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Paramodular Forms of Degree 2 and Level 3

by

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Abstract. Let $\Gamma_3 \subset \mathrm{Sp}_2(\mathbb{Q})$ be the paramodular group of level 3 and $\Gamma_3^* \subset \mathrm{Sp}_2(\mathbb{R})$ the maximal normal discrete extension of Γ_3 of index 2. Denote by $D\Gamma_3^*$ the commutator subgroup of Γ_3^*. The main goal of the present note is to determine the structure of the graded ring of paramodular forms for $D\Gamma_3^*$. Since all generators constructed here are actually modular forms for Γ_3^* with certain multiplier-systems, we can derive generators for the graded rings of paramodular forms for all groups Γ with $D\Gamma_3^* \subset \Gamma \subset \Gamma_3^*$, especially Γ_3.

1. Introduction

In [16], Igusa determined generators of the graded ring of modular forms for $\Gamma_1 = \mathrm{Sp}_2(\mathbb{Z})$, the paramodular group of degree 2 and level 1. This was the first example, where generators of the graded ring of paramodular forms of degree 2 are known. Later, Freitag [9] gave another proof of Igusa's result, using a distinguished Siegel modular form Θ_5 with (uniquely determined) nontrivial multiplier-system and known zero-divisor. Then, using similar techniques, Freitag [10] determined generators of the graded ring of modular forms of even weight for Γ_2^*, the maximal normal discrete extension of Γ_2 of index 2. Only recently, these results were extended to Γ_2 by Ibukiyama and Onodera [19]. As far as we know, these are the only results, where generators of the graded ring of modular forms for the paramodular group of degree 2 and level t are known explicitely. Apart from that, in degree 2, there is Ibukiyama's formula for the dimensions of the spaces of cusp-forms [17], which can be used to deduce some information about generators of the graded ring of modular forms for Γ_t (for small level t at least, see e.g. [18], [19]). The general result of Runge [27, Theorem 2.3], which describes the even part of the graded ring of modular forms for paramodular groups of arbitrary degree as invariants of a certain space of theta-constants, hardly gives any explicit information on generators.

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As another example, in this note we solve the case of degree 2 and level 3. More precisely we determine generators of the graded ring of modular forms for the commutator-subgroup $D\Gamma_3^*$ where Γ_3^* is the maximal normal discrete extension of Γ_3 (which is of index 2). Our method is in some sense the same as Freitag's [9]. By results of Borcherds [1], one can nowadays construct paramodular forms of degree 2 with known zero-divisor, so-called Borcherds-products. Θ_5 is an example of such a Borcherds-product. More examples were given in [15]. We construct Borcherds-products for Γ_3^* (following [1] or [3] more closely than [15]). These paramodular forms have non-trivial multiplier-systems in general and we have to consider modular forms for $D\Gamma_3^*$ quite naturally. The crucial point is, that we can find Borcherds-products with "minimal" zero-divisor. As in [9], [10] the only thing we have to do then, is to lift those modular forms on the divisor, that are restrictions of paramodular forms, to paramodular forms for Γ_3^*. Here we use "arithmetical liftings" (generalizations of Maaß's construction [24], [25]), introduced by Gritsenko [13], [14] and Gritsenko-Nikulin [15].

The Borcherds-products constructed in section 4, already appeared in [15] an it was noted there that these modular forms can be used "to construct all generators of the graded rings of modular forms for Γ_2 and Γ_3". In fact, for level 2 one can find all forms for the commutator-subgroup $D\Gamma_2^*$ (which has index 4 in Γ_2) by the very same method used here. There are good reasons to belive, that $t = 2$ and $t = 3$ are the only cases, where the problem is as easy as in the Siegel-case (see remark 4.4).

Now we give a short description of the following sections:

In section 2, we fix our notation concerning the paramodular group Γ_t and the extension Γ_t^*. From [20] we cite a special case of a general result on generators of paramodular groups, which we did not found anywhere else in the literature. Moreover we give a description of the character-groups of Γ_3 and Γ_3^* following [6].

In section 3 arithmetical liftings from vector-valued modular forms of half-integral weight for the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ to paramodular forms for Γ_3^* with multiplier-systems are defined. The main result, proposition 3.6, is essentially a reformulation of results from [14] and [15]. We explicitely calculate the dimensions of the associated Maaß-spaces, using a dimension-formula of Skoruppa [29], [7].

In section 4 we apply Borcherds theory in order to find paramodular forms for Γ_t^*. Borcherds theory is formulated in terms of orthogonal groups. The (well-known) connection with paramodular forms is cited from [15] and [3] mainly. The input for Borcherds lift are vector-valued modular forms of weight $-\frac{1}{2}$ for $\mathrm{Mp}_2(\mathbb{Z})$ with poles of small order at the cusp. Since we are looking for forms with "minimal" zero-divisor, we have to find vector-valued modular forms with poles of small order at the cusp. To this end, an obstruction-problem from [2] is solved.

In section 5 we use use (some of) the forms, constructed in the preceeding sections in order to prove our main result, theorem 5.2. It states, that the ring of paramodular forms for $D\Gamma_3^*$ is generated by tree Borcherds-products of weight 1, 6 and 12, together with four Maaß-lifts, needed to generate all the modular forms on the product of two upper half-planes with multiplier-systems of order 3. Moreover, we find all relations among the generators. In the same way, we find generators of the rings of paramodular forms for

Γ_3^* and Γ_3 and all relations among them. The formula for the dimensions of the spaces of paramodular forms for Γ_3, following from these results, coincides with Ibukiyama's formula [18].

In the remaining part of this section we set up basic notations:

For a ring R (always assumed to be commutative and with unity), we denote by $R^{n\times n}$ the set of n by n matrices with entries in R. Given $A, B\ in R^{n\times n}$, we write $A^{\rm tr}$ for the transpose of A and define $B[A] = A^{\rm tr}BA$. Let $I_n \in R^{n\times n}$ be the identity-matrix of dimension n (if n is obvious, we just write I instead of I_n). $\mathrm{Sp}_n(R)$, the symplectic group of degree n with entries in the ring R, is given by $\mathrm{Sp}_n(R) = \{M \in R^{2n\times 2n} \mid \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)[M] = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)\}$.

For $m \in \mathbb{N}$, $n \in \mathbb{Z}$ we write $m|n$, if m divides n.

For a group G the group of abelian characters of G is denoted by $G^{\rm ab}$ and the commutator-group is denoted by DG. If $G^{\rm ab}$ is finite (which will always be the the case later), one has $G^{\rm ab} \cong G/DG$. For $n \in \mathbb{N}$ we set $\mathcal{C}_n = \mathbb{Z}/n\mathbb{Z}$.

We use $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as generators of $\mathrm{SL}_2(\mathbb{Z})$. The principal congruence subgroup (of level n) is $\mathrm{SL}_2(\mathbb{Z})[n] := \{M \in \mathrm{SL}_2(\mathbb{Z}) \mid M \equiv I \bmod n\mathbb{Z}\}$. The space of (elliptic) modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is denoted by $[\mathrm{SL}_2(\mathbb{Z}), k, 1]$. For $4 \le k \in 2\mathbb{N}$ denote by $g_k \in [\mathrm{SL}_2(\mathbb{Z}), k, 1]$ the normalized elliptic Eisenstein-series of weight k. Explicitly, $g_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n\in\mathbb{N}} \sigma_{k-1}(n)e^{2\pi i n\tau}$, where B_k is the $k^{\rm th}$ Bernoulli-number and $\sigma_k(n) = \sum_{d\in\mathbb{N},\, d|n} d^k$. Let $\eta(\tau) = e^{2\pi i\tau/24}\prod_{n\in\mathbb{N}}(1 - e^{2\pi i n\tau})$ be the Dedekind-eta-function and v_η be the multiplier-system of η. v_η^2 is a generator of $\mathrm{SL}_2(\mathbb{Z})^{\rm ab} \cong \mathcal{C}_{12}$. $\Delta_{12} = \eta^{24} = \frac{1}{12^3}(g_4^3 - g_6^2)$ is the first non-trivial cusp-form for $\mathrm{SL}_2(\mathbb{Z})$ (up to normalization).

2. Paramodular groups of degree 2

We think of paramodular groups (of degree 2) as subgroups of the rational symplectic group $\mathrm{Sp}_2(\mathbb{Q})$. For $t \in \mathbb{N}$ define $P_t := \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$ (a polarization, chosen in normal form without restriction) and $D_t = \left(\begin{smallmatrix} I_2 & 0 \\ 0 & P_t \end{smallmatrix}\right)$. Later on we will specialize $t = 3$, but if possible, we give results for general t.

Definition 2.1. *The paramodular group Γ_t (of level t) is given by*

$$\Gamma_t := \{M \in \mathrm{Sp}_2(\mathbb{Q}) \mid D_t^{-1} M D_t \in \mathbb{Z}^{4\times 4}\}\,.$$

The conjugated group $\widehat{\Gamma_t} := D_t^{-1}\Gamma_t D_t \subset \mathbb{Z}^{4\times 4}$ is the integral paramodular group (of level t). Note that $\widehat{\Gamma_t}$ leaves the form $\left(\begin{smallmatrix} 0 & -P_t \\ P_t & 0 \end{smallmatrix}\right)$ invariant.

As is well known [22], [14], paramodular groups have non-trivial discrete extensions in $\mathrm{Sp}_2(\mathbb{R})$ for $t > 1$. In our special case (were t will be prime later on), we define a distinguished extension of index 2 of Γ_t. Set

$$V_t := \begin{pmatrix} U_t & 0 \\ 0 & U_t^{\rm tr} \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R}) \quad \text{with} \quad U_t := \begin{pmatrix} 0 & \sqrt{t} \\ 1/\sqrt{t} & 0 \end{pmatrix}\,.$$

Then $V_t^2 = I$ and $\gamma_t : M \mapsto V_t M V_t^{-1}$ is an involution in $\mathrm{aut}(\Gamma_t)$.

DEFINITION 2.2. *The extended paramodular group* Γ_t^* (*of level* t) *is the group, generated by* V_t *over* Γ_t, *i.e.*

$$\Gamma_t^* := \langle \Gamma_t \cup \{V_t\} \rangle = \Gamma_t \cup \Gamma_t V_t \subset \mathrm{Sp}_2(\mathbb{R}) .$$

Γ_t^* is an extension of index 2 of Γ_t for $t > 1$. In general, there is an even bigger maximal normal discrete extension $\Gamma_t^{\max} \supset \Gamma_t^*$, which is generated by (suitably defined elements) $V_d \in \mathrm{Sp}_2(\mathbb{R})$ for all $d \| t$ (see [15, 1.3] for details). If t is square-free, then $\Gamma_t^{\max}$ is maximal discrete, and if t is prime, then $\Gamma_t^* = \Gamma_t^{\max}$ is maximal discrete too (though in general, it is not). Typical elements of Γ_t are

$$J_t = \begin{pmatrix} 0 & -P_t^{-1} \\ P_t & 0 \end{pmatrix},$$

$$\mathrm{rot}(U) = \begin{pmatrix} U & 0 \\ 0 & U^{-\mathrm{tr}} \end{pmatrix} \qquad \text{for } U \in \Omega_t = \{M \in \mathrm{GL}_2(\mathbb{Z}) \mid P_t M P_t^{-1} \in \mathbb{Z}^{2\times 2}\},$$

$$\mathrm{trans}(S) = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \qquad \text{for } S \in \Sigma_t = \{M \in \mathbb{Q}^{2\times 2} \mid M = M^{\mathrm{tr}},\ M P_t \in \mathbb{Z}^{2\times 2}\},$$

$$M_1 \times_t M_2 = \begin{pmatrix} a_1 & 0 & b_2 & 0 \\ 0 & a_2 & 0 & b_2/t \\ c_1 & 0 & d_1 & 0 \\ 0 & t c_2 & 0 & d_2 \end{pmatrix} \qquad \text{for } M_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) .$$

We need generators of Γ_t. From [20, Satz 1.12] we cite

LEMMA 2.3. Γ_t *is generated by* J_t *and* trans(S) *for* $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1/t \end{smallmatrix}\right)$.

In the sequel, modular forms for Γ_3^* with arbitrary multiplier-systems will be considered. For the rest of this section we specialize to $t = 3$. Since multiplier-systems for Γ_3 and Γ_3^* have integral weight [5, Satz 14], multiplier-systems are just (abelian) characters of Γ_3 resp. Γ_3^*.

The groups Γ_t^{ab} and $\Gamma_t^{*\mathrm{ab}}$ are known by [6]. Characters of Γ_3 arise in the following way: There are surjective homomorphisms ($\mathbb{F}_p$ is the field with p elements)

$$\alpha_2 : \Gamma_3 \to \mathrm{Sp}_2(\mathbb{F}_2), \qquad M \mapsto D_3^{-1} M D_3 \bmod 2\mathbb{Z},$$

$$\beta_3 : \Gamma_3 \to \mathrm{SL}_2(\mathbb{F}_3)^2, \quad M \mapsto \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_4 & 3b_4 \\ c_4/3 & d_4 \end{pmatrix} \right) \bmod 3\mathbb{Z}.$$

Recall, that $D_3^{-1}\Gamma_3 D_3$ is the integral paramodular group and that $\left(\begin{smallmatrix} 0 & -P_3 \\ P_3 & 0 \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right) \bmod 2\mathbb{Z}$ Since $\mathrm{Sp}_2(\mathbb{F}_2)$ is isomorphic to the symmetric group on six elements (e.g. via the action on the six odd theta-characteristics in $\mathbb{F}_2^4$), there is a character $\widehat{\kappa}$ of $\mathrm{Sp}_2(\mathbb{F}_2)$ of order 2. The pull-back of $\widehat{\kappa}$ gives a character $\kappa := \widehat{\kappa} \circ \alpha_2 \in \Gamma_3^{\mathrm{ab}}$ of order 2 (in the same way, the nontrivial character of the Siegel modular group $\Gamma_1 = \mathrm{Sp}_2(\mathbb{Z})$ arises [23]).

As is well known, $\mathrm{SL}_2(\mathbb{F}_3)^{\mathrm{ab}}$ is (isomorphic to) a cyclic group of order 3, generated by a character $\widehat{\mu}$, which is uniquely determined by $\widehat{\mu}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = e^{2\pi i/3}$.

The corresponding characters of $\mathrm{SL}_2(\mathbb{F}_3)^2$, which arise by first projecting on the j^{th} component, are denoted by $\widehat{\mu}_j$. The pull-back of the characters $\widehat{\mu}_j$ gives two (independent) characters $\mu_j := \widehat{\mu}_j \circ \beta_3 \in \Gamma_3^{\mathrm{ab}}$ of order 3.

The following lemma is a special case of [6, Theorem 4.2].

LEMMA 2.4. *$\Gamma_3^{\mathrm{ab}} \cong \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_3$ is generated by κ, μ_1 and μ_2.*

Explicit values of κ and μ_1 on J_3, the subgroups of rotations and the subgroups of translations are given by

$$\mu_1(M) = \begin{cases} e^{2\pi i s_1/3} & \text{for } M = \operatorname{trans}\begin{pmatrix} s_1 & s_2 \\ s_2 & s_4/3 \end{pmatrix}, \\ 1 & \text{for } M = \operatorname{rot}\begin{pmatrix} u_1 & 3u_2 \\ u_3 & u_4 \end{pmatrix} \text{ and } M = J_3\,, \end{cases} \tag{2.1}$$

$$\kappa(M) = \begin{cases} (-1)^{s_1+s_2+s_4} & \text{for } M = \operatorname{trans}\begin{pmatrix} s_1 & s_2 \\ s_2 & s_4/3 \end{pmatrix}, \\ (-1)^{(1+u_1+u_4)(1+u_2+u_3)+u_1u_4} & \text{for } M = \operatorname{rot}\begin{pmatrix} u_1 & 3u_2 \\ u_3 & u_4 \end{pmatrix}, \\ 1 & \text{for } M = J_3\,. \end{cases} \tag{2.2}$$

(For more explicit formulas see e.g. [23], [15, Lemma 1.2], [6]). Note that $\mu_2 = \mu_1 \circ \gamma_3$, so explicit values of μ_2 can easily be read of (2.1) too.

The involution $\gamma_3 \in \operatorname{aut}(\Gamma_3)$ acts on Γ_3^{ab} by $\nu \mapsto \nu \circ \gamma_3$. If $\nu \circ \gamma_3 = \nu$, we say that ν is symmetric. $\nu \in \Gamma_3^{\mathrm{ab}}$ can be extended to a character of Γ_3^* if (and only if) ν is symmetric. In this case, ν is extended to a character of Γ_3^* by $\nu(V_3) = 1$. It was shown in [6, Sec. 5], that κ is symmetric. Since $\mu_1 \circ \gamma_3 = \mu_2$, the characters μ_1 and μ_2 are not symmetric, but on the other hand, $\mu := \mu_1\mu_2$ is symmetric. We extend κ and μ to characters of Γ_3^* as above (and denote this extended characters by the same symbols again), i.e. as characters of Γ_3^* we have $\kappa(V_3) = \mu(V_3) = 1$. Since Γ_3^* is an extension of index 2 of Γ_3, generated by V_3, another character χ of Γ_3^* is defined by $\chi(V_3) = -1$ and $\chi(\Gamma_3) = \{1\}$.

The following lemma is a special case of [6, Cor. 5.5].

LEMMA 2.5. *$\Gamma_3^{*\mathrm{ab}} \cong \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_3$ is generated by χ, κ and μ.*

Explicit values of μ on J_3, rotations $\operatorname{rot}(U)$, $U \in \Omega_t$, and translations $\operatorname{trans}(S)$, $S \in \Sigma_t$, are given by

$$\mu(M) = \begin{cases} e^{2\pi i (s_1+s_4)/3} & \text{for } M = \operatorname{trans}\begin{pmatrix} s_1 & s_2 \\ s_2 & s_4/3 \end{pmatrix}, \\ 1 & \text{for } M = \operatorname{rot}\begin{pmatrix} u_1 & 3u_2 \\ u_3 & u_4 \end{pmatrix} \text{ and } M = J_3\,. \end{cases} \tag{2.3}$$

Since $V_3 \notin D\Gamma_3^*$, we have $D\Gamma_3^* \subset \Gamma_3$. In fact, lemma 2.5 implies, that $[\Gamma_3 : D\Gamma_3^*] = 6$ and $\Gamma_3/D\Gamma_3^*$ is generated by the coset trans $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)D\Gamma_3^*$.

For $\nu \in \Gamma_3^{\mathrm{ab}}$ or $\nu \in \Gamma_3^{*\mathrm{ab}}$ we define $\widetilde{\nu} \in \mathrm{SL}_2(\mathbb{Z})^{\mathrm{ab}}$ via $\widetilde{\nu}(M) := \nu(M \times_t I)$.

3. Jacobi-forms with characters and Maaß-Lifts

In this section we use "arithmetical liftings", defined by Gritsenko [13, 14] and Gritsenko-Nikulin [15] to construct paramodular forms with certain multiplier-systems. This is the first of two fundamental methods, used to construct generators of the graded ring of modular forms for $D\Gamma_3^*$. The other one is Borcherds-products, being presented in the following section.

First we fix our notation concerning paramodular forms. Let $\mathbb{H}_n$ be the Siegel upper half-plane of degree n and $(M, Z) \mapsto M\cdot Z$ the usual action of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{H}_n$ (as biholomorphic transformations). The standard factor of automorphy is $j_n(M, Z) = \det(CZ + D)$, if $M = \left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$. The corresponding action of weight $k \in \mathbb{Z}$ on functions $f : \mathbb{H}_n \to \mathbb{C}$ is given by $f\big|_k M(Z) := j_n(M, Z)^{-k} f(M \cdot Z)$.

DEFINITON 3.1. *Assume that $\Gamma \subset \Gamma_t^*$ with finite index. Let ν be a character of Γ. A holomorphic function $f : \mathbb{H}_2 \to \mathbb{C}$ is a paramodular form of weight $k \in \mathbb{Z}$ with character ν for Γ if*

$$f\big|_k M(Z) = \nu(M) f \quad \text{for all } M \in \Gamma .$$

f is a cusp-form, if additionally $\lim_{y\to\infty} f\big|_k M\left(\begin{smallmatrix}z&0\\0&iy\end{smallmatrix}\right) = 0$ for all $M \in \mathrm{Sp}_2(\mathbb{Q})$ and $z \in \mathbb{H}_1$. The space of paramodular forms of weight k with character ν for Γ is denoted by $[\Gamma, k, \nu]$. The subspace of cusp-forms is denoted by $[\Gamma, k, \nu]_{\mathrm{cusp}}$.

