# *On symplectic Euler factors of genus two*

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This paper is an attempt to find a "genus two version" of Eichler's correspondence [4] [5]. The results have been announced in [10]. After [4], [5], several authors have studied the correspondence between automorphic forms belonging to discrete subgroups of $SU(2)$ and of $SL(2, \boldsymbol{R})$ which preserves $L$ functions, notably, [15], [26]. For the groups of higher rank, Ihara [13] studied automorphic forms on $USp(4)=\{g\in M_2(\boldsymbol{H})\,;\ g^t\bar{g}=1_2\}$ ($H$: the Hamilton quaternions), and, as a generalization of Eichler's correspondence, suggested to consider the correspondence between automorphic forms belonging to discrete subgroups of $USp(4)$ and of $Sp(2, \boldsymbol{R})$ (symplectic group of size four). This problem can be regarded as a special case of the problem of functoriality with respect to $L$ groups proposed later by Langlands (cf. [18], [19]). Let $\rho_\nu$ be the representation of $USp(4)$ corresponding to the Young diagram $\begin{array}{|c|c|c|}\hline 1 & \cdots & \nu \\ \hline 1 & \cdots & \nu \\ \hline\end{array}$. Ihara clarified, among others, that the weight of the Siegel modular forms which would correspond to automorphic forms on $USp(4)$ with 'weight $\rho_\nu$' should be $\nu+3$, by showing some character relations between $\rho_\nu$ and holomorphic discrete series representations of $Sp(2, \boldsymbol{R})$. But there has been no known example of such a correspondence at all, and we did not know either, which discrete subgroup of $USp(4)$ should correspond to which discrete subgroup of $Sp(2, R)$. In this note, we give some examples of pairs of automorphic forms of $Sp(2, R)$ and $USp(4)$ whose Euler 3-factors coincide with each other. This coincidence does not seem accidental, since the coefficients of the Euler factors are fairly large. These Euler 3-factors satisfy the Ramanujan Conjecture, and are obtained from 'new forms' (which can not be obtained as 'liftings' of the forms of one variable, and are not contained in the linear span of automorphic forms belonging to any 'larger' discrete subgroups). We also propose a conjecture which seems reasonable for the present. In § 1, we give examples of new forms and conjecture. To know which form is a new form, we must obtain all forms containing old ones. So, in § 2 and § 3, we give eigen space decomposition of whole space of forms with smaller weights, together with proofs.

## §1. Conjecture and examples

Let $D$ be a definite quaternion algebra over $\boldsymbol{Q}$ with the prime discriminant $p$, and $\mathcal{O}$ be a maximal order of $D$. Put $G=\{g\in M_2(D)\,;\, g\,{}^t\bar{g}=n(g)1_2,\ n(g)\in\boldsymbol{Q}^\times\}$. In the typical case of Eichler's correspondence, automorphic forms on the adelization $D_A^\times$ of $D^\times$ belonging to $\boldsymbol{H}^\times\prod_q \mathcal{O}_q^\times$ $(H=D\otimes\boldsymbol{R})$ correspond with those belonging to $\Gamma_0(p)\subset SL(2,\boldsymbol{R})$. But in the case of genus two, there are large gaps between the "main terms" (the contribution of the identity element to the dimension of automorphic forms by means of the trace formula) of 'level one' subgroups of $G$ and $\Gamma_0(p)$-type subgroups of $Sp(2,\boldsymbol{Q})$. On the other hand, we know that any reductive algebraic group over a local field has the unique minimal parahoric subgroup up to conjugation. So, it seems natural to consider the correspondence between automorphic forms belonging to (global) discrete subgroups which are obtained from open subgroups of the adelization whose $p$-components are minimal parahoric. This means that we should consider 'level $\pi$' discrete subgroups also for $G$, where $\pi$ is a prime element of $\mathcal{O}_p=\mathcal{O}\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$. More precisely, put

$$G_p^*=\left\{g\in M_2(D_p)\,;\, g\begin{pmatrix}0&1\\1&0\end{pmatrix}{}^t\bar{g}=n(g)\begin{pmatrix}0&1\\1&0\end{pmatrix},\ n(g)\in\boldsymbol{Q}_p^\times\right\},$$

where $D_p=D\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$. For any prime $q$, let $G_q$ be the $q$-component of the adelization $G_A$ of $G$. Then, $G_p^*$ is isomorphic to $G_p$, and we fix such an isomorphism. Put

$$U_p^*=\begin{pmatrix}\mathcal{O}_p&\mathcal{O}_p\\ \pi\mathcal{O}_p&\mathcal{O}_p\end{pmatrix}^\times\cap G_p^* \quad\text{and}\quad U_q=M_2(\mathcal{O}_q)^\times\cap G_q \quad\text{for}\quad q\neq p.$$

Put $U_0(D)=G_\infty U_p^*\prod_{q\neq p}U_q\subset G_A$, where $G_\infty$ is the infinite part of $G_A$. Now, we define the space $\mathfrak{M}_\nu(U_0(D))$ of automorphic forms of 'weight $\rho_\nu$' belonging to $U_0(D)$. Regard $(x,y)\in\boldsymbol{H}^2$ as the variable over eight dimensional vector space over $\boldsymbol{R}$. Denote by $\mathfrak{M}_\nu$ the $\boldsymbol{R}$ vector space of real valued homogeneous polynomial functions $f(x,y)$ on $\boldsymbol{H}^2$ of degree $2\nu$ which satisfy

(1) $f(ax,ay)=N(a)^\nu f(x,y)$ for any $a\in\boldsymbol{H}$, and
(2) $\Delta f=0$,

where $N$ is the reduced norm of $H$ and $\Delta$ is the usual Laplacian with respect to the metric $N(x)+N(y)$ of $\boldsymbol{H}^2$. Then, $G$ acts on $\mathfrak{M}_\nu$ as $f(x,y)\to f((x,y)g)$ for $g\in G$. This representation is an extension of $\rho_\nu$ to $G$, which will be also denoted by $\rho_\nu$. Then, $\mathfrak{M}_\nu(U_0(D))$ is the set of $\mathfrak{M}_\nu$-valued functions $f$ on $G_A$ such that

(1) $f(gh)=f(g)$ for all $h\in G$, and
(2) $f(ug)=\rho_\nu(u_\infty)f(g)$ for all $u\in U_0(D)$,

where $u_\infty$ is the infinite component of $u$. For an integer $n$ prime to $p$, put $T(n)=\bigcup_g U_0(D)gU_0(D)$, where $g$ runs through the elements of $G_A \cap G_\infty U_p^* \prod_{q\neq p} M_2(O_q)$ whose similitudes are $n$. Put $T(n)=\coprod_i g_i U_0(D)$ (disjoint). Then, the action of $T(n)$ on $\mathfrak{M}_\nu(U_0(D))$ is defined by:

$$(T(n)f)(g)=\sum_i \rho_\nu(g_i)f(g_i^{-1}g).$$

On the other hand, put

$$B(p)=Sp(2, \boldsymbol{Z})\cap\begin{pmatrix} * & * & * & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & p* & * & * \end{pmatrix},$$

where $*$ runs through any integers. Denote by $S_k(B(p))$ the space of Siegel cusp forms with weight $k$ belonging to $B(p)$. The Hecke operator $T(n)(p \nmid n)$ and its action on $S_k(B(p))$ are defined as usual.

CONJECTURE. *For each even integer $k \geq 4$, there exists a $\boldsymbol{C}$ linear isomorphism $i_k$ of 'new forms' of $\mathfrak{M}_{k-3}(U_0(D))$ to 'new forms' of $S_k(B(p))$ such that $L(s, i_k(f)) = L(s, f)$ up to Euler $p$-factors for any common eigen 'new form' $f$ of $\mathfrak{M}_{k-3}(U_0(D))$ of all the Hecke operators $T(n)(p \nmid n)$.*

Here, we define new forms of $\mathfrak{M}_\nu(U_0(D))$ (resp. $S_k(B(p))$) to be the elements of the orthogonal complement of the space spanned by automorphic forms of $G_A$ (resp. cusp forms of $Sp(2, \boldsymbol{Q}_A)$) belonging to any larger subgroups of $G_A$ (resp. $Sp(2, \boldsymbol{Q}_A)$) containing $U_0(D)$ (resp. $Sp(2, R)\prod_{q\neq p} Sp(2, Z_q)B(p)_p$, where $B(p)_p$ is the topological closure of $B(p)$ in $Sp(2, \boldsymbol{Q}_p)$). We denote by $L(s, *)$ the (denominator of the) $L$ function of Andrianov type.

