\cdot 5\mathbf{t}[v]\mathbf{r}\times\mathbf{s} + 5\cdot 4\mathbf{t}[v]\mathbf{r}\times\mathbf{t}\\
&= \mathbf{r}\times\mathbf{s}(7\cdot 6\mathbf{s}[v] - 7\cdot 5\mathbf{t}[v]) + \mathbf{r}\times\mathbf{t}(5\cdot 4\mathbf{t}[v] - 7\cdot 5\mathbf{s}[t]).
\end{aligned}$$

Now it is clear that $44\cdot Q(\mathbf{r},\mathbf{s},\mathbf{t},v) = -4(\mathbf{s}+\mathbf{t})\times\mathbb{D}P + 11\mathbf{r}\times\mathbb{D}P$ and therefore $G_{15} = -4\cdot 11\cdot\{F_{10},\varphi_4\}$.

Chapter 4

Generators for modules of Siegel modular forms

This chapter consists of two parts. In the first part we will give a proof of the structure theorem for $M_{(2,*)}(\Gamma_2)$ due to Satoh and Ibukiyama and in the second part we will complete Ibukiyama's structure theorem for $M_{(6,*)}(\Gamma_2)$.

4.1 Generators for the module of Siegel modular forms of weight $(2,k)$

Satoh and Ibukiyama found generators for the $M^0_*(\Gamma_2)$-modules $M^0_{(2,*)}(\Gamma_2)$ and $M^1_{(2,*)}(\Gamma_2)$ respectively. They both had to show non-vanishing of some determinant to complete their proofs and they did this by computing Fourier coefficients and using the theory of Jacobi forms [29, 17]. Aoki and Ibukiyama later showed that the determinant used in Satoh's proof is non-vanishing for a good reason: it is a non-zero multiple of the functional determinant of a coordinate map $\mathscr{H}_2/\Gamma_2 \mapsto \mathbb{C}^3$ [2]. This fact can not only be applied to Satoh's proof, but also to Ibukiyama's proof. This leads to a proof of the structure theorem for $M_{(2,*)}(\Gamma_2)$ without computing Fourier coefficients or using Jacobi forms. This is interesting, because if we want to find generators for the ring $\bigoplus_{m,k} M_{(m,k)}(\Gamma_2)$, then we should do this without computing Fourier coefficients for the simple reason that this would require infinitely many computations[1].

We start by stating Satoh's and Ibukiyama's structure theorems. We will use the Rankin-Cohen differential operators defined in Definition 3.3.1. Theorem 1.4.1 implies that

$$M^0_*(\Gamma_2) = \mathbb{C}[\varphi_4, \varphi_6, \chi_{10}, \chi_{12}].$$

Theorem 4.1.1 (Satoh)**.** *The $M^0_*(\Gamma_2)$-module $\mathfrak{F}$ generated by the sections*

$$\frac{d}{d\tau}\varphi_4, \quad \frac{d}{d\tau}\varphi_6, \quad \frac{d}{d\tau}\chi_{10} \quad \text{and} \quad \frac{d}{d\tau}\chi_{12}$$

is free and $M^1_{(2,*)}(\Gamma_2)$ is the submodule of $\mathfrak{F}$ generated by

$$G_{10} := [\varphi_4, \varphi_6], \quad G_{14} := [\varphi_4, \chi_{10}], \quad G_{16} := [\varphi_4, \chi_{12}],$$
$$G'_{16} := [\varphi_6, \chi_{10}], \quad G_{18} := [\varphi_6, \chi_{12}] \quad \text{and} \quad G_{22} := [\chi_{10}, \chi_{12}].$$

[1]The ring $\bigoplus_{m,k} M_{(m,k)}(\Gamma_2)$ is not finitely generated ([14] p. 234)

Theorem 4.1.2 (Ibukiyama)**.** *The image of $\Lambda^2\mathfrak{F}$ under the map induced by $\times$ is free. This image contains $M^1_{(2,*)}(\Gamma_2)$. The latter is generated by*

$$G_{21} := [\varphi_4, \varphi_6, \chi_{10}], \quad G_{23} := [\varphi_4, \varphi_6, \chi_{12}], \quad G_{27} := [\varphi_4, \chi_{10}, \chi_{12}] \quad \text{and} \quad G_{29} := [\varphi_6, \chi_{10}, \chi_{12}].$$

Remark 4.2. The modular forms $G_{10}, G_{14}, G_{16}, G'_{16}, G_{18}$ and G_{22} satisfy the relations

$$\begin{aligned} R_{20}: &\quad 10\chi_{10}G_{10} - 6\varphi_6 G_{14} + 4\varphi_4 G'_{16} = 0, & R_{22}: &\quad 12\chi_{12}G_{10} - 6\varphi_6 G_{16} + 4\varphi_4 G_{18} = 0,\\ R_{26}: &\quad 12\chi_{12}G_{14} - 10\chi_{10}G_{16} + 4\varphi_4 G_{22} = 0, & R_{28}: &\quad 12\chi_{12}G'_{16} - 10\chi_{10}G_{18} + 6\varphi_6 G_{22} = 0. \end{aligned}$$

The relations R_{20}, R_{22}, R_{26} and R_{28} themselves satisfy a relation:

$$R_{32}: \quad 12\chi_{12}R_{20} - 10\chi_{10}R_{22} + 6\varphi_6 R_{26} - 4\varphi_4 R_{28} = 0$$

and this relation is also satisfied by G_{21}, G_{23}, G_{27} and G_{29}:

$$R_{33}: \quad 12\chi_{12}G_{21} - 10\chi_{10}G_{23} + 6\varphi_6 G_{27} - 4\varphi_4 G_{29} = 0.$$

We have already seen a relation between $[\cdot,\cdot]$ and $[\cdot,\cdot,\cdot]$ in Remark 3.4. A similar identity is given by

$$4\pi i [f_1, f_2, f_3] = [f_1, f_2] \times \tfrac{d}{d\tau} f_3 - [f_1, f_3] \times \tfrac{d}{d\tau} f_2 + [f_2, f_3] \times \tfrac{d}{d\tau} f_1. \tag{4.1}$$

Note that for instance $[\varphi_4, \varphi_6, \varphi_{10}] = [\varphi_4, \varphi_6] \times \frac{d}{d\tau}\chi_{10} - [\varphi_4, \chi_{10}] \times \frac{d}{d\tau}\varphi_6 + [\varphi_6, \chi_{10}] \times \frac{d}{d\tau}\varphi_4$ and that according to R_{20} we have $[\varphi_4, \varphi_6]10\chi_{10} - [\varphi_4, \chi_{10}]6\varphi_6 + [\varphi_6, \chi_{10}]4\varphi_4 = 0$. This will be useful for the proof of Theorem 4.1.2.

4.2.1 The proofs of Theorems 4.1.1 and 4.1.2

Due to Tsushima [32] and Arakawa (Proposition 1.6.1) we know the dimensions of $M_{(2,k)}(\Gamma_2)$ for all k:

$$\sum_{k\equiv 0(2)} \dim M_{(k,2)}(\Gamma_2) t^k = \frac{t^{10} + t^{14} + 2t^{16} + t^{18} - t^{20} - t^{26} - t^{28} + t^{32}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} \tag{4.2}$$

$$\sum_{k\equiv 1(2)} \dim M_{(k,2)}(\Gamma_2) t^k = \frac{t^{21} + t^{23} + t^{27} + t^{29} - t^{33}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}. \tag{4.3}$$

If we now consider the relations in Remark 4.2, we see that in order to prove Theorems 4.1.1 and 4.1.2 we only have to prove the statements about $\mathfrak{F}$ and $\Lambda^2\mathfrak{F}$, i.e. we have to prove the following lemma:

Lemma 4.2.1. *(o) The sections*

$$\tfrac{d}{d\tau}\varphi_4, \quad \tfrac{d}{d\tau}\varphi_6, \quad \tfrac{d}{d\tau}\chi_{10} \quad \text{and} \quad \tfrac{d}{d\tau}\chi_{12}$$

are independent over the ring $M^0_*(\Gamma_2)$.

(i) The products

$$\tfrac{d}{d\tau}\varphi_4 \times \tfrac{d}{d\tau}\varphi_6, \quad \tfrac{d}{d\tau}\varphi_4 \times \tfrac{d}{d\tau}\chi_{10}, \quad \tfrac{d}{d\tau}\varphi_4 \times \tfrac{d}{d\tau}\chi_{12}, \quad \tfrac{d}{d\tau}\varphi_6 \times \tfrac{d}{d\tau}\chi_{10}, \quad \tfrac{d}{d\tau}\varphi_6 \times \tfrac{d}{d\tau}\chi_{12} \text{ and } \tfrac{d}{d\tau}\chi_{10} \times \tfrac{d}{d\tau}\chi_{12}$$

are also independent over $M^0_*(\Gamma_2)$.

The following proof is largely due to Satoh [29] and Aoki and Ibukiyama [2].