Since in the following we will have to consider (elliptic and Jacobi) modular forms of half-integral weight too, we need the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$. This is two-fold cover of $\mathrm{SL}_2(\mathbb{Z})$, consisting of pairs (M, ω), where $M \in \mathrm{SL}_2(\mathbb{Z})$ and $\omega : \mathbb{H}_1 \to \mathbb{C}$ is a holomorphic square-root of $j_1(M, \tau)$, i.e. we have $\omega(\tau)^2 = j_1(M, \tau)$ (see [7, Sec. 4.2] for some more details on $\mathrm{Mp}_2(\mathbb{Z})$). Standard generators of $\mathrm{Mp}_2(\mathbb{Z})$ are

$$\widehat{T} := \left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, 1\right), \quad \widehat{J} := \left(\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \sqrt{\tau}\right).$$

By $\sqrt{\tau}$ we always denote the principal value of the square root of τ, determined by $\Re(\sqrt{\tau}) > 0$ or $\Re(\sqrt{\tau}) = 0$ and $\Im(\sqrt{\tau}) \geq 0$. The subgroup $\mathrm{Mp}_2(\mathbb{Z})[n] := \{(M, \omega) \mid M \equiv I \bmod n\mathbb{Z}\}$ is the principal congruence subgroup (of level n) of the metaplectic group. $\mathrm{Mp}_2(\mathbb{Z})$ acts with weight $k \in \frac{1}{2}\mathbb{Z}$ on functions $f : \mathbb{H}_1 \to V$ (where V is a $\mathbb{C}$-vector-space) by

$$f\big|_k (M, \omega)(\tau) := \omega(\tau)^{-2k} f(M \cdot \tau) . \tag{3.1}$$

Let $H(\mathbb{Z})$ be the integral Heisenberg-group as in [15, Sec. 1]. We define the metaplectic Jacobi-group $\mathrm{MJ}_2(\mathbb{Z})$ to be

$$\mathrm{MJ}_2(\mathbb{Z}) := \mathrm{Mp}_2(\mathbb{Z}) \ltimes H(\mathbb{Z}),$$

where the action of $\mathrm{Mp}_2(\mathbb{Z})$ on $H(\mathbb{Z})$ is given by the action of the first component. The parabolic subgroup $\Gamma_{t,\infty} \subset \Gamma_t$ is defined by

$$\Gamma_{t,\infty} := \{M \in \Gamma_t \mid M \text{ has last row } (0,0,0,1)\}.$$

Note that $\Gamma_{1,\infty} \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$. Thus we can think of $\mathrm{MJ}_2(\mathbb{Z})$ as a two-fold cover of $\Gamma_{1,\infty}$ and $\mathrm{MJ}_2(\mathbb{Z})$ acts with weight $k \in \frac{1}{2}\mathbb{Z}$ on functions $f : \mathbb{H}_2 \to V$ (where V is again a $\mathbb{C}$-vector-space) by

$$\begin{aligned} &f\big|_k((M,\omega),[u,v;w])(Z) \\ &:= \omega(z_1)^{-2k} f\left((M \times_1 I)\,\mathrm{rot}\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\mathrm{trans}\begin{pmatrix}0 & v\\ v & w-uv\end{pmatrix}\cdot Z\right), \\ &Z = \begin{pmatrix} z_1 & z_2\\ z_2 & z_3\end{pmatrix}. \end{aligned}$$

We also write $f\big|_k(M,\omega)$ for $f\big|_k((M,\omega),[0,0;0])$ and $f\big|_k[u,v;w]$ for $f\big|_k((I,1),[u,v;w])$. Let v_H be the character of $H(\mathbb{Z})$, defined by

$$v_H([u,v;w]) := (-1)^{u+v+uv+w}. \tag{3.2}$$

Following [15, Lemma 3.1], all characters of $\mathrm{MJ}_2(\mathbb{Z})$ are of the form $v_{a,b} := v_\eta^a \times v_H^b$ with $a \in \mathbb{Z}/24\mathbb{Z}$ and $b \in \mathbb{Z}/2\mathbb{Z}$. $v_{a,b}$ factors over $\mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$, if and only if a is even (or equivalently if $v_{a,b}$ has order ≤ 12). In this case we see, that $v_{4,1}$ is the restriction of $\kappa\mu^2 \in \Gamma_3^{*\mathrm{ab}}$ to $\Gamma_{1,\infty} \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$. Therefore, precisely the characters $v_{4j,j}$, $j \in \mathbb{Z}/6\mathbb{Z}$, can be lifted (from $\Gamma_{1,\infty}$) into $\Gamma_3^{*\mathrm{ab}}$. This will be used frequently later.

For a function $\Phi : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$ we define $\widetilde{\Phi}_m$ on $\mathbb{H}_2$ by $\widetilde{\Phi}_m\left(\begin{smallmatrix} z_1 & z_2\\ z_2 & z_3\end{smallmatrix}\right) = \Phi(z_1,z_2)e^{2\pi i m z_3}$. We give a definition of Jacobi-forms with character which is suitable for our needs (compare [15, Def. 1.4]).

DEFINITION 3.2. *Let $v_{a,b}$ be a character of $\mathrm{MJ}_2(\mathbb{Z})$. A holomorphic function Φ : $\mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$ is a Jacobi-form of weight $k \in \frac{1}{2}\mathbb{Z}$, index $m \in \frac{1}{2}\mathbb{Z}$, with character $v_{a,b}$, if $\widetilde{\Phi}_m$ satisfies*

$$\widetilde{\Phi}_m\big|_k M = v_{a,b}(M)\widetilde{\Phi}_m \quad \textit{for all } M \in \mathrm{MJ}_2(\mathbb{Z}) \tag{3.3}$$

and Φ admits a Fourier-expansion

$$\Phi(z_1,z_2) = \sum_{\substack{n,l\in\mathbb{Q},\, n\geq 0\\ 4mn-l^2\geq 0}} \alpha(n,l)e^{2\pi i(nz_1+lz_2)}$$

(where n, l have bounded denominators, depending on $v_{a,b}$). Moreover, if $\alpha(n,l) \neq 0$ implies $4mn - l^2 > 0$, then Φ is a cusp-form. The space of Jacobi-forms of weight k and index m with character $v_{a,b}$ is denoted by $[\mathrm{MJ}_2(\mathbb{Z}),k,m,v_{a,b}]$. The subspace of cusp-forms is denoted by $[\mathrm{MJ}_2(\mathbb{Z}),k,m,v_{a,b}]_{\mathrm{cusp}}$.

A first simple observation is $2m \equiv b \bmod 2\mathbb{Z}$ or $[\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}] = \{0\}$. This holds because $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$ implies

$$e^{2\pi imw}\widetilde{\Phi}_m(Z) = \widetilde{\Phi}_m\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3+w\end{pmatrix} = \widetilde{\Phi}_m\big|_k((I,1)\,,[0,0;w])(Z) = (-1)^{bw}\widetilde{\Phi}_m(Z)\,.$$

Therefore we will frequently assume $2m \equiv b \bmod 2\mathbb{Z}$. Another consequence of the transformation (3.3) is that $f \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$ has non-zero Fourier-coefficients $\alpha(n,l)$ for $n \equiv \frac{a}{24} \bmod \mathbb{Z}$ and $l \equiv \frac{b}{2} \bmod \mathbb{Z}$ only.

If weight and index are integral and the character is trivial, our Jacobi-forms are just the Jacobi-forms from [8], such as the Eisenstein-series $E_{k,m}$ for even $k \geq 4$ and the first cusp-forms of index 1, $\phi_{10,1}$ and $\phi_{12,1}$ (in the notation of [8]). Examples of Jacobi-forms with non-trivial character are the Dedekind-function $\eta \in [\mathrm{MJ}_2(\mathbb{Z}), \frac{1}{2}, 0, v_\eta \times 1]$ (which does not depend on the second variable of course) and the theta-series

$$\vartheta_{1/2}(\tau,z) := \sum_{m\in\mathbb{Z}}\left(\frac{-4}{m}\right)e^{2\pi i(m^2\tau/8+mz/2)} \in [\mathrm{MJ}_2(\mathbb{Z})\,, \tfrac{1}{2}, \tfrac{1}{2}, v_\eta^3 \times v_H]\,,$$

$$\vartheta_{3/2}(\tau,z) := \sum_{m\in\mathbb{Z}}\left(\frac{12}{m}\right)e^{2\pi i(m^2\tau/24+mz/2)} \in [\mathrm{MJ}_2(\mathbb{Z}), \tfrac{1}{2}, \tfrac{3}{2}, v_\eta \times v_H]\,.$$

Note that $\vartheta_{1/2}(\tau,0) = 0$ for all $\tau \in \mathbb{H}_1$, whereas $\vartheta_{3/2}(\tau,0) = 2\eta(\tau) \neq 0$ for all $\tau \in \mathbb{H}_1$ (as can be seen from the well-known product-expansions [15]). More examples of Jacobi-forms with non-trivial character can be constructed in the following way: If $\varphi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, 1]_{\mathrm{cusp}}$ is a cusp-form with trivial character, then (we may assume $m \in \mathbb{Z}$ without restriction and)

$$\text{(3.4)} \qquad \varphi\,\eta^{-j} \in [\mathrm{MJ}_2(\mathbb{Z}), k - j/2, m, v_\eta^{-j} \times 1]\,, \qquad \text{if} \quad j \in \mathbb{N} \text{ with } jm \leq 18\,.$$

This is because $\eta^{-j}(\tau) = e^{-2\pi ij\tau/24}(1 + \mathcal{O}(e^{2\pi i\tau}))$ and for a cusp-form in $[\mathrm{MJ}_2(\mathbb{Z}), k, m, 1]_{\mathrm{cusp}}$, non-trivial Fourier-coefficients $\alpha(n,l)$ exist for $4nm - l^2 \geq 3$ only. Then $4(n - j/24)m - l^2 \geq 3 - jm/6 \geq 0$ for $18 \geq jm$.

As is well known, Jacobi-forms appear as Fourier-Jacobi-coefficients of paramodular forms. In the reverse, Jacobi-forms can be lifted to paramodular forms (by so called "arithmetical liftings", i.e. generalizations of Maaß's construction [25]), as described in [13, 14] (for trivial character) and [15] (for nontrivial character).

We take a slightly different point of view so far, as we use the correspondence of Jacobi-forms to vector-valued modular forms for $\mathrm{Mp}_2(\mathbb{Z})$ in order to lift such vector-valued forms. In this way, "arithmetical liftings" (for trivial character) were described in [1, Th. 14.3] in a more general context.

As in [8], it is easily seen, that the Fourier-coefficients $\alpha(n,l)$ of a Jacobi-form of weight m depend on $4mn - l^2$ and $l \bmod 2m\mathbb{Z}$ only, essentially. If non-trivial characters are to be taken into account, this has to be changed slightly.

LEMMA 3.3. *Assume* $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$. *Then* $e^{-\pi ibl/2m}\alpha(n,l)$ *depends on* $4mn - l^2$ *and* $l \bmod 2m\mathbb{Z}$ *only.*

Proof. We may assume $m \equiv b/2 \bmod \mathbb{Z}$ (or $\Phi = 0$). It follows from the definition, using the transformation-formula (3.3) for rot $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right)$ and trans $\left(\begin{smallmatrix} 0 & v \\ v & 0 \end{smallmatrix}\right)$, that

$$e^{2\pi i m(u^2\tau+2uz)}\Phi(\tau, z + u\tau + v) = (-1)^{b(u+v)}\Phi(\tau, z)\,.$$

This implies for $n, l \in \mathbb{Q}$

$$\alpha(n + mu^2 + lu, l + 2mu) = e^{-2\pi i l v}(-1)^{b(u+v)}\alpha(n, l) = (-1)^{bu}\alpha(n, l)\,, \tag{3.5}$$

since $l \equiv b/2 \bmod \mathbb{Z}$ (or $\alpha(n, l) = 0$). With $(n', l') = (n + mu^2 + lu, l + 2mu)$, we have $u = (l' - l)/2m$ and $4mn' - l'^2 = 4mn - l^2$. Now (3.5) can be formulated as

$$e^{-\pi i b l'/2m}\alpha(n', l') = e^{-\pi i b l'/2m}(-1)^{bu}\alpha(n, l) = e^{-\pi i b l/2m}\alpha(n, l)\,.$$

■

For $m, x \in \frac{1}{2}\mathbb{Z}$ and $b \in \mathbb{Z}$ with $m \equiv x \equiv \frac{b}{2} \bmod \mathbb{Z}$ define a theta-series $\theta_{m,x,b}$ on $\mathbb{H}_1 \times \mathbb{C}$ by

$$\theta_{m,x,b}(\tau, z) = \sum_{l \equiv x \bmod 2m\mathbb{Z}} e^{\pi i b l/2m} e^{2\pi i (l^2\tau/4m + lz)}\,.$$

Obviously, $\theta_{m,x,b}$ depends on $x \bmod 2m\mathbb{Z}$ only. For $b = 0$ (and m integral), these are just the theta-series $\theta_{m,x}$ from [8, §5, (4)]. $\theta_{m,x,b}$ can be reduced to $\theta_{m,x}$ as follows: If m is integral, then $\theta_{m,x,b}(\tau, z) = \theta_{m,x}(\tau, z + b/4m)$. In the case $m \in \frac{1}{2} + \mathbb{Z}$, we have

$$\begin{aligned}\theta_{m,x,b}(\tau, z) &= \sum_{l \equiv x \bmod 2m\mathbb{Z}} e^{2\pi i (l^2\tau/4m + lz + lb/4m)} = \sum_{2l \equiv 2x \bmod 4m\mathbb{Z}} e^{2\pi i ((2l)^2 \frac{\tau}{2}/8m + 2l(\frac{z}{2} + b/8m))} \\ &= \theta_{2m,2x}(\tau/2, z/2 + b/8m)\,.\end{aligned}$$

Especially we find the following transformation law for $\theta_{m,x,b}$ under the generators of $\mathrm{Mp}_2(\mathbb{Z})$ (compare [8,§5], [1, Sec. 4]).

$$\left(\theta_{m,x,b}\right)^{\sim}_{m} |_{1/2}\widehat{T} = e^{2\pi i x^2/4m} \left(\theta_{m,x,b}\right)^{\sim}_{m}\,, \tag{3.6}$$

$$\begin{aligned}\left(\theta_{m,x,b}\right)^{\sim}_{m} |_{1/2}\widehat{J} = & e^{2\pi i b x/4m} \frac{1}{\sqrt{2mi}} \sum_{x' : (m+\mathbb{Z})/2m\mathbb{Z}} \\ & \times e^{-2\pi i x' x/2m} e^{-2\pi i b x'/4m} \left(\theta_{m,x',b}\right)^{\sim}_{m}\,.\end{aligned} \tag{3.7}$$

Let $\mathbb{V}_m := \{f : (m + \mathbb{Z})/2m\mathbb{Z} \to \mathbb{C}\}$ be the vector-space of complex functions on $(m + \mathbb{Z})/2m\mathbb{Z}$. The characteristic functions $f^c_x \in \mathbb{V}_m$, $x \in (m + \mathbb{Z})/2m\mathbb{Z}$, form a basis of $\mathbb{V}_m$.

We define the $\mathbb{V}_m$-valued theta-series $\Theta_{m,b} : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{V}_m$ by $\Theta_{m,b}(\tau, z)(x) = \theta_{m,x,b}(\tau, z)$. Now the formulas (3.6) and (3.7) imply, that $\Theta_{m,b}$ is a $\mathbb{V}_m$-valued modular form for $\mathrm{Mp}_2(\mathbb{Z})$ with a multiplier-system $\rho_{m,b}$ (i.e. a representation $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{V}_m)$), determined by (3.6) and (3.7) (more precisely, $\Theta_{m,b}$ is a $\mathbb{V}_m$-valued metaplectic Jacobi-form; see the following definition 3.4). Explicitly, for $f \in \mathbb{V}_m$ we have

$$(\rho_{m,b}(\widehat{T})f)(x) = e^{2\pi i x^2/4m} f(x)\,,$$
$$(\rho_{m,b}(\widehat{J})f)(x) = e^{2\pi i bx/4m} \frac{1}{\sqrt{2mi}} \sum_{x':(m+\mathbb{Z})/2m\mathbb{Z}} e^{-2\pi i x'x/2m} e^{-2\pi i bx'/4m} f(x')\,.$$

Up to isomorphy, $\rho_{m,b}$ does not depend on b. Given $n \in 2\mathbb{Z}$ we define $A_n \in \mathrm{GL}(\mathbb{V}_m)$ by $(A_n f)(x) = e^{2\pi i nx/4m} f(x)$ (note that $e^{2\pi i nx/4m}$ depends on x modulo $2m$ only, since n is even). Now if $b, b' \in \mathbb{Z}$ satisfy $b \equiv b' \bmod 2\mathbb{Z}$, then we see $\rho_{m,b'}(M) = A_{b'-b}\rho_{m,b}(M)A_{b'-b}^{-1}$, or, in other words, $\rho_{m,b'}$ and $\rho_{m,b}$ are conjugate by $A_{b'-b}$.

DEFINITION 3.4. *Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be a finite-dimensional representation, such that ρ factors over a principal congruence group $\mathrm{Mp}_2(\mathbb{Z})[n]$. A holomorphic function $f : \mathbb{H}_1 \to V$ is a (V-valued) meromorphic modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ with multiplier-system ρ, if*

$$f\big|_k M = \rho(M) f \quad \text{for all} \quad M \in \mathrm{Mp}_2(\mathbb{Z})$$

(here the action of $\mathrm{Mp}_2(\mathbb{Z})$ is defined as in (3.1)) and f has at most a pole at the cusp $i\infty$. If in addition f is bounded in any region $\Im(\tau) \ge y_0 > 0$ (i.e. if there is no pole at the cusp), then f is a (holomorphic) modular form. Moreover, if $\lim_{\Im(\tau)\to\infty} f(\tau) = 0$, then f is cusp-form. The space of meromorphic modular forms of weight k with multiplier-system ρ is denoted by $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]_{\mathrm{mer}}$, the subspace of (holomorphic) modular forms is denoted by $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$ and the subspace of cusp-forms by $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]_{\mathrm{cusp}}$.

Of course, if k is integral, the action of $\mathrm{Mp}_2(\mathbb{Z})$ factors over $\mathrm{SL}_2(\mathbb{Z})$ and we can think of $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$ as a space of (V-valued) elliptic modular forms. In our case, the vector-space V will almost always be $\mathbb{V}_m$ for some $m \in \frac{1}{2}\mathbb{Z}$. For $f : \mathbb{H}_1 \to \mathbb{V}_m$, we define the components $f_x : \mathbb{H}_1 \to \mathbb{C}$ for $x \in (m+\mathbb{Z})/2m\mathbb{Z}$ by $f(\tau) = \sum_{x:(m+\mathbb{Z})/2m\mathbb{Z}} f_x(\tau) f_x^c$.

On $\mathbb{V}_m$ there is a scalar product, defined by

$$(f, g) = \sum_{x:(m+\mathbb{Z})/2m\mathbb{Z}} f(x)g(x)$$

(this pairing is *not* hermitian, but respects holomorhy instead). For a representation ρ of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{V}_m$, denote by ρ^* the dual representation of ρ with respect to the pairing above, i.e. ρ^* satisfies $(\rho^*(M)f, \rho(M)g) = (f, g)$ for all $f, g \in \mathbb{V}_m$ and $M \in \mathrm{Mp}_2(\mathbb{Z})$.

LEMMA 3.5. *Let $k \in \frac{1}{2}\mathbb{Z}$, $v_{a,b} \in \mathrm{MJ}_2(\mathbb{Z})^{\mathrm{ab}}$ and $m \in \frac{1}{2}\mathbb{Z}$ with $2m \equiv b \bmod 2\mathbb{Z}$. Then*

$$[\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{1}{2}, v_\eta^a \rho_{m,b}^*] \to [\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]\,,$$
$$f = (f_x)_{x:(m+\mathbb{Z})/2m\mathbb{Z}} \mapsto (f, \Theta_{m,b}) = \sum_{x:(m+\mathbb{Z})/2m\mathbb{Z}} f_x \theta_{m,x,b}$$

is an isomorphism of the vector-spaces.