Now, we give some examples for $p=2$. Put $D=\boldsymbol{Q}+\boldsymbol{Q}i+\boldsymbol{Q}j+\boldsymbol{Q}k$, $i^2=-1$, $j^2=-1$, $ij=-ji=k$, and $\mathcal{O}=\boldsymbol{Z}+\boldsymbol{Z}i+\boldsymbol{Z}j+\boldsymbol{Z}(1+i+j+k)/2$. Then, the discriminant of $D$ is two and $\mathcal{O}$ is a maximal order of $D$. We can show that the 'class number' of $U_0(D)$ (that is, the number of the double cosets in $G\backslash G_A/U_0(D)$) is one. Then, $\mathfrak{M}_\nu(U_0(D))$ can be identified with

$$\mathfrak{M}_\nu(\Gamma_0)=\{f\in\mathfrak{M}_\nu ;\ f((x, y)\gamma)=f(x, y) \quad \text{for all } \gamma\in\Gamma_0\},$$

where

$$\Gamma_0=\left\{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}, \begin{pmatrix} 0 & a \\ d & 0\end{pmatrix} ;\ a,\ d\in\mathcal{O}^\times,\ N(a-d)\equiv 0 \bmod 2\right\}.$$

Under this identification, the Hecke operator $T(n)(2\nmid n)$ acts on $\mathfrak{M}_\nu(\Gamma_0)$ as

$$f(x, y) \longrightarrow (T(n)f)(x, y)=\sum_{g\in\Delta_n/\Gamma_0} f((x, y)g),$$

where

$$\Delta_n=\left\{g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G\cap M_2(\mathcal{O})\,;\ n(g)=n \text{ and } N(a-d)\equiv N(b-c)\equiv 0 \bmod. 2\right\}.$$

On the other hand, by using Igusa [12], the graded ring $A(B(2))$ of modular forms belonging to $B(2)$ and the ideal of cusp forms in $A(B(2))$ has been given explicitly in terms of theta constants, together with an explicit dimension formula (cf. [11]). Now, let $f\in\mathfrak{M}_\nu(U_0(D))$ or $S_k(B(2))$ be a common eigen form of all $T(n)(2\nmid n)$. Then, the Hecke polynomial of $f$ at a prime $q\neq 2$ is defined by

$$H_q(T,\ f)=T^4-\lambda(q)T^3+(\lambda(q)^2-\lambda(q^2)-q^{2m-4})T^2-\lambda(q)q^{2m-4}T+q^{4m-6},$$

where $m=k$ or $\nu+3$ for $f\in S_k(B(2))$ or $\mathfrak{M}_\nu(\Gamma_0)$, respectively, and $\lambda(q)$ or $\lambda(q^2)$ is the eigen value of $T(q)$ or $T(q^2)$ on $f$, respectively. Denote by $\mathfrak{M}_\nu^0(\Gamma_0)$ or $S_k^0(B(2))$ the space of new forms of $\mathfrak{M}_\nu(\Gamma_0)$ or $S_k(B(2))$, respectively. For small odd $\nu$ and even $k$, we obtain the following table:

| | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| dim $\mathfrak{M}_\nu(\Gamma_0)$ | 0 | 0 | 1 | 1 | 2 |
| dim $\mathfrak{M}_\nu^0(\Gamma_0)$ | 0 | 0 | 0 | 1 | 1 |

| | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| dim $S_k(B(2))$ | 0 | 0 | 1 | 3 | 6 | 12 |
| dim $S_k^0(B(2))$ | 0 | 0 | 0 | 0 | 1 | 1 |

Define the real valued functions $x_i=x_i(x)$ $(i=1,\cdots,4)$ on $\boldsymbol{H}$, by $x=x_1+x_2i+x_3j+x_4k$. Put

$$f_7(x,\ y)=(N(y)-N(x)(N(x)^2-3N(x)N(y)+N(y)^2)\prod_{i=1}^{4}(\bar{y}x)_i, \text{ and}$$

$$f_9(x,\ y)=(N(y)-N(x))\Big(153N(x)^4-1122N(x)^3N(y)+2618N(x)^2N(y)^2$$
$$-1122N(x)N(y)^3+153N(y)^4-1292\sum_{i=1}^{4}(\bar{y}x)_i^4\Big)\prod_{i=1}^{4}(\bar{y}x)_i^4.$$

Put also

$$X=(\theta_{0000}^4+\theta_{0001}^4+\theta_{0010}^4+\theta_{0011}^4)/4,$$
$$Y=(\theta_{0000}\theta_{0010}\theta_{0001}\theta_{0011})^2,$$
$$Z=(\theta_{0100}^4-\theta_{0110}^4)^2/16384, \qquad T=(\theta_{0100}\theta_{0110})^4/256,$$
$$K=(\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2/4096, \quad \text{and}$$
$$R=(X^2-Y-1024Z-64T)/64,$$

where $\theta_m(\tau)$ is a theta constant on the Siegel upper half space of genus two given by

$$\theta_m(\tau)=\sum_{p\in Z^2}\exp 2\pi i[{}^t(p+m'/2)\tau(p+m'/2)/2+{}^t(p+m'/2)m''/2]$$

for any $m={}^t(m', m'')$, $m', m'' \in \boldsymbol{Z}^2$. Put

$$F_{10}=12XTR-2XYZ+X^2K+YK+1024ZK+96RK, \quad \text{and}$$

$$F_{12}=36YTR+36864ZTR+3840TR^2-1920RYZ+12X^2TR$$
$$-21Y^2Z-21504YZ^2+XYK+1024XZK-3840K^2+13X^2YZ+7X^3K.$$

THEOREM. *Bases of* $\mathfrak{M}_\nu^0(\Gamma_0)$ *or* $S_k^0(B(2))$ *for* $\nu=7, 9$, *and* $k=10, 12$, *are given respectively as follows*:

$$M_7^0(\Gamma_0)=\boldsymbol{C}f_7(x, y), \qquad M_9^0(\Gamma_0)=\boldsymbol{C}f_9(x, y),$$

$$S_{10}^0(B(2))=\boldsymbol{C}F_{10}, \qquad S_{12}^0(B(2))=\boldsymbol{C}F_{12}.$$

*The Hecke polynomials of these automorphic forms at* $q=3$ *are given by*:

$$H_3(T, f_7)=H_3(T, F_{10})=T^4+18360T^3+297016470T^2+3^{17}\cdot 18360T+3^{34}$$
$$=(T^2+108(85-8\sqrt{61})T+3^{17})(T^2+108(85+8\sqrt{61})T+3^{17}),$$

*and*

$$H_3(T, f_9)=H_3(T, F_{12})=T^4+14760T^3-9330332490T^2+3^{21}\cdot 14760T+3^{42}$$
$$=(T^2+36(205+2\sqrt{5845969})T+3^{21})(T^2+36(205-2\sqrt{5845969})T+3^{21}).$$

*The absolute values of the zeros of these polynomials are equal to* $3^{17/2}$ *and* $3^{21/2}$, *respectively.*

## § 2. Eigen space decomposition in case of $USp(4)$

In § 2, we shall give Euler 3-factors of the Dirichlet series defined by Hecke operators acting on $\mathfrak{M}_\nu(U_0(D))$ for some small $\nu$ when the discriminant of $D$ is two.

**2.1.** As before, let $D$ be a definite quaternion algebra over $\boldsymbol{Q}$ with a prime discriminant $p$. (Very soon, we take $p=2$.) Introduce two more subgroups of $G_A$:

$$U_{1,p}^*=M_2(\mathcal{O}_p)^\times \cap G_p^*, \qquad U_{2,p}^*=\begin{pmatrix} \mathcal{O}_p & \pi^{-1}\mathcal{O}_p \\ \pi\mathcal{O}_p & \mathcal{O}_p \end{pmatrix}^\times \cap G_p^*.$$

Put

$$U_i(D)=G_\infty \prod_{q\neq p} U_q\cdot U_{i,p}^* \qquad \text{for} \quad i=1, 2,$$

and let $U_0(D)$ be as in 1.1. Decompose $G_A$ into disjoint union:

$$G_A=\coprod_{j=1}^{h_i} U_i(D)x_{ij}G \qquad (i=0, 1, 2).$$

Call $h_i$ *the class number* of $U_i(D)$. Put $\Gamma_{ij}=x_{ij}^{-1}U_i(D)x_{ij}\cap G$. The main volume

(Mass) of $U_i(D)$ is defined as

$$M_G(U_i(D))=\sum_{j=1}^{h_j}|\Gamma_{ij}|^{-1}.$$

Then, $M_G(U_i(D))$ ($i=1, 2, 0$) are given respectively as

$$(p-1)(p^2+1)/5760,\quad (p^2-1)/5760,^{1)}\quad (p^4-1)/5760.$$

In fact, the assertion for $U_1(D)$ is a special case of [9] Prop. 9. The rest is easy if we note that

$$[U^*_{2,p}:U^*_p]=p^2+1,\quad [U^*_{1,p}:U^*_p]=p+1.$$

**2.2.** From now on, assume $p=2$, and present $D$ and $\mathcal{O}$ as in §1. Put $r=i-k$ and $g=\begin{pmatrix}1 & -1\\ 0 & r\end{pmatrix}$. Define subgroups $\Gamma_i$ ($i=0, 1, 2$) of $G$ by:

$$\Gamma_1=GL_2(\mathcal{O})\cap G,\quad \Gamma_2=g^{-1}GL_2(\mathcal{O})g\cap G,\quad \text{and}\quad \Gamma_0=\Gamma_1\cap\Gamma_2.$$

Then, $\Gamma_i$ is $G$-conjugate to $U_i(D)\cap G$ and $h_i=1$ for each $i=0, 1, 2$. In fact, the first assertion can be easily proved by giving local 'integral' conjugations. On the other hand, we have $|\Gamma_i|=1152, 1920, 384$ for $i=1, 2, 0$, by direct calculation, so $h_i=1$ by Mass formula. (The fact that $h_1=h_2=1$ has been also contained in Theorems in [9].) More precisely, $\Gamma_2$ consists of the following elements of $G$:

(1) $\begin{pmatrix}ar^{-1} & -aa_0r^{-1}\\ ar^{-1} & aa_0r^{-1}\end{pmatrix}$, (2) $\begin{pmatrix}ar^{-1} & aa_0r^{-1}\\ -ar^{-1} & aa_0r^{-1}\end{pmatrix}$,

(3) $\begin{pmatrix}a & 0\\ 0 & aa_0\end{pmatrix}$, (4) $\begin{pmatrix}0 & aa_0\\ a & 0\end{pmatrix}$,

(5) $\begin{pmatrix}(1+r^{-1}x)a & r^{-1}xaa_0\\ r^{-1}xa & (1+r^{-1}x)aa_0\end{pmatrix}$,

where $a\in\mathcal{O}^\times$, $a_0\in\{\pm1, \pm i, \pm j, \pm k\}$ and $x\in\{-i, k, (\pm1-i\pm j+k)/2\}$. The group $\Gamma_0$ consists of elements of the forms (3) and (4), and

$$\Gamma_1=\left\{\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}, \begin{pmatrix}0 & a\\ d & 0\end{pmatrix};\ a, d\in\mathcal{O}^\times\right\}.$$

Now, since $h_i=1$, we can identify each $\mathfrak{M}_\nu(U_i(D))$ with

$$\mathfrak{M}_\nu(\Gamma_i)=\{f(x, y)\in\mathfrak{M}_\nu;\ f((x, y)\gamma)=f(x, y)\ \text{ for all } \gamma\in\Gamma_i\}\qquad (i=0, 1, 2).$$

PROPOSITION 2.1. *Notations being as above, the dimensions of automorphic*

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1) This was first pointed out to me by K. Hashimoto who thought that $U_2(D)$ is important as well as $U_1(D)$, but did not consider on $U_0(D)$.