Proof. Let $g_1, \ldots, g_4$ be algebraically independent modular forms of weight $k_1, \ldots, k_4$ and define $\psi_\mu := g_\mu^{k_1} g_1^{-k_\mu}$. The functions ψ_2, ψ_3, ψ_4 are local parameters of the variety $\Gamma_2 \backslash \mathscr{H}_2$. This implies that the Jacobian

$$\frac{\partial(\psi_2, \psi_3, \psi_4)}{\partial(\tau_1, z, \tau_2)} \neq 0. \tag{4.4}$$

We will now give Satoh's argument to prove *(o)*. Let $g \in M_{k,2}(\Gamma_2)$ and pick a $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_2$, then

$$\begin{aligned}
\left(\tfrac{d}{d\tau} g\right)(\gamma\tau) &= j(\gamma,\tau) \tfrac{d}{d\tau} g(\gamma\tau) j(\gamma,\tau)' \qquad (4.5)\\
&= j(\gamma,\tau) \left(g(\tau) \tfrac{d}{d\tau} \det(j(\gamma,\tau))^k + \det(j(\gamma,\tau))^k \tfrac{d}{d\tau} g(\tau) \right) j(\gamma,\tau)' \\
&= j(\gamma,\tau) \left(g(\tau) k \det(j(\gamma,\tau))^k j(\gamma,\tau)^{-1} c + \det(j(\gamma,\tau))^k \tfrac{d}{d\tau} g(\tau) \right) j(\gamma,\tau)' \\
&= \mathrm{Sym}^2 \otimes \det^k(j(\gamma,\tau)) \cdot \left(k g(\tau) j(\gamma,\tau)^{-1} c + \tfrac{d}{d\tau} g(\tau) \right).
\end{aligned}$$

So if we take for instance $a, d = 0$ and $b = -c = 1$, then $(\frac{d}{d\tau} g)(-\tau^{-1}) = \det(\tau)^k \tau (k g(\tau) \tau^{-1} + \frac{d}{d\tau} g(\tau))\tau$. Let $k \in \mathbb{Z}$ and choose $f_\ell \in M_{k-\ell}(\Gamma_2)$. Suppose that $f = f_4 \frac{d}{d\tau}\varphi_4 + f_6 \frac{d}{d\tau}\varphi_6 + f_{10}\frac{d}{d\tau}\chi_{10} + f_{12}\frac{d}{d\tau}\chi_{12} = 0$, then $f(-\tau^{-1}) = 0$ and hence

$$\left(4\varphi_4(\tau) f_4(\tau) + 6\varphi_6(\tau) f_6(\tau) + 10\chi_{10}(\tau) f_{10}(\tau) + 12\chi_{12}(\tau) f_{12}(\tau)\right) \tau^{-1} + f(\tau) = 0,$$

which implies that $4\varphi_4(\tau) f_4(\tau) + 6\varphi_6(\tau) f_6(\tau) + 10\chi_{10}(\tau) f_{10}(\tau) + 12\chi_{12}(\tau) f_{12}(\tau) = 0$.

Let

$$X = \begin{pmatrix} 4\varphi_4 & 6\varphi_6 & 10\chi_{10} & 12\chi_{12} \\ \frac{\partial}{\partial\tau_{11}}\varphi_4 & \frac{\partial}{\partial\tau_{11}}\varphi_6 & \frac{\partial}{\partial\tau_{11}}\chi_{10} & \frac{\partial}{\partial\tau_{11}}\chi_{12} \\ \frac{\partial}{\partial\tau_{22}}\varphi_4 & \frac{\partial}{\partial\tau_{22}}\varphi_6 & \frac{\partial}{\partial\tau_{22}}\chi_{10} & \frac{\partial}{\partial\tau_{22}}\chi_{12} \\ \frac{\partial}{\partial\tau_{12}}\varphi_4 & \frac{\partial}{\partial\tau_{12}}\varphi_6 & \frac{\partial}{\partial\tau_{12}}\chi_{10} & \frac{\partial}{\partial\tau_{12}}\chi_{12} \end{pmatrix}$$

then $X(f_4, f_6, f_{10}, f_{12})' = 0$. Satoh proved non-vanishing of $\det(X)$ by computing a non-zero Fourier coefficient. Aoki and Ibukiyama used (4.4) with $g_1 = \varphi_4$, $g_2 = \varphi_6$, $g_3 = \chi_{10}$ and $g_4 = \chi_{12}$ to prove that $\det(X)$ actually equals a non-zero scalar multiple of χ_{35}. This proves *(o)*.

Now we will proceed with *(i)*. As we have mentioned before, Ibukiyama used Jacobi forms to prove linear independence of his generators. We will basically repeat the argument used in *(o)* above. Define

$$Z := \left(\tfrac{d}{d\tau}\psi_2 \times \tfrac{d}{d\tau}\psi_3, \tfrac{d}{d\tau}\psi_2 \times \tfrac{d}{d\tau}\psi_4, \tfrac{d}{d\tau}\psi_3 \times \tfrac{d}{d\tau}\psi_4 \right),$$

then $\det(Z) = 4\left(\frac{\partial(\psi_2,\psi_3,\psi_4)}{\partial(\tau_1,z,\tau_2)}\right)^2 \neq 0$.

We can express the products $\frac{d}{d\tau}\psi_\mu \times \frac{d}{d\tau}\psi_\nu$ in terms of the functions $g_1, \ldots, g_4$ as follows:

$$\begin{aligned}
\tfrac{d}{d\tau}\psi_\mu \times \tfrac{d}{d\tau}\psi_\nu &= \frac{k_1 (g_\mu g_\nu)^{k_1-1}}{g_1^{k_\mu+k_\nu+1}} \left(k_1 g_1 \tfrac{d}{d\tau} g_\mu \times \tfrac{d}{d\tau} g_\nu - k_\mu g_\mu \tfrac{d}{d\tau} g_1 \times \tfrac{d}{d\tau} g_\nu + k_\nu g_\nu \tfrac{d}{d\tau} g_1 \times \tfrac{d}{d\tau} g_\mu \right) \\
&= (2\pi i)^2 \frac{k_1 (g_\mu g_\nu)^{k_1-1}}{g_1^{k_\mu+k_\nu+1}} [g_1, g_\mu, g_\nu].
\end{aligned}$$

This shows that $\det([g_1, g_\mu, g_\nu])_{1<\mu<\nu} \neq 0$.

Now choose again $g_1 = \varphi_4$, $g_2 = \varphi_6$, $g_3 = \chi_{10}$, $g_4 = \chi_{12}$ and define

$$Y_1 := \left(G_{10}, G_{14}, G_{16}, G'_{16}, G_{18}, G_{22}\right)$$

and

$$Y_2 := \Big(\tfrac{d}{d\tau}\varphi_4 \times \tfrac{d}{d\tau}\varphi_6, \tfrac{d}{d\tau}\varphi_4 \times \tfrac{d}{d\tau}\chi_{10}, \tfrac{d}{d\tau}\varphi_4 \times \tfrac{d}{d\tau}\chi_{12}, \\ \tfrac{d}{d\tau}\varphi_6 \times \tfrac{d}{d\tau}\chi_{10}, \tfrac{d}{d\tau}\varphi_6 \times \tfrac{d}{d\tau}\chi_{12}, \tfrac{d}{d\tau}\chi_{10} \times \tfrac{d}{d\tau}\chi_{12} \Big)$$

and let $Y = (Y_1', Y_2')'$. Using the relations R_i and (4.1) in Remark 4.2, we can show that $\det(Y)$ is a non-zero multiple of

$$\det(G_{10}, G_{14}, G_{16}) \cdot \det(G_{21}, G_{23}, G_{27})$$

so that $\det(Y) \neq 0$. Now let $f_{\mu,\nu}$ be modular forms in $M_{k-k_\mu-k_\nu}(\Gamma_2)$ and suppose that

$$\sum_{\mu,\nu} f_{\mu,\nu} \tfrac{d}{d\tau} g_\mu \times \tfrac{d}{d\tau} g_\nu = 0.$$

With similar calculations as in (4.5) we see that the transformation $\tau \mapsto -\tau^{-1}$ causes

$$\det(\tau)^{k+1}\mathrm{Sym}^2(\tau) \sum_{\mu,\nu} f_{\mu,\nu}(\tau) \left(\tfrac{d}{d\tau} g_\mu \times \tfrac{d}{d\tau} g_\nu + \tau^{-1} \times \left(k_\nu g_\nu \tfrac{d}{d\tau} g_\mu - k_\mu g_\mu \tfrac{d}{d\tau} g_\nu \right) \right) = 0$$

and hence

$$\sum_{\nu,\mu} f_{\nu,\mu} [g_\mu, g_\nu] = 0.$$

Let $f = (f_{\mu,\nu})_{1 \le \mu < \nu \le 4}$, then $Yf = 0$, which shows that $f = 0$, because $\det(Y) \neq 0$. $\square$