Proof. If $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, v_\eta^a \rho_{m,b}^*]$, then $F = (f, \Theta_{m,b})$ transforms as a Jacobi-form of weight k, index m with character $v_{a,b}$. For the Heisenberg-part of $\mathrm{MJ}_2(\mathbb{Z})$ this follows from

$$\begin{aligned}\widetilde{F}_m\big|_k[u,v;w](Z) &= \widetilde{F}_m\begin{pmatrix} z_1 & z_2+uz_1+v \\ z_2+uz_1+v & z_3+u^2z_1+2uz_2+uv+w\end{pmatrix} \\ &= \big(f(z_1),\,\Theta_{m,b}(z_1,z_2+uz_1+v)\big)\,e^{2\pi i m(z_3+u^2z_1+2uz_2+uv+w)} \\ &= (-1)^{b(u+v)}e^{2\pi i m(uv+w)}\,\big(f(z_1),\,\Theta_{m,b}(z_1,z_2)\big)\,e^{2\pi i m z_3} \\ &= v_H([u,v;w])^b\,\widetilde{F}_m(Z)\end{aligned}$$

for $Z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3\end{smallmatrix}\right) \in \mathbb{H}_2$ (note that $\theta_{m,x,b}(\tau, z+u\tau+v) = (-1)^{b(u+v)}e^{-2\pi i m(u^2\tau+2uz)} \times\theta_{m,x,b}\,(\tau,z)$ for $u, v \in \mathbb{Z}$ since $2m \equiv b \bmod 2\mathbb{Z}$). Given $M \in \mathrm{Mp}_2(\mathbb{Z})$ we have

$$\begin{aligned}\widetilde{F}_m\big|_k M(Z) &= \big(f\big|_{k-1/2}M(z_1)\,,\,(\Theta_{m,b})^{\sim}_{m}\big|_{1/2}M(Z)\big) \\ &= v_\eta^a(M)\,\big(\rho^*_{m,b}(M)f(z_1)\,,\,\rho_{m,b}(M)(\Theta_{m,b})^{\sim}_{m}(Z)\big) \\ &= v_\eta^a(M)\,\big(f(z_1),\,\Theta_{m,b}(z_1,z_2)\big)\,e^{2\pi i m z_3} = v_\eta^a(M)\widetilde{F}_m(Z)\,,\end{aligned}$$

since $\rho^*_{m,b}$ is the dual of $\rho_{m,b}$ with respect to the given scalar product on $\mathbb{V}_m$. Moreover F satisfies the cusp-condition (since all f_x and $\theta_{m,x,b}$ do). The mapping is injective, since for any fixed $\tau \in \mathbb{H}_1$, the theta-series $\theta_{m,x,b}(\tau,\,\cdot\,)$, $x \in (m+\mathbb{Z})/2m\mathbb{Z}$, are linearly independent (as functions of the second variable). Finally we show that the mapping is surjective. Let $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$ with Fourier-development

$$\Phi(\tau,z) = \sum_{\substack{n\equiv n_0,\, l\equiv l_0 \bmod \mathbb{Z},\\ 4mn-l^2\geq 0}} \alpha(n,l)e^{2\pi i(n\tau+lz)}$$

(where $n_0 = \frac{a}{24}$ and $l_0 = \frac{b}{2}$, of course). By lemma 3.3 we know, that $c_l(4mn-l^2) := e^{-\pi i bl/2m}\alpha(n,l)$ depends on $4mn-l^2$ and $l \bmod 2m\mathbb{Z}$ only. Therefore we have

$$\begin{aligned}\Phi(\tau,z) &= \sum_{l\equiv l_0 \bmod \mathbb{Z}}\ \sum_{\substack{n\equiv n_0 \bmod \mathbb{Z},\\ 4mn-l^2\geq 0}} c_l(4mn-l^2)e^{2\pi i(4mn-l^2)\tau/4m}e^{\pi i bl/2m}e^{2\pi i(l^2\tau/4m+lz)} \\ &= \sum_{l':(l_0+\mathbb{Z})/2m\mathbb{Z}}\ \sum_{\substack{N\equiv 4mn_0-l'^2 \bmod \mathbb{Z},\\ N\geq 0}} c_{l'}(N)e^{2\pi i N\tau/4m} \sum_{l\equiv l' \bmod \mathbb{Z}} e^{\pi i bl/2m}e^{2\pi i(l^2\tau/4m+lz)} \\ &= \sum_{l':(l_0+\mathbb{Z})/2m\mathbb{Z}} f_{l'}(\tau)\theta_{m,l',b}(\tau,z) = \big(f,\,\Theta_{t,b}\big)\,(\tau,z)\,,\end{aligned}$$

(note that $l' \in \frac{1}{2}\mathbb{Z}$, thus $l \equiv l' \bmod \mathbb{Z}$ implies $l^2 \equiv l'^2 \bmod \mathbb{Z}$) with $f = (f_x)_{x:(m+\mathbb{Z})/2m\mathbb{Z}}$ where

$$f_x(\tau) = \sum_{0\leq N\equiv 4mn_0-x^2 \bmod \mathbb{Z}} c_{l'}(N)e^{2\pi i N\tau/4m}\,.$$

Now $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$ implies $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, v_\eta^a \rho^*_{m,b}]$ (using the linear independence of the theta-series once more). ■

The arithmetical lifting uses Hecke-operators on Jacobi-forms of integral weight with characters (since paramodular forms have integral weight). In this case, the character factors over $\mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$, i.e. it is of the form $v_{a,b}$ with a even. Thus let $\Phi \in [\mathrm{MJ}_2(\mathbb{Z}), k, m, \xi \times v_H^\varepsilon]$ be a Jacobi-form of integral weight k, where $\xi \in \mathrm{SL}_2(\mathbb{Z})^{\mathrm{ab}}$. Let $Q \in \mathbb{N}$ satisfy $\mathrm{SL}_2(\mathbb{Z})[Q] \subset \mathrm{kern}(\xi)$. Given $l \in \mathbb{N}$ we define the Hecke-operator $\mathcal{T}^{(Q)}(l)$ on Φ as in [15, (1.12)] by

$$\widetilde{\Phi}_m\big|_k \mathcal{T}^{(Q)}(l)(z_1, z_2, z_3) := l^{2k-3} \sum_{ad=l,\, b \bmod d} d^{-k} \xi(\sigma_a) \Phi\left(\tfrac{az_1+bQ}{d}, az_2\right) e^{2\pi i l m z_3}\,.$$

Here, $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ has to satisfy $\sigma_a \equiv \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \bmod Q$. As was proved in [15, Lemma 1.7], if $\gcd(l, 2^\varepsilon Q) = 1$, one has $\Phi\big|_k \mathcal{T}^{(Q)}(l) \in [\mathrm{MJ}_2(\mathbb{Z}), k, lm, \xi_l \times v_H^\varepsilon]$, where ξ_l is a twist of ξ (especially, $\xi_l = \xi$, if $l \equiv 1 \bmod Q$, the only case we will need later on).

Now we can formulate the main result of this section:

PROPOSITION 3.6. *Assume $k \in \mathbb{Z}$. Let d be a divisor of 6 and $\nu = \mu^2\kappa \in \Gamma_3^{*\mathrm{ab}}$. Set $Q = \frac{6}{d}$. Then there is an injective homomorphism*

$$\mathcal{M} : [\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{1}{2}, \widetilde{\nu}^d \rho^*_{d/2,d}] \to [\Gamma_3^*, k, \nu^d \chi^k]\,,$$

defined by ($c_l(n)$ is the n^{th} Fourier-coefficient of $\tau \mapsto f(\tau)(l)$)

$$\mathcal{M}(f)(Z) = c_0(0)\frac{-B_k}{2k} g_k(z_1) + \sum_{m \in \mathbb{N},\, m \equiv 1 \bmod Q\mathbb{Z}} m^{2-k} \left(f, \Theta_{d/2,d}\right)^{\sim}_{d/2} \big|_k \mathcal{T}^{(Q)}(m)(Z)$$

Cusp-forms are mapped to cusp-forms by $\mathcal{M}$.

Proof. Recall that $\nu = v_\eta^4 \times v_H$ as characters of the Jacobi-group $\mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$, thus $\widetilde{\nu} = v_\eta^4$. Therefore by lemma 3.5 we have $\Phi := f \cdot \Theta_{d/2,d} \in [\mathrm{MJ}_2(\mathbb{Z}), k, \frac{d}{2}, v_{4d,d}]$.

First assume $d < 6$. Then $\nu^d \neq 1$ is non-trivial and every $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \widetilde{\nu}^d \rho^*_{d/2,d}]$ is a cusp-form, i.e. $c_0(0) = 0$. Thus in the notation of [15, Th. 1.12] we have $\mathcal{M}(f) = \mathrm{Lift}_1(\Phi)$. Note that the series defining $\mathcal{M}(f)$ converges for all $k \geq 0$ in this case, since Φ is a cusp-form. Now [15, Th. 1.12] implies the claim (because of $Q = \frac{6}{d}$, the lift has level $Q\,d/2 = 3$).

Now let $d = 6$. In this case the character ν^d is trivial and the Hecke-operators $\mathcal{T}^{(1)}(m)$ are just the Hecke-operators $T_-(m)$ from [14]. Thus in the notation of [14, Hauptsatz 2.1] we have $\mathcal{M}(f) = F_\Phi$. For a cusp-form Φ, this series is $\mathrm{Lift}_1(\Phi)$ as in [15] again and converging for $k \geq 0$. In general, Φ is cuspidal if and only if $\Phi(\tau, 0) \in [\mathrm{SL}_2(\mathbb{Z}), k, 1]$ is cuspidal. Thus if Φ is not cuspidal, we may assume $k \geq 4$ and the series defining $\mathcal{M}(f)$ converges in this case too. Now [14, Hauptsatz 2.1, Satz 3.8] imply the claim (note that Jacobi-forms of prime index p are necessarily eigenforms of the operator W_p from [8]). ■

$\mathcal{M}$ is the Maaß-lift and the image $\mathcal{M}_{k,d} := \mathcal{M}([\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \widetilde{\nu}^d \rho^*_{d/2,d}])$ is the Maaß-space with character $\nu^d \chi^k$.

Using the arithmetical lifting Lift_{-1} from [15, Th. 1.12], one can define homomorphisms $[\mathrm{Mp}_2(\mathbb{Z}), k-\frac{1}{2}, \widetilde{v}^d \rho^*_{d/2,d}] \to [\Gamma_3^*, k, \overline{v}^d \chi^k]$ too, but these are not necessarily injective (and they are not injective in our case, as can be seen, combining results from section 5 with the dimension-formulas for the spaces $[\mathrm{Mp}_2(\mathbb{Z}), k-\frac{1}{2}, \widetilde{v}^d \rho^*_{d/2,d}]$, given in this section).

Note that the representation $\rho_{m,b}$ is reducible in almost all cases, i.e. except for $m=\frac{1}{2}$ and $m=1$. For our needs, the decomposition is given as follows: $\widehat{J}^2 = (-I, i)$ is a central element in $\mathrm{Mp}_2(\mathbb{Z})$. On $\mathbb{V}_m$, the element $\widehat{J}^2$ acts via $\rho_{m,b}$ as

$$(\rho_{m,b}(\widehat{J}^2) f)(x) = -i e^{2\pi i b x/2m} f(-x)\,. \tag{3.8}$$

Define $W(-1) := \rho_{m,b}(\widehat{J}^2)$. Since $\widehat{J}^2$ is central, $W(-1)$ commutes with $\rho_{m,b}$. Thus all eigenspaces of $W(-1)$ are invariant under $\rho_{m,b}$. Because $W(-1)^2 = -\mathrm{id}_{\mathbb{V}_m}$, $W(-1)$ has order 4 and non-trivial eigenspaces for eigenvalues $\pm i$ only. In this case denote by $\mathbb{V}_{m,s} \subset \mathbb{V}_m$ the eigenspace with eigenvalue s of $W(-1)$. The restriction of $\rho_{m,b}$ to $\mathbb{V}_{m,s}$ is denoted by $\rho_{m,b,s}$. In our special cases (where $2m|6$), it turns out, that $\rho_{m,b,s}$ is always irreducible (though it is not in general; compare [8] and [29]). Note that $W(-1)$ acts as a scalar (that is, $W(-1) = c\,\mathrm{id}_{\mathbb{V}_m}$ for some $c \in \mathbb{C}^\times$), if and only if $m \le 1$. Thus $\rho_{m,b}$ is reducible, if $m > 1$ (and easily seen to be irreducible for $m \le 1$). The decomposition $\rho_{m,b} = \rho_{m,b,i} \oplus \rho_{m,b,-i}$ induces a decomposition

$$[\mathrm{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho^*_{t,b}] = [\mathrm{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho^*_{t,b,i}] \oplus [\mathrm{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho^*_{t,b,-i}]\,,$$

and via the isomorphism from lemma 3.5 a decomposition

$$[\mathrm{MJ}_2(\mathbb{Z}), k, t, v_{a,b}] = [\mathrm{MJ}_2(\mathbb{Z}), k, t, v_{a,b}, i] \oplus [\mathrm{MJ}_2(\mathbb{Z}), k, t, v_{a,b}, -i]\,.$$

Via the Maaß-lift $\mathcal{M}$, we also get a decomposition

$$\mathcal{M}_{k,d} = \mathcal{M}_{k,d,i} \oplus \mathcal{M}_{k,d,-i}\,,$$

where $\mathcal{M}_{k,d,s} := \mathcal{M}([\mathrm{Mp}_2(\mathbb{Z}), k-\frac{1}{2}, \widetilde{v}^d \rho^*_{d/2,d,s}])$.

The decomposition of $\rho_{m,b}$ with respect to $W(-1)$ has to be known, if we want to evaluate Skoruppa's dimension-formula [29, 7] for $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$, since for this formula, $\rho(\widehat{J}^2)$ has to act as a scalar.

LEMMA 3.7 ([7, TH. 4.2], [29, SATZ 5.1]). *Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be a representation of dimension n of $\mathrm{Mp}_2(\mathbb{Z})$ with $\mathrm{Mp}_2(\mathbb{Z})[N] \subset \mathrm{kern}(\rho)$ for some $N \in \mathbb{N}$ and $\rho(\widehat{J}^2) = \zeta\,\mathrm{id}_V$ with a fourth root of unity ζ. Let $l_j \in \mathbb{R}$, $j = 1, \cdots, n$, be such that $e^{2\pi i l_j}$ runs through the eigenvalues of $\rho(\widehat{T})$. Define*

$$A(\rho) := \sharp\{j \mid l_j \equiv 0 \bmod \mathbb{Z}\}\,, \qquad B(\rho) := \sum_{j=1}^{n} \mathbb{B}_1(l_j)$$

(here $\mathbb{B}_1$ is given by $\mathbb{B}_1(x) = 0$ for $x \in \mathbb{Z}$ and $\mathbb{B}_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$ for $x \in \mathbb{R} \setminus \mathbb{Z}$). Then for $k \in \frac{1}{2}\mathbb{Z}$ one has

$$\dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho] - \dim[\mathrm{Mp}_2(\mathbb{Z}), 2-k, \rho^*]_{\text{cusp}}$$
$$= \begin{cases} 0, & \textit{if} \quad \rho(\widehat{-1}) \neq i^{-2k}\,\mathrm{id}_V, \\ \dfrac{k-1}{12}\dim(\rho) + \dfrac{1}{2}A(\rho) - B(\rho) + \dfrac{1}{4}\Re(e^{2\pi i k/4}\operatorname{trace}\rho(\widehat{J})) & \\ +\dfrac{2}{3\sqrt{3}}\Re(e^{2\pi i(k+\frac{1}{2})/6}\operatorname{trace}\rho(\widehat{J}\widehat{T}))\,, & \textit{if} \quad \rho(\widehat{-1}) = i^{-2k}\,\mathrm{id}_V. \end{cases}$$

Note that for $k \geq 2$, the formula gives an explicit expression for $\dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$, since $\dim[\mathrm{Mp}_2(\mathbb{Z}), 2-k, \rho^*]_{\text{cusp}} = 0$ in this case. For $k = \frac{1}{2}$ and $k = \frac{3}{2}$ there is an explicit formula for $\dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$ in [29] also.

Let $\mathcal{R} = \mathbb{C}[g_4, g_6]$ be the graded ring of elliptic modular forms. Using the dimension-formula from lemma 3.7 as in [29, Satz 7.3], we see that

$$[\mathrm{Mp}_2(\mathbb{Z}), \tfrac{1}{2}\mathbb{Z}, \rho] := \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} [\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$$

always is a free module of rank $\dim(\rho)$ over $\mathcal{R}$. By lemma 3.5, the same is true for the analogously defined spaces $[\mathrm{MJ}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, m, v_{a,b}]$ of Jacobi-forms.

A basis for the eigenspace $\mathbb{V}_{m,s}$ is given as follows (recall $f_x^c \in \mathbb{V}_m$ being the characteristic function of $x \in (m+\mathbb{Z})/2m\mathbb{Z}$ and $W(-1)^2 = -\mathrm{id}_{\mathbb{V}_m}$): For $s \in \{\pm i\}$ let

$$f_{x,s}^c := f_x^c - s\,W(-1) f_x^c \in \mathbb{V}_{m,s}\,.$$

If $2x \not\equiv 0 \bmod 2m$ then $f_{x,s}^c \neq 0$. If on the other hand $2x \equiv 0 \bmod 2m$ then we have $W(-1) f_x^c = -i(-1)^{2m} f_x^c$, i.e. $f_x^c \in \mathbb{V}_{m,-i(-1)^{2m}}$. Now let

$$\mathcal{B}_{0,s} = \left\{ f_x^c \,\middle|\, x \in (m+\mathbb{Z})/2m\mathbb{Z},\ 2x \equiv 0 \bmod 2m,\ s = -i(-1)^{2m} \right\},$$
$$\mathcal{B}_{1,s} = \left\{ f_{x,s}^c \,\middle|\, x \in \pm(m+\mathbb{Z})/2m\mathbb{Z},\ 2x \not\equiv 0 \bmod 2m \right\}.$$

Then $\mathcal{B}_s := \mathcal{B}_{0,s} \cup \mathcal{B}_{1,s}$ is a basis of $\mathbb{V}_{m,s}$. Using this basis, one can calculate all the parameters in the dimension-formula from lemma 3.7 for $\rho_{m,b,s}$. For example one has

$$\operatorname{trace}(\rho_{m,b,s}(\widehat{J})) = \sum_{\substack{x:\pm(m+\mathbb{Z})/2m\mathbb{Z},\\ 2x \not\equiv 0 \bmod 2m}} \frac{e^{-2\pi i x^2/2m} + i s e^{2\pi i x^2/2m}}{\sqrt{2mi}} + \sum_{\substack{x:(m+\mathbb{Z})/2m\mathbb{Z},\\ 2x \equiv 0 \bmod 2m}} \frac{e^{-2\pi i x^2/2m}}{\sqrt{2mi}}\,,$$
$$\operatorname{trace}(\rho_{m,b,s}(\widehat{J}\widehat{T})) = \sum_{\substack{x:\pm(m+\mathbb{Z})/2m\mathbb{Z},\\ 2x \not\equiv 0 \bmod 2m}} \frac{e^{-2\pi i x^2/4m} + i s e^{2\pi i x^2 3/4m}}{\sqrt{2mi}} + \sum_{\substack{x:(m+\mathbb{Z})/2m\mathbb{Z},\\ 2x \equiv 0 \bmod 2m}} \frac{e^{-2\pi i x^2/4m}}{\sqrt{2mi}}\,.$$

Of course it is possible to find explicit formulas for the traces for all the irreducible constituents of $\rho_{m,b}$, as was shown for $m \in \mathbb{Z}$ in [29].