*forms belonging to* $U_i(D)$ *are described by the formulae*

$$\sum_{\nu=0}^{\infty} \dim \mathfrak{M}_\nu(U_1(D))t^\nu = \frac{(1+t^{13})(1-t^4+t^8)}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$$

$$\sum_{\nu=0}^{\infty} \dim \mathfrak{M}_\nu(U_2(D))t^\nu = \frac{(1+t^5)(1-t^4+t^8-t^{12}+t^{16})}{(1-t^4)^2(1-t^6)(1-t^{10})}$$

$$\sum_{\nu=0}^{\infty} \dim \mathfrak{M}_\nu(U_0(D))t^\nu = \frac{(1+t^5)(1-t^4+t^8)}{(1-t^2)(1-t^4)^2(1-t^6)},$$

*where t is an indeterminant.*

PROOF. It is obvious that $\dim \mathfrak{M}_\nu(U_i(D)) = \Big(\sum_{\gamma\in\Gamma_i} \mathrm{tr}\, \rho_\nu(\gamma)\Big)/|\Gamma_i|$. So, use the character formula of Weyl. (As for $i=1, 2$, an explicit formula for $\mathfrak{M}_\nu(U_i(D))$ for general $D$ has been given in [9].) q.e.d.

Now, recall that the space of new forms $\mathfrak{M}_\nu^0(U_0(D))$ is the orthogonal complement of $\mathfrak{M}_\nu(U_1(D))+\mathfrak{M}_\nu(U_2(D))$ in $\mathfrak{M}_\nu(U_0(D))$ with respect to its natural inner metric.

*Numerical examples.*

| $\nu$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dim \mathfrak{M}_\nu(U_1(D))$ | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 3 | 0 | 3 |
| $\dim \mathfrak{M}_\nu(U_2(D))$ | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 1 | 2 |
| $\dim \mathfrak{M}_\nu(U_0(D))$ | 1 | 0 | 1 | 0 | 2 | 1 | 3 | 1 | 5 | 2 | 6 |
| $\dim \mathfrak{M}_\nu^0(U_0(D))$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |

| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 6 | 1 | 7 | 1 | 8 | 1 | 11 | 2 | 14 | 3 | 15 | 3 | 21 | 6 |
| 1 | 3 | 2 | 3 | 2 | 5 | 3 | 5 | 3 | 8 | 5 | 7 | 5 | 11 | 8 |
| 3 | 10 | 5 | 12 | 6 | 17 | 10 | 21 | 12 | 28 | 17 | 33 | 21 | 43 | 28 |
| 2 | 1 | 2 | 2 | 3 | 4 | 6 | 5 | 7 | 6 | 9 | 11 | 13 | 11 | 14 |

THEOREM 2.2. *Common eigen basis of* $\mathfrak{M}_\nu(\Gamma_0)$ *and* $\mathfrak{M}_\nu(\Gamma_2)$, *and their Hecke polynomials at* 3, *for small* $\nu$, *are given as follows*:

$\nu=5$

$$\mathfrak{M}_5(\Gamma_0)=\mathfrak{M}_5(\Gamma_2)=\boldsymbol{C}f_5,$$

*where* $f_5(x, y)=(\bar{y}x)_1(\bar{y}x)_2(\bar{y}x)_3(\bar{y}x)_4(N(y)-N(x))$. *We have* $\lambda(3)=1080$ *and* $\lambda(9)=2800089$ *for* $f_5$, *and* $H_3(T, f_5)=(T-3^6)(T-3^7)(T^2+1836T+3^{13})$.

$\nu=7$

$$\mathfrak{M}_7(\Gamma_2)=\{0\}, \quad \mathfrak{M}_7(\Gamma_0)=\boldsymbol{C}f_7,$$

*where* $f_7(x, y)=f_5(x, y)(N(x)^2-3N(x)N(y)+N(y)^2)$. *We have* $\lambda(3)=-18360$ *and* $\lambda(9)=-2973591$ *for* $f_7$, *and* $H_3(T, f_7)=T^4+18360T^3+297016470T^2+3^{17}\cdot 18360T+3^{34}$.

$\nu=9$

$$\mathfrak{M}_9(\Gamma_2)=\boldsymbol{C}f_9^{(1)}, \quad \mathfrak{M}_9(\Gamma_0)=\boldsymbol{C}f_9^{(1)}+\boldsymbol{C}f_9,$$

*where* $f_9^{(1)}=34g_1+g_2$, $f_9=51g_2-646g_1$,

$$g_1=f_5(x, y)\left(14\sum_{i=1}^{4}(\bar{y}x)_i^4-5N(x)^2N(y)^2\right), \qquad \text{and}$$

$$g_2=f_5(x, y)\times(21N(x)^4-154N(x)^3N(y)+296N(x)^2N(y)^2-154N(x)N(y)^3+21N(y)^4).$$

*The polynomials* $f_9^{(1)}$ *and* $f_9$ *are common eigen forms of* $T(n)(2\nmid n)$ *and we have* $\lambda(3)=307800$ *and* $\lambda(9)=3^8\cdot 8142169$ *for* $f_9^{(1)}$, *and* $\lambda(3)=-14760$ *and* $\lambda(9)=6061405689$ *for* $f_9$. *The Hecke polynomials are*

$$H_3(T, f_9^{(1)})=(T-3^{10})(T-3^{11})(T^2-71604T+3^{21}),$$

*and*

$$H_3(T, f_9)=T^4+14760T^3-9330332490T^2+3^{21}\cdot 14760T+3^{42}.$$

PROOF. By routine calculation, we can show that $f_5$, $f_7$, $f_9^{(1)}$ and $f_9$ are automorphic forms belonging to $\Gamma_0$. By virtue of Prop. 2.1, these span $\mathfrak{M}_\nu(\Gamma_0)$ for $\nu=5, 7, 9$. Next, we calculate the eigen values of $T(3)$ and $T(9)$. We have $\deg T(3)=40$ and $\deg T(9)=1201$. Let $\Delta_n$ be as in 1.1, then the representatives of cosets $\Delta_n/\Gamma_0$ for $n=3, 9$ are given as follows:

$\Delta_3/\Gamma_0$

(1) $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$, $\quad z_1, z_2\in\{1\pm i\pm j\}$, and

(2) $\begin{pmatrix} 1 & \beta \\ -\bar{\beta} & 1 \end{pmatrix}$, $\quad \beta\in O$, $N(\beta)=2$.

The numbers of elements of the form (1) and (2) are 16 and 24, respectively.

$\Delta_9/\Gamma_0$

(1) $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $\quad$ where $a$ and $d$ range over the set

$\{3\}\cup\{z_1z_2\,;\ z_1\neq\bar{z}_2,\ z_1,\ z_2\in\{1\pm i\pm j\}\}$.

(2) $\begin{pmatrix}1 & b\\ -\bar{b} & 1\end{pmatrix}$, $b\in\mathcal{O},\ N(b)=8$.

(3) $\begin{pmatrix}r & b\\ r\bar{b}r^{-1} & r\end{pmatrix}$, where $r=i-k$ and $b$ ranges over the elements of $\mathcal{O}$ such that $N(b)=7$.

(4) $\begin{pmatrix}2 & b\varepsilon\\ -\bar{b} & 2\varepsilon\end{pmatrix}$, where $b$ ranges over the elements of $\mathcal{O}$ such that $N(b)=5$, and for each $b$, we take an element $\varepsilon\in\mathcal{O}^{\times}$ such that $b\varepsilon\equiv\bar{b}$ mod. $r$. (Such $\varepsilon$ always exists for each $b$.)

(5) $\begin{pmatrix}z_1 & \beta_1z_2\\ -\bar{\beta}_1z_1 & z_2\end{pmatrix}$, $\begin{pmatrix}z_1 & z_1\beta_2\\ -z_2\beta_2 & z_2\end{pmatrix}$, where $z_1,\ z_2\in\{1\pm i\pm j\}$, $\beta_1$ ranges over the elements of $\mathcal{O}$ such that $N(\beta_1)=2$, and $\beta_2$ ranges over the elements of $\mathcal{O}$ such that $N(\beta_2)=2$ and $z_1\beta_2z_2^{-1}\notin\mathcal{O}$.

The numbers of elements of the form (1), $\cdots$, (5) are 16, 24, 192, 144, and 672, respectively. Then it can be seen that

$$(4.6)\qquad (T(3)^2-T(9))f=39f(3x,\ 3y)+3\sum_{g\in A}f((x,\ y)g)+6\sum_{g\in B}f((x,\ y)g)$$

for any $f\in\mathfrak{M}_\nu(\Gamma_0)$, where

$$A=\left\{\begin{pmatrix}z_1 & \beta z_2\\ -\bar{\beta}z_1 & z_2\end{pmatrix};\ z_1,\ z_2\in\{1\pm i\pm j\},\text{ and }\beta\text{ is an element of }\mathcal{O}\text{ such that }N(\beta)=2\text{ and }z_1^{-1}\beta z_2\in\mathcal{O}\right\},$$

and

$$B=\left\{\begin{pmatrix}3 & 0\\ 0 & a\end{pmatrix};\ a\in\{z_1z_2\,;\ z_1\neq\bar{z}_2,\ z_1,\ z_2\in\{1\pm i\pm j\}\}\right\}.$$