4.3 Generators for $\bigoplus_{k\equiv 1(2)} M_{(6,k)}(\Gamma_2)$

Ibukiyama and Satoh have given generators for the $M_*^0(\Gamma_2)$-modules $\bigoplus_{k\equiv 0(2)} M_{(m,k)}(\Gamma_2)$ for $m = 2,4,6$ and $\bigoplus_{k\equiv 1(2)} M_{(m,k)}(\Gamma_2)$ for $k = 2,4$. Here we will give generators for the module

$$M^1_{(6,*)}(\Gamma_2) := \bigoplus_{k\equiv 1(2)} M_{(6,k)}(\Gamma_2).$$

Using the differential operators studied in Section 3.5, we can construct vector-valued Siegel modular forms of weight $(6,k)$ with k odd. However, the smallest k for which such a modular form is possibly non-zero equals 15, whereas $\dim M_{(6,11)}(\Gamma_2) = \dim M_{(6,13)}(\Gamma_2) = 1$. It is therefore impossible to find generators for $\bigoplus_{k\equiv 1(2)} M_{(6,k)}(\Gamma_2)$ using RC-operators on classical Siegel modular forms alone[2]. A similar problem exists for the module $M^0_{(6,*)}(\Gamma_2)$ and Ibukiyama solved this by using vector-valued theta series and Klingen-Eisenstein series. We will use these modular forms and the bracket $\{\cdot,\cdot\}$ to construct non-zero modular forms of weight $(6,11)$ and $(6,13)$. We also find non-zero modular forms of 'higher' weight using RC-operators on triples of classical Siegel modular forms. We then show that our forms generate $M^1_{(6,*)}(\Gamma_2)$.

[2]Unless we would be able to see that such a modular form is divisible by a classical Siegel modular form. For instance, if a cusp form vanishes on the locus $\{z = 0\}$, then it will be divisible by χ_{10}.

4.3.1 Hecke operators on vector-valued Siegel modular forms

Let ρ denote the representation $\mathrm{Sym}^m \otimes \det^k$ and let $\mathsf{M}^2_\mu = \{\gamma \in \mathrm{Mat}_4(\mathbb{Z}) \mid \gamma J \gamma' = \mu J\}$ for $\mu \in \mathbb{N}$. The Hecke operator $T(\mu)$ on $M_{(m,k)}(\Gamma_2)$ is defined by

$$T(\mu)f = \mu^{2k+m-3} \sum_{\gamma \in \Gamma_2 \backslash \mathsf{M}^2_\mu} f|_\rho \gamma, \qquad f \in M_{(m,k)}(\Gamma_2).$$

This operator $T(\mu)$ is well defined on $M_{(m,k)}(\Gamma_2)$ and its image is again contained in $M_{(m,k)}(\Gamma_2)$. Suppose that $f \in M_{(m,k)}(\Gamma_2)$ and write

$$f(\tau) = \sum_{n \succeq 0} a(n) q^n$$

for the Fourier series of f and

$$T(\mu)f(\tau) = \sum_{n \succeq 0} a(\mu, n) q^n$$

for the Fourier series of $T(\mu)f$. We are interested in the relation between $a(n)$ and $a(\mu, n)$. For some choices of n, this relation is relatively easy. This result is due to Arakawa [3] and we formulate it in Theorem 4.3.2 below. First we need a definition.

Let $n = \begin{pmatrix} n_1 & r/2 \\ r/2 & n_2 \end{pmatrix}$ be a half-integer symmetric matrix with $(n_1, n_2, r) = 1$ and write $D = r^2 - 4n_1 n_2$. Let $K = \mathbb{Q}(\sqrt{D})$. Suppose that K is an imaginary quadratic field with class number 1 and discriminant D. This means that $D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}$ by the Stark-Heegner Theorem [31].

Definition 4.3.1. Let $\omega = \frac{r - \sqrt{D}}{2n_1}$, then $\{1, \omega\}$ is a basis for K. We define the matrix $L(\alpha)$ via $L(\alpha)(1, \omega)' = (\alpha, \alpha\omega)'$ and write $\mathrm{Sym}^m(\alpha) := \mathrm{Sym}^m(L(\alpha))$.

Theorem 4.3.2 (Arakawa)**.** *Let $f \in M_{(m,k)}(\Gamma_2)$ and let n be as above, i.e. $(n_1, n_2, r) = 1$ and $D = r^2 - 4n_1 n_2$ is the discriminant of the imaginary quadratic number field $K = \mathbb{Q}(\sqrt{D})$ that has class number* 1*. Let p be a prime number, ℓ a positive integer such that $(p, \ell) = 1$ and b any positive integer. Consider the ideal $(p) \subset \mathcal{O}_K$. We have 3 possibilities.*

1. *The ideal (p) splits in $\mathcal{O}_K$, that is $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p} \neq \overline{\mathfrak{p}}$. Choose α_1 and α_2 in $\mathcal{O}_K$ such that $(\alpha_1) = \mathfrak{p}$ and $(\alpha_2) = \overline{\mathfrak{p}}$.*[3] *In this case we have*

$$a(p^b, \ell n) = a(p^b \ell n) + \sum_{a=1}^{b} p^{a(k+m-2)} \left(\mathrm{Sym}^m(\alpha_1)^{-a} a(p^{b-a} \ell n) + \mathrm{Sym}^m(\alpha_2)^{-a} a(p^{b-a} \ell n) \right).$$

2. *The ideal (p) ramifies in $\mathcal{O}_K$, that is $(p) = \mathfrak{p}^2$. Choose an $\alpha \in \mathcal{O}_K$ such that $(\alpha) = \mathfrak{p}$. We then have*

$$a(p^b, \ell n) = a(p^b \ell n) + p^{k+m-2} \mathrm{Sym}^m(\alpha)^{-1} a(p^{b-1} \ell n).$$

3. *The ideal (p) in $\mathcal{O}_K$ is a prime ideal. In this case the following holds:*

$$a(p^b, \ell n) = a(p^b \ell n).$$

[3] this is possible, since $\mathcal{O}_K$ is a principal ideal domain.

Remark 4.4. Note that especially case 3 of Theorem 4.3.2 is useful. If, for instance, $f = \sum a(n)q^n \in M_{(m,k)}(\Gamma_2)$ is an eigenform of $T(2)$, then we can find the corresponding eigenvalue of $T(2)$ by comparing $a\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ with $a\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, since (2) is a prime ideal in $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. Similarly, we can find the eigenvalue of $T(3)$ by comparing $a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with $a\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ since (3) is prime in $\mathbb{Z}[\sqrt{-1}]$.
The Fourier coefficients of $T(p)f$ can also be calculated directly from the definition of the Hecke operator. In addition to Arakawa's paper we also refer to Andrianov [1] who treats the classical case.

Example 4.4.1. We know that the modular form $2\pi i F := 6\varphi_6 \frac{d}{d\tau}\varphi_4 - 4\varphi_4 \frac{d}{d\tau}\varphi_6 \in M_{2,10}(\Gamma_2)$ is an eigenform for all Hecke operators (F lives in a 1-dimensional space). Write $F = \sum a(n)q^n$, $\varphi_4 = \sum a_4(n)q^n$ and $\varphi_6 = \sum a_6(n)q^n$. We have

$$a(n) = \sum_{n_1+n_2=n} (6n_1 - 4n_2)a_4(n_1)a_6(n_2),$$

so let us calculate the Fourier coefficient of F at $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$:

$$\begin{aligned} a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} 2^5 3^3 5 \cdot 7 - \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} 2^7 3^3 5 \cdot 7 \\ &- \begin{pmatrix} -4 & 0 \\ 0 & 6 \end{pmatrix} 2^7 3^3 5 \cdot 7 + \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} 2^4 3^3 \cdot 5 \cdot 7 \cdot 11 \\ &= -2^8 3^4 5 \cdot 7 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} a\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix} 2^5 3^3 5 \cdot 7 \cdot 41 - \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix} 2^9 3^6 5 \cdot 7^2 - \begin{pmatrix} 12 & 0 \\ 0 & -8 \end{pmatrix} 2^7 3^6 5 \cdot 7 \cdot 11 \\ &- \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix} 2^9 3^6 5 \cdot 7^2 - \begin{pmatrix} 2 & 10 \\ 10 & 2 \end{pmatrix} 2^7 3^3 5 \cdot 7 + \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} 2^{13} 3^3 5 \cdot 7^2 11 \\ &+ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} 2^9 3^6 5 \cdot 7 \cdot 11 + \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix} 2^{13} 3^3 5 \cdot 7^2 11 - \begin{pmatrix} 2 & -10 \\ -10 & 2 \end{pmatrix} 2^7 3^3 5 \cdot 7 \\ &+ \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix} 2^9 3^5 5 \cdot 7 \cdot 11 \cdot 19 - \begin{pmatrix} -8 & 0 \\ 0 & 12 \end{pmatrix} 2^7 3^6 5 \cdot 7 \cdot 11 + \begin{pmatrix} -8 & 0 \\ 0 & 2 \end{pmatrix} 2^9 3^5 5 \cdot 7 \cdot 11 \cdot 19 \\ &+ \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix} 2^4 3^3 5^2 7 \cdot 11 \cdot 109 \quad = \quad 2^{11} 3^4 5^2 7 \cdot 167 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Let $n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $D = -4$ and $K = \mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{-1})$ is a number field with class number 1 and discriminant D. The ideal (2) ramifies in $\mathcal{O}_K$ as $(2) = (1+\sqrt{-1})^2$. We have $L(1+\sqrt{-1}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Write $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By Theorem 4.3.2, we have

$$\begin{aligned} a(2, \mathbf{1}) &= a(2 \cdot \mathbf{1}) + 2^{10+2-2} \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} a(\mathbf{1}) \tfrac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= a(2 \cdot \mathbf{1}) + 2^9 a(\mathbf{1}) \quad = \quad 2^{11} 3^5 5 \cdot 7 \cdot 257 \cdot \mathbf{1} \end{aligned}$$

If we write $T(2)F = \lambda(2)F$, then we see that $\lambda(2) = -2^3 3 \cdot 257 = -24 \cdot (1+2^8)$.