The following table lists the parameters of the dimension formula for $[\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \rho]$ with $\rho = \widetilde{\nu}^d \rho^*_{d/2,d,s}$, where d is a divisor of 6 and $\nu = \mu^2\kappa \in \Gamma_3^{*\mathrm{ab}}$:

d	s	$A(\rho)$	$B(\rho)$	$\dim(\rho)$	trace $\rho(\widehat{J})$	trace $\rho(\widehat{JT})$	ρ
1	i	0	$\frac{-11}{24}$	1	$\frac{1-i}{\sqrt{2}}$	$\frac{\sqrt{3}-i}{2}$	$\widetilde{\nu}\rho^*_{1/2,i}$
2	$-i$	0	$\frac{-7}{12}$	2	0	$\frac{-i-\sqrt{3}}{2}$	$\widetilde{\mu}\rho^*_{1,-i}$
3	i	0	$\frac{-5}{12}$	2	0	$\frac{-\sqrt{3}+i}{2}$	$\widetilde{\kappa}\rho^*_{3/2,i}$
3	$-i$	0	$\frac{-1}{24}$	1	$-\frac{1+i}{\sqrt{2}}$	$\frac{\sqrt{3}+i}{2}$	$\widetilde{\kappa}\rho^*_{3/2,-i}$
6	i	0	$\frac{7}{12}$	2	0	$\frac{-\sqrt{3}+i}{2}$	$\rho^*_{3,i}$
6	$-i$	1	$\frac{1}{3}$	4	0	$\frac{\sqrt{3}+i}{2}$	$\rho^*_{3,-i}$

Using these parameters, we can calculate the dimensions of the Maaß-spaces $\mathcal{M}_{k,d,s}$. The following table lists the dimensions of $\mathcal{M}_{k,d,s}$ for $k \leq 14$. There are no nontrivial forms of weight $k \leq 0$. The last two columns list the character ν of the forms in the Maaß-space $\mathcal{M}_{k,d,s}$ and a basis of the module $[\mathrm{MJ}_2(\mathbb{Z}), \mathbb{Z}, d/2, (v_{4,1})^d, s]$ over $\mathcal{R}$. We use the following abbreviations:

$$\phi_{1,1/2} := \eta\vartheta_{1/2} \in [\mathrm{MJ}_2(\mathbb{Z}), 1, \tfrac{1}{2}, v_{4,1}, i]\,,$$

$$\phi_{4,1} := \frac{\phi_{12,1}}{\eta^{16}} \in [\mathrm{MJ}_2(\mathbb{Z}), 4, 1, v_{8,0}, -i]\,,$$

$$\phi_{6,3/2} := \eta^{11}\vartheta_{3/2} \in [\mathrm{MJ}_2(\mathbb{Z}), 6, \tfrac{3}{2}, v_{12,1}, -i]\,,$$

$$\phi_{8,3} := \frac{\phi_{12,1}}{\eta^{12}}\vartheta_{1/2}^4 \in [\mathrm{MJ}_2(\mathbb{Z}), 8, 3, 1, -i]\,,$$

$$\phi_{9,3} := \eta^{14}\vartheta_{1/2}^3\vartheta_{3/2} \in [\mathrm{MJ}_2(\mathbb{Z}), 9, 3, 1, i]\,,$$

$$\phi_{11,3} := \frac{\phi_{12,1}}{\eta^4}\vartheta_{1/2}\vartheta_{3/2} \in [\mathrm{MJ}_2(\mathbb{Z}), 11, 3, 1, i]\,.$$

d	s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	ν	
1	i	1	0	0	0	1	0	1	0	1	0	1	0	2	0	$\mu^2\kappa$	$\phi_{1,1/2}$
2	$-i$	0	1	0	1	0	1	0	2	0	2	0	2	0	3	μ	$\phi_{1,1/2}^2$, $\phi_{4,1}$
3	i	0	0	1	0	1	0	1	0	2	0	2	0	2	0	κ	$\phi_{1,1/2}^3$, $\phi_{1,1/2}\phi_{4,1}$
3	$-i$	0	0	0	0	0	1	0	0	0	1	0	1	0	1	κ	$\phi_{6,3/2}$
6	i	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	$\phi_{9,3}$, $\phi_{11,3}$
6	$-i$	0	0	0	1	0	2	0	2	0	3	0	4	0	4	1	$E_{4,3}$, $E_{6,3}$, $\phi_{1,1/2}^6$, $\phi_{8,3}$

Note that it is easy to deduce the dimensions of $\mathcal{M}_{k,d,s}$ for $k \geq 15$ from the values given in the table, since for $k \geq 2$ we have by the dimension formula $\dim[\mathrm{Mp}_2(\mathbb{Z}), k + 12, \rho] = \dim(\rho) + \dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$.

For $k = 1$ and $k = 2$ lemma 3.7 does not give the dimensions of $\mathcal{M}_{k,d,s}$ explicitly and the results from [29, Satz 5.2] have to be used. Alternatively, some ad hoc arguments can be given as follows: Since $[\mathrm{Mp}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, \rho]$ is a free module of rank $\dim(\rho)$ over $\mathcal{R}$, we find $\dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho] \leq \dim[\mathrm{Mp}_2(\mathbb{Z}), k + l, \rho]$ for all $4 \leq l \in 2\mathbb{N}$. Especially $\dim \mathcal{M}_{k,d,s} \leq r_{d,s}(k) := \min\{\dim \mathcal{M}_{k+4,d,s}, \dim \mathcal{M}_{k+6,d,s}\}$. For $k = 1$ or $k = 2$ this minimum is 0 except for the following cases:
$\dim \mathcal{M}_{1,1,i} = 1$, since $\widetilde{\mu}^2\widetilde{\kappa}\rho^*_{1/2,i}$ is the multiplier-system of the η-function and $r_{1,i}(1) = 1$.
$\dim \mathcal{M}_{2,2,-i} = 1$, since $r_{2,-i}(2) = 1$ and $\phi^2_{1,1/2} \in [\mathrm{MJ}_2(\mathbb{Z}), 2, 1, v_{8,0}, -i]$.
$\dim \mathcal{M}_{2,6,-i} = 0$, since $\dim[\mathrm{MJ}_2(\mathbb{Z}), 2, 3, 1] = 0$ was proved in [8, Th. 9.1].
$\dim \mathcal{M}_{1,3,i} = 0$, since $\dim \mathcal{M}_{2,6,-i} = 0$ and $0 \neq \varphi \in [\mathrm{MJ}_2(\mathbb{Z}), 1, \frac{3}{2}, v_{12,1}, i]$ would imply $0 \neq \varphi^2 \in [\mathrm{MJ}_2(\mathbb{Z}), 2, 3, 1, -i]$.

In section 5 we need the restriction of Maaß-lifts to the diagonal, which in the case of paramodular forms is properly defined by a certain Witt-operator (see (5.1)).

If $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \widetilde{\nu}^d\rho^*_{d/2,d}]$, then we associate with f the “Nullwert” of the Jacobi-form $(f, \Theta_{d/2,d})$, which we denote by $\Psi_0 f$, i.e.

$$(\Psi_0 f)(z) = (f, \Theta_{d/2,d})(z, 0)\,.$$

Obviously we have $\Psi_0 f \in [\mathrm{SL}_2(\mathbb{Z}), k, \widetilde{\nu}^d]$. The mapping $f \mapsto \Psi_0 f$ is a surjective homomorphism (as we will prove immediately). The importance of this surjectivity is, that we can show now, how to lift certain modular forms from the diagonal as Maaß-lifts to paramodular forms. Although the following lemma is needed only in a very few special cases later on, we give a slightly more general statement here.

LEMMA 3.8. 1) *Let d be a divisor of 6 and $\nu = \mu^2\kappa \in \Gamma_3^{*\mathrm{ab}}$.*

$$[\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{1}{2}, \widetilde{\nu}^d\rho^*_{d/2,d}] \to [\mathrm{SL}_2(\mathbb{Z}), k, \widetilde{\nu}^d]\,, \qquad f \mapsto \Psi_0 f$$

is a surjective homomorphism.

2) *Let $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \rho^*_{3,6}]$ and assume that $(\Psi_0 f)(\tau) = \sum_{n\in\mathbb{N}_0} \alpha(n)e^{2\pi i n\tau}$ is an simultaneous eigenform of all Hecke-operators $\mathcal{T}^{(1)}(m)$. Then*

$$W_3(\mathcal{M}(f))(z_1, z_3) = \frac{1}{\alpha(1)}\Psi_0 f(z_1)\,\Psi_0 f(z_3)\,. \tag{3.9}$$

Proof. 1) $[\mathrm{SL}_2(\mathbb{Z}), k, \widetilde{\nu}^d] = \{0\}$, if k is odd, since $\widetilde{\nu}(-I) = 1$. Thus we may assume $k \equiv 0 \bmod 2\mathbb{Z}$. Then there are three cases left:

Case 1: $d = 2$. Since $[\mathrm{SL}_2(\mathbb{Z}), k, \widetilde{\nu}^2] = \eta^8[\mathrm{SL}_2(\mathbb{Z}), k - 4, 1]$ by lemma 5.1 and $\phi_{4,1}(\tau, 0) = \phi_{12,1}(\tau, 0)/\eta^{16}(\tau) = 12\eta^8(\tau)$, a pre-image of $\eta^8 f$ in $[\mathrm{MJ}_2(\mathbb{Z}), k, 1, (v_{4,1})^2]$ is given by $\frac{1}{12}\phi_{4,1} f$.

Case 2: $d = 3$. Since $[\mathrm{SL}_2(\mathbb{Z}), k, \widetilde{\nu}^3] = \eta^{12}[\mathrm{SL}_2(\mathbb{Z}), k - 6, 1]$ by lemma 5.1 and $\phi_{6,3/2}(\tau, 0) = 2\eta^{12}(\tau)$, a pre-image of $\eta^{12} f$ in $[\mathrm{MJ}_2(\mathbb{Z}), k, \frac{3}{2}, (v_{4,1})^3]$ is given by $\frac{1}{2}\phi_{6,3/2} f$.

Case 3: $d = 6$. Since a basis of $[\mathrm{SL}_2(\mathbb{Z}), k, 1]$ is given by $g_4^a g_6^b$ for all $(a, b) \in \mathbb{N}_0^2$ with $4a + 6b = k$, it is sufficient to find pre-images of these forms. A pre-image of $g_4^a g_6^b$ in $[\mathrm{MJ}_2(\mathbb{Z}), k, 3, 1]$ is $g_4^{a-1} g_6^b E_{4,3}$, if $a > 0$, or $g_4^a g_6^{b-1} E_{6,3}$, if $b > 0$.

2) Assume $f \in [\mathrm{Mp}_2(\mathbb{Z}), k-\frac{1}{2}, \rho_{3,6}^*]$ is such that $\Psi_0 f$ is an simultaneous eigenform of all Hecke-operators $\mathcal{T}^{(1)}(m)$ (which are the usual Hecke-operators on elliptic modular forms as in [26] or [21]), i.e. $\Psi_0 f\big|_k \mathcal{T}^{(1)}(l) = l^{k-2}\frac{\alpha(l)}{\alpha(1)}\Psi_0 f$ for all $l \in \mathbb{N}$ (because of the normalization of $\mathcal{T}^{(1)}(l)$ chosen here). From the definition we derive for $\Phi = (f, \Theta_{3,6})$

$$\widetilde{\Phi}_3\big|_k \mathcal{T}^{(1)}(m)(z_1, 0, z_3) = m^{2k-3} \sum_{ad=m,\, b \bmod d} d^{-k}\Phi(\tfrac{az_1+b}{d}, 0)e^{2\pi i 3 m z_3}$$
$$= (\Psi_0 f)\big|_k \mathcal{T}^{(1)}(m)(z_1)e^{2\pi i 3mz_3} = m^{k-2}\frac{\alpha(m)}{\alpha(1)}(\Psi_0 f)(z_1)e^{2\pi i3mz_3}\,.$$

Finally we find

$$\begin{aligned} W_3(\mathcal{M}(f))(z_1, z_3) &= \mathcal{M}(f)\begin{pmatrix} z_1 & 0 \\ 0 & z_3/3\end{pmatrix} = c_0(0)\frac{-B_k}{2k}g_k(z_1) \\ &\quad + \sum_{m\in\mathbb{N}} m^{2-k}\widetilde{\Phi}_3\big|_k\mathcal{T}^{(1)}(m)(z_1, 0, z_3/3) \\ &= \alpha(0)\frac{-B_k}{2k}g_k(z_1) + \sum_{m\in\mathbb{N}}\frac{\alpha(m)}{\alpha(1)}(\Psi_0 f)(z_1)\, e^{2\pi i 3mz_3/3} \\ &= \alpha(0)\frac{-B_k}{2k}g_k(z_1) + (\Psi_0 f)(z_1)\frac{1}{\alpha(1)}\sum_{m\in\mathbb{N}}\alpha(m)e^{2\pi i m z_3} \\ &= \alpha(0)\left(\frac{-B_k}{2k}g_k(z_1) - \frac{1}{\alpha(1)}(\Psi_0 f)(z_1)\right) + \frac{1}{\alpha(1)}(\Psi_0 f)(z_1)(\Psi_0 f)(z_3) \\ &= \frac{1}{\alpha(1)}(\Psi_0 f)(z_1)(\Psi_0 f)(z_3)\,, \end{aligned}$$

since $\alpha(0) = 0$ or $\Psi_0 f$ is a multiple of the Eisenstein-series g_k (in which case $\frac{1}{\alpha(1)}\Psi_0 f(z_1) = \frac{-B_k}{2k}g_k(z_1)$). ■

If a suitable Hecke-theory for $[\mathrm{SL}_2(\mathbb{Z}), k, \widetilde{\nu}]$ with nontrivial characters $\widetilde{\nu}$ had been developed, one might prove formulas for the restrictions of the Maaß-lifts to the diagonal in the same way in general.

4. Divisors and Borcherds-products

Using results of R. Borcherds [1], it is possible to find paramodular forms (of degree 2) with known (zero-)divisors, so called Borcherds-products. Borcherds theory is formulated in terms of orthogonal groups and paramodular forms arise from these using well-known isomorphisms of the underlying groups. A description can be found e.g. in [15, Sec. 1.3] and, more general, in [12].

Throughout this section we assume $t \in \mathbb{N}$. Consider the lattice $L := \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}$ equipped with the quadratic form $q_t((l_1, l_2, l_3, l_4, \beta)) = l_1 l_2 + l_3 l_4 - t\beta^2$. L has signature $(2, 3)$. Let

$$K = \{\lambda \in M \mid l_1 = l_2 = 0\},$$
$$\mathcal{K}_+ = \{Y = (y_1, y_3, y_2) \in K \otimes \mathbb{R} \mid q_t(Y), y_1 > 0\}.$$

Associated with L is the half-space $\mathbb{H}_L = K \otimes \mathbb{R} + i\mathcal{K}_+$, which is essentially the Siegel-half-space $\mathbb{H}_2$ of degree 2 via the biholomorphic transformation

$$\omega_t : \mathbb{H}_2 \to \mathbb{H}_L, \quad \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mapsto (z_1, tz_3, z_2).$$

Let $\mathrm{O}(L)^+ = \mathrm{O}(L) \cap \mathrm{SO}(L \otimes \mathbb{R})^+$, where $\mathrm{SO}(L \otimes \mathbb{R})^+$ is the connected component of the identity in the special orthogonal group of $L \otimes \mathbb{R}$. $\mathrm{PO}(L)^+ = \mathrm{O}(L)^+/\{\pm \mathrm{id}\}$ acts on $\mathbb{H}_L$ as a group of biholomorphic transformations.

As described in [15], the arithmetic structure $\mathrm{PO}(L)^+$ corresponds to the group $\mathrm{P}\Gamma_t^{\max} = \Gamma_t^{\max}/\{\pm I\}$ in the symplectic setting, i.e. there is an isomorphism $\Omega_t : \mathrm{P}\Gamma_t^{\max} \to \mathrm{PO}(L)^+$, which is compatible with the identification of the associated half-spaces via ω_t. In other words, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{P}\Gamma_t^{\max} \times \mathbb{H}_2 & \xrightarrow{(\Omega_t, \omega_t)} & \mathrm{PO}(L)^+ \times \mathbb{H}_L \\ \downarrow & & \downarrow \\ \mathbb{H}_2 & \xrightarrow{\omega_t} & \mathbb{H}_L \end{array}$$

where the vertical arrows indicate the action of $\Gamma_t^{\max}$ resp. $\mathrm{O}(L)^+$ on the corresponding half-space. Explicit formulas for Ω_t and the action of $\mathrm{O}(L)^+$ on $\mathbb{H}_2$ can be found in [12, Prop. 2.6] and [15, Sec. 1.3].

Using the automorphic embedding (Ω_t, ω_t) we can think of modular forms for (subgroups of) $\mathrm{O}(L)^+$ as paramodular forms. Note that the weight of the forms is the same on both sides (see [3, Sec. 3.3] for details on the translation of factors of automorphy). We do not worry about how multiplier-systems correspond exactly, since they are left unspecified in the next theorem and we will determine them in the symplectic setting.

Let L' be the dual of L (with respect to the bilinear form $b_t(x, y) = q_t(x + y) - q_t(x) - q_t(y)$ associated with q_t). Explicitly $L' = \mathbb{Z}^2 \times \mathbb{Z}^2 \times \frac{1}{2t}\mathbb{Z}$, thus we can identify L'/L with $\frac{1}{2t}\mathbb{Z}/\mathbb{Z}$ (in the obvious way). For $\lambda = (l_1, l_2, l_3, l_4, \beta) \in L'$ with $q_t(\lambda) < 0$ define

$$\lambda^\perp := \left\{ Z \in \mathbb{H}_2 \mid l_1 - tl_2 \det(Z) + \operatorname{trace}\left(\begin{pmatrix} l_3 & -t\beta \\ -t\beta & tl_4 \end{pmatrix} Z \right) = 0 \right\}.$$

This is a rational quadratic divisor. The discriminant of $\lambda^\perp$ is $\delta(\lambda^\perp) = -4tq_t(\lambda)$, if $\lambda \in L'$ is primitive. As is easily seen, $\Gamma_t^{\max}$ acts on the set of rational quadratic divisors of fixed discriminant: Given $M \in \Gamma_t^{\max}$ and $\lambda^\perp$ a rational quadratic divisor, the set $M\lambda^\perp := \{M \cdot$

$Z \mid Z \in \lambda^\perp\}$ is again a rational quadratic divisor with $\delta(\lambda^\perp) = \delta(M\lambda^\perp)$ (in fact this action realizes a homomorphism $\Gamma_t^{\max} \to \mathrm{PO}(L)^+$, which is essentially the isomorphism Ω_t). Following Freitag and Hermann [12, Lemma 4.4], one can show:

LEMMA 4.1. *All rational quadratic divisors of fixed discriminant are equivalent under* $\Gamma_t^{\max}$.

Note that the Γ_t-orbits of rational quadratic divisors of fixed discriminant are distinguished by their image in the discriminant-group L'/L. The importance of lemma 4.1 is, that if t is prime, any modular form for Γ_t^*, that vanishes on a rational quadratic divisor $\lambda^\perp$, vanishes on the orbit $\Gamma_t^*\lambda^\perp = \bigcup_{M\in\Gamma_t^*} M\lambda^\perp$. This will be used extensively in section 5 to determine generators of graded rings of modular forms for Γ_3^* (with multiplier-systems).

Set $\mathbb{V}_L := \{f : L'/L \to \mathbb{C}\}$ and let $\rho_L : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{V}_L)$ be the Weil-representation associated with the quadratic module $(L'/L, q_t \bmod \mathbb{Z})$ as in [1, Sec. 4]. Then $\rho_L \cong \rho_{t,0}^*$ (where $\rho_{t,0}^*$ is the representation of $\mathrm{Mp}_2(\mathbb{Z})$, which already appeared in the preceeding section) via the identification $\frac{x}{2t} \mapsto x$ of $L'/L = \frac{1}{2t}\mathbb{Z}/\mathbb{Z}$ and $\mathbb{Z}/2t\mathbb{Z}$. Again we have a decomposition $\rho_L = \rho_{L,i} \oplus \rho_{L,-i}$, where $\rho_{L,s}$ is the restriction of ρ_L to the eigenspace of $\rho_L(\widehat{J}^2)$ with eigenvalue s. Note that $\rho_{L,s} \cong \rho_{t,0,-s}^*$ because of the dual. $\mathrm{O}(L)^+$ acts on $\mathbb{V}_L$ by $(Mf)(l) = f(M^{-1}l)$. The discriminant-kernel $\mathrm{O}(L)_d^+ \subset \mathrm{O}(L)^+$ is the subgroup, fixing $\mathbb{V}_L$ pointwise. Via the action on $\mathbb{V}_L$, there is an induced action of $\mathrm{O}(L)^+$ on the space $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho_L]_{\mathrm{mer}}$. The crucial point is, that $\mathrm{O}(L)^+/\mathrm{O}(L)_d^+ \cong \mathrm{O}(L/L')$ is the orthogonal group of $(L'/L, q_t \bmod \mathbb{Z})$. Therefore, the action of $\mathrm{O}(L)^+$ commutes with the action of $\mathrm{Mp}_2(\mathbb{Z})$ via ρ_L on $\mathbb{V}_L$ (and, for square-free t at least, $\mathrm{O}(L/L')$ decomposes ρ_L into irreducible constituents).

$f \in [\mathrm{Mp}_2(\mathbb{Z}), k, \rho_L]_{\mathrm{mer}}$ has a Fourier-expansion

$$f = \sum_{l\in L'/L} \sum_{-\infty \ll n \in q_t(l)+\mathbb{Z}} c(l,n) e^{2\pi i n\tau} f_l^c\,.$$

Note that $c(l,n)$ is defined for $l \in \frac{1}{2t}\mathbb{Z}$ also (via the identification of $\frac{1}{2t}\mathbb{Z}/\mathbb{Z}$ with L'/L).

The following fundamental theorem is a special case of [1, Th. 13.3] or [3, Th. 3.19] for lattices of signature $(2,3)$.