Incidentally, $|A|=96$ and $|B|=12$. Now, we calculate $\lambda(3)$ and $\lambda(9)$ for $f_7$. Note that, to obtain $\lambda(n)$, we need only calculate the coefficient of the monomial $x_1x_2x_3x_4$ in the polynomial $(T(n)f_7)(x,\ 1)$, since $\mathfrak{M}_7(\Gamma_0)$ is one dimensional. For polynomials $f$ and $g$ in four variables $x_1,\ x_2,\ x_3,\ x_4$, we write $f\sim g$ if their coefficients of $x_1x_2x_3x_4$ are equal. Then, for $z_1,\ z_2\in\{1\pm i\pm j\}$, we have

$$f_7(xz_1,\ z_2)=27\Big(\prod_{i=1}^{4}(\bar{z}_1xz_2)_i\Big)(1-N(x))(1-3N(x)+N(x)^2)\sim 27\prod_{i=1}^{4}(\bar{z}_1xz_2)_i.$$

Put $z_0=1+i+j$. Then, we have $iz_0i^{-1}=1+i-j$, $jz_0j^{-1}=1-i+j$, and $kz_0k^{-1}=1-i-j$. Then, it is easy to see that

$$\prod_{i=1}^{4}(\bar{z}_1xz_2)_i\sim\prod_{i=1}^{4}(\bar{z}_0xz_0)_i=3x_1(x_2+2x_3-2x_4)(2x_2+x_3+2x_4)(2x_2-2x_3-x_4)$$
$$\sim 9x_1x_2x_3x_4.$$

So, the contribution of the cosets of the form (1) of $T(3)$ is $3888x_1x_2x_3x_4$. Next, we calculate $f_7(x-\bar{\beta}, x\beta+1)$ for $\beta\in\mathcal{O}$ such that $N(\beta)=2$. First, we show that $f_7(x-\bar{\beta}_1, x\beta_1+1)\sim f_7(x-\bar{\beta}_2, x\beta_2+1)$ for any $\beta_1, \beta_2\in\mathcal{O}$ such that $N(\beta_1)=N(\beta_2)=2$. Put $Q_8=\{\pm 1, \pm i, \pm j, \pm k\}$. Then, it is easy to see that, for any $\beta\in O$ and $\varepsilon\in Q_8$, we have

$$f_7(x-\bar{\beta}\bar{\varepsilon}, x\beta\varepsilon+1)=f_7(x-\bar{\varepsilon}\bar{\beta}, x\beta+\bar{\varepsilon})=f_7(\varepsilon x-\bar{\beta}, \varepsilon x\beta+1)\sim f_7(x-\bar{\beta}, x\beta+1).$$

Note that $\{\beta\in\mathcal{O}; N(\beta)=2\}=(1+i)Q_8\cup(1+j)Q_8\cup(1+k)Q_8$. So, we can assume that $\beta=1+i$, $1+j$, or $1+k$. But, we have $a^{-1}(1+i)a=1+j$ and $a^{-1}(1+j)a=1+k$ for $a=(1-i-j-k)/2$. It is obvious that

$$f_7(a^{-1}(x-\bar{\beta})a, \quad a^{-1}(x\beta+1)a)=f_7(x-\bar{\beta}, x\beta+1) \qquad \text{and}$$
$$f_7(a^{-1}xa-\bar{\beta}, \quad a^{-1}xa\beta+1)\sim f_7(x-\bar{\beta}, x\beta+1).$$

So, we need calculate $f_7(x-\bar{\beta}, x\beta+1)$ only for $\beta=1+i$. For $\beta=1+i$, we have

$$\begin{aligned}f_7(x-\bar{\beta}, x\beta+1)&\sim\Big(\prod_{i=1}^{4}(x-\bar{\beta}-\bar{\beta}\bar{x}\bar{\beta})_i\Big)(2\mathrm{tr}(x\beta)-1)(5(\mathrm{tr}(x\beta))^2-5\mathrm{tr}(x\beta)-1)\\&\sim 9x_3x_4(x_1+2x_2-1)(x_2+2x_1+1)(4x_1-4x_2-1)(40x_1x_2+10x_1-10x_2+1)\\&\sim -927x_1x_2x_3x_4.\end{aligned}$$

So, the contribution of the cosets in $T(3)$ of the form (2) is $-22248x_1x_2x_3x_4$. So, $T(3)f_7=-18360f_7$. In the same way, we can show that $T(3)f_5=1080f_5$. Next, we calculate $\lambda(9)$ for $f_7$. First, we calculate the eigen value of $T(3)^2-T(9)$. For $a\in\{z_1z_2; z_1\neq\bar{z}_2, z_1, z_2\in\{1\pm i\pm j\}\}$, we have $f_7(3x, a)\sim f_7(3x, z_0\varepsilon z_0)$, where $\varepsilon=1$, $i$, or $j$. For these $\varepsilon$, we can show that $f_7(3x, z_0\varepsilon z_0)\sim -3^{11}\cdot 5x_1x_2x_3x_4$. For example,

$$\begin{aligned}f_7(3x, z_0^2)&\sim 3^{10}\prod_{i=1}^{4}(\bar{z}_0^2x)_i\\&\sim 3^{10}(x_1+2x_2+2x_3)(2x_1-x_2+2x_4)(2x_1-x_3-2x_4)(2x_2-2x_3+x_4)\\&\sim -3^{11}\cdot 5x_1x_2x_3x_4.\end{aligned}$$

So, $6\sum_{g\in B}f_7((x, y)g)\sim -63772920x_1x_2x_3x_4$. Next, for $z_1, z_2\in\{1\pm i\pm j\}$ and $\beta\in\mathcal{O}$ such that $N(\beta)=2$ and $z_1^{-1}\beta z_2\in\mathcal{O}$, we get the following equivalence:

$$\begin{aligned}&f_7((x-\bar{\beta})z_1, (x\beta+1)z_2)\\&\sim 3^6\Big(\prod_{i=1}^{4}(\bar{z}(x-\bar{\beta}-\bar{\beta}\bar{x}\bar{\beta})z_1)_i\Big)(2\mathrm{tr}(x\beta)-1)(5(\mathrm{tr}(x\beta))^2-5\mathrm{tr}(x\beta)-1).\end{aligned}$$

As we can put $z_1=\varepsilon_1z_0\varepsilon_1^{-1}$, $z_2=\varepsilon_2z_0\varepsilon_2^{-1}$, for some $\varepsilon_1, \varepsilon_2\in\{1, i, j, k\}$, it is easy to see that $f_7((x-\bar{\beta})z_1, (x\beta+1)z_2)\sim f_7((x-\bar{\beta}_1)z_0, (x\beta_1+1)z_0)$, where $\beta_1=\bar{\varepsilon}_1\beta\varepsilon_2$. But, we have $z_1^{-1}\beta z_2 \in \mathcal{O}$ if and only if $z_0^{-1}\beta_1 z_0 \in \mathcal{O}$. So, we need calculate

$$f_7((x-\bar{\beta}_1)z_0,\ (x\beta_1+1)z_0)$$

only for

$$\beta_1 \in \{\pm(i+j),\ \pm(j+k),\ \pm(i-k)\}.$$

But direct calculation shows that

$$f_7((x-\bar{\beta}_1)z_0,\ (x\beta_1+1)z_0) \sim 3^8 \cdot 5 \cdot 23 x_1x_2x_3x_4$$

for all these $\beta_1$. So,

$$3\sum_{g\in A} f_7((x,\ y)g) \sim 217300320 x_1x_2x_3x_4.$$

These calculations show that

$$(T(3)^2 - T(9))f_7 = 340063191 f_7.$$

In the same way, we have

$$(T(3)^2 - T(9))f_5 = -1633689 f_5.$$

So, we obtain $\lambda(9)$ for $f_5$ and $f_7$.

Similar calculations show that

$$T(3)f_9^{(1)} = 307800 f_9^{(1)} \quad \text{and}$$

$$(T(3)g_2)(x,\ 1) = 1528632\ x_1x_2x_3x_4 + 59547312\ x_1^5x_2x_3x_4 + \cdots.$$

So, we have

$$\begin{pmatrix} T(3)g_2 \\ T(3)f_9^{(1)} \end{pmatrix} = \begin{pmatrix} -14760 & 87552 \\ 0 & 307800 \end{pmatrix} \begin{pmatrix} g_2 \\ f_9^{(1)} \end{pmatrix}.$$

So, the polynomials $f_9^{(1)}$, $f_9$ are common eigen functions of $T(n)(2 \nmid n)$, and $\lambda(3) = 307800,\ -14760$, respectively. Similar routine calculations show that

$$\sum_{g\in B} f_9^{(1)}((x,\ y)g) \sim -860934420 \cdot 21\ x_1x_2x_3x_4,$$

$$\sum_{g\in A} f_9^{(1)}((x,\ y)g) \sim 10458758880 \cdot 21\ x_1x_2x_3x_4,$$

$$\sum_{g\in B} f_9((x,\ y)g) \sim -860934420 \cdot 21 \cdot 51\ x_1x_2x_3x_4, \qquad \text{and}$$

$$\sum_{g\in A} f_9((x,\ y)g) \sim 5262446880 \cdot 21 \cdot 51\ x_1x_2x_3x_4.$$

So, we obtain $\lambda(9)$ for $f_9^{(1)}$ and $f_9$ easily from these data. q. e. d.

## § 3. Eigen space decomposition in case of $Sp(2, \boldsymbol{R})$

In this section, we decompose the space $S_k(B(2))$ into common eigen spaces of all the Hecke operators $T(n)(2 \nmid n)$ for even small $k$ and calculate their eigen values $\lambda(n)$ for $T(n)$ for small $n$. The examples of common eigen forms in this paper are divided into two classes:

(1) Those which satisfy the Ramanujan Conjecture (at least) at their Euler 3-factors.

(2) Those which do not satisfy the Ramanujan Conjecture.