The function F in the above example is a scalar multiple of the Klingen-Eisenstein series $E_{2,10}(\Delta)$ (cf. Example 1.6.3). We also have $\Phi F = c \cdot \Delta$ for some constant $c \neq 0$. With similar calculations as above, we can also calculate $T(3)F$ and then we find that $T(3)F = 252 \cdot (1+2^8)F$. Note that $T(2)\Delta = -24\Delta$ and $T(3)\Delta = 252\Delta$. This is not a coincidence:

Proposition 4.4.2. *Suppose that $f \in N_{(m,k)}(\Gamma_2)$ and that $\Phi f = \chi \in S_{k+m}(\Gamma_1)$, then f is an eigenform for all Hecke operators if and only if χ is an eigenform for all Hecke operators. If for all primes p we have $T(p)\chi = \lambda_0(p)\chi$ and $T(p)f = \lambda(p)f$, then*

$$\lambda(p) = \lambda_0(p)\cdot(1+p^{k-2}).$$

For a proof of this proposition, we refer to Arakawa's article [3].

4.4.1 Computing Fourier coefficients of $E_{6,6}$

Using a RC-operator and a Hecke operator, we will find two linearly independent functions in $M_{(6,10)}(\Gamma_2)$. Then we will determine a linear combination of these two functions that is divisible by φ_4 in $M_{(6,*)}(\Gamma_2)$. For convenience, we will write 'frequencies' $\begin{pmatrix} n_1 & r/2 \\ r/2 & n_2 \end{pmatrix}$ as triples (n_1, n_2, r) and Fourier coefficients as row vectors.

Define a polynomial P as follows. Let $p(r,s) = \frac{1}{840}(5s^3 - 30s^2r + 42sr^2 - 14r^3)$ and define $P(\mathbf{r}, \mathbf{s}, v) = p(\mathbf{r}[v], \mathbf{s}[v])$. The polynomial P is homogeneous of weight $(6,0)$ and $\mathbf{L}_{(4,6)}P = 0$. Define the modular form $F_{10} \in M_{(6,10)}(\Gamma_2)$ by $F_{10} = D_P(\varphi_4, \varphi_6)$. Write $F_{10} = \sum_n a(n)q^n$ and $T(2)F_{10} = \sum_n a(2,n)q^n$. The function F_{10} is not an eigenform, since

$$a(1,1,0) = (-1782, 0, 270, 0, 270, 0, -1782)$$

and

$$a(2,(1,1,0)) = (-88678800, 0, 40597200, 0, 40597200, 0, -88678800) \notin \mathbb{C}\cdot a(1,1,0).$$

We have $\dim M_{(6,10)}(\Gamma_2) = 2$ and $\dim S_{(6,10)}(\Gamma_2) = 1$. The space $M_{(6,10)}(\Gamma_2)$ is therefore spanned by F_{10} and $T(2)F_{10}$.[4] We can find the eigenforms—an Eisenstein series and a cusp form—in $M_{(6,10)}(\Gamma_2)$ as follows. Since

$$a(1,0,0) = (7,0,0,0,0,0,0) \quad \text{and} \quad a(2,(1,0,0)) = (388584, 0, 0, 0, 0, 0, 0),$$

the function $2560896 \cdot \Theta_{6,10} := 55512F_{10} - T(2)F_{10}$ is a cusp form. Write $\Theta_{6,10} = \sum_n c(n)q^n$ and $T(2)\Theta_{6,10} = \sum c(2,n)q^n$, then

$$c(1,1,1) = (2,6,5,0,5,6,2).$$

Since $-4\det\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} = -3$ and 2 is inert in $\mathbb{Q}(\sqrt{-3})$, we get

$$c(2,(1,1,1)) = c(2,2,2) = (3360, 10080, 8400, 0, 8400, 10080, 3360) = 1680\cdot c(1,1,1).$$

Therefore, the eigenvalue of $T(2)$ on $S_{6,10}$ equals 1680. We now have

$$T(2)\left\{\begin{array}{rcl} F_{10} & = & 55512F_{10} - 2506890\cdot\Theta_{6,10} \\ \Theta_{6,10} & = & 1680\cdot\Theta_{6,10} \end{array}\right. \implies C\cdot E_{6,10} = 2232F_{10} - 106704\cdot\Theta_{6,10}$$

for some constant C. We have $\Phi C\cdot E_{6,10} = 7\cdot 2243(q_1 + 216q_1^2 - 3384q_1^3 + \cdots) = 7\cdot 2243\Delta E_6 \in S_{18}(\Gamma_1)$, where $q_1 = e^{2\pi i\tau_1}$. Hence, we find that $C = 7\cdot 2243$.

[4] It can be expected that modular forms constructed via RC-operators and the forms $\varphi_4, \varphi_6, \chi_5, \chi_{10}$ and χ_{12} are rarely eigenforms for the Hecke operators. In the case of elliptic modular forms there are for instance only finitely many Rankin-Cohen brackets of eigenforms that are eigenforms themselves [22].

n	(1,0,0)	(1,1,0)	(1,1,1)	(2,1,0)	(2,1,1)	(2,2,0)	(2,2,1)	(2,2,2)
	1	$-2\cdot 3\cdot 5\cdot 11$	$-2^3 11$	$2^5 3^2 19$	$2^{10}3$	$2^4 3\cdot 5^2 109$	$2^{10}3\cdot 31$	$2^6 3\cdot 11\cdot 17$
	0	0	$-2^3 3\cdot 11$	0	$2^9 3^2$	0	$2^9 3^2 31$	$2^6 3^2 11\cdot 17$
	0	$2\cdot 3^2 5^2$	$-2^4 3\cdot 5$	$-2^4 3^5 5$	$2^8 3^2 5$	$-2^4 3^3 5^2$	$-2^8 3^2 5\cdot 11$	$2^7 3^2 5\cdot 17$
	0	0	$-2^3 5$	0	$2^7 3\cdot 5^2$	0	$-2^7 3\cdot 5\cdot 97$	$2^6 3\cdot 5\cdot 17$
	0	$2\cdot 3^2 5^2$	$-2^4 3\cdot 5$	$2^3 3^3 5\cdot 19$	$-2^7 3^2 5$	$-2^4 3^3 5^2$	$-2^8 3^2 5\cdot 11$	$2^7 3^2 5\cdot 17$
	0	0	$-2^3 3\cdot 11$	0	$-2^7 3^2 11$	0	$2^9 3^2 31$	$2^6 3^2 11\cdot 17$
	0	$-2\cdot 3\cdot 5\cdot 11$	$-2^3 11$	$-2^2 3^2 11\cdot 19$	$-2^7 3\cdot 11$	$2^4 3\cdot 5^2 109$	$2^{10}3\cdot 31$	$2^6 3\cdot 11\cdot 17$

Table 4.1: Fourier coefficients of $E_{6,6} \in M_{(6,6)}(\Gamma_2)$. The Fourier coefficients $a(n) = a_0x^6 + a_1x^5y + \cdots + a_6y^6$ are represented as column vectors $(a_0,\ldots,a_6)'$. The frequencies $\begin{pmatrix} n_1 & r/2 \\ r/2 & n_2 \end{pmatrix}$ are written as (n_1, n_2, r).

Lemma 4.4.3. *Define* $\Theta_{6,10}$ *as above, then* $E_{6,6}\varphi_4 = E_{6,10} - \frac{77043}{2243}\Theta_{6,10} \in M_{(6,10)}(\Gamma_2)$.

