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ON AN ALGEBRA OF SIEGEL MODULAR FORMS ASSOCIATED WITH THE THETA GROUP $\Gamma_2(1,2)$

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Abstract. In this note, we shall calculate some homogeneous polynomials explicitly based on B. Runge's conjecture in [4] p. 203 and give the explicit structure of the graded ring $A(\Gamma_2(1,2))$ of Siegel modular forms of genus two belonging to the discrete subgroup $\Gamma_2(1,2)$ of $Sp(2,\boldsymbol{R})$.

§ 1. Notations and preliminaries

Throughout this note, we will use the following notations.
We denote by $\boldsymbol{H}_g$ the Siegel upper half space of genus g defined by

$$\boldsymbol{H}_g := \{Z \in M(g,\boldsymbol{C}) \mid Z : \text{symmetric, } \mathrm{Im}(Z) > 0\}.$$

We denote by $Sp(g,\boldsymbol{R})$ the usual real symplectic group of size $2g$ defined by

$$Sp(g,\boldsymbol{R}) := \left\{ X \in M(2g,\boldsymbol{R}) \,\middle|\, {}^tXJX = J, J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} \right\}.$$

Furthermore we consider the following discrete subgroup of $Sp(g,\boldsymbol{R})$,

$$\Gamma_g := Sp(g,\boldsymbol{Z}).$$

We call this Γ_g the Siegel modular group, and we put

$$\Gamma_g(n) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g \,\middle|\, \begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv \begin{bmatrix} I_g & 0 \\ 0 & I_g \end{bmatrix} \bmod n \right\}$$

which is called the principal congruence subgroup of level n.

We put

$$\Gamma_g(n,2n) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g(n) \,\middle|\, (A^tB)_0 \equiv (D^tC)_0 \equiv 0 \bmod 2n \right\},$$

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where for $M = (m_{ij}) \in M(g, \boldsymbol{C})$, M_0 denotes the column vector $M_0 = \begin{bmatrix} m_{11} \\ \vdots \\ m_{gg} \end{bmatrix}$ of diagonal elements of M.

$\Gamma_g(1,2)$ is called the theta group.

The symplectic group $Sp(g, \boldsymbol{R})$ acts transitively on the Siegel upper half space $\boldsymbol{H}_g$ by

$$X(Z) := (AZ + B)(CZ + D)^{-1} \quad \text{for } X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(g, \boldsymbol{R}),\ Z \in \boldsymbol{H}_g.$$

We assume that $g \geq 2$.

For any non-negative integer k, we call $f(Z)$ a Siegel modular form of weight k if

(i) f is a holomorphic function on $\boldsymbol{H}_g$,

(ii) $f(X(Z)) = \det(CZ + D)^k f(Z)$ for any $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g$.

The space of all modular forms of the same weight k has a structure of a vector space and we denote it by $[\Gamma_g, k]$. The direct sum of them becomes the finite generated graded algebra and we denote it by $A(\Gamma_g) := \oplus_k [\Gamma_g, k]$.

We use the classical notation for the thetas, i.e. for $(\tau, z) \in \boldsymbol{H}_g \times \boldsymbol{C}^g$,

$$\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau, z) = \sum_{x \in \boldsymbol{Z}^g} \exp 2\pi i \left(\frac{1}{2} \tau\left[x + \frac{1}{2}\alpha\right] + \left\langle x + \frac{1}{2}\alpha, z + \frac{1}{2}\beta \right\rangle \right)$$

where $\langle x, y \rangle$ denotes the standard scalar product, $\tau[x] = {}^t x \tau x$ and i denotes $\sqrt{-1}$.

The elements α and β of $\boldsymbol{Z}^g$ are also regarded as elements in $\boldsymbol{F}_2^g$ with entries 0 and 1. Here we remark that $\theta\begin{bmatrix} \alpha + 2p \\ \beta + 2q \end{bmatrix} = \theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(-1)^{\langle \alpha, q \rangle}$.

It is well known that $\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau, z)$ is an analytic function on $\boldsymbol{H}_g \times \boldsymbol{C}^g$.

The $\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau, 0)$ are called the theta constants.

In the case of $g = 2$, there exists the following 10 even pairs of α, β, i.e., $(-1)^{\langle \alpha, \beta \rangle} = 1$:

$$\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \theta\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \theta\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \theta\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \theta\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \theta\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \theta\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\theta\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \theta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \theta\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Now we define $\Theta := \prod_{even} \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$. Here we note that the degree of Θ is 10.