THEOREM 4.2 *Let* $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ *with Fourier-coefficients* $c(l,n)$ *for* $l \in L'/L$ *and* $-\infty \ll n \in q(l)+\mathbb{Z}$. *Assume* $c(l,n) \in \mathbb{Z}$ *for all* $n \le 0$ *and all* $l \in L'/L$. *Then there is a* (*meromorphic*) *modular form* $\mathcal{B}(f)$ *of weight* $c(0,0)/2$ *for the subgroup of* $\mathrm{O}(L)^+$ *fixing* f *with some multiplier-system*, *such that all zeros and poles of* $\mathcal{B}(f)$ *are along rational quadratic divisors* $\lambda^\perp$, $\lambda \in L'$ *primitive*, $q_t(\lambda) < 0$, *with multiplicity*

$$\sum_{r\in\mathbb{N}} c(r\lambda, r^2 q_t(\lambda))\,.$$

We will refer to the functions $\mathcal{B}(f)$ as Borcherds-products. As the name suggests, these functions have product-expansions, but these are converging only “near cusps” in general. As explained above, we think of Borcherds-products $\mathcal{B}(f)$ as paramodular forms also. A description of the product-expansions of Borcherds-products in the symplectic setting was given in our special case in [15, Th. 2.1] (there the input are "weak" Jacobi-forms instead of meromorphic vector-valued modular forms, which is essentially the same by arguments analogous to lemma 3.5).

Every $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ is fixed by the discriminant-kernel $\mathrm{O}(L)_d^+$ at least. But the irreducible constituents of ρ_L are fixed by nontrivial subgroups of $\mathrm{O}(L)^+ / \mathrm{O}(L)_d^+$ in general. For weight $-\frac{1}{2}$, non-trivial contributions to $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ only come from $\rho_{L,i}$ since $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,-i}]_{\mathrm{mer}}$ is trivial. This implies (using the explicit formula for $\rho^*_{t,0}(\widehat{J^2})$ given in (3.8)), that if $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$, then $f(\tau)(-l) = f(\tau)(l)$, i.e. f is invariant under $M \in \mathrm{O}(L)^+$, if M acts as multiplication by -1 on the discriminant-group (at least). Note that $\Omega_t(V_t)$ acts as -1 on the discriminant-group. Thus Borcherds-products are always paramodular forms for Γ_t^* at least. In general, there is a distinguished irreducible constituent of ρ_L, which is invariant under the full group $\mathrm{O}(L)^+$.

We summarize results from [1, Th. 13.3], [3, Th. 3.19] and [15, Th. 2.1] in our special case (in the symplectic setting):

COROLLARY 4.3. *Let* $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ *with Fourier-coefficients* $c(l,n)$ *for* $l \in L'/L$ *and* $-\infty \ll n \in q_t(l) + \mathbb{Z}$. *Assume* $c(l,n) \in \mathbb{Z}$ *for all* $n \leq 0$ *and all* $l \in L'/L$.

1) *There is a* (*meromorphic*) *modular form* $\mathcal{B}(f)$ *of weight* $c(0,0)/2$ *for* Γ_t^* *with some multiplier-system*, *such that all zeros and poles of* $\mathcal{B}(f)$ *are along rational quadratic divisors* $\lambda^\perp$, $\lambda \in L'$ *primitive*, $q_t(\lambda) < 0$, *with multiplicity*

$$\sum_{r \in \mathbb{N}} c(r\lambda, r^2 q_t(\lambda))\,.$$

2) *Let* $n_0 := -\min\{n \in \frac{1}{4t}\mathbb{Z} \mid c(l,n) \neq 0$ *for some* $l \in L'/L\}$, *define*

$$A = \frac{1}{24}\sum_{l \in \mathbb{Z}} c(l/2t, -l^2/4t)\,, \qquad B = \frac{1}{2}\sum_{l \in \mathbb{N}} l\, c(l/2t, -l^2/4t)\,,$$

$$C = \frac{1}{4}\sum_{l \in \mathbb{Z}} l^2 c(l/2t, -l^2/4t)\,, \quad D = \sum_{n \in \mathbb{N},\, l \in \mathbb{Z}} \sigma_1(n) c(l/2t, -n - l^2/4t)$$

and set $\lambda_W = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$ (*this is essentially the Weyl-vector from* [1, *Th.* 13.3]). *Then* $\mathcal{B}(f)$ *has a product-expansion, converging for* $\Im(Z) > n_0$, *of the form*

$$\mathcal{B}(f)(Z) = e^{2\pi i\, \mathrm{trace}(\lambda_W Z)} \prod_{(m,n,l)>0} (1 - e^{2\pi i(nz_1 + lz_2 + tmz_3)})^{c(l/2t, mn - l^2/4t)}$$

(*here* $(m,n,l) > 0$ *means* $m, n \in \mathbb{N}_0$, $l \in \mathbb{Z}$ *and* $l < 0$ *or* $m + n > 0$). *Moreover, the Borcherds-product satisfies* $\mathcal{B}(f)(V_t \cdot Z) = (-1)^D \mathcal{B}(f)(Z)$.

Borcherds lift is multiplicative: For f and g satisfying the assumptions from corollary 4.3, one finds $\mathcal{B}(f + g) = \mathcal{B}(f)\mathcal{B}(g)$.

In the remaining part of this section, we give some explicit examples of Borcherds-products for Γ_3^*. For applications in the following section, we are interested in forms with "minimal" divisor (and "small" weight). From Borcherds theorem it is heuristically clear, that we have to determine meromorphic forms in $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,i}]_{\mathrm{mer}}$ with poles of minimal order. The following table lists representatives λ_j of all the orbits $\Omega_3(\Gamma_3^*)\lambda_j$ with primitive $\lambda_j \in L'$ such that $-1 \leq q_3(\lambda_j) = \frac{-j}{12} < 0$. The 4th column gives the defining equation of the rational quadratic divisor $\lambda_j^\perp$ with discriminant $\delta(\lambda_j^\perp) = j \leq 12$. In the 5th column, the order ord_ψ of the Borcherds-product $\psi = \mathcal{B}(f)$, associated with a form $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,i}]_{\mathrm{mer}}$ with Fourier-coefficients $c(l,n)$, along $\lambda_j^\perp$ is given, if f has a pole of order ≤ 1 at the cusp $i\infty$.

j	λ_j	$q(\lambda_j)$	Z	$\mathrm{ord}_\psi(\lambda_j^\perp)$
1	$(0,0,0,0,\frac{1}{6})$	$-\frac{1}{12}$	$z_2 = 0$	$c(\frac{1}{6}, -\frac{1}{12}) + c(\frac{2}{6}, -\frac{4}{12}) + c(\frac{3}{6}, -\frac{9}{12})$
4	$(1,0,0,0,\frac{2}{6})$	$-\frac{1}{3}$	$z_2 = \frac{1}{2}$	$c(\frac{1}{3}, -\frac{1}{3})$
9	$(1,0,0,0,\frac{3}{6})$	$-\frac{3}{4}$	$z_2 = \frac{1}{3}$	$c(\frac{1}{2}, -\frac{3}{4})$
12	$(0,0,1,-1,0)$	-1	$z_3 = \frac{1}{3}z_1$	$c(0,-1)$

In order to construct Borcherds-products explicitly, the only question that remains is: How to find modular forms $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,i}]_{\mathrm{mer}}$ with explicitly given singularities? There are (at least) two possibilities:

1) The first method is to find holomorphic modular forms of weight $12n - 1/2$, $n \in \mathbb{N}$ with multiplier-system $\rho_{L,i}$ and divide by Δ_{12}^n. In other words, we can use

$$[\mathrm{Mp}_2(\mathbb{Z}), -\tfrac{1}{2}, \rho_{L,i}]_{\mathrm{mer}} = \sum_{n\in\mathbb{N}} \Delta_{12}^{-n} [\mathrm{Mp}_2(\mathbb{Z}), 12n - \tfrac{1}{2}, \rho_{L,i}]\,.$$

Since for $t = 3$ we have $\rho_{L,i} \cong \rho^*_{3,0,-i}$, and we gave an explicit basis of the module $[\mathrm{MJ}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, 3, 1, i] \cong [\mathrm{Mp}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, \rho^*_{3,0,-i}]$ over $\mathcal{R}$ in section 3 on page 13, all forms in the spaces $[\mathrm{Mp}_2(\mathbb{Z}), 12n - \frac{1}{2}, \rho_{L,i}]$ can be calculated explicitly in principle.

2) There is a second method given by Borcherds in [2]. The result from [2, Th. 3.1] states, that there is a "simple" criterion for a given singularity of type $(\sum_{\substack{n\in q_t(l)+\mathbb{Z} \\ -\infty \ll n \leq 0}} h(l,n)q^n)_{l\in L'/L}$ (at $i\infty$) to be extensible to a meromorphic form in $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$: There exists $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ (with Fourier-coefficients $c_f(l,n)$), such that $h(l,n) = c_f(l,n)$ for all l and $n \leq 0$, if and only if every $g \in [\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*]$ (with Fourier-coefficients $c_g(l,n)$) satisfies

$$\sum_{l\in L'/L} \sum_{n\in q_t(l)+\mathbb{Z},\, n\leq 0} h(l,n)c_g(l,-n) = 0\,. \tag{4.1}$$

Note that in general for $f \in [\mathrm{Mp}_2(\mathbb{Z}), 2-k, \rho]_{\mathrm{mer}}$ and $g \in [\mathrm{Mp}_2(\mathbb{Z}), k, \rho^*]$, we have $(f,g) \in [\mathrm{SL}_2(\mathbb{Z}), 2, 1]_{\mathrm{mer}}$ and therefore $\sum_{l,n\leq 0} c_f(l,n)c_g(l,-n) = 0$.

Speaking informally, we can say that all obstructions for a given singularity to be extensible to a meromorphic form in $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ come from forms in $[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*]$. Therefore we call $[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*]$ the obstruction-space for $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$.

Using the dimension-formula from lemma 3.7 (and the parameters from the table on page 13 as well as $\rho_L^* \cong \rho_{3,0}$), we find $\dim[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*] = 1$. A generator of the obstruction-space can be realized as a vector-valued Eisenstein-series $E_{\frac{5}{2}}$ as given in [4, Th. 4.8]. The Fourier-development of $E_{\frac{5}{2}}$ is given by (here we set $q = e^{2\pi i\tau}$)

$$\begin{aligned} E_{\frac{5}{2}}(\tau)(0) &= 1 - 24q^1 - 72q^2 + \mathcal{O}(q^3) \\ E_{\frac{5}{2}}(\tau)(1/6) = E_{\frac{5}{2}}(\tau)(5/6) &= -1q^{1/12} - 12q^{13/12} + \mathcal{O}(q^{25/12}) \\ E_{\frac{5}{2}}(\tau)(2/6) = E_{\frac{5}{2}}(\tau)(4/6) &= -7q^{1/3} - 55q^{4/3} + \mathcal{O}(q^{7/3}) \\ E_{\frac{5}{2}}(\tau)(3/6) &= -34q^{3/4} - 48q^{7/4} + \mathcal{O}(q^{11/4})\,. \end{aligned}$$

Note that $E_{\frac{5}{2}}$ actually lies in $[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_{L,i}^*]$ thus for $l \in L'/L$ we find

$$-iE_{\frac{5}{2}}(\tau)(l) = \left(\rho_L^*(\widehat{J^2})E_{\frac{5}{2}}\right)(\tau)(l) = -iE_{\frac{5}{2}}(\tau)(-l)\,.$$

If we restrict the order of the pole at $i\infty$ to be ≤ 1, the obstruction-problem (4.1) admits the following singularities $H_l = \sum_{n \in q_t(l)+\mathbb{Z},\, -1 \leq n \leq 0} h(l,n)q^n$, $l \in L'/L$, as solutions:

$H_0 =$	$0\,q^{-1} + 2\,q^0$	$0\,q^{-1} + 12\,q^0$	$0\,q^{-1} + 32\,q^0$	$1\,q^{-1} + 24\,q^0$
$H_{1/6} = H_{5/6} =$	$1\,q^{-1/12}$	$-1\,q^{-1/12}$	$-1\,q^{-1/12}$	$0\,q^{-1/12}$
$H_{2/6} = H_{4/6} =$	$0\,q^{-1/3}$	$1\,q^{-1/3}$	$0\,q^{-1/3}$	$0\,q^{-1/3}$
$H_{3/6} =$	$0\,q^{-3/4}$	$q^{-3/4}$	$1\,q^{-3/4}$	$0\,q^{-3/4}$.

The corresponding Borcherds-products are denoted by ψ_1, ψ_6, ψ_{16} and ψ_{12} (the index always indicates the weight of the Borcherds-product). The following table lists the weight k and the order of the Borcherds-product ψ_k along the rational quadratic divisors $\lambda_j^\perp$ of discriminant ≤ 12 (there are no zeros along rational quadratic divisors of discriminant > 12). The 6th column gives the multiplier-system ν of the Borcherds-products. In the last four columns, the parameters A, B, C and D from corollary 4.3 (originally [15, Theorem 2.1]) are listed.

k	$\lambda_1^\perp$	$\lambda_4^\perp$	$\lambda_9^\perp$	$\lambda_{12}^\perp$	ν	A	B	C	D
1	1	0	0	0	$\chi\kappa\mu^2$	1/6	1/2	1/2	0
6	0	1	0	0	κ	1/2	1/2	3/2	0
16	0	0	1	0	μ	4/3	1	4	0
12	0	0	0	1	χ	1	0	0	1

Since the multiplier-system of Borcherds-products is left undetermined by theorem 4.2, we have to give additional arguments. Again, there are several possibilities:

1) Let ν be the multiplier-system of the Borcherds-product ψ_k. From the parameter A and the product-expansion of ψ_k in corollary 4.3, one can easily deduce $\psi_k\left(Z+\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right)$ $= e^{2\pi i A}\psi_k(Z)$, thus $\nu\left(\mathrm{trans}\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right) = e^{2\pi i A} = (\mu^2\kappa)\left(\mathrm{trans}\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right)^{6A}$. Furthermore, $\psi_k\big|_k V_3 = (-1)^{k+D}\psi_k$ follows directly from corollary 4.3, thus $\nu(V_3) = (-1)^{k+D}$. Since each character $\nu \in \Gamma_3^{*\mathrm{ab}}$ is uniquely determined by the values $\nu\left(\mathrm{trans}\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right)$ and $\nu(V_3)$, we have $\nu = (\mu^2\kappa)^{6A}\chi^{k+D}$.

2) Alternatively, some ad hoc arguments gan be given for some of the Borcherds-products at least. By the way, we get other useful information as a side-effect. For example, we can identify some Borcherds-products with certain Maaß-lifts.

Note that modular forms of weight k with multiplier-system ν necessarily have zeros along certain rational quadratic divisors $\lambda^\perp$, if this divisor is fixed pointwise by a transformation $M_\lambda \in \Gamma_3^*$, such that $j_2(M_\lambda, Z)^{-k} \neq \nu(M_\lambda)$ for all $Z \in \lambda^\perp$. Here are some examples of such transformations for (some of) the rational quadratic divisors of discriminant ≤ 1:

$$Z \in \lambda_1^\perp \Longrightarrow Z = Z\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right],$$

$$Z \in \lambda_4^\perp \Longrightarrow Z = Z\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$Z \in \lambda_{12}^\perp \Longrightarrow Z = Z\left[\begin{pmatrix}0 & \sqrt{3}\\ 1/\sqrt{3} & 0\end{pmatrix}\right].$$

In all cases we have $j_2(M_\lambda, Z) = -1$. Thus for $f \in [\Gamma_3^*, k, \chi^j(\mu^2\kappa)^l]$ we find

$$f(Z) = 0 \ \text{ on } \begin{cases}\lambda_1^\perp, & \text{if } (-1)^k \neq 1\,,\\ \lambda_4^\perp, & \text{if } (-1)^k \neq (-1)^l\,,\\ \lambda_{12}^\perp, & \text{if } (-1)^k \neq (-1)^j\,.\end{cases}$$

Some special cases frequently needed later are

(4.2) $\quad f = 0$ on $\lambda_1^\perp$ for $f \in [\Gamma_3^*, 2k+1, \nu]$ $\quad$ (all $k \in \mathbb{Z}$, $\nu \in \Gamma_3^{*\mathrm{ab}}$),

(4.3) $\quad f = 0$ on $\lambda_4^\perp$ for $f \in [\Gamma_3^*, 2k, \kappa(\chi\mu)^j]$ $\quad$ (all $k, j \in \mathbb{Z}$),

(4.4) $\quad f = 0$ on $\lambda_{12}^\perp$ for $f \in [\Gamma_3^*, 2k, \chi(\kappa\mu)^j]$ $\quad$ (all $k, j \in \mathbb{Z}$).

Now we can determine the multiplier-system of the Borcherds-products ψ_1 and ψ_6.

Let $0 \neq f_1 \in \mathcal{M}_{1,1,i}$ be a generator of the Maaß-space of weight 1 with character $\chi\mu^2\kappa$. It follows from (4.2), that $f_1 = 0$ on $\lambda_1^\perp$. Then f_1/ψ_1 is a non-trivial paramodular form in $[D\Gamma_3^*, 0, 1] \cong \mathbb{C}$. Thus we have $\psi_1 = cf_1$ for some $c \in \mathbb{C}^\times$. Especially, ψ_1 has multiplier-system $\chi\mu^2\kappa$.

Let $0 \neq f_6 \in \mathcal{M}_{6,3,-i}$ be a generator of the Maaß-space of weight 6 with character κ. It follows from (4.3), that $f_6 = 0$ on $\lambda_4^{\perp}$. Then f_6/ψ_6 is a non-trivial paramodular form in $[D\Gamma_3^*, 0, 1] \cong \mathbb{C}$. Thus we have $\psi_6 = cf_6$ for some $c \in \mathbb{C}^\times$. Especially, ψ_6 has multiplier-system κ.

The other Borcherds-products can't be dealt with in the same way. But for ψ_{12} at least, we can find some other arguments (we do not need ψ_{16} in section 5). Note that the zeros of ψ_{12} imply, that the multiplier-system of ψ_{12} is of the form $\nu = \chi^j \mu^l$ for some $j, l \in \mathbb{Z}$ (if κ would appear in ν, then we had $\psi_{12} = 0$ on $\lambda_4^{\perp}$ using (4.3)). We want it to be χ. ψ_{12} cannot be a Maaß-lift in this case and we cannot give an argument analogous to ψ_1 and ψ_6 but have to find another realization of ψ_{12}. The idea is the following: If we find a non-trivial form $f \in [\Gamma_3, k, \xi]$, where ξ is a *non-symmetric* character in Γ_3^{ab} (such as μ_1) of order n, then $f^n - f^n|_{nk} V_3$ is a non-trivial form with character χ for $\Gamma_3^* f^n - f^n|_k V_3$ is non-trivial, since otherwise we would have $f^n(Z) = f^n|_{nk} V_3(Z) = (-1)^{-nk} f^n(V_3 \cdot Z)$ for all $Z \in \mathbb{H}_2$. Because $\mathbb{H}_2$ is simply connected, this would imply $f(Z) = (-1)^{-k} \zeta\, f(V_3 \cdot Z)$ for all $Z \in \mathbb{H}_2$ with ζ a fixed n^{th} root of unity. From this we could derive for all $M \in \Gamma_3$

$$\begin{aligned}\xi(M) f(Z) = f|_k M(Z) &= \zeta \left(f|_k V_3 M\right)(Z) = \zeta\, \xi(V_3 M V_3^{-1}) f|_k V_3(Z) \\ &= \xi(V_3 M V_3^{-1}) f(Z)\,.\end{aligned}$$

Thus ξ would be symmetric, in contradiction to our assumption on ξ.