Proof. The function $\varphi_4 E_{6,6}$ should be a linear combination of $E_{6,10}$ and $\Theta_{6,10}$. Write $E_{6,6} = \sum_n b_6(n)q^n$, $E_{6,10} = \sum_n b_{10}(n)q^n$ and $\varphi_4 = \sum_n a_4(n)q^n$. Then we have

$$\widehat{E_{6,6}\varphi_4}(1,1,1) = b_6(1,1,1)a_4(0,0,0) = \beta b_{10}(1,1,1) + \alpha c(1,1,1) \tag{4.6}$$

and we can choose $\beta = 1$, since $\Phi E_{6,6}\varphi_4 = \Phi E_{6,10}$. We also have

$$\widehat{E_{6,6}\varphi_4}(1,1,0) = b_6(1,1,0) + a_4(1,0,0)(b_6(1,0,0) + b_6(0,1,0)) = b_{10}(1,1,0) + \alpha c(1,1,0) \tag{4.7}$$

and

$$\widehat{E_{6,6}\varphi_4}(2,2,2) = \sum_{n_1+n_2=(2,2,2)} b_6(n_1)a_4(n_2) = b_{10}(2,2,2) + \alpha c(2,2,2). \tag{4.8}$$

The above sum contains Fourier coefficients $b_6(unu')$ with $u \in \mathsf{GL}_2(\mathbb{Z})$ and

$$n \in \{(1,0,0),(1,1,0),(1,1,1),(2,2,2)\}.$$

We can now use equations (4.6), (4.7) and (4.8), Remark 1.3 and the fact that $b_6(2,2,2) = b_6(2,(1,1,1)) = -24\cdot(1+2^4)b_6(1,1,1)$ to find that $\alpha = -\frac{77043}{2243}$. $\square$

We are now able to compute Fourier coefficients of $E_{6,6}$ (see Table 4.1).

4.4.2 Ibukiyama's theta series

In order to find generators for $M_{(6,*)}(\Gamma_2)$, Ibukiyama constructed a theta series $\Theta_{6,8}$ with harmonic coefficients of weight $(6,8)$ [18]. He showed that it is non-vanishing by way of computing a few Fourier coefficients [15]. We will construct the form $\Theta_{6,8}$ using similar methods as in Section 4.4.1, but now we will be able to use the Fourier coefficients computed by Ibukiyama.

We start again by constructing a RC-polynomial. We will use the polynomial P from Example 3.4.4 which is homogeneous of weight $(m,2)$. We take $m = 6$ and let $(4,6)$ be the type. Then we have after rescaling

$$P(\mathbf{r},\mathbf{s},v) = 2\cdot 11\cdot 9\det(\mathbf{r})P_0(\mathbf{r},\mathbf{s},v) - 11\cdot 7p_1(\mathbf{r},\mathbf{s})P_1(\mathbf{r},\mathbf{s},v) + 2\cdot 7\cdot 7\det(\mathbf{s})P_2(\mathbf{r},\mathbf{s},v),$$

n	$(1,1,0)$	$(1,1,1)$	$(2,1,0)$	$(2,1,1)$	$(2,2,0)$	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$	$(3,3,0)$
	0	0	$-2^3 3$	2^4	2^8	$-2^4 3^2$	0	2^4	0
	0	0	0	$2^3 3$	0	$-2^3 3^3$	0	$2^3 3^2$	0
	-2	1	$2^2 3$	2^3	-2^8	0	0	2^7	$2^4 3^3 5^3$
	0	2	0	-2^5	0	$2^4 3^2 5$	0	$2^4 3^2$	0
	-2	1	$2^2 3^2$	-2^4	-2^8	0	0	2^7	$2^4 3^3 5^3$
	0	0	0	0	0	$-2^3 3^3$	0	$2^3 3^2$	0
	0	0	0	0	2^8	$-2^4 3^2$	0	2^4	0

Table 4.2: Fourier coefficients of $\Theta_{6,8} \in S_{(6,8)}(\Gamma_2)$. Note that the eigenvalue $\lambda(p)$ of $T(p)$ on $S_{(6,8)}(\Gamma_2)$ equals 0 for $p=2$ and $-2^3 3^3 5^3 = -27000$ for $p=3$.

with

$$\begin{aligned}
P_0(\mathbf{r},\mathbf{s},v) &= 8\mathbf{s}[v]^3 - 32\mathbf{s}[v]^2\mathbf{r}[v] + 32\mathbf{s}[v]\mathbf{r}[v]^2 - 8\mathbf{r}[v]^3,\\
P_1(\mathbf{r},\mathbf{s},v) &= 5\mathbf{s}[v]^3 - 27\mathbf{s}[v]^2\mathbf{r}[v] + 36\mathbf{s}[v]\mathbf{r}[v]^2 - 12\mathbf{r}[v]^2,\\
P_2(\mathbf{r},\mathbf{s},v) &= \tfrac{1}{7}(20\mathbf{s}[v]^3 - 150\mathbf{s}[v]^2\mathbf{r}[v] + 270\mathbf{s}[v]\mathbf{r}[v]^2 - 120\mathbf{r}[v]^3) \quad \text{and}\\
p_1 &= \det(\mathbf{r}+\mathbf{s}) - \det(\mathbf{r}) - \det(\mathbf{s}).
\end{aligned}$$

Then we define

$$F_{12} = \tfrac{1}{6652800} D_P(\varphi_4, \varphi_6) \in S_{(6,12)}(\Gamma_2).$$

Since F_{12} is a cusp form[5], $T(2)F_{12}$ is also a cusp form. Write $F_{12} = \sum_n a(n)q^n$ and $T(2)F_{12} = \sum_n a(2,n)q^n$, then

$$a(1,1,0) = (-10,0,12,0,12,0,-10)$$

and

$$a(2,(1,1,0)) = (-74880,0,-23040,0,-23040,0,-74880)$$

and this shows that F_{12} and $T(2)F_{12}$ span $S_{(6,12)}(\Gamma_2)$. Write $\Theta_{(6,8)} = \sum_n c(n)q^n$, then[6]

$$c(1,1,0) = (0,0,-2,0,-2,0,0)$$

which must also be the Fourier coefficient of $\varphi_4\Theta_{6,8}$ at $(1,1,0)$. This shows that $7488F_{12} - T(2)F_{12}$ is divisible by φ_4 and that

$$\Theta_{6,8} = \frac{1}{112896}(7488F_{12} - T(2)F_{12}).$$

Now we can compute more Fourier coefficients of $\Theta_{6,8}$ (Table 4.2).

4.4.3 A full set of generators for $M^1_{(6,*)}(\Gamma_2)$

We will now define 7 modular forms and prove that they generate $M^1_{(6,*)}(\Gamma_2)$. We use the forms $E_{6,6}$, $\Theta_{6,8}$, the brackets $\{\cdot,\cdot\}$ and RC-polynomials constructed via the operator $\mathbf{M}$. We calculated many Fourier coefficients and not only of the forms we present here. Needless to say, a set of generators

[5] The Fourier coefficients $a(n)$ of F_{12} are linear combinations of $P(n_1,n_2)$ with $n_1+n_2=n$. If $\det(n)=\det(n_1+n_2)=0$, then also $\det(n_1)=\det(n_2)=0$.

[6] We rescaled Ibukiyama's coefficients. Also Ibukiyama uses an isomorphic, but slightly different representation Sym^6. The Fourier coefficients we give correspond to the representation on $\mathbb{C}[x,y]^{(6)}$.

is not unique. Our choice is therefore necessarily somewhat arbitrary. Many times we rescale in order to get smaller Fourier coefficients. The coefficients we calculated are integer valued, but we did not check integrality for all coefficients.

First we define the following polynomials in $\mathbb{C}[r,s,t]$:

$$\begin{aligned}
p_{15} &= \frac{1}{160}(5r^2 - 14rt + 7t^2) \\
p_{17} &= \frac{1}{192}(4r^2 - 8rt + 3t^2) \\
p_{19} &= \frac{1}{1920}(22r^2 - 24rt + 5t^2) \\
p_{21} &= \frac{1}{2880}(22r^2 - 24rt + 5t^2) \\
p_{23} &= \frac{1}{16}(13r^2 - 14rt + 3t^2).
\end{aligned}$$

These polynomials p_i are harmonic meaning that they lie in the kernel of $\mathbf{L}_{\ell_i}$ where

$$\ell_{15} = (6,6,5), \quad \ell_{17} = (6,6,7), \quad \ell_{19} = (5,5,11), \quad \ell_{21} = (5,7,11), \quad \ell_{23} = (6,6,13).$$

Define $k_i = \ell_i - (1,1,1)$, and define polynomials $q_i \in \mathbb{C}[\mathbf{r},\mathbf{s},\mathbf{t},v]$ by $q_i = \mathbf{M}_{k_i}(p_i)$. The polynomials q_i are harmonic meaning that

$$\mathbf{L}_{k_i} q_i = 0, \qquad i \in \{15,17,19,21,23\}.$$

Using the polynomials q_i, we can define the RC-operators $D_i := D_{q_i}$ that act on triples of classical Siegel modular forms of appropriate weights. We will define vector-valued Siegel modular forms F_i of weight $(6,i)$ as follows:

$$\begin{aligned}
F_{15} &= D_{15}(\chi_5, \chi_5, \varphi_4), \\
F_{17} &= D_{17}(\chi_5, \chi_5, \varphi_6), \\
F_{19} &= D_{19}(\varphi_4, \varphi_4, \chi_{10}), \\
F_{21} &= D_{21}(\varphi_4, \varphi_6, \chi_{10}), \\
F_{23} &= D_{23}(\chi_5, \chi_5, \chi_{12}).
\end{aligned}$$

These five modular forms are non-zero and a few of their Fourier coefficients are given in Table 4.3.