We use this Θ to describe our problem (posed by B. Runge).

Finally we have to prepare the definition of a code. A code C means a linear subspace C of $\boldsymbol{F}_2^n$, when its dimension k and the minimal weight d, it is denoted by $[n,k,d]$. Where the weight d means the number of entries 1 in an element α of a code C. A code is called doubly-even if the weight of every element of the code is divisible by 4. A code C is called the self-dual when it coinsides with its dual $C^{\perp} = \{x \in \boldsymbol{F}_2^n \mid \langle x,y\rangle = 0 \textit{ for any } y \in C\}$, where $\langle x,y\rangle$ means the standard inner product.

§ 2. The relation between the Siegel modular group Γ_g and the corresponding finite group

We can choose the following generator of Γ_g:

$$\Gamma_g = \left\langle \begin{bmatrix} I_g & S \\ 0 & I_g \end{bmatrix}, \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}, {}^tS = S \in M(g,\boldsymbol{Z}) \right\rangle.$$

On the other hand, the thetas of second order are given by

$$f_a(\tau) := \theta\begin{bmatrix} a \\ 0 \end{bmatrix}(2\tau) = \sum_{x \in \boldsymbol{Z}^g} \exp 2\pi i \left(\tau\left[x + \frac{1}{2}a\right]\right)$$

for $a \in \boldsymbol{Z}^g$, where i denotes $\sqrt{-1}$.

The function f_a depends only on a *mod2*, here a is considered also as the element of $\boldsymbol{F}_2^g$.

The action of Γ_g on thetas of second order $f_a = \theta\begin{bmatrix} a \\ 0 \end{bmatrix}(2\tau, 0)$ is the following.

PROPOSITION 2.1 (see [3], p. 60). (1) *For* $\sigma = \begin{bmatrix} I_g & S \\ 0 & I_g \end{bmatrix} \in \Gamma_g$, $\sigma(f_a) = i^{S[a]} f_a$ *with* $S[a] = {}^t aSa$.

(2) *We define the* $2^g \times 2^g$*-matrix* T_g *by* $T_g := \left(\frac{1+i}{2}\right)^g ((-1)^{\langle a,b\rangle})_{a,b} \in \boldsymbol{F}_2^g$. *Then* $J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$ *acts on* f_a *as follows*:

$$J(f_a) = \sqrt{\det(-\tau)} \sum_{b \in \boldsymbol{F}_2^g} (T_g)_{a,b} f_a$$

for $f_a = \theta\begin{bmatrix} a \\ 0 \end{bmatrix}(2\tau, 0)$.

Then we can consider the following $2^g \times 2^g$-matrix D_S:

$D_S = diag(i^{S[a]})$ for $a \in \boldsymbol{F}_2^g$, and we define $H_g := \langle T_g, D_S \rangle$.

Here H_g acts faithfully on the vector space generated by f_a, $a \in \boldsymbol{F}_2^g$ and $H_g/\{\pm 1\}$ acts on the algebra $\boldsymbol{C}[f_a f_b]_{a,b \in \boldsymbol{F}_2^g}$.

Then the following theorem is known.

PROPOSITION 2.2 (see [3], p. 60). *There exists the surjective group homomorphism* $\phi : \Gamma_g \to H_g/\{\pm 1\}$ *satisfying* $\phi(J) = T_g$ *and* $\phi(\sigma) = \sigma\left(\begin{bmatrix} I_g & S \\ 0 & I_g \end{bmatrix}\right) = D_S$ *with the notations in proposition* 2.1.

By the surjective group homomorphism ϕ in above proposition 2.2, the subgroup $\Gamma_2(1,2)$ of Γ_2 corresponds to the following finite group G:

$$G = \left\langle S_4, diag(\pm 1, \pm 1, \pm 1, \pm 1), M_1 = \left(\frac{1+i}{2}\right)\begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix} \right\rangle$$

where S_4 denotes the symmetric group of degree 4.

§ 3. Runge's conjecture

From above proposition 2.1 and proposition 2.2 § 2, B. Runge proved the following theorem 3.1 about the correspondence between the ring of modular forms and the invariant ring of some finite group by using the theta functions (see [3]).