For each example of the common eigen form $f_\nu$ or $f_\nu^{(1)}$ in $\mathfrak{M}_\nu(\Gamma_0)$ in § 2, there exists a common eigen form $F$ in $S_{\nu+3}(B(2))$ such that $H_3(T, F)=H_3(T, f_\nu)$ or $H_3(T, f_\nu^{(1)})$. Some of these $F$ are of type (1) and some are of type (2). We also note that there exist many forms of type (1) or (2) whose Euler 3-factors do not coincide with those of any forms in $\mathfrak{M}_{k-3}(\Gamma_0)$. By virtue of Oda [24] and Kojima [16], or Yoshida [31], [32], we can show that the examples of the forms of type (2) in this paper are obtained by lifting cusp forms of one variable with weight $2k-2$. Namely, for each examples $F$ of type (2) with weight $k$, there exists a form $f$ belonging to $\Gamma_0^1(2)=\left\{g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z})\,;\ c\equiv 0 \bmod. 2\right\}$ with weight $2k-2$ such that $L(s, F)=\zeta(s-k+1)\zeta(s-k+2)L(s, f)$, up to Euler 2-factors. But we shall omit the proof of this fact in this paper, since it should be proved in general situation. Instead, for the convenience of the readers, we give here the common eigen new cusp forms of $S_{2k-2}(\Gamma_0^1(2))$ and their eigen values $\omega(n)$ for small $k$ and $n$. (As for the examples of the eigen values of the cusp forms belonging to $SL(2, Z)$, see [17].) Put $v=(2\theta_{01}^4+\theta_{10}^4)/2$ and $u=\theta_{01}^8+\theta_{01}^4\theta_{10}^4+\theta_{10}^8$. Then, it is easy to see that the graded ring $A(\Gamma_0^1(2))$ of modular forms belonging to $\Gamma_0^1(2)$ is generated by these algebraic independent forms $u$ and $v$. The ideal of cusp forms of $A(\Gamma_0^1(2))$ is generated by $(u-v^2)(u-4v^2)$. We denote by $S_{2k-2}^0(\Gamma_0^1(2))$ the space of new forms of $S_{2k-2}(\Gamma_0^1(2))$. Let $\Gamma_0^*(p)$ denote the group generated by $\Gamma_0^1(p)$ and $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$. Now, the eigen forms and the eigen values are give as follows:

$$\dim S_{10}^0(\Gamma_0^1(2))=1$$

$$h_{10}=(u-v^2)(u-4v^2)v \quad \cdots \quad \omega(3)=-156$$

$$\dim S_{14}^0(\Gamma_0^1(2))=2$$

$$h_{14}^{(1)}=(u-v^2)(u-4v^2)v\{(u-v^2)+(u-4v^2)\} \quad \cdots \quad \omega(3)=-1836$$

$$h_{14}^{(2)}=(u-v^2)(u-4v^2)v\{(u-v^2)-(u-4v^2)\} \quad \cdots \quad \omega(3)=1236$$

The former is also the form belonging to $\Gamma_0^*(2)$.

$$\dim S_{18}^0(\Gamma_0^1(2))=1$$

$$h_{18}=4v(u-v^2)(u-4v^2)\{(u-v^2)^2+(u-4v^2)^2\}-17v(u-v^2)^2(u-4v^2)^2 \quad \cdots \ \omega(3)=6084$$

$$\dim S_{22}^0(\Gamma_0^1(2))=2$$

$$h_{22}^{(1)}=2v(u-v^2)(u-4v^2)\{(u-v^2)^3+(u-4v^2)^3\} +31v(u-v^2)^2(u-4v^2)^2\{(u-v^2)+(u-4v^2)\} \quad \cdots \ \omega(3)=71604$$

$$h_{22}^{(2)}=2v(u-v^2)(u-4v^2)\{(u-4v^2)^3-(u-v^2)^3\} +33v(u-v^2)^2(u-4v^2)^2\{(u-v^2)-(u-4v^2)\} \quad \cdots \ \omega(3)=59316.$$

The former is also the form belonging to $\Gamma_0^*(2)$.

**3.1.** Now, we explain the general procedure of calculation of the eigen values of common eigen forms belonging to $B(2)$. For any cusp form $f(Z)=\sum_{T>0} a(T)e^{2\pi i\,\mathrm{tr}(TZ)}$ in $S_k(B(p))$, put $(f|T(n))(Z)=\sum_{T>0} a(n;T)e^{2\pi i\,\mathrm{tr}(TZ)} (p \nmid n)$. Then, for a prime $q\neq p$ and a positive integer $\delta$, $a(q^\delta;T)$ can be easily described as follows as in [1], [6], [21]: For a positive integer $q$, define an equivalence relation of pairs of coprime integers by $(u_1, u_2)\sim(u_1', u_2')\longleftrightarrow au_1\equiv u_1'$ and $au_2\equiv u_2'$ mod. $q$ for some $a\in \boldsymbol{Z}$. Fix a complete set of representatives $(u_1, u_2)$ of the above equivalence classes so that $p\nmid u_1$, and for each $(u_1, u_2)$, fix a vector $(v_1, v_2)\in \boldsymbol{Z}^2$ so that $\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2\end{pmatrix}\in {}^t\Gamma_0^1(p)=\{{}^tg\,;\, g\in\Gamma_0^1(p)\}$. Let $R(q^\beta)$ be the set of these matrices $\begin{pmatrix} u_1, & u_2 \\ v_1, & v_2\end{pmatrix}$. Put $T=\begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix}$ and $UT{}^tU=\begin{pmatrix} a_u & b_u/2 \\ b_u/2 & c_u\end{pmatrix}$ for any $U\in SL(2, \boldsymbol{Z})$. Then, we have

$$(3.1)\qquad a(q^\delta;T)=\sum_{\substack{\alpha+\beta+\gamma=\delta\\ \alpha,\beta,\gamma\geq 0}} q^{(k-2)\beta+(2k-3)\gamma} \sum_{\substack{U\in R(q^\beta)\\ a_u\equiv 0\,(q^{\beta+\gamma})\\ b_u\equiv c_u\equiv 0\,(q^\gamma)}} a\left(q^\alpha\begin{pmatrix} a_uq^{-\beta-\gamma} & b_uq^{-\gamma}/2 \\ b_uq^{-\gamma}/2 & c_uq^{\beta-\gamma}\end{pmatrix}\right),$$

where $\alpha$, $\beta$, and $\gamma$ are integers.

Therefore, we can calculate the representation matrix of $T(n)$, as far as sufficiently many Fourier coefficients are available. The Fourier coefficients of $X$, $Y$, $T$ and $K$ in §1 were calculated by using an electronic computer. We used the table of coefficients of the Eisenstein series with weight four in Resnikoff and Saldaña [25] to obtain those of $Z$. Here, we write down some of the relations (3.1) which will be used later. We denote by $(t_1, t_2, t)$ the half integral positive definite matrix $\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2\end{pmatrix}$.

(3.1)

$$a(3;(1,1,0))=a(3,3,0)$$
$$a(3;(1,1,1))=a(3,3,3)+3^{k-2}a(1,1,1)$$
$$a(3;(1,2,0))=a(3,6,0)+2\cdot 3^{k-2}a(3,1,2)$$
$$a(3;(1,2,1))=a(3,6,3)$$
$$a(3;(1,3,0))=a(3,9,0)+3^{k-2}a(3,1,0)$$
$$a(3;(1,3,1))=a(3,9,3)+2\cdot 3^{k-2}a(3,1,1)$$
$$a(3;(2,2,0))=a(6,6,0)$$
$$a(3;(2,2,1))=a(6,6,3)+3^{k-2}a(4,1,1)$$
$$a(3;(2,2,2))=a(6,6,6)+3^{k-2}a(2,2,2).$$

Moreover, we have

$$a(9;(1,1,0))=a(9,9,0)$$
$$a(3;(3,3,0))=a(9,9,0)+3^{k-2}\{a(1,9,0)+a(9,1,0)+2a(5,2,2)\}$$
$$+3^{2k-3}a(1,1,0)$$
$$a(9;(1,1,1))=a(9,9,9)+3^{k-2}a(3,3,3)$$
$$a(3;(3,3,3))=a(9,9,9)+3^{k-2}a(3,3,3)$$
$$+3^{k-2}a(1,7,1)+3^{k-2}\cdot 2a(7,1,1)+3^{2k-3}a(1,1,1).$$

So, if $F$ is an eigen form of $T(3)$ and $T(9)$, we have

$$\lambda(9)a(1,1,0)=\lambda(3)a(3,3,0)-3^{2k-3}a(1,1,0)$$
$$-3^{k-2}\{a(1,9,0)+a(9,1,0)+2a(5,2,2)\},$$
$$\lambda(9)a(1,1,1)=\lambda(3)a(3,3,3)-3^{2k-3}a(1,1,1)$$
$$-3^{k-2}\{a(1,7,1)+2a(7,1,1)+a(3,3,3)\}.$$

Now, to clarify the meaning of the new forms of $S_k(B(p))$, we consider some more discrete subgroups of $Sp(2, \boldsymbol{R})$. Put

$$\rho=\begin{pmatrix} & & & -1 \\ & & -1 & \\ & p & & \\ p & & & \end{pmatrix},$$

$$\Gamma_0'(p)=\left\{g\in Sp(2,\boldsymbol{Z})\,;\ g\equiv\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \bmod. p\right\},$$

$$\Gamma_0'(p)=\left\{g\in Sp(2,\boldsymbol{Z})\,;\ g\equiv\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \bmod. p\right\},$$

$\Gamma_0''(p)=\rho\Gamma_0'(p)\rho^{-1}$, and

$$K(p)=\left\{g\in Sp(2,\boldsymbol{Q})\,;\ g=\begin{pmatrix} * & * & *p^{-1} & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & * & * & * \end{pmatrix}\right\},$$

where $*$ runs through integers. The topological closures in $Sp(2,\boldsymbol{Q}_p)$ of these subgroups and $Sp(2,\boldsymbol{Z})$ and $\rho Sp(2,\boldsymbol{Z})\rho^{-1}$ make up all proper parahoric subgroups of $Sp(2,\boldsymbol{Q}_p)$ containing $B(p)_p$. By definition, $S_k^0(B(p))$ is the orthogonal complement of $S_k(\Gamma_0(p))+S_k(\Gamma_0'(p))+S_k(\Gamma_0''(p))$ in $S_k(B(p))$ with respect to the Petersson metric. Now, put

$$S_1=B(p)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}B(p),\quad S_2=B(p)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}B(p),$$

and

$$S_0=B(p)\begin{pmatrix} 0 & 0 & -p^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}B(p).$$

Then, $\Gamma_0(p)=B(p)\cup S_1$, $\Gamma_0'(p)=S_2\cup B(p)$, and $\Gamma_0''(p)=B(p)\cup S_0$. By virtue of Iwahori and Matsumoto [14], we have $S_i^2=p\cdot 1+(p-1)S_i$ $(i=0, 1, 2)$ as elements in the Hecke algebra, where 1 denotes the identity element, that is, $B(p)$ as a double coset. The next lemma is easy and the proof will be omitted.