Let Θ_8 denote the theta series that was defined in Section 4.4.2 and define a modular form $F_{13} \in M_{(6,13)}(\Gamma_2)$ as follows:

$$2\pi i\, F_{13} = \frac{1}{14400}\{\Theta_{6,8}, \varphi_4\},$$

where $\{\cdot,\cdot\}$ is the RC-operator defined in Section 3.6. The function F_{13} is again a non-zero modular form and this fact is illustrated by some non-zero Fourier coefficients shown in Table 4.4.

Finally, let $E_{6,6}$ denote the Klingen-Eisenstein series of weight $(6,6)$. Then we define the function $F_{11} \in M_{(6,11)}(\Gamma_2)$ as

$$2\pi i\, F_{11} = \frac{1}{576}\{E_{6,6}, \varphi_4\}.$$

The function F_{11} is non-zero. Some Fourier coefficients are given in Table 4.5.

Now we can formulate the following Theorem:

n	$(1,1,0)$	$(1,1,1)$	$(2,1,0)$	$(2,1,1)$	$(2,2,0)$	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$
F_{15}	0	0	0	$2\cdot3\cdot5$	0	$2\cdot3^25\cdot73$	0	$-2\cdot3\cdot5$
	1	0	$2^33\cdot13$	2^219	2^617^2	$2^33\cdot5\cdot11\cdot29$	$2^53\cdot19$	-2^313
	0	0	0	2^25	0	$2\cdot3\cdot5^2397$	$2^43\cdot5\cdot19$	$-2\cdot3^25$
	0	0	$2^23^2\cdot5$	-2^45	0	0	0	0
	0	0	0	$-2^25\cdot7$	0	$-2\cdot3\cdot5^2397$	$-2^43\cdot5\cdot19$	$2\cdot3^25$
	-1	0	$-2\cdot3\cdot17$	-2^37	-2^617^2	$-2^33\cdot5\cdot11\cdot29$	$-2^5\cdot3\cdot19$	2^313
	0	0	0	0	0	$-2\cdot3^25\cdot73$	0	$2\cdot3\cdot5$
F_{17}	0	0	0	$-2\cdot3\cdot11$	0	$2\cdot3^25^27\cdot11$	0	$1\cdot3\cdot11$
	1	0	0	$-2^25\cdot7$	$2^55\cdot17\cdot23$	2^63^279	-2^43^27	2^8
	0	0	0	2^25	0	$2\cdot3^25\cdot719$	$-2^33^25\cdot7$	$2\cdot3^35$
	0	0	$-2^23\cdot5^2$	2^55	0	0	0	0
	0	0	0	$2^25\cdot11$	0	$-2\cdot3^25\cdot719$	$2^33^25\cdot7$	$-2\cdot3^35$
	-1	0	$2\cdot3\cdot59$	2^311	$-2^55\cdot17\cdot23$	-2^63^279	2^43^27	2^8
	0	0	0	0	0	$-2\cdot3^25^27\cdot11$	0	$2\cdot3\cdot11$
F_{19}	0	0	0	1	0	3^317	0	-1
	0	0	-2^2	$2\cdot7$	2^4677	2^329	-2^323	2^3
	0	0	0	2^23^2	0	$-5\cdot1181$	$-2^25\cdot23$	19
	0	0	-2^43	2^33	0	0	0	0
	0	0	0	0	0	$5\cdot1181$	$2^25\cdot23$	-19
	0	0	0	0	-2^4677	-2^329	2^323	-2^3
	0	0	0	0	0	3^317	0	1
F_{21}	0	0	0	-5	0	$3\cdot5^219$	0	5
	0	0	2^25	$-2\cdot5$	2^4101	-2^2149	-2^329	2^25
	0	0	0	$-2\cdot3$	0	-229	-2^2529	31
	0	0	0	-2^33	0	0	0	0
	0	0	0	$-2\cdot3\cdot5$	0	229	2^2529	-31
	0	0	2^33	$-2\cdot3$	-2^4101	2^2149	2^329	-2^25
	0	0	0	0	0	$-3\cdot5^219$	0	-5
F_{23}	0	0	0	0	0	3	0	0
	0	0	0	0	$-2\cdot83$	-37	0	0
	0	0	0	0	0	$-2\cdot5^2$	0	0
	0	0	0	0	0	0	0	0
	0	0	0	0	0	$2\cdot5^2$	0	0
	0	0	0	0	$2\cdot83$	37	0	0
	0	0	0	0	0	-3	0	0

Table 4.3: Fourier coefficients of some modular forms $F_k = \sum a(n)q^n$ of weight $(6,k)$.

n	$(1,1,0)$	$(1,1,1)$	$(2,1,0)$	$(2,1,1)$	$(2,2,0)$	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$
	0	0	0	0	0	0	0	0
	2^2	-2	$-2^5 3\cdot 17$	$2^6 13$	$-2^8 7\cdot 17$	$-2^6 3\cdot 5^2 7$	$2^7 3\cdot 5^3$	$2^6 13$
	0	-5	0	$2^5 5\cdot 17$	0	$2^5 3\cdot 5\cdot 7\cdot 97$	$2^6 3\cdot 5^4$	$2^5 3^2 5$
	0	0	$-2^4 3^2 5\cdot 7$	$2^9 5$	0	0	0	0
	0	5	0	$2^5 5\cdot 7$	0	$-2^5 3\cdot 5\cdot 7\cdot 97$	$-2^6 3\cdot 5^4$	$-2^5 3^2 5$
	-2^2	2	$-2^3 3\cdot 37$	$2^6 7$	$2^8 7\cdot 17$	$2^6 3\cdot 5^2 7$	$-2^7 3\cdot 5^3$	$-2^6 13$
	0	0	0	0	0	0	0	0

Table 4.4: Fourier coefficients of $F_{13} = c\cdot\{\Theta_{6,8}, \varphi_4\}$.

n	$(1,1,0)$	$(1,1,1)$	$(2,1,0)$	$(2,1,1)$	$(2,2,0)$	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$
	0	0	0	$-2^4 3^2 5$	0	$2^4 3^3 5\cdot 7$	0	$2^4 3^2 5$
	$-2^2 5$	-2	$-2^9 3$	$-2^4 7\cdot 17$	$2^7 5^2 47$	$-2^4 3\cdot 5\cdot 7\cdot 23$	$2^6 3\cdot 11^2$	$2^4 151$
	0	-5	0	$-2^3 5\cdot 37$	0	$-2^3 3\cdot 5^2 11^2$	$2^5 3\cdot 5\cdot 11^2$	$2^3 3\cdot 5\cdot 23$
	0	0	$2^4 3^3 5$	$-2^8 5$	0	0	0	0
	0	5	0	$-2^3 5\cdot 11$	0	$2^3 3\cdot 5^2 11^2$	$-2^5 3\cdot 5\cdot 11^2$	$-2^3 3\cdot 5\cdot 23$
	$2^2 5$	2	$-2^3 3\cdot 11$	$-2^4 11$	$-2^7 5^2 47$	$2^4 3\cdot 5\cdot 7\cdot 23$	$-2^6 3\cdot 11^2$	$-2^4 151$
	0	0	0	0	0	$-2^4 3^3 5\cdot 7$	0	$-2^4 3^2 5$

Table 4.5: Fourier coefficients of $F_{11} = c\cdot\{E_{6,6}, \varphi_4\}$

Theorem 4.4.4. *Define the modular forms F_i for $i \in I := \{11,13,15,17,19,21,23\}$ as above. Then we have*

$$M^1_{(6,*)}(\Gamma_2) = \bigoplus_{i\in I} M^0_*(\Gamma_2)\cdot F_i$$

as a module over $M^0_*(\Gamma_2)$.

Remark 4.5. Let $U \to \Gamma_2\backslash\mathscr{H}_2$ be the Hodge bundle corresponding to the factor of automorphy j, then a modular form $F \in M_{(m,k)}(\Gamma_2)$ defines a section of $\mathrm{Sym}^m(U)\otimes L^{\otimes k}$. Here L denotes the determinant line bundle of U.