As an example of such a form for Γ_3 with non-symmetric character, we want to define an Eisenstein-series $E_4(\mu_1) \in [\Gamma_3, 4, \mu_1]$ of Klingen-type for Γ_3 with (non-symmetric) character μ_1 by

$$E_4(\mu_1)(Z) := \sum_{M : \pm \Gamma_{3,\infty} \backslash \Gamma_3} \mu_1(M)^{-1} \eta^8((M \cdot Z)^*) 1|_4 M(Z)$$

(here $(M \cdot Z)^*$ is the upper left entry of $M \cdot Z$). The summation is well-defined, since $\widetilde{\mu}_1$ is the multiplier-system of η^8. Moreover, $\lim_{y \to \infty} E_4(\mu_1)\left(\begin{smallmatrix} \tau & 0 \\ 0 & iy \end{smallmatrix}\right) = \eta^8(\tau) \neq 0$, thus $E_4(\mu_1)$ is non-trivial. Since μ_1 has order 3, the argument given above, now would lead to $\psi_{12} \in [\Gamma_3^*, 12, \chi]$, if the sum defining $E_4(\mu_1)$ would converge absolutely. But this is not the case. Probably the convergence can be fixed using some sort of Hecke-trick. We avoid this minor problem in the following way. We define an Eisenstein-series $E_8(\mu_1^2) \in [\Gamma_3, 8, \mu_1^2]$ of Klingen-type for Γ_3 with (non-symmetric) character μ_1^2 by

$$E_8(\mu_1^2)(Z) := \sum_{M : \pm \Gamma_{3,\infty} \backslash \Gamma_3} \mu_1(M)^{-2} \eta^{16}((M \cdot Z)^*) 1|_8 M(Z)\,.$$

Note that the sum converges absolutely now and is well-defined again, i.e. we have $0 \neq E_8(\mu_1^2) \in [\Gamma_3, 8, \mu_1^2]$. In this case, the argument given above leads to $0 \neq f_{24} := E_8(\mu_1^2)^3 + E_8(\mu_1^2)^3|_{24} V_3 \in [\Gamma_3^*, 24, \chi]$. From (4.4) we get $f_{24} = 0$ on $\lambda_{12}^{\perp}$. Thus $f_{12} := f_{24}/\psi_{12}$ is a non-trivial form of weight 12 for Γ_3^* (with some multiplier-system). $f_{12}|_{12} V_3 = -f_{12}$ would imply $f_{12}/\psi_{12} \in [D\Gamma_3^*, 0, 1]$ and $f_{24} = c\,\psi_{12}^2$ with some constant $c \in \mathbb{C}^\times$. As a square, ψ_{12}^2 can have multiplier-systems of 3-power-order only, but the multiplier-system of f_{24} has order 2. Thus we must have $f_{12}|_{12}V_3 = f_{12}$ and $\psi_{12}|_{12}V_3 = -\psi_{12}$, i.e. the multiplier-system of ψ_{12} is of the form $\nu = \chi\mu^l$ for some $l \in \mathbb{Z}$. Now if $\mu^l \neq 1$, then $W_3(\psi_{12})(z_1, z_3) = \psi_{12}\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_3/3 \end{smallmatrix}\right)$ would be a form of weight 12 on $\mathbb{H}_1 \times \mathbb{H}_1$ with (non-trivial) multiplier-system $v_\eta^{8l} \times v_\eta^{8l}$. With $l \in \{1, 2\}$, lemma 5.1 leads to $W_3(\psi_{12})(z_1, z_3) = c\,\eta^{8l}(z_1)\eta^{8l}(z_3)g_{12-4l}(z_1)g_{12-4l}(z_3)$ for some $c \in \mathbb{C}^\times$. On the other hand we have

$$W_3(f|_k V_3)(z_1, z_3) = (-1)^k f\left(V_3 \cdot \begin{pmatrix} z_1 & 0 \\ 0 & z_3/3 \end{pmatrix}\right) = (-1)^k f\begin{pmatrix} z_3 & 0 \\ 0 & z_1/3 \end{pmatrix}$$
$$= (-1)^k W_3(f)(z_3, z_1)$$

in general. Since $\psi_{12}|_{12}V_3 = -\psi_{12}$, this implies $W_3(\psi_{12})(z_1, z_3) = -W_3(\psi_{12})(z_3, z_1)$. But obviously $\eta^{8l}(z_1)\eta^{8l}(z_3)g_{12-4l}(z_1)g_{12-4l}(z_3)$ is invariant under interchanging z_1 and z_3. Thus in this case $W_3(\psi_{12}) = 0$ would follow, but ψ_{12} does not vanish on $\lambda_1^\perp$ (identically). All together, ψ_{12} has multiplier-system χ.

REMARK 4.4. 1) There is no $M \in \Gamma_3^*$, which fixes $\lambda_9^\perp$ pointwise. Thus vanishing along $\lambda_9^\perp$ can't be correlated to certain combinations of characters/weights as with the other rational-quadratic divisors (of norm ≤ 12) above.

2) The characters μ^j cannot be used to derive zeros of forms along any rational-quadratic divisor in general. For example, both ψ_1^2 and ψ_{16} have character μ, but their zero-divisors are disjoint.

3) All four Borcherds-products ψ_k already appeared in [15] (Th. 2.6, (3.22), Exam. 1.17 and (4.8) in [15] resp.).

4) In some sense the problem of finding generators for $[D\Gamma_3^*, \mathbb{Z}, 1]$ is as easy as in the Siegel case (that is $[D\Gamma_1, \mathbb{Z}, 1]$), since in both cases the rational-quadratic divisors can be separated by Borcherds-products (i.e. for any rational-quadratic divisor $\lambda^\perp$ there is a Borcherds-product vanishing exactly along the orbit $\Gamma_t^*\lambda^\perp$ with order 1). The crucial point is, that there are no cusp-forms in the obstruction-space in both cases, since by results of Bruinier [3] the Eisenstein-series in the obstruction-space determines the weight of a Borcherds-product only. By the same reason it can be expected, that the analogous problem for Γ_2^* can be solved in the same way (though everything is known in this case by [19], for Γ_2 and Γ_2^* at least). For $t \geq 4$ the obstruction-space seems to have dimension ≥ 2 (the obstruction-space is in fact contained in the subspace associated with $\rho_{L,i}^*$). The following table lists $D = \dim[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*]$ for some small t.

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
D	1	1	1	2	2	2	2	2	3	3	3	3	4	3	4	4	5	4	5	4	6

In general, finding generators for $[D\Gamma_t^*, \mathbb{Z}, 1]$ gets more and more involved, as the dimension of the obstruction-space increases.

5. Graded rings of modular forms

The divisor $\lambda_1^{\perp}$ can be used for a reduction process in the same way as it was used by Freitag in the case of Siegel modular forms of degree two. More precisely, if we can lift all (generators for the ring of) automorphic forms on $\lambda_1^{\perp}$, that arise from paramodular forms of level 3 by restriction to $\lambda_1^{\perp}$, then for any paramodular form a suitable linear combination with this lifts is divisible by the Borcherds-product ψ_1 from the preceding section. It will turn out, that it is sufficient to lift generators of a certain subring only, if we make use of some of the other Borcherds-products too.

First we introduce some notation: For $n \in \mathbb{N}$ let $\mathbb{C}_n := \mathbb{C}[X_1, \cdots, X_n]$ be the ring of polynomials in the n (independent) indeterminants $X_1, \cdots, X_n$. If $l \leq n$ we have a natural inclusion $\mathbb{C}_l \subset \mathbb{C}_n$. For $P \in \mathbb{C}_n$ and $j \in \{1, \cdots, n\}$ let $\deg_j(P)$ denote the degree of P with respect to X_j.

We define the Witt-Operator W_3 on functions $f : \mathbb{H}_2 \to \mathbb{C}$ by

$$W_3(f)(z_1, z_3) := f\begin{pmatrix} z_1 & 0 \\ 0 & \frac{1}{3}z_3 \end{pmatrix}. \tag{5.1}$$

From the embedding of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \to \Gamma_t$, defined in section 2, we see, that $f \in [\Gamma_3^*, k, v]$ implies $W_3(f) \in [\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), k, \widetilde{v}]$, i.e. $W_3(f)$ is an elliptic modular form of weight k with character $\widetilde{v}$ in each of the two variables separately. Moreover from the transformation of f under V_3 it follows that $W_3(f)(z_3, z_1) = (-1)^k v(V_3) W_3(f)(z_1, z_3)$.

First we need more information about $[\mathrm{SL}_2(\mathbb{Z}), k, \widetilde{v}]$. The following lemma 5.1 is well-known. On the other hand it is prototypical for analogous statements in higher dimensions, such as our main theorem 5.2 and the following lemmata. Therefore we sketch a proof of lemma 5.1 also.

For even $n \in \mathbb{N}$ set $\Gamma(n) := \mathrm{kern}(v_\eta^n) \subset \mathrm{SL}_2(\mathbb{Z})$ (this is the invariance group of η^n).

LEMMA 5.1. *Let $n \in \mathbb{N}$ be an even divisor of* 24 *and* $m = \frac{24}{n}$.

1) $$[\Gamma(n), \mathbb{Z}, 1] := \bigoplus_{k \in \mathbb{Z}} [\Gamma(n), k, 1] = \mathbb{C}[\eta^n, g_4, g_6].$$

More precisely, for $j \in \{0, \cdots, m-1\}$ we have

$$[\mathrm{SL}_2(\mathbb{Z}), k, v_\eta^{nj}] = \eta^{nj} [\mathrm{SL}_2(\mathbb{Z}), k - nj/2, 1].$$

2) *The generators of $[\Gamma(n), \mathbb{Z}, 1]$ satisfy the relation* $(\eta^n)^m = \frac{1}{12^3}(g_4^3 - g_6^2)$.

3) *Let $P_n = X_1^m - \frac{1}{12^3}(X_2^3 - X_3^2) \in \mathbb{C}_3$ and $I = (P_n) \subset \mathbb{C}_3$ be the ideal, generated by P_n. Then $[\Gamma(n), \mathbb{Z}, 1] \cong \mathbb{C}_3/I$.*

Proof. 1) If $n \in \mathbb{N}$ is an even divisor of 24, then $D\mathrm{SL}_2(\mathbb{Z}) \subset \Gamma(n)$ and $\Gamma(n)/D\mathrm{SL}_2(\mathbb{Z})$ acts on $[\Gamma(n), k, 1]$ as a group of commuting operators. Thus if $f \in [\Gamma(n), k, 1]$, then we can decompose $f = \sum_{j:\mathbb{Z}/m\mathbb{Z}} f_j$ with $f_j \in [\mathrm{SL}_2(\mathbb{Z}), k, v_\eta^{nj}]$. Therefore without restriction, we can assume $f \in [\mathrm{SL}_2(\mathbb{Z}), k, v_\eta^{nj}]$ with $j \in \{0, \cdots, m-1\}$. Then

$$f(\tau+1) = v_\eta^{nj}(T)f(\tau) = e^{2\pi i j/m} f(\tau)$$

and f has a zero at $i\infty$ of order $\frac{j}{m}$ at least. This implies $f_j/\eta^{nj} \in [\mathrm{SL}_2(\mathbb{Z}), k - nj/2, 1]$.

2) follows from $(\eta^n)^m = \Delta_{12} = \frac{1}{12^3}(g_4^3 - g_6^2)$, as is well-known.

3) The relation stated in 2) shows $P_n(\eta^n, g_4, g_6) = 0$ on $\mathbb{H}_1$. Now assume that $Q \in \mathbb{C}_3$ satisfies $Q(\eta^n, g_4, g_6) = 0$ on $\mathbb{H}_1$. We have to show, that $Q \in I$. After reducing Q modulo I, we may assume $\deg_1(Q) < m$. Write $Q = \sum_{j=0}^{m-1} X_1^j R_j$ with $R_j \in \mathbb{C}[X_2, X_3]$ for all j. From our assumption $\sum_{j=0}^{m-1} (\eta^n)^j R_j(g_4, g_6) = 0$ on $\mathbb{H}_1$ follows. The summand $(\eta^n)^j R_j(g_4, g_6)$ has character v_η^{nj}. Since these characters are all different, all summands have to vanish separately. Thus we get $R_j(g_4, g_6) = 0$ on $\mathbb{H}_1$ for all j. Because g_4 and g_6 are algebraically independent, $R_j = 0$ follows. Thus we arrive at $Q = 0 \in I$. ■

Using lemma 3.8 we can choose Maaß-lifts $E_k \in \mathcal{M}_{k,6,-i} \subset [\Gamma_3^*, k, 1]$ for $k \in \{4, 6, 12\}$, such that

$$W_3(E_k)(z_1, z_3) = g_k(z_1) g_k(z_3)\,.$$

Especially, $W_3(E_4)$, $W_3(E_6)$ and $W_3(E_{12})$ generate the ring $[\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), 2\mathbb{Z}, 1]_{\mathrm{symm}}$ of symmetric modular forms of even weight on $\mathbb{H}_1 \times \mathbb{H}_1$ (here symmetric means $f(z_1, z_3) = f(z_3, z_1)$ for all $(z_1, z_3) \in \mathbb{H}_1 \times \mathbb{H}_1$; see e.g. [11, III, Folg. 4.1]). Moreover these forms are algebraically independent on $\mathbb{H}_1 \times \mathbb{H}_1$. Note that, despite the notation, E_k is not necessarily an Eisenstein-series for Γ_3^* of Siegel-type. From dimension-formulas we will prove later, it can be seen, that in fact E_k is an Eisenstein-series (up to a non-zero factor) for $k \in \{4, 6\}$.

Now we choose a Maaß-lift $f_4 \in \mathcal{M}_{4,2,-i} \subset [\Gamma_3^*, 4, \mu]$, such that

$$W_3(f_4)(z_1, z_3) = \eta^8(z_1)\eta^8(z_3)\,.$$

In fact, f_4 is uniquely determined by this property, since $\dim \mathcal{M}_{4,2,-i} = 1$. Note, that if $f \in \mathcal{M}_{4,2,-i}$, then $W_3(f) \in \mathbb{C}\eta^8(z_1)\eta^8(z_3)$ by lemma 5.1. Note too, that $W_3(f) = 0$ implies $f = 0$: If $W_3(f) = 0$, we have $f/\psi_1^2 \in [\Gamma_3^*, 2, 1]$ and $W_3(f/\psi_1^2) = 0$ again, thus $f/\psi_1^4 \in [\Gamma_3^*, 0, \mu^2] = \{0\}$ and $f = 0$.

Let ψ_6, ψ_{12} be the Borcherds-products from section 4. Without loss of generality we can scale these products in such a way, that

$$W_3(\psi_6)(z_1, z_3) = \eta^{12}(z_1)\eta^{12}(z_2)\,,$$

$$W_3(\psi_{12})(z_1, z_3) = \Delta_{12}(z_1) g_6^2(z_2) - g_6^2(z_1)\Delta_{12}(z_2)\,.$$

Note that $[\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), 12, 1]_{\mathrm{anti}}$ (the space of anti-symmetric modular forms of weight 12 on $\mathbb{H}_1 \times \mathbb{H}_1$) has dimension 1 and is generated by $\Delta_{12}(z_1) g_6(z_2)^2 - g_6(z_1)^2 \Delta_{12}(z_2)$ (and 12 is the smallest weight k such that $\dim[\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), k, 1]_{\mathrm{anti}} \neq 0$).

With the Maaß-lifts just chosen and the Borcherds-products, constructed in section 4, we have found enough forms in order to generate the graded ring of paramodular forms for $D\Gamma_3^*$. We summarize the properties of these forms (the lower index always indicates the weight of the modular form):

$$E_k \in [\Gamma_3^*, k, 1] \quad \text{for } k = 4, 6, 12, \qquad f_4 \in [\Gamma_3^*, 4, \mu]\,,$$

$$\psi_1 \in [\Gamma_3^*, 1, \chi\kappa\mu^2]\,, \quad \psi_6 \in [\Gamma_3^*, 6, \kappa]\,, \quad \psi_{12} \in [\Gamma_3^*, 12, \chi]\,.$$

For later use we define some forms (the lower index is the weight again):

$$h_6 = \psi_1^6 \in [\Gamma_3^*, 6, 1]\,, \quad h_8 = \psi_1^4 f_4 \in [\Gamma_3^*, 8, 1]\,, \quad h_{10} = \psi_1^2 f_4^2 \in [\Gamma_3^*, 10, 1]\,,$$

$$h_9 = \psi_1^3 \psi_6 \in [\Gamma_3^*, 9, \chi]\,, \quad h_{11} = \psi_1 \psi_6 f_4 \in [\Gamma_3^*, 11, \chi]\,,$$

$$h_{21} = \psi_1^3 \psi_6 \psi_{12} \in [\Gamma_3^*, 21, 1]\,, \quad h_{23} = \psi_1 f_4 \psi_6 \psi_{12} \in [\Gamma_3^*, 23, 1]\,.$$

Now we state our main result.

THEOREM 5.2.

1) $$[D\Gamma_3^*, \mathbb{Z}, 1] := \bigoplus_{k\in\mathbb{Z}} [D\Gamma_3^*, k, 1] = \mathbb{C}[f_4, \psi_6, \psi_{12}, \psi_1, E_4, E_6, E_{12}]\,.$$

2) *ψ_1, E_4, E_6 and E_{12} are algebraically independent.*

3) *The generators of $[D\Gamma_3^*, \mathbb{Z}, 1]$ satisfy relations of the form* (*with certain constants $c_j \in \mathbb{C}$ and polynomials p_1, $p_2 \in \mathbb{C}_4$ and $p_3 \in \mathbb{C}_6$*)

$$f_4^3 - c_1\psi_1^4 f_4 E_4 = p_1(\psi_1^6, E_4, E_6, E_{12})\,,$$

$$\psi_6^2 - c_2\psi_1^4 f_4 E_4 = p_2(\psi_1^6, E_4, E_6, E_{12})\,,$$

$$\psi_{12}^2 = p_3(\psi_1^4 f_4, \psi_1^2 f_4^2, \psi_1^6, E_4, E_6, E_{12})\,.$$

4) *$[D\Gamma_3^*, \mathbb{Z}, 1] \cong \mathbb{C}_7/I$, where $I = (P_1, P_2, P_3) \subset \mathbb{C}_7$ is the ideal, generated by*

$$P_1 = X_1^3 - c_1 X_4^4 X_1 X_5 - p_1(X_4^6, X_5, X_6, X_7)\,,$$

$$P_2 = X_2^2 - c_2 X_4^4 X_1 X_5 - p_2(X_4^6, X_5, X_6, X_7) \quad \textit{and}$$

$$P_3 = X_3^2 - p_3(X_4^4 X_1, X_4^2 X_1^2, X_4^6, X_5, X_6, X_7)\,.$$

Proof. 1) $\Gamma_3^*/D\Gamma_3^*$ acts on $[D\Gamma_3^*, k, 1]$ as a group of commuting operators. The decomposition of $[D\Gamma_3^*, k, 1]$ into the eigenspaces of these operators is

$$[D\Gamma_3^*, k, 1] = \bigoplus_{\nu\in\Gamma_3^{*\mathrm{ab}}} [\Gamma_3^*, k, \nu]\,. \tag{5.2}$$

Now let $f \in [D\Gamma_3^*, k, 1]$. Using the decomposition (5.2), we can write $f = \sum_{\nu\in\Gamma_3^{*\mathrm{ab}}} f_\nu$ with $f_\nu \in [\Gamma_3^*, k, \nu]$. Therefore we can assume $f \in [\Gamma_3^*, k, \nu]$ without restriction.

Case 1: $k \equiv 1 \bmod 2\mathbb{Z}$. Then $W_3(f) = 0$ by (4.2) and $f/\psi_1 \in [\Gamma_3^*, k-1, \nu\chi\kappa\mu]$.

Case 2: $k \equiv 0 \bmod 2\mathbb{Z}$, $\nu(V_3) = -1$. Then $f = 0$ on $\lambda_{12}^\perp$ by (4.4), thus $f/\psi_{12} \in [\Gamma_3^*, k-12, \nu\chi]$.

Case 3: $k \equiv 0 \bmod 2\mathbb{Z}$, $\nu = \kappa(\chi\mu)^j$ for some $j \in \mathbb{Z}$. Then $f = 0$ on $\lambda_4^\perp$ by (4.3), thus $f/\psi_6 \in [\Gamma_3^*, k-6, \nu\kappa]$.

Case 4: $k \equiv 0 \bmod 2\mathbb{Z}$, $\nu = \mu^j$, $j \in \{0,1,2\}$. Now, $W_3(f) \in [\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), k, v_\eta^{8j}]_{\mathrm{symm}}$, i.e. $W_3(f)$ is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ with multiplier-system v_η^{8j} in each of the two variables. Using lemma 5.1 we find $W_3(f)(z_1,z_2) = \eta^{8j}(z_1)\eta^{8j}(z_2)h(z_1,z_2)$ with a symmetric modular form $h \in [\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), k-4j, 1]_{\mathrm{symm}}$. Thus there is a polynomial P such that

$$h(z_1,z_3) = P(g_4(z_1)g_4(z_3), g_6(z_1)g_6(z_3)\,, g_{12}(z_1)g_{12}(z_3))\,.$$

This shows, that $W_3(f)$ can be lifted as $f_4^j\; P(E_4,E_6,E_{12})$ to a paramodular form on $\mathbb{H}_2$. Now we have $f - f_4^j\; P(E_4,E_6,E_{12}) = 0$ on $\lambda_1^\perp$ and $(f - f_4^j\; P(E_4,E_6,E_{12}))/\psi_1 \in [\Gamma_3^*, k-1, \chi\kappa\mu^{j+1}]$.

2) Assume $Q \in \mathbb{C}_4$ satisfies $Q(\psi_1,E_4,E_6,E_{12}) = 0$ on $\mathbb{H}_2$. Write $Q = \sum_{t\in\mathbb{N}_0} X_1^t R_t$, where $R_t \in \mathbb{C}[X_2,X_3,X_4]$ for all t. Restricting to $\mathbb{H}_1 \times \mathbb{H}_1$, we find

$$0 = W_3(Q(\psi_1,E_4,E_6,E_{12})) = R_0(W_3(E_4), W_3(E_6), W_3(E_{12}))\,.$$

Since $W_3(E_4)$, $W_3(E_6)$ and $W_3(E_{12})$ are algebraically independent (on $\mathbb{H} \times \mathbb{H}$), $R_0 = 0$ follows. Then $Q = X_1(\sum_{t\in\mathbb{N}_0} X_1^t R_{t+1})$ is divisible by X_1 and $\widetilde{Q} = \sum_{t\in\mathbb{N}_0} X_1^t R_{t+1}$ satisfies $\widetilde{Q}(\psi_1,E_4,E_6,E_{12}) = 0$ on $\mathbb{H}_2$ again. Inductively $R_t = 0$ for all t, and $Q = 0$ altogether, follows. Therefore ψ_1, E_4, E_6 and E_{12} are algebraically independent.