LEMMA 3.2. *For any $f\in S_k(B(p))$, we have $f|(1+S_1)\in S_k(\Gamma_0(p))$, $f|(1+S_2)\in S_k(\Gamma_0'(p))$, and $f|(1+S_0)\in S_k(\Gamma_0''(p))$. Put $V_i=S_k(B(p))|(p-S_i)$ for $i=0, 1, 2$. Then, $V_i=\{f\in S_k(B(p))\,;\, f|(1+S_i)=0\}$ and the orthogonal complement of $S_k(\Gamma_0(p))$, $S_k(\Gamma_0'(p))$, or $S_k(\Gamma_0''(p))$ in $S_k(B(p))$ with respect to the Petersson metric is given by $V_1$, $V_2$, or $V_0$, respectively. Besides, we have $S_k^0(B(p))=V_0\cap V_1\cap V_2$.*

Now, for the sake of convenience, put $U=(\theta_{1000}^{8}+\theta_{1001}^{8}+\theta_{1100}^{8}+\theta_{1111}^{8})/512$. We have $U=R+64Z$. Bases of $S_k(B(2))$ and $S_k(\Gamma_0(2))$ have been given in [11], and bases of $S_k(K(2))$, $S_k(\Gamma_0'(2))$, and $S_k(\Gamma_0''(2))$ for small $k$ will be obtained in the following way:

(1) The dimension of the space $S_k(\Gamma_0''(2))=S_k(\rho\Gamma_0'(2)\rho^{-1})$ has been known ([11]), and

$$S_k(\Gamma_0''(2))=\{F\in S_k(B(2))\,;\ a(t_1, t_2, t)=0 \text{ if } t_1 \text{ is odd}\}.$$

So, we can get a basis of $S_k(\Gamma_0''(2))$ if sufficiently many Fourier coefficients are available.

(2) We have $S_k(\Gamma_0'(2))=S_\rho(S_k(\Gamma_0''(2)))$, where $S_\rho=B(2)\rho$. As we have

$$\rho=\begin{pmatrix} & & & -1\\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1\end{pmatrix},$$

it is not difficult to give the representation matrix of $\rho$ by using the theta transformation formula for the action of $\begin{pmatrix} & & & -1\\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$. Actually, we have

$$X|[\rho]_2=X,\quad Y|[\rho]_4=1024Z,\quad Z|[\rho]_4=Y/1024,$$

$$U|[\rho]_4=(Y-1024Z+16U)/16,\quad K|[\rho]_6=K,\quad T|[\rho]_4=T,$$

and $R|[\rho]_4=R$.

(3) The space $S_k(K(2))$ is obtained as the intersection $S_k(\Gamma_0'(2))\cap S_k(\Gamma_0''(2))$.

**3.2.** First, we shall give examples and proofs for $k=6$, 8 and 10, and later for $k=12$. As for the cusp forms belonging to $Sp(2, \boldsymbol{Z})$, the non-vanishing of the Saito-Kurokawa or the Oda lifting and some examples of the eigen values have been known ([2], [17], [20], [33]).

THEOREM 3.3. *Common eigen basis of* $S_k(B(2))$ *and their eigen values* $\lambda(n)$ *for* $k=6$, 8, 10, *and small* $n$, *are given as follows*:

$k=6$

$$S_6(B(2))=S_6(\Gamma_0(2))=\langle K\rangle$$

$$S_6(\Gamma_0'(2))=S_6(\Gamma_0''(2))=S_6(K(2))=\{0\}.$$

$$\lambda(3)=168,\quad \lambda(9)=32841,$$

*and we have*

$$L(s, K)=L(s, h_{10})\zeta(s-4)\zeta(s-5)$$

*up to Euler* 2 *factors.*

$k=8$

$$S_8(B(2))=\langle XK,\ YZ,\ TR\rangle$$
$$S_8(\Gamma_0(2))=\langle XK,\ YZ\rangle$$
$$S_8(\Gamma_0'(2))=S_8(\Gamma_0''(2))=S_8(K(2))=\langle F_8\rangle,$$

*where* $F_8=(YZ-XK-4TR)/4$.

*Eigen forms are* $3YZ-XK$, $YZ-XK$, *and* $F_8$.

$$\lambda(3)=4152,\quad \lambda(9)=9914841,\quad \text{for } 3YZ-XK,$$
$$\lambda(3)=1080,\quad \lambda(9)=2800089,\quad \text{for } YZ-XK \text{ and } F_8.$$

*We have*

$$L(s,\ 3YZ-XK)=L(s,\ h_8^{(1)})\zeta(s-6)\zeta(s-7),\qquad \text{and}$$
$$L(s,\ YZ-XK)=L(s,\ F_8)=L(s,\ h_8^{(2)})\zeta(s-6)\zeta(s-7),$$

*up to Euler* 2 *factors.*

$k=10$

$$S_{10}(B(2))=\langle XYZ,\ XTR,\ X^2K,\ YK,\ ZK,\ UK\rangle$$
$$S_{10}(\Gamma_0(2))=\langle XYZ,\ X^2K,\ YK,\ ZK\rangle$$
$$S_{10}(Sp(2,\ \mathbf{Z}))=\langle YK\rangle$$
$$S_{10}(\Gamma_0''(2))=\langle XYZ-X^2K+48UK-4XTR,\ ZK\rangle$$
$$S_{10}(\Gamma_0'(2))=\langle XYZ-X^2K+3YK-3072ZK+48UK-4XTR,\ YK\rangle$$
$$S_{10}(K(2))=\langle XYZ-X^2K+48UK-4XTR-3072ZK\rangle$$
$$S_{10}^0(B(2))=\langle F_{10}\rangle,$$

*where*

$$F_{10}=96UK+12XTR-2XYZ+X^2K+YK-5120ZK\,.$$

*Put*

$$F_{10}^{(2)}=5XYZ-2X^2K-YK-1024ZK\,.$$

*Then, eigen forms are* $YZ$, $ZK$, $XYZ-X^2K+48UK-4XTR$, $2XYK+X^2K$, $F_{10}^{(2)}$, *and* $F_{10}$. *We have*

$\lambda(3)=21960$ *for the first four eigen forms,*
$\lambda(3)=32328$ *for* $F_{10}^{(2)}$, *and*
$\lambda(3)=-18360$, $\lambda(9)=-2973591$, $\lambda(5)=741900$, *for* $F_{10}$.

*We have* $L(s,F)=L(s,\ \Delta_{18})\zeta(s-8)\zeta(s-9)$, *up to Euler* 2 *factors, where* $F$ *is any eigen form of* $S_{10}(B(2))$ *such that* $\lambda(3)=21960$, *and* $\Delta_{18}$ *is the unique normalized cusp form belonging to* $SL(2,\ \mathbf{Z})$ *with weight* 18. *We also have* $L(s,\ F_{10}^{(2)})=L(s,\ h_{18})\zeta(s-8)\zeta(s-9)$ *up to Euler* 2 *factors. As for* $F_{10}$, *we have*

$$H_3(T, F_{10})=T^4+18360T^3+297016470T^2+3^{17}\cdot 18360T+3^{34}$$
$$=(T^2+108(85-8\sqrt{61})T+3^{17})(T^2+108(85+8\sqrt{61})T+3^{17}),$$

*and this* 3 *factor satisfies the Ramanujan Conjecture.*

PROOF. We shall prove this theorem for each $k$.

$k=6$

We have $a(1, 1, 1; K)=1$, $a(3, 3, 3; K)=87$, $a(1, 7, 1; K)=-156$. So, by virtue of (3.1), it is easy to see that $\lambda(3)=168$ and $\lambda(9)=32841$. The Oda lifting from $h_{10}$ does not vanish. So, $L(s, K)$ is decomposed as in the Theorem.

$k=8$

We have the following table of the Fourier coefficients:

| | $XK$ | $YZ$ | $TR$ |
|---|---|---|---|
| (1, 1, 0) | −2 | 2 | 1 |
| (1, 1, 1) | 1 | 1 | 0 |
| (3, 3, 0) | 912 | 5232 | |
| (3, 3, 3) | 3423 | 3423 | |
| (5, 2, 2) | 5286 | 858 | |
| (1, 9, 1) | −3290 | −3290 | |

So, we can see that a form, whose coefficients at (1, 1, 0) and (1, 1, 1) vanish, is a constant multiple of $F_8$. So, $S_8(\Gamma_0''(2))=\langle F_8\rangle$. But, $F_8|[\rho]_8=F_8$. So $S_8(\Gamma_0'(2))=S_8(K(2))=\langle F_8\rangle$. It is also easy to see that

$$T(3)\begin{pmatrix} XK \\ YZ \end{pmatrix}=\begin{pmatrix} 1848 & 2304 \\ 768 & 3384 \end{pmatrix}\begin{pmatrix} XK \\ YZ \end{pmatrix}.$$

So, we get easily $\lambda(3)$ and $\lambda(9)$ for the forms in $S_8(\Gamma_0(2))$. But, the coefficients of $(S_1+1)(TR)$ at (1, 1, 0) or (1, 1, 1) are $2a(1, 1, 0; TR)+a(2, 1, 2; TR)$ or $3a(1, 1, 1; TR)$, that is, 2 or 0, respectively, since $a(2, 1, 2; TR)=0$. As noted in Lemma 3.2, $(S_1+1)(TR)\in S_8(\Gamma_0(2))$. So, we get $(S_1+1)(TR)=(YZ-XK)/2$. So, we have $L(s, TR)=L(s, YZ-XK)$ up to Euler 2 factors. The Oda lifting does not vanish also in this case.