Proof (of Theorem 4.4.4). The dimensions of $M_{(6,k)}(\Gamma_2)$ for odd k are given by the Hilbert-Poincaré series

$$\sum_{k\equiv 1(2)} \dim M_{(6,k)}(\Gamma_2)t^k = \frac{t^{11}+t^{13}+t^{15}+t^{17}+t^{19}+t^{21}+t^{23}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

Hence, we can prove the theorem by finding seven modular forms of weights $(6,11)$, $(6,13)$, $(6,15)$, $(6,17)$, $(6,19)$, $(6,21)$ and $(6,23)$ that are independent over $M^0_*(\Gamma_2)$. We claim that the above defined functions F_i satisfy this condition. Let k be an odd integer and let f_ℓ be a classical Siegel modular form of weight $k-\ell$. We have to show that

$$F_{11}f_{11} + F_{13}f_{13} + F_{15}f_{15} + F_{17}f_{17} + F_{19}f_{19} + F_{21}f_{21} + F_{23}f_{23} = 0$$

implies that $f_\ell = 0$ for all ℓ. The forms $f_\ell F_\ell$ are sections of $\mathrm{Sym}^6(U)\otimes L^{\otimes k}$. We can now show that the forms F_ℓ are independent by showing that the section $F_{11}\wedge\cdots\wedge F_{23}$ of $\Lambda^7(\mathrm{Sym}^6(U))\otimes L^{\otimes 119} \cong L^{\otimes 140}$

is non-vanishing or, equivalently, that the determinant

$$\chi_{140} := \det(F_{11}, F_{13}, F_{15}, F_{17}, F_{19}, F_{21}, F_{23})$$

is a non-zero function on $\mathscr{H}_2$. This determinant is a classical Siegel modular form of weight $11 + 13 + \cdots + 23 + \frac{6(6+1)}{2} = 140$. We can show that χ_{140} is non-vanishing by finding a non-zero Fourier coefficient.

Write $\chi_{140} = \sum_{n \succ 0} c(n) q^n$. The determinant of a 7×7 matrix (f_{ij}) is given by $\det(f_{ij}) = \sum_{\sigma \in S_7} \varepsilon(\sigma) \prod_{i=0}^{6} f_{\sigma(i),i}$. Hence, if $(f_{0j}, \ldots, f_{6j})'$ is a vector-valued periodic function on $\mathscr{H}_2$ with Fourier coefficients $(a_{0j}(n), \ldots, a_{6j}(n))'$, then $\det(f_{ij})$ has a Fourier series

$$\sum_{\sigma \in S_7} \varepsilon(\sigma) \prod_{i=0}^{6} \sum_{n_i} a_{\sigma(i)i}(n_i) q^{n_i} = \sum_n q^n \sum_{n = n_0 + \cdots + n_6} \det(a_{ij}(n_j)).$$

Hence, if we denote the Fourier transform by $\hat{}$, we get

$$c_{140}(n) = \sum_{n = n_0 + \cdots + n_6} \det(\hat{F}_{11}(n_1), \hat{F}_{13}(n_2), \ldots, \hat{F}_{23}(n_6)). \tag{4.9}$$

The functions $F_{11}, F_{13}, \ldots, F_{23}$ are cusp forms and therefore we will have no luck finding a non-zero Fourier coefficient $c_{140}(n)$ for $n_1, n_2 < 7$. This means that the number of decompositions of n into 7 positive definite matrices will be rather large. We can reduce the number of calculations by choosing a convenient n for which many of the decompositions lead to zero determinants in the sum of (4.9).

Let $n = \begin{pmatrix} 12 & 2 \\ 2 & 8 \end{pmatrix}$, then if we take for instance the decomposition

$$n = n_0 + \cdots + n_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1/2 \\ 1/2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1/2 \\ 1/2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix},$$

then

$$\det(\hat{F}_{11}(n_0), \hat{F}_{13}(n_1), \ldots, \hat{F}_{23}(n_6)) = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & -5 & 3 \\ -20 & -2 & 312 & 0 & 14 & -10 & -37 \\ 0 & -5 & 0 & 0 & 36 & -6 & -50 \\ 0 & 0 & 180 & -300 & 24 & -24 & 0 \\ 0 & 5 & 0 & 0 & 0 & -30 & 50 \\ 20 & 2 & -102 & 354 & 0 & -12 & 37 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{vmatrix}$$

which equals $2^{14}3^5 5^3 11$. Using a computer, we then showed that $c_{140}(n) = -2^{18}3^7 5^2$. We checked this by also computing $c_{140}(unu')$ for $u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and found that $c_{140}(unu') = c_{140}(n)$ which must hold for the Fourier coefficients of a classical Siegel modular form of even weight (Remark 1.3). This proves the Theorem. □

Remark 4.6. Since we now know all modular forms of weight $(6, k)$ with $k \in \mathbb{Z}$, we can calculate the eigenvalues of the Hecke operators $T(p)$. We did this for $p = 2, 3$ and some k (Table 4.6 and 4.7).

At our request, G. van der Geer computed some of these eigenvalues using a completely independent method that is based on counting points on hyperelliptic curves over finite fields [14, 10, 11].

k	$\lambda(2)$ on $N_{(6,k)}(\Gamma_2)$	$\lambda(2)$ on $S_{(6,k)}(\Gamma_2)$
6	$-24\cdot(1+2^4)$	—
8	—	0
10	$216\cdot(1+2^8)$	1680
11	—	-11616
12	$-528\cdot(1+2^{10})$	$X^2-22368X+57231360$
13	—	-24000
15	—	$X^2+68256X+593510400$
17	—	$X^3+363264X^2+136028160X-4603543289856000$
19	—	$X^4+1202400X^3-1311202861056X^2$ $-179858880190218240X-1566691549034368204800$

Table 4.6: Eigenvalues of the Hecke operator $T(2)$ on $M_{(6,k)}(\Gamma_2)$ for some values of k. If a polynomial in X is given, the eigenvalues $\lambda(2)$ are the roots of this polynomial.

k	$\lambda(3)$ on $S_{(6,k)}(\Gamma_2)$
8	-27000
10	-6120
11	-106488
12	$X^2+335664X-14832719455680$
13	-8505000
15	$X^2+228022128X+8319716602228800$
17	$X^3+1086146712X^2-341960280255362880X-188775313801934579676864000$

Table 4.7: Eigenvalues of the Hecke operator $T(3)$ on $M_{(6,k)}(\Gamma_2)$ for some values of k.

Our values agree with these. Also note that the characteristic polynomials of $T(2)$ and $T(3)$ on $S_{(6,k)}(\Gamma_2)$ with $k=12$ and 15 have the following discriminant:

k	$\Delta(\det(T(2)-X))$	$\Delta(\det(T(3)-X))$
12	$2^{10}3^2 7^2 601$	$2^{14}3^6 7^2 13^2 601$
15	$2^{10}3^2 29\cdot 83\cdot 103$	$2^{12}3^8 29\cdot 53^2 83\cdot 103$

This shows for $p=2,3$ that the eigenvalues of $T(p)$ on $S_{(6,12)}(\Gamma_2)$ and $S_{(6,15)}(\Gamma_2)$ are elements of the same quadratic number field $\mathbb{Q}(\sqrt{601})$ and $\mathbb{Q}(\sqrt{29\cdot 83\cdot 103})$ respectively. We also verified that the characteristic polynomials of $T(2)$ and $T(3)$ on $S_{(6,17)}(\Gamma_2)$ define the same number field.

* * *

Samenvatting voor de eerstejaars student

In deze scriptie geef ik enkele expliciete voorbeelden van *vectorwaardige Siegel modulaire vormen van geslacht* 2. Om enig inzicht te geven wat dit zijn en waarom iemand zich hier mee bezig zou houden, is het mogelijk beter om bij Siegel modulaire vormen van geslacht 1 te beginnen. Deze zijn beter bekend als *elliptische* modulaire vormen. Elliptische modulaire vormen zijn holomorfe functies[7] f op het bovenhalfvlak $\mathscr{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ die aan twee functionaalvergelijkingen voldoen:

$$f(\tau+1) = f(\tau) \qquad \text{en} \qquad f(-\tau^{-1}) = \tau^k f(\tau).$$

Hier is $k \in \mathbb{Z}$ het *gewicht* van f. Daarnaast moet f begrensd zijn in omgevingen van ∞.

Dit soort functies duikt op bij allerlei getaltheoretische problemen. Een beroemd voorbeeld is de Laatste Stelling van Fermat, maar gelukkig zijn er ook eenvoudige voorbeelden. In deze samenvatting zal ik er een bespreken.

Gehele getallen schrijven als de som van twee kwadraten

Een bekend probleem in de getaltheorie luidt als volgt:

"Welke gehele getallen zijn te schrijven als de som van twee kwadraten?"

We kunnen een poging doen om een vermoeden te formuleren door naar enkele voorbeelden te kijken. De volgende getallen zijn de som van twee kwadraten:

$$1 = 1^2+0^2, \quad 2 = 1^2+1^2, \quad 5 = 1^2+2^2, \quad 13 = 2^2+3^2, \quad 17 = 1^2+4^2$$

terwijl bijvoorbeeld de volgende getallen dit niet zijn:

$$3, \quad 7, \quad 11, \quad 19, \quad 23.$$

Als we goed zoeken binnen de *priemgetallen*, dan kunnen we een patroon vinden. Het blijkt dat:

Stelling 1. *Een priemgetal $p > 2$ is te schrijven als de som van twee kwadraten dan en slechts dan als $p-1$ deelbaar is door* 4.