3) Obviously, $f_4^3, \psi_6^2 \in [\Gamma_3^*, 12, 1]$ and $\psi_{12}^2 \in [\Gamma_3^*, 24, 1]$. From lemma 5.3, 1) it follows, that f_4^3, ψ_6^2 and ψ_{12}^2 are polynomials in h_8, h_{10}, h_6, E_4, E_6 and E_{12}. Thus the claim is true for ψ_{12}^2. If $P \in \mathbb{C}_6$ is a polynomial, such that $P(h_8,h_{10},h_6,E_4,E_6,E_{12}) \in [\Gamma_3^*, 12, 1]$, then P is of the form $P = cX_1X_4 + p$ with $c \in \mathbb{C}$, $p \in \mathbb{C}[X_3,X_4,X_5,X_6]$, since all monomials of weight 12 containing h_8 or h_{10} are given by h_8E_4. Therefore the polynomials for f_4^3 and ψ_6^2 are of the form given in the lemma.

4) The relations stated in 3) show $P_j(f_4,\psi_6,\psi_{12},\psi_1,E_4,E_6,E_{12}) = 0$ on $\mathbb{H}_2$ for $j = 1,2,3$. We have to prove, that if $Q \in \mathbb{C}_7$ is such that $Q(f_4,\psi_6,\psi_{12},\psi_1,E_4,E_6,E_{12}) = 0$ on $\mathbb{H}_2$, then $Q \in I := (P_1,P_2,P_3)$. Therefore assume $Q(f_4,\cdots,E_{12}) = 0$ on $\mathbb{H}_2$. Moreover, after reducing Q modulo I, we can assume $\deg_3(Q) \le 1$, $\deg_2(Q) \le 1$ and $\deg_1(Q) \le 2$. Write $Q = \sum_{t\in\mathbb{N}_0} X_4^t R_t$, where $R_t \in \mathbb{C}_7$ satisfies $\deg_4(R_t) = 0$ for all t (i.e. X_4 does not appear in R_t). Now we show $R_0 = 0$. First write

$$R_0 = \sum_{\substack{0\le r_1\le 2,\\ 0\le r_2,r_3\le 1}} X_1^{r_1}X_2^{r_2}X_3^{r_3}U_{r_1r_2r_3} \quad \text{with} \quad U_{r_1r_2r_3} \in \mathbb{C}[X_5,X_6,X_7] \text{ for all } r_1,r_2,r_3\,.$$

Restricting to $\lambda_1^\perp$, we find (here $f\otimes g$ is defined by $(f\otimes g)(z_1,z_2) = f(z_1)g(z_2)$)

$$\begin{aligned}
0 &= W_3(Q(f_4,\cdots,E_{12}))\\
&= R_0(\eta^8\otimes\eta^8, \eta^{12}\otimes\eta^{12}, W_3(\psi_{12})\,, 0, g_4\otimes g_4, g_6\otimes g_6, g_{12}\otimes g_{12})\\
&= \sum_{0\le r_1\le 2,\, 0\le r_2,r_3\le 1} (\eta\otimes\eta)^{8r_1+12r_2} W_3(\psi_{12})^{r_3} U_{r_1r_2r_3}(g_4\otimes g_4, g_6\otimes g_6, g_{12}\otimes g_{12})
\end{aligned}$$

on $\mathbb{H}_1 \times \mathbb{H}_1$. Since the characters of the summands come from the factors $(\eta \otimes \eta)^{8r_1+12r_2}$ and are therefore all different, and because each summand is $(-1)^{r_3}$-symmetric (under $(z_1, z_4) \mapsto (z_4, z_1)$), the summands vanish one for one. Thus we have $U_{r_1r_2r_3}(g_4 \otimes g_4, g_6 \otimes g_6, g_{12} \otimes g_{12}) = 0$ on $\mathbb{H}_1 \times \mathbb{H}_1$ and, by the argument applied in the proof of 2), $U_{r_1r_2r_3} = 0$ for all r_1, r_2, r_3. This proves $R_0 = 0$. Now $Q = X_4(\sum_{t \in \mathbb{N}_0} X_4^t R_{t+1})$ follows, i.e. Q is divisible by X_4, and we can apply the same argument to Q/X_4 again. Inductively we derive $R_t = 0$ for all t. Therefore $Q = 0 \in I$, as had to be shown.

Since all generators for $[D\Gamma_3^*, \mathbb{Z}, 1]$ are modular forms for Γ_3^* (with multiplier-systems), we can in principle find generators for the graded rings of paramodular forms for all groups Γ with $D\Gamma_3^* \subset \Gamma \subset \Gamma_3^*$. We give three examples. The following lemma will be useful for a reduction-process from forms on a 4-dimensional half-space, e.g. the hermitian half-space of degree 2. An analogous example was given by Freitag [11], where he used paramodular forms in $[\Gamma_2^*, 2\mathbb{Z}, 1]$ to determine generators for the ring of symmetric Hermitian modular forms of degree 2 for $\mathbb{Q}(\sqrt{-1})$. Actually, this was our prime motivation for the present note and we hope to use the results presented here in order to find generators for the ring of Hermitian modular forms of degree 2 for $\mathbb{Q}(\sqrt{-2})$ soon.

LEMMA 5.3.

1) $$[\Gamma_3^*, 2\mathbb{Z}, 1] := \bigoplus_{k \in 2\mathbb{Z}} [\Gamma_3^*, k, 1] = \mathbb{C}[h_8, h_{10}, h_6, E_4, E_6, E_{12}],$$

2) *h_6, E_4, E_6 and E_{12} are algebraically independent.*

3) *The generators of $[\Gamma_3^*, 2\mathbb{Z}, 1]$ satisfy the following relations (here $c_1 \in \mathbb{C}$ and $p_1 \in \mathbb{C}_4$ are the same as in theorem 5.2, 3))*

$$h_8^2 = h_6 h_{10},$$
$$h_{10}^2 = h_8(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})),$$
$$h_8 h_{10} = h_6(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})).$$

4) *$[\Gamma_3^*, 2\mathbb{Z}, 1] \cong \mathbb{C}_6/I$, where $I = (P_1, P_2, P_3) \subset \mathbb{C}_6$ is the ideal, generated by*

$$P_1 = X_1^2 - X_2 X_3,$$
$$P_2 = X_2^2 - X_1(c_1 X_1 X_4 + p_1(X_3, X_4, X_5, X_6)) \quad \textit{and}$$
$$P_3 = X_1 X_2 - X_3(c_1 X_1 X_4 + p_1(X_3, X_4, X_5, X_6)).$$

Proof. 1) Let $f \in [\Gamma_3^*, k, 1]$ and $k \equiv 0 \bmod 2\mathbb{Z}$. We apply the reduction process from lemma 5.2. First there is a polynomial $Q_1 \in \mathbb{C}_3$, such that $f - Q_1(E_4, E_6, E_{12}) = 0$ along $\lambda_1^\perp$ of order 2 (since k is even). Then we have $g_1 = (f - Q_1(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-2, \mu_1^2]$. Now there is a polynomial $Q_2 \in \mathbb{C}_3$, such that $g_1 - f_4^2 Q_2(E_4, E_6, E_{12}) = 0$ along $\lambda_1^\perp$ of order 2 again. Thus $g_2 = (g_1 - Q_2(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-4, \mu_1]$ and once more, there is a polynomial $Q_3 \in \mathbb{C}_3$, such that $g_2 - f_4 Q_3(E_4, E_6, E_{12}) = 0$ along $\lambda_1^\perp$ of order 2. Finally $g_3 = (g_2 - Q_3(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-6, 1]$ follows. Summarizing, we have

$$f = g_1\psi_1^2 + P_1(E_4, E_6, E_{12}) = (g_2\psi_1^2 + f_4^2 P_2(E_4, E_6, E_{12}))\psi_1^2 + P_1(E_4, E_6, E_{12})$$
$$= (g_3\psi_1^2 + f_4 P_3(E_4, E_6, E_{12}))\psi_1^4 + \psi_1^2 f_4^2 P_2(E_4, E_6, E_{12}) + P_1(E_4, E_6, E_{12})$$
$$= g_3\,\psi_1^6 + f_4\,\psi_1^4\,P_3(E_4, E_6, E_{12}) + f_4^2\,\psi_1^2\,P_2(E_4, E_6, E_{12}) + P_1(E_4, E_6, E_{12})\,.$$

Inductively $f \in \mathbb{C}[\psi_1^6, \psi_1^4 f_4, \psi_1^2 f_4^2, E_4, E_6, E_{12}]$ follows.

2) follows from theorem 5.2, 2).

3) The first relation follows directly from the definition of the h_j's. For the other two relations remember $f_4^3 = c_1 h_8\, E_4 + p_1(h_6, E_4, E_6, E_{12})$ from theorem 5.2.

4) The relations stated in 3) show $P_j(h_8, h_{10}, h_6, E_4, E_6, E_{12}) = 0$ for $j = 1, 2, 3$. Now assume that $Q \in \mathbb{C}_6$ satisfies $Q(h_8, h_{10}, h_6, E_4, E_6, E_{12}) = 0$ on $\mathbb{H}_2$. Reducing Q modulo I, we can assume that the degree of Q as a polynomial in X_1 and X_2 is 1 at most, i.e. Q is of the form

$$Q = U_0 + X_1U_1 + X_2U_2 \quad \text{with} \quad U_j \in \mathbb{C}[X_3, X_4, X_5, X_6] \text{ for } j = 0, 1, 2\,.$$

Write $U_j = \sum_{k\geq 0} X_3^k R_{j,k}$ with $R_{j,k} \in \mathbb{C}[X_4, X_5, X_6]$, $j = 0, 1, 2$, $k \in \mathbb{N}_0$. We show, that $R_{0,0} = R_{1,0} = R_{2,0} = 0$. Note that h_8 and h_{10} vanish along $\lambda_1^\perp$ of order 4 and 2 resp. Thus by restriction to $\lambda_1^\perp$ we have

$$0 = W_3(Q(h_8, h_{10}, h_6, E_4, E_6, E_{12})) = W_3(U_0(h_6, E_4, E_6, E_{12}))$$
$$= R_{0,0}(W_3(E_4), W_3(E_6), W_3(E_{12}))$$

on $\mathbb{H}_1^2$. This implies $R_{0,0} = 0$. Now, on $\mathbb{H}_2$, we have

$$0 = \sum_{k\geq 1} h_6^k R_{0,k}(E_4, E_6, E_{12}) + \sum_{k\geq 0} h_8 h_6^k R_{1,k}(E_4, E_6, E_{12})$$
$$+ \sum_{k\geq 0} h_{10} h_6^k R_{2,k}(E_4, E_6, E_{12})$$
$$= \psi_1^2 \Bigg(\psi_1^4 \sum_{k\geq 0} h_6^k R_{0,k+1}(E_4, E_6, E_{12}) + \sum_{k\geq 0} \psi_1^2 f_4^2 h_6^k R_{1,k}(E_4, E_6, E_{12})$$
$$+ \sum_{k\geq 0} f_4^2 h_6^k R_{2,k}(E_4, E_6, E_{12}) \Bigg)\,.$$

Now the term in the brackets has to vanish on $\mathbb{H}_2$ and by restriction to $\lambda_1^\perp$ we get $0 = W_3(f_4)^2 R_{2,0}(W_3(E_4), W_3(E_6), W_3(E_{12}))$ on $\mathbb{H}_1^2$. Since f_4 does not vanish along $\lambda_1^\perp$, this implies $R_{2,0} = 0$. Now, on $\mathbb{H}_2$, we have

$$0 = \sum_{k\geq 1} h_6^k R_{0,k}(E_4, E_6, E_{12}) + \sum_{k\geq 0} h_8 h_6^k R_{1,k}(E_4, E_6, E_{12})$$
$$+ \sum_{k\geq 1} h_{10} h_6^k R_{2,k}(E_4, E_6, E_{12})$$
$$= \psi_1^4 \left(\psi_1^2 \sum_{k\geq 0} h_6^k R_{0,k+1}(E_4, E_6, E_{12}) + \sum_{k\geq 0} f_4^2 h_6^k R_{1,k}(E_4, E_6, E_{12}) \right.$$
$$\left. + \sum_{k\geq 0} f_4^2 \psi_1^4 h_6^k R_{2,k+1}(E_4, E_6, E_{12}) \right).$$

Again the term in the brackets has to vanish on $\mathbb{H}_2$ and restriction to $\lambda_1^\perp$ leads to $0 = W_3(f_4)^2 R_{1,0}(W_3(E_4), W_3(E_6), W_3(E_{12}))$ this time. As before $R_{1,0} = 0$ follows. Altogether we see, that

$$Q = X_3 \left(\sum_{k\geq 0} X_3^k R_{0,k+1} + X_1 \sum_{k\geq 0} X_3^k R_{1,k+1} + X_2 \sum_{k\geq 0} X_3^k R_{2,k+1} \right)$$

is divisible by X_3. Now the same argument can be applied to Q/X_3 again. Inductively, $R_{j,k} = 0$ for all j, k follows. Therefore $Q = 0$ and $Q \in I$ is proved. ■

LEMMA 5.4.

1) $$[\Gamma_3^*, \mathbb{Z}, 1] := \bigoplus_{k\in\mathbb{Z}} [\Gamma_3^*, k, 1] = \mathbb{C}[h_{21}, h_{23}, h_8, h_{10}, h_6, E_4, E_6, E_{12}]\,,$$

2) *The generators of* $[\Gamma_3^*, \mathbb{Z}, 1]$ *satisfy the following relations* (*here* $c_j \in \mathbb{C}$, p_1, $p_2 \in \mathbb{C}_4$ *and* $p_3 \in \mathbb{C}_6$ *are as in theorem* 5.2, 3))

$$h_8^2 = h_6 h_{10}\,,$$
$$h_{10}^2 = h_8(c_1 h_8\, E_4 + p_1(h_6, E_4, E_6, E_{12}))\,,$$
$$h_8 h_{10} = h_6(c_1 h_8\, E_4 + p_1(h_6, E_4, E_6, E_{12}))\,,$$
$$h_{21}^2 = h_6(c_2 h_8\, E_4 + p_2(h_6, E_4, E_6, E_{12}))p_3(h_8, h_{10}, h_6, E_4, E_6, E_{12})\,,$$
$$h_{23}^2 = h_{10}(c_2 h_8\, E_4 + p_2(h_6, E_4, E_6, E_{12}))p_3(h_8, h_{10}, h_6, E_4, E_6, E_{12})\,,$$
$$h_{21} h_{23} = h_8(c_2 h_8\, E_4 + p_2(h_6, E_4, E_6, E_{12}))p_3(h_8, h_{10}, h_6, E_4, E_6, E_{12})\,,$$
$$h_{21} h_8 = h_{23} h_6\,,$$
$$h_{21} h_{10} = h_{23} h_8\,,$$
$$h_{23} h_{10} = h_{21}(c_1 h_8\, E_4 + p_1(h_6, E_4, E_6, E_{12}))\,.$$

3) $[\Gamma_3^*, \mathbb{Z}, 1] \cong \mathbb{C}_8/I$, *where* $I = (P_j \mid j = 1, \cdots, 9) \subset \mathbb{C}_8$ *is the ideal*, *generated by*

$$P_1 = X_3^2 - X_4X_5,$$
$$P_2 = X_4^2 - X_3\left(c_1X_3X_6 + p_1(X_5, X_6, X_7, X_8)\right),$$
$$P_3 = X_3X_4 - X_5\left(c_1X_3X_6 + p_1(X_5, X_6, X_7, X_8)\right),$$
$$P_4 = X_1^2 - p_4(X_3, X_4, X_5, X_6, X_7, X_8)X_5\,,$$
$$P_5 = X_2^2 - p(X_3, X_4, X_5, X_6, X_7, X_8)X_4\,,$$
$$P_6 = X_1X_2 - p(X_3, X_4, X_5, X_6, X_7, X_8)X_3\,,$$
$$P_7 = X_1X_3 - X_2X_5\,,$$
$$P_8 = X_1X_4 - X_2X_3\,,$$
$$P_9 = X_2X_4 - X_1\left(c_1X_3X_6 + p_1(X_5, X_6, X_7, X_8)\right),$$

with $P_4 = (c_2X_3X_6 + p_2(X_5, X_6, X_7, X_8)) \cdot p_3(X_5, X_6, X_7, X_8)$.

Proof. 1) Let $f \in [\Gamma_3^*, k, 1]$. If $k \equiv 0 \bmod 2\mathbb{Z}$, then $f \in \mathbb{C}[h_8, h_{10}, h_6, E_4, E_6, E_{12}]$ by lemma 5.3. Therefore assume $k \equiv 1 \bmod 2\mathbb{Z}$. The reduction-process from lemma 5.2 now leads to $f/\psi_1\psi_6\psi_{12} \in [\Gamma_3^*, k-19, \mu]$ and the existence of $P \in \mathbb{C}_3$, such that $f/\psi_1\psi_6\psi_{12} - f_4P(E_4, E_6, E_{12})$ vanishes along $\lambda_1^\perp$. Now $\widetilde{f} = (f/\psi_1\psi_6\psi_{12} - f_4P(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-23, 1]$ follows. We arrive at

$$f = \psi_1\psi_6\psi_{12}(\psi_1^2\widetilde{f} + f_4P(E_4, E_6, E_{12})) \in h_{21}[\Gamma_3^*, k-21, 1] + h_{23}[\Gamma_3^*, k-23, 1]\,.$$

Since $k-21$ and $k-23$ are even, lemma 5.3 again implies the claim.

2) The relations involving h_8^2, h_{10}^2 and h_8h_{10} are the same as in lemma 5.3. The remaining relations follow from the definitions of the h_k together with the relations from theorem 5.2.

3) The relations stated in 2) show $P_j(h_{21}, h_{23}, h_8, h_{10}, h_6, E_4, E_6, E_{12}) = 0$ for $j = 1, \cdots, 9$. Now assume that $Q \in \mathbb{C}_8$ satisfies $Q(h_{21}, h_{23}, h_8, h_{10}, h_6, E_4, E_6, E_{12}) = 0$ on $\mathbb{H}_2$. Reducing Q modulo I, we can assume that the degree of Q as a polynomial in $X_1, \cdots, X_4$ is 2 at most. Furthermore all terms of degree 2 in $X_1, \cdots, X_4$ can be reduced to X_2X_3. Then Q is of the form (here we set $X_0 = 1$)

$$Q = \sum_{0\le j\le 4} X_jU_j + X_2X_3U_{23} \quad \text{with} \quad U_j \in \mathbb{C}[X_5, X_6, X_7, X_8] \text{ for } j = 0, \cdots, 4, 23\,.$$

Write $U_j = \sum_{k\ge 0} X_5^k R_{j,k}$ with $R_{j,k} \in \mathbb{C}[X_6, X_7, X_8]$, $j = 0, \cdots, 4, 23$, $k \in \mathbb{N}_0$. We show, that $R_{j,0} = 0$ for $j = 0, \cdots, 4, 23$. Note that 1, $h_{21} = \psi_1^3\psi_6\psi_{12}$, $h_{23} = \psi_1 f_4\psi_6\psi_{12}$, $h_8 = \psi_1^4 f_4$, $h_{10} = \psi_1^2 f_4^2$ and $h_{23}h_8 = \psi_1^5 f_4^2\psi_6\psi_{12}$ vanish along $\lambda_1^\perp$ of order 0, 3, 1, 4, 2, 5 resp. Since these orders are all different, we can proceed exactly as in the proof of lemma 5.3, i.e. by extracting the highest power of ψ_1 and restriction to $\lambda_1^\perp$ of $Q(h_{21}, \cdots, E_{12})$, successively $R_{0,0} = R_{2,0} = R_{4,0} = R_{1,0} = R_{3,0} = R_{5,0} = 0$ follows. Altogether we see, that Q is divisible by X_5 and the same argument can be applied to Q/X_5 again. Inductively, $R_{j,k} = 0$ for all j, k follows. Therefore $Q = 0$ and $Q \in I$ is proved. ■

Since the dimensions $\dim[\Gamma_3, k, 1]$ are known by results of Ibukiyama ([17], [18]), we determine generators for $[\Gamma_3, \mathbb{Z}, 1]$ in order to compare the dimension formulas. Note that $\Gamma_3 = \mathrm{kern}(\chi)$.

LEMMA 5.5.