$k=10$

We have the following table of the Fourier coefficients:

| | $XYZ$ | $X^2K$ | $YK$ | $ZK$ | $UK$ | $XTR$ |
|---|---|---|---|---|---|---|
| (1, 1, 0) | 2 | −2 | −2 | 0 | 0 | 1 |
| (1, 1, 1) | 1 | 1 | 1 | 0 | 0 | 0 |
| (1, 2, 1) | 16 | 48 | −16 | 0 | 0 | −8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| (3, 1, 2) | 28 | −92 | 36 | 0 | −2 | 6 |
| (2, 2, 0) | −288 | −1760 | 32 | −2 | −136 | 288 |
| (2, 1, 2) | 2 | −2 | −2 | 0 | 0 | 0 |
| (2, 2, 2) | 400 | 112 | 240 | 1 | 52 | 48 |
| (3, 3, 0) | 62352 | −80784 | −43920 | 0 | −1056 | 23112 |
| (3, 3, 3) | 17703 | 10791 | 15399 | 0 | −48 | 1152 |
| (3, 6, 3) | 351360 | 1054080 | −351360 | 0 | 16128 | 17856 |
| (6, 6, 0) | −1605888 | −48086784 | 702720 | −43920 | −2998848 | 9715968 |
| (6, 3, 6) | 62352 | −80784 | −43920 | 0 | 960 | 25344 |
| (6, 6, 6) | 6749424 | 545040 | 3695760 | 15399 | 788460 | 1034064 |
| (1, 9, 0) | 22986 | −41418 | −4554 | 0 | 0 | 16101 |
| (9, 1, 0) | 22986 | −41418 | −4554 | 0 | −576 | 9189 |
| (5, 2, 2) | 22986 | −41418 | −4554 | 0 | −288 | 12645 |
| (5, 5, 0) | 5128340 | −9183380 | −1073300 | 0 | −94080 | 2448970 |

Using the table, we can see that the space of those cusp forms whose Fourier coefficients at (1, 1, 0), (1, 1, 1), (1, 2, 1) and (3, 1, 2) vanish, is two dimensional. So, it coincides with $S_{10}(\Gamma_0''(2))$. A basis of $S_{10}(\Gamma_0'(2))$ and $S_{10}(K(2))$ are also easily obtained. Now, we note that the matrix of the coefficients at (1, 1, 0), (1, 1, 1), (1, 2, 1), (2, 1, 2), (2, 2, 0) and (2, 2, 2) of $UK$, $XTR$, $XYZ$, $X^2K$, $YK$ and $ZK$ is non singular. So, calculating $a(3; (1, 1, 0))$ etc., we get the representation matrix of the action of $T(3)$:

$$T(3)^t(UK, XTR, XYZ, X^2K, YK, ZK)=A^t(UK, XTR, XYZ, X^2K, YK, ZK),$$

where

$$A=\begin{pmatrix} 5832 & -2016 & 216 & -120 & -144 & 884736 \\ -193536 & -2232 & 6912 & -3168 & -2592 & 9732096 \\ 0 & 0 & 27720 & -2304 & -1152 & -1179648 \\ 0 & 0 & -11520 & 26568 & 2304 & 2359296 \\ 0 & 0 & 0 & 0 & 21960 & 0 \\ 0 & 0 & 0 & 0 & 0 & 21960 \end{pmatrix}.$$

So, we can get an eigen basis of $S_{10}(B(2))$ and the eigen values of $T(3)$. The eigen space for the eigen value $\lambda(3)=21960$ is four dimensional, and the generators are given as in the Theorem. Now, it is easy to show the following relations:

$$U(2)(YK)=64(2XYZ+X^2K)+48YK+65536ZK,$$

$$U(2)^2(YK)=-1024(2XYZ+X^2K)+138496YK+66060288ZK, \quad \text{and}$$

$$16U(2)(YK)+U(2)^2(YK)=139264YK+67108864ZK.$$

where $U(2)=\Gamma_0(2)\begin{pmatrix}1&&&\\&1&&\\&&2&\\&&&2\end{pmatrix}\Gamma_0(2)$. Besides, $XYZ-X^2K+48UK-4TR-3072ZK \in S_{10}(K(2))$, and its image by $1+S_1$ does not vanish, since the coefficients of the image at (2, 1, 0) is equal to $-72$. By virtue of these relations, we see that these four cusp forms are common eigen forms of all the Hecke operators $T(n)(2\nmid n)$, and their $L$ functions coincide with each other up to Euler 2-factors. It is well known that $S_{10}(Sp(2, \boldsymbol{Z}))=YK$ ([12]), and that $L(s, YK)=L(s, \Delta_{18})\zeta(s-8)\zeta(s-9)$ ([2], [17], [20], [33]). So, we have the assertion for the forms with $\lambda(3)=21960$. It is also easy to see that $L(s, F_{10}^{(2)})=L(s, h_{18})\zeta(s-8)\zeta(s-9)$, up to Euler 2-factors. Now, we get $a(1, 1, 0; F_{10})=4\neq 0$. So, we get $\lambda(9)$ and $H_3(T, F_{10})$ easily. q.e.d.

Next, we give the examples for $k=12$.

THEOREM 3.4. *A basis of cusp forms belonging to each discrete subgroups is given as follows:*

$$S_{12}(B(2))=\langle Y^2Z,\ YZ^2,\ XYK,\ XZK,\ K^2,\ X^2YZ,\ X^3K,\ YTR,\ ZTR,\ UTR,\ YZU,\ X^2TR\rangle,$$

$$S_{12}(\Gamma_0(2))=\langle Y^2Z,\ YZ^2,\ XYK,\ XZK,\ K^2,\ X^2YZ,\ X^3K\rangle,$$

$$S_{12}(\Gamma_0'(2))=\langle A,\ B,\ C,\ D\rangle,$$

$$S_{12}(\Gamma_0''(2))=\langle A',\ B',\ C',\ D'\rangle,$$

$$S_{12}(Sp(2, \boldsymbol{Z}))=\langle 3Y^2Z-2XYK+3072K^2\rangle,$$

$$S_{12}(K(2))=\langle F_{12}^{(2)},\ 67584A-1344B+18C-23D\rangle,$$

$$S_{12}^0(B(2))=\langle F_{12}\rangle,$$

*where*

$$A=XZK+12ZTR,$$

$$B=-48YZ^2+384ZTR+8UTR+YZU+XUK,$$

$$C=Y^2Z-1792YZ^2-XYK-4YTR-2048ZTR+64UTR+32YZU,$$

$$D=4608YZ^2+X^2YZ-X^3K-192UTR-96YZU-4X^2TR,$$

$$A'=A|[\rho]_{12},\quad B'=B|[\rho]_{12},\quad C'=C|[\rho]_{12},\quad D'=D|[\rho]_{12},$$

$$F_{12}^{(2)}=-18432A+240B+3C+D,$$

$$F_{12}=36YTR-208896ZTR+3840UTR-1920UYZ+12X^2TR-21Y^2Z$$
$$+101376YZ^2+XYK+1024XZK-3840K^2+13X^2YZ+7X^3K.$$

*An eigen space of $T(3)$ in $S_{12}(B(2))$ for each an eigen value of $T(3)$ is at the same time a common eigen space of all the Hecke operators $T(n)(2\nmid n)$. The eigen values and eigen spaces are given as follows:*

(i) $\lambda(3)=107352$ *for the four dimensional space spanned by* $3X^2Z-2XYK+3072K^2$, $3YZ^2+3K^2-2XZK$, $-9Y^2Z+14X^2YZ+10X^3K-23040K^2-6144XZK$, *and* $67584A-1344B+18C-23D$. *For any element $F$ of this space, we have the equality:*

$$L(s, F)=L(s, \Delta_{22})\zeta(s-10)\zeta(s-11) \text{ up to Euler 2-factors.}$$

(ii) $\lambda(3)=295512$ *for* $9XYZ-X^3K-3456K^2+9216XZK$ *and its $L$ function coincides with*

$$L(s, h_{22}^{(2)})\zeta(s-10)\zeta(s-11) \text{ up to Euler 2-factors.}$$

(iii) $\lambda(3)=307800$, $\lambda(9)=3^8\cdot 8142169$ *for the two dimensional space spanned by* $F_{12}^{(2)}$ *and* $12Y^2Z+3XYK-16X^2YZ+X^3K+12288YZ^2-5760K^2+3072XZK$. *For any element $F$ of this space, we have the equality:*

$$L(s, F)=L(s, h_{22}^{(1)})\zeta(s-10)\zeta(s-11) \text{ up to Euler two factors.}$$

(iv) $\lambda(3)=-88488$, $\lambda(9)=-1563802119$ *for the four dimensional space spanned by* $Y^2Z+5XYK-1024YZ^2-5120XZK$, $13XYK-2X^2YZ+X^3K-4608YZ^2+5760K^2-9728XZK$, $27136A-496B+11C-7D$, and $27136A'-496B'+11C'-7D'$. *The $L$ functions of any elements $F$ of this space coincide with each other up to Euler 2 factors. The Hecke polynomial at 3 is given by*

$$\begin{aligned}H_3(T, F)&=T^4+88488T^3+5907143862T^2+88488\cdot 3^{21}T+3^{42}\\&=(T^2+36(1229+2\sqrt{3273745})T+3^{21})\\&\quad\times(T^2+36(1229-2\sqrt{3273745})T+3^{21}),\end{aligned}$$

*which satisfies the Ramanujan Conjecture.*

(v) $\lambda(3)=-14760$ *for $F_{12}$. The Hecke polynomial at 3 is given by*

$$\begin{aligned}H_3(T, F_{12})&=T^4+14760T^3-9330332490T^2+14760\cdot 3^{21}T+3^{42}\\&=(T^2+36(205+\sqrt{23383876})T+3^{21})\\&\quad\times(T^2+36(205-\sqrt{23383876})T+3^{21}),\end{aligned}$$