Deze stelling kent vele bewijzen en de implicatie '$\Longrightarrow$' is heel eenvoudig aan te tonen. Voor de andere implicatie (als $p \equiv 1 \pmod 4$, dan $p = a^2+b^2$) is meer techniek nodig. Een van die bewijzen maakt gebruik van modulaire vormen.

[7] een holomorfe functie f op $U \subseteq \mathbb{C}$ is rond ieder punt z in zijn domein U te ontwikkelen als machtreeks: $f(w) = \sum_{n=0}^{\infty} a_n (w-z)^n$ voor w dicht bij z.

Ik heb hierboven voornamelijk naar priemgetallen gekeken, maar we hoeven ons natuurlijk niet tot priemgetallen te beperken. Zo is bijvoorbeeld 25 gelijk aan 5^2+0^2. Voor het getal 25 is er bovendien nog een andere oplossing: $25 = 3^2+4^2$. Dit suggereert dat het handig is om voor elke n de oplossingen van de vergelijking $n = a^2 + b^2$ te 'tellen'. We definieren voor $n \in \mathbb{Z}$ het *representatiegetal* $r_2(n)$ als volgt:

$$r_2(n) := \#\{(a,b) \in \mathbb{Z}^2 \mid a^2 + b^2 = n\},$$

dus $r_2(n)$ is het aantal paren van gehele getallen (a,b) zo dat $a^2 + b^2 = n$. Zo is bijvoorbeeld $r_2(3)$ gelijk aan 0 en $r_2(5)$ gelijk aan 8.[8]

We kunnen deze getallen $r_2(n)$ bestuderen door ze als coëfficiënten van een machtreeks te gebruiken. Deze machtreeks heet de *voortbrengende functie* van $r_2(n)$:

$$\theta^2(q) = \sum_{n=0}^{\infty} r_2(n) q^n = 1 + 4q + 4q^2 + 0q^3 + 4q^4 + 8q^5 + \cdots$$

en als we nu op een of andere manier de coëfficiënten van θ^2 kunnen vinden, dan kunnen we mogelijk onze vraag beantwoorden. Ik heb de voortbrengende functie 'θ^2' genoemd omdat deze functie het kwadraat is van een andere machtreeks, de Riemann-thèta-functie die gedefinieerd is als

$$\theta(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Dit is eenvoudig in te zien:

$$\theta(q)^2 = \sum_{n,m=-\infty}^{\infty} q^{n^2+m^2} = \sum_{\ell=0}^{\infty} q^{\ell} \cdot \#\{(n,m) \in \mathbb{Z}^2 \mid n^2 + m^2 = \ell\} = \theta^2(q).$$

De warmtevergelijking op de cirkel

Deze functie θ heeft mooie eigenschappen, maar om deze te vinden, is het beter om θ net iets anders te definiëren, namelijk als functie op het bovenhalfvlak:

$$\theta(\tau) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \qquad \tau \in \mathscr{H}$$

en het blijkt dat deze machtreeks een holomorfe functie definieert op $\mathscr{H}$. De mooie eigenschappen van θ waar ik hier boven op doelde zijn

$$\theta(\tau + 2) = \theta(\tau) \quad \text{en} \quad \theta(-\tau^{-1}) = \sqrt{\tau / i}\,\theta(\tau) \tag{$*$}$$

en dus voldoet θ (bijna) aan de definitie van een modulaire vorm.

De linker eigenschap van θ volgt onmiddellijk uit de definitie. De rechter eigenschap volgt uit de warmtevergelijking op de cirkel[9]:

$$\tfrac{\partial}{\partial t} u(x,t) = \frac{1}{4\pi} \tfrac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}_{\geq 0}, \quad u(x+1,t) = u(x,t), \quad u(x,0) = \delta_0$$

[8] Pas op: we tellen alle *paren* van *gehele* getallen, dus in het geval $n = 5$ hebben we de oplossingen $(1,5)$, $(1,-5)$, $(-1,-5)$, $(-1,5)$, $(5,1)$, $(5,-1)$, $(-5,-1)$ en $(-5,1)$. Dit doen we alleen maar om technische redenen.

[9] Dit argument is ook te vinden in [7].

waarbij $\delta_0(x)$ Diracs deltafunctie is[10]. De warmtevergelijking beschrijft de verandering in de verdeling van warmte in een object. In dit geval kunnen we denken aan een ring waarbij we op tijdstip $t = 0$ in één punt warmte toevoegen. Op de lijn heeft de warmtevergelijking een bekende oplossing, namelijk de Gaussische functie

$$H_t(x) = \frac{1}{\sqrt{t}} e^{-\pi x^2/t}$$

en we kunnen hier een oplossing voor de warmtevergelijking op de cirkel mee construeren. Natuurlijk voldoet H_t *niet* aan de vergelijking $u(x+1,t) = u(x,t)$, maar als we definiëren

$$G_t(x) := \sum_{n=-\infty}^{\infty} H_t(x+n),$$

dan voldoet G nog steeds aan de warmtevergelijking en G is bovendien periodiek. Met behulp van Fourieranalyse kunnen we nog een tweede oplossing voor de warmtevergelijking op de cirkel construeren en deze tweede oplossing is gegeven door

$$F_t(x) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} e^{2\pi i n x}.$$

We hebben nu 2 oplossingen voor bovenstaande differentiaalvergelijking gevonden. De klassieke mechanica is echter deterministisch en daarom moet gelden dat $F_t(x) = G_t(x)$ voor alle x en t. In het bijzonder geldt $F_t(0) = G_t(0)$ en daaruit volgt

$$\theta(it) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = F_t(0) = G_t(0) = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} = \frac{1}{\sqrt{t}} \theta(i/t).$$

Daarom geldt voor alle $\tau \in i\mathbb{R}_{\geq 0}$ dat

$$\theta(\tau) = \frac{1}{\sqrt{\tau/i}} \theta(-\tau^{-1}).$$

Uit de theorie van holomorfe functies volgt nu dat bovenstaande vergelijking moet gelden voor alle $\tau \in \mathscr{H}$.

De ruimte van modulaire vormen

Ik heb in het begin van deze samenvatting de definitie van een modulaire vorm gegeven, maar we kunnen deze definitie uitbreiden zo dat we functies die aan de vergelijkingen (∗) voldoen—bijvoorbeeld de functie θ—ook modulaire vormen mogen noemen en, belangrijker, zo dat we de theorie van modulaire vormen kunnen gebruiken voor de functie θ en dus voor onze vraag over het representatiegetal $r_2(n)$. Als we schrijven $q = e^{\pi i \tau}$, dan geldt namelijk $\theta^2(\tau) = \sum_n r_2(n) q^n$ en θ^2 is een modulaire vorm van gewicht 1. Wat de theorie van modulaire vormen ons nu leert is dat er niet zo gek veel modulaire vormen van gewicht 1 zijn. Alle modulaire vormen van gewicht 1—en dus ook θ^2—zijn op een multiplicatieve scalar na gelijk aan een zogenaamde Eisensteinreeks:

$$E_1(\tau) = \sum_{n=0}^{\infty} a(n) q^n,$$

[10] De deltafunctie is geen functie, maar een *distributie*. De 'functie' δ_0 heeft de eigenschap dat $\int_{-\infty}^{\infty} \delta_0(x) dx = 1$ en $\delta_0(x) = 0$ wanneer $x \neq 0$.

waarbij de coëfficiënten $a(n)$ berekend kunnen worden, onafhankelijk van het 'som van twee kwadraten' probleem. In dit geval geldt $a(0)=1$ en voor $n>1$

$$a(n)=4\sum_{d}(-1)^{\frac{d-1}{2}}.$$

Hier wordt de som genomen over de oneven delers d van n. We kunnen nu Stelling 1 moeiteloos bewijzen:

Bewijs (van Stelling 1). Er geldt dat $r_2(p) = 4\sum_{d|p,d\equiv 1(2)}(-1)^{\frac{d-1}{2}}$. De delers van p zijn gelijk aan 1 en p en $(-1)^0 = 1$. Als $p-1$ deelbaar is door 4, dan is $(p-1)/2$ deelbaar door 2 en dus geldt $(-1)^{\frac{p-1}{2}}=1$. We vinden $r_2(p)=8$.

Als $p-1$ niet deelbaar is door 4, dan is $\frac{p-1}{2}$ oneven en dus geldt geldt $(-1)^{\frac{p-1}{2}} = -1$. In dit geval vinden we $r_2(p)=0$. □

Er zijn vele generalisaties van modulaire vormen mogelijk. In het geval van de 'elliptische' modulaire vormen kunnen we stellingen als Stelling 1 bewijzen. Dit kan omdat we de functie E_1 kennen. Voor Siegel modulaire vormen hebben we in de meeste gevallen dergelijke kennis nog niet. In mijn scriptie heb ik een specifiek voorbeeld behandeld en voor dit specifieke voorbeeld 'alle' (Siegel) modulaire vormen expliciet kunnen beschrijven.

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