THEOREM 3.1 (B. Runge). *For any element* $a = (a_1, \ldots, a_g)$ *in* $\boldsymbol{F}_2^g$, *we define the function*

$$Y_a : \overbrace{\boldsymbol{F}_2^n \times \cdots \times \boldsymbol{F}_2^n}^{g} \to \boldsymbol{Z}_+$$

by

$$Y_a(\gamma_1, \ldots, \gamma_g) := \#\{i \mid a = (\gamma_{i,1}, \ldots, \gamma_{i,g})\}$$

where

$$(\gamma_1, \ldots, \gamma_g) = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1g} \\ \vdots & & \vdots \\ \gamma_{n1} & \cdots & \gamma_{ng} \end{bmatrix}.$$

Then for a self-dual doubly-even code C, *the polynomial of* 2^g*-variables*

$$P_g(C) := \sum_{(\gamma_1,\ldots,\gamma_g)\in C^g} \prod_{a\in F_2^g} f_a^{Y_a(\gamma_1,\ldots,\gamma_g)},$$

with the thetas of second order

$$f_a = \theta\begin{bmatrix} a \\ 0 \end{bmatrix}(2\tau, 0)$$

becomes a modular form (for the full modular group) of weight $k = n/2$.

From the fact in [1], p. 177, the fact in [2], theorem 7.1 in p. 200, the statement of p. 201 and Lemma 2.1 of p. 180 in [4], the space generated by the polynomials $P_2(C)$ for $C = C^{\perp}$ becomes the invariant ring $\boldsymbol{C}[f_a; a \in \boldsymbol{F}_2^2]^{G'}$ for $G' = \left\langle S_4, diag(\pm 1, \pm 1, \pm 1, \pm 1), W = 1/\sqrt{2}\begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix} \right\rangle$, where the matrix M_1 of $G = \phi(\Gamma_2(1,2))$ and W of G' differ by an eight root of unity. Furthermore, from the arguement in p. 202, p. 203 in [4], for our case of $G = \phi(\Gamma_2(1,2))$, $A(\Gamma_2(1,2)) = \oplus_k [\Gamma_2(1,2), k]$ corresponds to a certain subring of $\boldsymbol{C}[f_a, a \in \boldsymbol{F}_2^2, \Theta]$ where $\Theta = \Pi_{even}\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

Here, let us recall the related facts from the invariant theory.

PROPOSITION 3.2 (in [3], p. 76). *Let G be a finite group acting on a polynomial ring $A = \boldsymbol{C}[X_1, \ldots, X_r]$ and if $R = A^G$ is the invariant ring, it is known that R is of the type*:

$$R = \boldsymbol{C}[g_1, \ldots, g_r] \oplus \boldsymbol{C}[g_1, \ldots, g_r]g_{r+1} \oplus \cdots \oplus \boldsymbol{C}[g_1, \ldots, g_r]g_n \quad (r \leq n)$$

where each g_i is a homogeneous polynomial of degree d_i.

We denote by $R_{(l)}$ the homogeneous part of degree l in R. Then we have

$$\Phi_G(\lambda) = \sum_{l\geq 0} dim_{\boldsymbol{C}} R_{(l)}\lambda^l = \frac{1 + \sum_{i=r+1,\ldots,n} \lambda^{d_i}}{\prod_{i=1,\ldots,r}(1 - \lambda^{d_i})}.$$

In [4], p. 203, B. Runge calculated $\Phi_{\phi(\Gamma_2(1,2))}(\sqrt{\lambda})$ as follows:

$$\Phi_{\phi(\Gamma_2(1,2))}(\sqrt{\lambda}) = \frac{1 + \lambda^6 + \lambda^8 + \lambda^{10} + \lambda^{19} + \lambda^{21} + \lambda^{23} + \lambda^{29}}{(1 - \lambda^4)^2(1 - \lambda^6)(1 - \lambda^{12})}$$

and he remarked that this Poincaré series accords with [2], p. 405. Then he described the following conjecture:

CONJECTURE (Runge) (in [4], p. 203). *The ring* $\oplus_k[\Gamma_2(1,2),k]$ *has the following graded ring structure.*

$$\begin{aligned}\oplus_k[\Gamma_2(1,2),k] &= \boldsymbol{C}[P_2^4, P_2^3P_6, P_2^2P_6^2, P_2P_6^3, P_6^4, P_8, P_{12}, P_{28}\Theta, P_{32}\Theta, P_{36}\Theta]\\ &= \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^3P_6\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^2P_6^2\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2P_6^3\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{28}\Theta\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{32}\Theta\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{36}\Theta\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]Q_{58}\end{aligned}$$

where $\Theta = \Pi_{even}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, P_i *is a certain homogeneous polynomial of degree* i *and* Q_{58} *is a certain polynomial of degree* 58, *product of such* P_i*'s and* Θ.