1) $$[\Gamma_3, \mathbb{Z}, 1] := \bigoplus_{k\in\mathbb{Z}} [\Gamma_3, k, 1] = \mathbb{C}[h_9, h_{11}, h_8, h_{10}, \psi_{12}, h_6, E_4, E_6, E_{12}]\,,$$

2) *The generators of* $[\Gamma_3, \mathbb{Z}, 1]$ *satisfy the following relations* (*here* $c_j \in \mathbb{C}$, p_1, $p_2 \in \mathbb{C}_4$ *and* $p_3 \in \mathbb{C}_6$ *are as in theorem* 5.2, 3) *again*)

$$\begin{aligned}
h_8^2 &= h_6 h_{10}\,,\\
h_{10}^2 &= h_8(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12}))\,,\\
h_8 h_{10} &= h_6(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12}))\,,\\
h_9^2 &= h_6(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12}))\,,\\
h_{11}^2 &= h_{10}(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12}))\,,\\
h_9 h_{11} &= h_8(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12}))\,,\\
h_9 h_8 &= h_{11} h_6\,,\\
h_9 h_{10} &= h_{11} h_8\,,\\
h_{11} h_{10} &= h_9(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12}))\,,\\
\psi_{12}^2 &= p_3(h_8, h_{10}, h_6, E_4, E_6, E_{12})\,.
\end{aligned}$$

3) $[\Gamma_3, \mathbb{Z}, 1] \cong \mathbb{C}_9/I$, *where* $I = (P_j \mid j = 1, \cdots, 10) \subset \mathbb{C}_9$ *is the ideal*, *generated by*

$$\begin{aligned}
P_1 &= X_3^2 - X_4 X_6\,,\\
P_2 &= X_4^2 - X_3\,(c_1 X_3 X_7 + p_1(X_6, X_7, X_8, X_9))\,,\\
P_3 &= X_3 X_4 - X_6\,(c_1 X_3 X_7 + p_1(X_6, X_7, X_8, X_9))\,,\\
P_4 &= X_1^2 - X_6(c_2 X_3 X_7 + p_2(X_6, X_7, X_8, X_9))\,,\\
P_5 &= X_2^2 - X_4(c_2 X_3 X_7 + p_2(X_6, X_7, X_8, X_9))\,,\\
P_6 &= X_1 X_2 - X_3(c_2 X_3 X_7 + p_2(X_6, X_7, X_8, X_9))\,,\\
P_7 &= X_1 X_3 - X_2 X_6\,,\\
P_8 &= X_1 X_4 - X_2 X_3\,,\\
P_9 &= X_2 X_4 - X_1(c_1 X_3 X_7 + p_1(X_6, X_7, X_8, X_9))\,,\\
P_{10} &= X_5^2 - p_3(X_3, X_4, X_6, X_7, X_8, X_9)\,.
\end{aligned}$$

Proof. 1) Let $f \in [\Gamma_3, k, 1]$. We can decompose f into eigenfunctions of V_3, i.e. $f = f_1 + f_\chi$ with $f_\nu \in [\Gamma_3^*, k, \nu]$. Thus without restriction assume $f \in [\Gamma_3^*, k, \nu]$,

$\nu \in \{1, \chi\}$. By lemma 5.3 it is sufficient to show, that f is a polynomial in h_9, h_{11} and ψ_{12} over $[\Gamma_3^*, 2\mathbb{Z}, 1]$. In order to prove this, we apply the reduction process from theorem 5.2.

Case 1: $k \equiv 0 \bmod 2\mathbb{Z}$, $\nu = 1$. Then $f \in [\Gamma_3^*, 2\mathbb{Z}, 1]$.

Case 2: $k \equiv 0 \bmod 2\mathbb{Z}$, $\nu = \chi$. Then $f/\psi_{12} \in [\Gamma_3^*, k-12, 1]$ and $f \in \psi_{12}[\Gamma_3^*, 2\mathbb{Z}, 1]$.

Case 3: $k \equiv 1 \bmod 2\mathbb{Z}$, $\nu = 1$. Then $f/\psi_1\psi_6\psi_{12} \in [\Gamma_3^*, k-19, \mu]$ and there exists $P \in \mathbb{C}_3$, such that $f/\psi_1\psi_6\psi_{12} - f_4 P(E_4, E_6, E_{12})$ vanishes along $\lambda_1^\perp$ of order 2. Now $\widetilde{f} = (f/\psi_1\psi_6\psi_{12} - f_4 P(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-21, 1]$ and $f = \psi_1^3\psi_6\psi_{12}\widetilde{f} + \psi_1\psi_6\psi_{12} f_4 P(E_4, E_6, E_{12}) \in h_9\psi_{12}[\Gamma_3^*, 2\mathbb{Z}, 1] + h_{11}\psi_{12}[\Gamma_3^*, 2\mathbb{Z}, 1]$ follows.

Case 4: $k \equiv 1 \bmod 2\mathbb{Z}$, $\nu = \chi$. Then $f/\psi_1\psi_6 \in [\Gamma_3^*, k-7, \mu]$ and there exists $P \in \mathbb{C}_3$, such that $f/\psi_1\psi_6 - f_4 P(E_4, E_6, E_{12})$ vanishes along $\lambda_1^\perp$ of order 2. Now $\widetilde{f} = (f/\psi_1\psi_6 - f_4 P(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-9, 1]$ and $f = \psi_1^3\psi_6\widetilde{f} + \psi_1\psi_6 f_4 P(E_4, E_6, E_{12}) \in h_9[\Gamma_3^*, 2\mathbb{Z}, 1] + h_{11}[\Gamma_3^*, 2\mathbb{Z}, 1]$ follows.

2) As in the previous cases (lemma 5.3, lemma 5.4), all relations follow from the definitions of the h_j's and the relations, stated in theorem 5.2.

3) The relations stated in 2) show $P_j(h_9, h_{11}, h_8, h_{10}, \psi_{12}, h_6, E_4, E_6, E_{12}) = 0$ for $j = 1, \cdots, 10$. Now assume $Q \in \mathbb{C}_9$ satisfies $Q(h_9, h_{11}, h_8, h_{10}, \psi_{12}, h_6, E_4, E_6, E_{12}) = 0$ on $\mathbb{H}_2$. If we reduce Q modulo (P_{10}), we can assume that $\deg_5(Q) \leq 1$. Note that X_5 does not appear in any of the other generators of I. If we reduce Q modulo $(P_1, \cdots, P_9)$, we can assume, that the degree of Q as a polynomial in $X_1, \cdots, X_4$ is 2 at most. Furthermore all terms of degree 2 in $X_1, \cdots, X_4$ can be reduced to X_2X_3. Then Q is of the form (we set $X_0 = 1$ as before)

$$Q = \sum_{0 \leq l \leq 1} X_5^l \Bigg(\sum_{0 \leq j \leq 4} X_j U_{j,l} + X_2 X_3 U_{23,l} \Bigg) \quad \text{with} \quad U_{j,l} \in \mathbb{C}[X_6, X_7, X_8, X_9]$$

for $l = 1, 2$, and $j = 0, \cdots, 4, 23$. Write $U_{j,l} = \sum_{k \geq 0} X_6^k R_{j,l,k}$ with $R_{j,l,k} \in \mathbb{C}[X_7, X_8, X_9]$, for $l = 1, 2$, $j = 0, \cdots, 4, 23$, $k \in \mathbb{N}_0$. We show, that $R_{j,l,0} = 0$ for $l = 1, 2$, $j = 0, \cdots, 4, 23$. Principally, we can now proceed as in lemma 5.4, since 1, $h_9 = \psi_1^3\psi_6$, $h_{11} = \psi_1 f_4 \psi_6$, $h_8 = \psi_1^4 f_4$, $h_{10} = \psi_1^2 f_4^2$ and $h_{23}h_8 = \psi_1^5 f_4^2 \psi_6 \psi_{12}$ again vanish along $\lambda_1^\perp$ of order 0, 3, 1, 4, 2, 5 resp.. The only difference is, that we now always get a sum of two terms (since Q is linear in X_5 by assumption)

$$R_{j,0,0}(W_3(E_4), W_3(E_6), W_3(E_{12})) + W_3(\psi_{12}) R_{j,1,0}(W_3(E_4), W_3(E_6), W_3(E_{12}))\,,$$

which has to vanish on $\mathbb{H}_1^2$. But since $W_3(\psi_{12})$ is antisymmetric (under $(z_1, z_3) \mapsto (z_1, z_3)$), whereas all $W_3(E_k)$ are symmetric, the summands have to vanish one for one. The proof now runs as before again, i.e. since $R_{j,l,0} = 0$ for all j, l, Q is divisible by X_6 and, inductively, $Q = 0$ as well as $Q \in I$ follows.

From theorem 5.2, lemma 5.4 and lemma 5.5 we can deduce the generating functions for the dimensions $\dim [\Gamma, k, 1]$ for $\Gamma = D\Gamma_3^*, \Gamma_3^*$ and Γ_3. We find

COROLLARY 5.6.

$$\sum_{k\in\mathbb{N}_0} \dim\,[D\Gamma_3^*, k, 1]\, t^k = \frac{(1+t^4+t^8)(1+t^6)(1+t^{12})}{(1-t)(1-t^4)(1-t^6)(1-t^{12})}\,,$$

$$\sum_{k\in\mathbb{N}_0} \dim\,[\Gamma_3^*, k, 1]\, t^k = \frac{1+t^8+t^{10}+t^{21}+t^{23}+t^{31}}{(1-t^4)(1-t^6)^2(1-t^{12})}\,,$$

$$\sum_{k\in\mathbb{N}_0} \dim\,[\Gamma_3, k, 1]\, t^k = \frac{(1+t^{12})(1+t^8+t^9+t^{10}+t^{11}+t^{19})}{(1-t^4)(1-t^6)^2(1-t^{12})}\,,$$

In the case of Γ_3, the same generating function was already deduced by Ibukiyama [18] from his general formula for $\dim\,[\Gamma_t, k, 1]_{\text{cusp}}$ for $k \geq 5$, given in [17].

Modular functions on $\Gamma_3^*\backslash\mathbb{H}_2$ are quotients of modular forms. Thus lemma 5.4 allows us to determine the function-field of the Satake-compactification $\overline{\Gamma_3^* \backslash \mathbb{H}_2}$ of $\Gamma_3^* \backslash \mathbb{H}_2$. Let $\mathcal{K} = \mathbb{C}\left[\frac{h_8}{E_4^2}, \frac{h_6}{E_6}, \frac{h_6^2}{E_4^3}\right]$. From the relations in lemma 5.4, 3) we deduce

$$\left(\frac{h_8}{E_4^2}\right)^2 = \frac{h_6 h_{10}}{E_4^4} = \frac{h_6^2}{E_4^3}\frac{E_6}{h_6}\frac{h_{10}}{E_4E_6} \in \mathcal{K}$$

and therefore $\frac{h_{10}}{E_4E_6} \in \mathcal{K}$. The polynomial p_1 in theorem 5.2, 3) satisfies $p_1(h_6, E_4, E_6, E_{12}) \in [\Gamma_3^*, 12, 1]$. This implies

$$p_1(X_1, X_2, X_3, X_4) = c_{11}X_1^2 + c_{12}X_1X_3 + c_{13}X_3^2 + c_{14}X_2^3 + c_{15}X_4$$

for some constants $c_j \in \mathbb{C}$. Recall that p_1 was chosen in such a way that $f_4^3 - c_1 h_8 E_4 = p_1(h_6, E_4, E_6, E_{12})$. Restricting to $\lambda_1^\perp$ we find

$$\begin{aligned} W_3(f_4^3)(z_1, z_3) &= (\eta_8(z_1)\eta_8(z_3))^3 = \Delta_{12}(z_1)\Delta_{12}(z_3) \\ &= c_{13}\left(g_6(z_1)g_6(z_3)\right)^2 + c_{14}\left(g_4(z_1)g_4(z_3)\right)^3 + c_{15}g_{12}(z_1)g_{12}(z_3) \end{aligned}$$

(because of $W_3(h_6) = W_3(h_8) = 0$). This implies $c_{15} \neq 0$, since $\Delta_{12} \otimes \Delta_{12} \notin \mathbb{C}[g_4 \otimes g_4, g_6 \otimes g_6]$. Using the relation for h_{10}^2 in lemma 5.4, 3) we deduce

$$\begin{aligned} \left(\frac{h_{10}}{E_4E_6}\right)^2 =& \frac{c_1E_4h_8^2 + h_8\left(c_{11}h_6^2 + c_{12}h_6E_6 + c_{13}E_6^2 + c_{14}E_4^3 + c_{15}E_{12}\right)}{E_4^2E_6^2} \\ =& c_1\left(\frac{h_8}{E_4^2}\right)^2\frac{E_4^3}{h_6^2}\left(\frac{h_6}{E_6}\right)^2 + c_{11}\frac{h_8}{E_4^2}\left(\frac{h_6}{E_6}\right)^2 + c_{12}\frac{h_8}{E_4^2}\frac{h_6}{E_6} + c_{13}\frac{h_8}{E_4^2} \\ &+ c_{14}\frac{h_8}{E_4^2}\frac{E_4^3}{h_6^2}\left(\frac{h_6}{E_6}\right)^2 + c_{15}\frac{h_8}{E_4^2}\frac{E_{12}}{E_6^2}\,. \end{aligned}$$

Therefore $\frac{E_{12}}{E_6^2} \in \mathcal{K}$. Now we can prove

COROLLARY 5.7. $\overline{\Gamma_3^* \setminus \mathbb{H}_2}$ is a rational variety. The function-field is given by $\mathbb{C}\left[\frac{h_8}{E_4^2}, \frac{h_6}{E_6}, \frac{h_6^2}{E_4^3}\right]$.

Proof. Let $\mathcal{K} = \mathbb{C}\left[\frac{h_8}{E_4^2}, \frac{h_6}{E_6}, \frac{h_6^2}{E_4^3}\right]$ as above. Assume f is a modular function on $\Gamma_3^* \setminus \mathbb{H}_2$. Then $f = g/g'$ with $g, g' \in [\Gamma_3^*, k, 1]$ is a quotient of modular forms of some weight $k \in \mathbb{N}$. We may assume $k \in 4\mathbb{N}$ (since $g/g' = g^4/g^3g'$ as modular functions). In this case, every monomial $h_6^{n_1} E_4^{n_2} E_6^{n_3} E_{12}^{n_4}$ of weight $k = 6n_1 + 4n_2 + 6n_3 + 12n_4$ satisfies

$$E_4^{-k/4} h_6^{n_1} E_4^{n_2} E_6^{n_3} E_{12}^{n_4} = \left(\frac{h_6}{E_6}\right)^{n_1} \left(\frac{E_{12}}{E_6^2}\right)^{n_4} \left(\frac{E_6^2}{E_4^3}\right)^{(n_3+n_1)/2+n_4} \in \mathcal{K}.$$

Since k is even, lemma 5.4, 3) implies, that g is of the form $g = U_0 + h_8 U_1 + h_{10} U_3$ with polynomials $U_j \in \mathbb{C}[h_6, E_4, E_6, E_{12}]$. Therefore $E_4^{-k/4} U_0, E_4^{-k/4}(E_4^2 U_1), E_4^{-k/4}(E_4 E_6 U_3) \in \mathcal{K}$ and

$$E_4^{-k/4} g = E_4^{-k/4} U_0 + \frac{h_8}{E_4^2} E_4^{2-k/4} U_1 + \frac{h_{10}}{E_4 E_6} E_4^{1-k/4} E_6 U_3 \in \mathcal{K}$$

follows from the remarks above. Since the same argument applies to g', we find $f = g/g' = E_4^{-k/4} g / E_4^{-k/4} g' \in \mathcal{K}$.

Since the field of modular functions on $\overline{\Gamma_3^* \setminus \mathbb{H}_2}$ has transcendence-degree 3, the generators given in the corollary have to be algebraically independent. ∎

References

[1] R. Borcherds, Automorphic forms with singularities on Grassmannians, Invent. math. **132**, 491–562 (1998).

[2] R. Borcherds, The Gross-Kohnen-Zagier theorem in higher dimensions. Duke Math. J. **97**, 219–233 (1999).

[3] J.-H. Bruinier, Borcherds products on O(2, l) and Chern classes of Heegner divisors. Preprint, May 2000 (http://www.mathi.uni-heidelberg.de/ bruinier/chern.ps)

[4] J.-H. Bruinier, M. Kuß, Eisenstein series attached to lattices and modular forms on orthogonal groups, Preprint, March 2000.
(http://www.mathi.uni-heidelberg.de/ bruinier/eis.ps)

[5] U. Christian, Einführung in die Theorie der paramodularen Gruppen. Math. Ann. **168**, 59–104 (1967).

[6] T. Dern and A. Marschner, Characters of Paramodular Groups of Degree 2 and some Extensions. Arbeitsbericht Lehrstuhl A f. Mathematik, Nr. 500, RWTH Aachen (October 2000).

[7] W. Eholzer and N.-P: Skoruppa, Modular invariance and uniqueness of conformal characters. Commun. Math. Phys. **174**, 117–136 (1995).

[8] M. Eichler and D. Zagier, The theory of Jacobi forms. Progress in Mathematics, Vol. 55. Boston-Basel-Stuttgart: Birkhäuser (1985).

[9] E. Freitag, Zur Theorie der Modulformen zweiten Grades. Nachr. Akad. Wiss. Göttingen, II. Math.-Phys. Kl. 1965, 151–157 (1965).

[10] E. Freitag, Modulformen zweiten Grades zum rationalen und Gaußschen Zahlkörper. Sitzungsber. Heidelberger Akad. Wiss., Math.-Naturw. Kl. 1967, 1. Abh. 1–50 (1967).

[11] E. Freitag, Siegelsche Modulfunktionen. Grundlehren der mathematischen Wissenschaften, 254. Berlin etc.: Springer-Verlag (1983).

[12] E. Freitag and C. F. Hermann, Some modular varieties of low dimension. Adv. Math. **152**, 203–287 (2000).

[13] V. A. Gritsenko, Irrationality of the moduli spaces of polarized abelian surfaces. In W. Barth (ed.) et al.: Abelian varieties. Proceedings of the international conference held in Egloffstein, Germany, October 3–8, 1993. Berlin: Walter de Gruyter. 63–81 (1995).

[14] V. A. Gritsenko, Modulformen zur Paramodulgruppe und Modulräume der abelschen Varietäten. Math. Gottingensis **12**, 1–89 (1995).

[15] V. A. Gritsenko and V. V. Nikulin, Automorphic forms and Lorentzian Kac-Moody algebras, Part II. Int. J. Math. **9**, 201–275 (1998).

[16] J.-I. Igusa, On Siegel modular forms of genus two I, II. Am. J. Math. **84**, 175–200 (1962); **86**, 392–412 (1964).

[17] T. Ibukiyama, On Relations of Dimensions of Automorphic Forms of Sp(2, R) and Its Compact Twist Sp(2) (I). Automorphic forms and number theory, Proc. Symp., Sendai/Jap. 1983, Adv. Stud. Pure Math. **7**, 7–30 (1985).

[18] T. Ibukiyama, Memorandum on explicit dimension formulas of paramodular forms. April 2000 (private communication).

[19] T. Ibukiyama and F. Onodera, On the graded ring of modular forms of the Siegel paramodular group of level 2. Abh. Math. Semin. Univ. Hamb. **67**, 297–305 (1997).

[20] F. O. Kappler, Über die Charaktere Siegelscher Stufengruppen. Ph.D.-thesis, Freiburg, 1977.

[21] M. Koecher and A. Krieg, Elliptische Funktionen und Modulformen. Berlin: Springer-Verlag (1998).

[22] G. Köhler, Erweiterungsfähigkeit paramodularer Gruppen. Nachr. Akad. Wiss. Gött., II Math.-Phys. Kl. **19**, 229–238 (1967).

[23] H. Maaß, Die Multiplikatorsysteme zur Siegelschen Modulgruppe. Nachr. Akad. Wiss. Gött., II Math.-Phys. Kl. 1964, 125–135.

[24] H. Maaß, Über eine Spezialschar von Modulformen zweiten Grades. I, II, III. Invent. Math. **52**, 95–104 (1979); **53**, 249–253, 255–265 (1979).

[25] H. Maaß, Über ein Analogon zur Vermutung von Saito-Kurokawa. Invent. Math. **60**, 85–104 (1980).

[26] T. Miyake, Modular forms. Berlin etc.: Springer-Verlag (1989).

[27] B. Runge, On symmetric Hilbert modular forms. Abh. Math. Sem. Univ. Hamburg **66**, 75–88 (1996).

[28] H. Rademacher, Über die Erzeugenden von Kongruenzuntergruppen der Modulgruppe. Abh. Math. Sem. Univ. Hamburg **7**, 134–148 (1929).

[29] N.-P. Skoruppa, Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts. Bonn. Math. Schr. **159** (1985).

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