In this note, we give an affirmative answer about the conjecture.

Actually B. Runge showed in the case of even weight related to this conjecture that:

THEOREM 3.3 (Runge) (see [4], p. 202). *The ring* $\oplus_{2|k}[\Gamma_2(1,2),k]$ *has the following graded ring structure.*

$$\begin{aligned}\oplus_{2|k}[\Gamma_2(1,2),k] &= \boldsymbol{C}[P_2^4, P_2^3P_6, P_2^2P_6^2, P_2P_6^3, P_6^4, P_8, P_{12}]\\ &= \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^3P_6\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^2P_6^2\\ &\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2P_6^3\end{aligned}$$

where each P_i *are the following polynomials*:

$$P_2 = (2),$$

$$P_6 = (6) - 5(4,2) + 30(2,2,2),$$

$$P_8 = (8) + 14(4,4) + 168(2,2,2,2),$$

and

$$P_{12} = (12) - 33(8,4) + 330(4,4,4) + 792(6,2,2,2)$$

where

$$(a_1, a_2, a_3, a_4) := \sum_{\sigma \in S_4 / Stab(a_1,a_2,a_3,a_4)} \prod_{i=1,\ldots,4} (f_{i-1}^{a_i})^{\sigma}.$$

Here we put $f_0 = f_{\begin{bmatrix}0\\0\end{bmatrix}}$, $f_1 = f_{\begin{bmatrix}0\\1\end{bmatrix}}$, $f_2 = f_{\begin{bmatrix}1\\0\end{bmatrix}}$, $f_3 = f_{\begin{bmatrix}1\\1\end{bmatrix}}$.

In this case, the corresponding finite group $G = \phi(\Gamma_2(1,2))$ is given as follows.

$$G = \left\langle S_4, \begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \pm 1 \end{bmatrix}, M_1 = \frac{1+i}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \right\rangle \subset GL(4, \boldsymbol{C}).$$

By theorem 3.3, what we have to consider are homogeneous polynomials of odd weight.

§ 4. The calculations and the result

By the invariance of the action of S_4 and diag(± 1, ± 1, ± 1, ± 1), we may put

$$P_{28} = \sum_{\substack{a_1+a_2+a_3+a_4=28 \\ a_1,a_2,a_3,a_4 \text{ are different odd integers}}} C_{a_1,a_2,a_3,a_4} (a_1, a_2, a_3, a_4)^A,$$

$$P_{32} = \sum_{\substack{a_1+a_2+a_3+a_4=32 \\ a_1,a_2,a_3,a_4 \text{ are different odd integers}}} C_{a_1,a_2,a_3,a_4} (a_1, a_2, a_3, a_4)^A$$

and

$$P_{36} = \sum_{\substack{a_1+a_2+a_3+a_4=36 \\ a_1,a_2,a_3,a_4 \text{ are different odd integers}}} C_{a_1,a_2,a_3,a_4} (a_1, a_2, a_3, a_4)^A$$

where C_{a_1,a_2,a_3,a_4}'s are the constants of rational integers and

$$(a_1,a_2,a_3,a_4)^A := \sum_{\sigma\in S_4} sign(\sigma) \prod_{i=1,\dots,4} (f_{i-1}^{a_i})\sigma.$$

By the invariance of the action of M_1, we calculate using the computer, nonzero alternating polynomials P_{28}, P_{32} and P_{36} as follows:

$$\begin{aligned} P_{28} = {} & (19,5,3,1)^A - 3(17,7,3,1)^A + (15,9,3,1)^A \\ & - 6(15,7,5,1)^A + 5(13,11,3,1)^A + 16(13,9,5,1)^A \\ & - 54(13,7,5,3)^A + 13(11,9,7,1)^A - 39(11,9,5,3)^A, \end{aligned}$$

$$\begin{aligned} P_{32} = {} & (21,7,3,1)^A - 7(19,9,3,1)^A + 21(17,11,3,1)^A - 63(17,7,5,3)^A \\ & - 35(15,13,3,1)^A - 18(15,9,7,1)^A + 203(15,9,5,3)^A + 63(13,11,7,1)^A \\ & - 294(13,11,5,3)^A - 322(13,9,7,3)^A + 2457(11,9,7,5)^A \end{aligned}$$

and

$$\begin{aligned} P_{36} = {} & 15(25,7,3,1)^A - 75(23,9,3,1)^A + 46(23,7,5,1)^A + 120(21,11,3,1)^A \\ & - 276(21,9,5,1)^A - 1074(21,7,5,3)^A + 644(19,13,3,1)^A - 135(19,9,7,1)^A \\ & + 1610(19,9,5,3)^A - 210(17,15,3,1)^A - 644(17,13,5,1)^A \\ & + 266(17,11,5,3)^A - 1665(17,9,7,3)^A + 1254(15,13,7,1)^A \\ & - 1330(15,13,5,3)^A + 1956(15,11,9,1)^A - 8736(15,11,7,3)^A \\ & + 34690(15,9,7,5)^A - 15750(13,11,9,3)^A + 20930(13,11,7,5)^A. \end{aligned}$$

So we have $\oplus_k[\Gamma_2(1,2),k]$ contains $\boldsymbol{C}[P_2^4, P_2^3P_6, P_2^2P_6^2, P_2P_6^3, P_6^4, P_8, P_{12}, P_{28}\Theta, P_{32}\Theta, P_{36}\Theta]$ as a subring.

Furthermore we remark that

$$\begin{aligned} \Phi_{\phi(\Gamma_2(1,2))}(\sqrt{\lambda}) &= \frac{1+\lambda^6+\lambda^8+\lambda^{10}+\lambda^{19}+\lambda^{21}+\lambda^{23}+\lambda^{29}}{(1-\lambda^4)^2(1-\lambda^6)(1-\lambda^{12})} \\ &= 1+2\lambda^4+2\lambda^6+4\lambda^8+5\lambda^{10}+9\lambda^{12}+9\lambda^{14}+15\lambda^{16}+17\lambda^{18}+\lambda^{19}+23\lambda^{20} \\ &\quad +\lambda^{21}+27\lambda^{22}+3\lambda^{23}+36\lambda^{24}+3\lambda^{25}+39\lambda^{26}+6\lambda^{27}+7\lambda^{29}+\dots\dots, \end{aligned}$$

and we checked that

$$\begin{aligned}
Q_{48} := P_2^3 P_6 P_{36} &= 15(37,7,3,1)^A - 105(35,9,3,1)^A + 16(35,7,5,1)^A \\
&+ 15(33,11,3,1)^A - 128(33,9,5,1)^A - 821(33,7,5,3)^A \\
&+ 615(31,13,3,1)^A + 144(31,11,5,1)^A + 3590(31,9,7,1)^A \\
&+ 5681(31,9,5,3)^A - 405(29,15,3,1)^A + 512(29,13,5,1)^A \\
&- 2633(29,11,7,1)^A + 7258(29,11,5,3)^A - 21320(29,9,7,3)^A \\
&- 1485(27,17,3,1)^A - 944(27,15,5,1)^A + 103(27,13,7,1)^A \\
&+ 182(27,13,5,3)^A - 31106(27,11,9,1)^A - 111872(27,11,7,3)^A \\
&- 68689(27,9,7,5)^A + 1275(25,19,3,1)^A - 640(25,17,5,1)^A \\
&+ 18826(25,15,7,1)^A + 377(25,15,5,3)^A - 12800(25,13,9,1)^A \\
&+ 88000(25,13,7,3)^A - 136786(25,11,9,3)^A - 104221(25,11,9,5)^A \\
&+ 1875(23,21,3,1)^A + 2000(23,19,5,1)^A - 24412(23,17,7,1)^A \\
&- 29837(23,17,5,3)^A + 11122(23,15,9,1)^A + 346624(23,15,7,3)^A \\
&+ 57819(23,13,11,1)^A + 207430(23,13,9,3)^A - 296200(23,15,7,3)^A \\
&+ 688701(23,11,9,5)^A - 36913(21,19,7,1)^A - 30788(21,19,5,3)^A \\
&- 85376(21,17,9,1)^A + 82478(21,17,7,3)^A + 178738(21,15,11,1)^A \\
&+ 209085(21,15,9,3)^A - 1453960(21,15,7,5)^A - 417746(21,13,11,3)^A \\
&+ 243712(21,13,9,5)^A + 4563377(21,11,9,7)^A + 92575(19,17,11,1)^A \\
&- 108025(19,17,9,3)^A - 1194129(19,17,7,5)^A + 230501(19,15,13,1)^A \\
&- 874752(19,15,11,3)^A - 1942435(19,15,9,5)^A - 2084558(19,13,11,5)^A \\
&+ 8801090(19,13,9,7)^A - 319143(17,15,13,3)^A - 2492914(17,15,11,5)^A \\
&+ 4411534(17,15,9,7)^A + 4772515(17,13,11,7)^A + 1161508(15,13,11,9)^A
\end{aligned}$$

is different from any linear combinations of the following polynomials:

$$P_2^4P_{12}P_{28} = (39,5,3,1)^A + (37,7,3,1)^A - 38(35,9,3,1)^A + 813(35,7,5,1)^A$$
$$- 38(33,11,3,1)^A + 760(33,9,5,1)^A + 733(33,7,5,3)^A$$
$$+ 142(31,13,3,1)^A + \ldots\ldots,$$

$$P_8P_{12}P_{28} = (39,5,3,1)^A - 3(37,7,3,1)^A - 18(35,9,3,1)^A + 805(35,7,5,1)^A$$
$$+ 62(33,11,3,1)^A - 2360(33,9,5,1)^A - 2319(33,7,5,3)^A$$
$$- 518(31,13,3,1)^A + \ldots\ldots,$$

$$P_8P_2^4P_{32} = (37,7,3,1)^A - 3(35,9,3,1)^A + 4(35,7,5,1)^A + 13(33,11,3,1)^A$$
$$- 16(33,9,5,1)^A - 71(33,7,5,3)^A - 31(31,13,3,1)^A + \ldots\ldots,$$

$$P_8^2P_{32} = (37,7,3,1)^A - 7(35,9,3,1)^A + 49(33,11,3,1)^A - 63(33,7,5,3)^A$$
$$- 231(31,13,3,1)^A + 28(31,13,3,1)^A + \ldots\ldots,$$

$$P_2^8P_{32} = (37,7,3,1)^A + (35,9,3,1)^A - 7(33,11,3,1)^A + 56(33,9,5,1)^A$$
$$- 63(33,7,5,3)^A - 596(31,13,3,1)^A + \ldots\ldots,$$

and

$$P_{12}P_{36} = 15(37,7,3,1)^A - 75(35,9,3,1)^A + 46(35,7,5,1)^A - 375(33,11,3,1)^A$$
$$- 276(33,9,5,1)^A - 1074(33,7,5,3)^A + 2475(31,13,3,1)^A + \ldots\ldots.$$

Furthermore we get these 7 polynomials $\times\Theta$ as a basis of the 7-dimensional vector space R_{29}, i.e.

$$R_{29} = \langle Q_{48}\Theta = P_2^3P_6P_{36}\Theta, P_2^4P_{12}P_{28}\Theta, P_8P_{12}P_{28}\Theta, P_8P_2^4P_{32}\Theta,$$
$$P_8^2P_{32}\Theta, P_2^8P_{32}\Theta, P_{12}P_{36}\Theta\rangle.$$

(We checked that $P_2^2P_6^2P_{32}\Theta$ and $P_2P_6^3P_{28}\Theta$ are linear combinations of above 7 polynomials $\times\Theta$.)

So we have the following decomposition:

$$
\begin{aligned}
&\boldsymbol{C}[P_2^4, P_2^3P_6, P_2^2P_6^2, P_2P_6^3, P_6^4, P_8, P_{12}, P_{28}\Theta, P_{32}\Theta, P_{36}\Theta] \\
&\quad = \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}] \\
&\qquad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^3P_6 \\
&\qquad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^2P_6^2 \\
&\qquad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2P_6^3 \\
&\qquad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{28}\Theta \\
&\qquad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{32}\Theta \\
&\qquad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{36}\Theta \\
&\qquad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]Q_{48}\Theta \\
&\qquad \oplus \cdots\cdots
\end{aligned}
$$

and using this result, we see that the Poincaré series of $\oplus_k[\Gamma_2(1,2), k]$ and that of this subring coinside. This implies that

$$
\begin{aligned}
\oplus_k[\Gamma_2(1,2), k] &= \boldsymbol{C}[P_2^4, P_2^3P_6, P_2^2P_6^2, P_2P_6^3, P_6^4, P_8, P_{12}, P_{28}\Theta, P_{32}\Theta, P_{36}\Theta] \\
&= \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}] \\
&\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^3P_6 \\
&\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^2P_6^2 \\
&\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2P_6^3 \\
&\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{28}\Theta \\
&\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{32}\Theta \\
&\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_{36}\Theta \\
&\quad \oplus \boldsymbol{C}[P_2^4, P_6^4, P_8, P_{12}]P_2^3P_6P_{36}\Theta.
\end{aligned}
$$

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