*which satisfies the Ramanujan Conjecture.*

PROOF. We use the following table of the Fourier coefficients:

| | $Y^2Z$ | $YZ^2$ | $XYK$ | $XZK$ | $K^2$ | $X^2YZ$ |
|---|---|---|---|---|---|---|
| (1, 1, 0) | 2 | 0 | −2 | 0 | 0 | 2 |
| (1, 1, 1) | 1 | 0 | 1 | 0 | 0 | 1 |
| (1, 2, 0) | −52 | 0 | −12 | 0 | 0 | 76 |
| (1, 2, 1) | −24 | 0 | 8 | 0 | 0 | 40 |
| (2, 1, 0) | −52 | 0 | −12 | 0 | 0 | 76 |
| (2, 1, 1) | −24 | 0 | 8 | 0 | 0 | 40 |
| (2, 1, 2) | 2 | 0 | −2 | 0 | 0 | 2 |
| (2, 2, 0) | 960 | 6 | 1856 | −2 | 6 | 1728 |
| (2, 2, 1) | 600 | 4 | −1224 | 0 | −4 | 1560 |
| (2, 2, 2) | 96 | 1 | 288 | 1 | 1 | 736 |
| (3, 1, 0) | 608 | 0 | 544 | 0 | 0 | 352 |
| (3, 1, 1) | 251 | 0 | −261 | 0 | 0 | 251 |
| (3, 1, 2) | −52 | 0 | −12 | 0 | 0 | 76 |
| (3, 2, 0) | −6984 | −16 | −7992 | −48 | −16 | −7752 |
| (3, 2, 1) | −6648 | −4 | 3240 | 0 | 4 | 8520 |
| (3, 2, 3) | 600 | 4 | −1224 | 0 | −4 | 1560 |
| (3, 3, 0) | 54960 | −336 | −183984 | −240 | 176 | −37200 |
| (3, 3, 3) | 8367 | 96 | 37551 | 192 | 32 | 118959 |
| (3, 6, 0) | −4430136 | −7392 | 3311544 | −672 | 6944 | 12515784 |
| (3, 6, 3) | −4124736 | −1152 | 305856 | 0 | 1152 | 11593152 |
| (4, 1, 1) | −1448 | 0 | 1848 | 0 | 0 | −3048 |
| (4, 2, 4) | 960 | 6 | 1856 | −2 | 6 | 1728 |
| (6, 6, 0) | −32208384 | 834576 | −264749568 | −183984 | 474128 | −8744448 |
| (6, 6, 3) | 51782184 | 117216 | 91807560 | 0 | −117216 | −567587736 |
| (6, 6, 6) | 140689824 | 77487 | 162153696 | 37551 | 11951 | 187528992 |
| (5, 2, 2) | −127302 | −96 | 10566 | −288 | −96 | −78150 |
| (1, 9, 0) | −373062 | 0 | −210618 | 0 | 0 | −188742 |

| | $X^3K$ | $YTR$ | $ZTR$ | $UTR$ | $YZU$ | $X^2TR$ |
|---|---|---|---|---|---|---|
| (1, 1, 0) | −2 | 1 | 0 | 0 | 0 | 1 |
| (1, 1, 1) | 1 | 0 | 0 | 0 | 0 | 0 |
| (1, 2, 0) | −140 | −10 | 0 | 0 | 0 | 54 |
| (1, 2, 1) | 72 | −8 | 0 | 0 | 0 | −8 |
| (2, 1, 0) | −140 | −16 | 0 | 1 | 2 | 48 |
| (2, 1, 1) | 72 | −4 | 0 | 0 | 1 | −4 |
| (2, 1, 2) | −2 | 0 | 0 | 0 | 0 | 0 |
| (2, 2, 0) | −6080 | −64 | 2 | 132 | 344 | 1728 |
| (2, 2, 1) | 2808 | 356 | 1 | 48 | 231 | −156 |

| | | | | | | |
|---|---|---|---|---|---|---|
| (2, 2, 2) | 160 | −32 | 0 | 2 | 60 | 96 |
| (3, 1, 0) | −3296 | 112 | 0 | 8 | −4 | 624 |
| (3, 1, 1) | 1787 | 64 | 0 | −4 | 0 | −192 |
| (3, 1, 2) | −140 | 6 | 0 | 0 | 2 | 6 |
| (3, 2, 0) | −56376 | 92 | 4 | 272 | −1204 | 9564 |
| (3, 2, 1) | 15336 | −2728 | 0 | −24 | −208 | −168 |
| (3, 2, 3) | 2808 | 200 | 0 | −8 | 208 | −312 |
| (3, 3, 0) | 252240 | 47192 | 20 | 80 | −19264 | −936 |
| (3, 3, 3) | 105135 | −4864 | −16 | −808 | 6272 | 2304 |
| (3, 6, 0) | −6064968 | −1287004 | 56 | 27104 | −383524 | 4033188 |
| (3, 6, 3) | −1228608 | −305856 | 0 | −7488 | 50688 | 1021248 |
| (4, 1, 1) | 16632 | 412 | 0 | −32 | −25 | 2460 |
| (4, 2, 4) | −6080 | −320 | 0 | 12 | 296 | 448 |
| (6, 6, 0) | −875260416 | 106703360 | 184496 | 9920864 | 43333928 | 263950848 |
| (6, 6, 3) | 1231366536 | 20824668 | 29304 | −1459296 | −1375767 | −291100068 |
| (6, 6, 6) | −172954272 | −3577312 | 6656 | 882014 | 4620548 | 60080544 |
| (5, 2, 2) | −333498 | −26019 | 24 | 1248 | −5280 | 20061 |
| (1, 9, 0) | −1722 | −40611 | 0 | 0 | 0 | −46755 |
| (9, 1, 0) | −1722 | −29859 | 0 | −768 | 2880 | −79011 |

We see that the space of the cusp forms whose Fourier coefficients at (1, 1, 0), (1, 1, 1), (1, 2, 0), (1, 2, 1), (3, 1, 0), (3, 1, 1), (3, 2, 0), (3, 2, 1) vanish is spanned by $A$, $B$, $C$ and $D$. But, $\dim S_{12}(\Gamma_0''(2))=4$. So, $S_{12}(\Gamma_0''(2))=\langle A, B, C, D\rangle$. So, it is easy to see that $S_{12}(\Gamma_0'(2))$ and $S_{12}(K(2))$ are given as in the theorem. By calculation, we can show that the matrix of the Fourier coefficients at (1, 1, 0), (1, 1, 1), (1, 2, 0), (1, 2, 1), (2, 2, 0), (2, 2, 1), (2, 2, 2) of $Y^2Z$, $YZ^2$, $XYK$, $XZK$, $K^2$, $X^2YZ$, $X^3K$, is non singular. So, by using the Fourier coefficients at (3, 3, 0), (3, 3, 3), (3, 6, 0), (3, 6, 3), (6, 6, 0), (6, 6, 3), (6, 6, 6), and (4, 1, 1), we get the representation matrix of $T(3)$ on $S_{12}(\Gamma_0(2))$. That is, we have

$$T(3)^t(Y^2Z,\ XYK,\ X^2YZ,\ X^3K,\ YZ^2,\ K^2,\ XZK)$$
$$=T^t(Y^2Z,\ XYK,\ X^2YZ,\ X^3K,\ YZ^2,\ K^2,\ XZK),$$

where

$$T=\begin{pmatrix} 102744 & 13824 & -55296 & 6144 & 18874368 & 21233664 & 132120576 \\ -6912 & 109656 & 9216 & -15360 & 28311552 & -63700992 & 179306496 \\ -124416 & 103680 & 204120 & -5376 & -127401984 & 84934656 & 106168320 \\ 186624 & -55296 & -41472 & 74328 & 191102976 & -281346048 & -56623104 \\ 18 & 126 & -54 & 6 & 102744 & 20736 & 13824 \\ 0 & -12 & 60 & -16 & 0 & 45144 & -12288 \\ 27 & 171 & 9 & -15 & -6912 & -62208 & 109656 \end{pmatrix}.$$

Then, the eigen values and the eigen spaces of $T$ is given as follows:

$\lambda(3)=107352$ for

$$\langle 3Y^2Z-2XYK+3072K^2,\ 3YZ^2+3K^2-2XZK,$$
$$-9Y^2Z+14X^2YZ+10X^3K-23040K^2-6144XZK\rangle.$$

$\lambda(3)=295512$ for

$$9XYZ-X^3K-3456K^2+9216XZK.$$

$\lambda(3)=307800$ for

$$12Y^2Z+3XYK-16X^2YZ+X^3K+12288YZ^2-5760K^2+3072XZK.$$

$\lambda(3)=-88488$ for

$$\langle Y^2Z+5XYK-1024YZ^2-5120XZK,$$
$$13XYK-2X^2YZ+X^3K-4608YZ^2+5760K^2-9728XZK\rangle.$$

Next, let $V_-$ be the eigen space of $S_1$ in $S_{12}(B(2))$ corresponding with the eigen value $-1$. Since $T(n)$ commutes with $S_1$ for odd $n$, the space $V_-$ is invariant by all the Hecke operators $T(n)(2\nmid n)$. By Lemma 3.2

$$V_-=\langle (2-S_1)(YTR),\ (2-S_1)(ZTR),\ (2-S_1)(UTR),$$
$$(2-S_1)(YZU),\ (2-S_1)(X^2TR)\rangle.$$

As we have seen in Lemma 3.2, $(S_1+1)f\in S_k(\Gamma_0(2))$ for any $f\in S_k(B(2))$, and the coefficients of $(S_1+1)f$ at $(t_1, t_2, t)$ is given by

$$a(t_1, t_2, t; f)+a(t_2, t_1, t; f)+a(t_1+t_2-t, t_1, 2t_1-t; f).$$

So, we get the following table of the Fourier coefficients:

| | $(S_1+1)YTR$ | $(S_1+1)ZTR$ | $(S_1+1)UTR$ | $(S_1+1)YZU$ | $(S_1+1)X^2TR$ |
|---|---|---|---|---|---|
| (1, 1, 0) | 2 | 0 | 0 | 0 | 2 |
| (1, 1, 1) | 0 | 0 | 0 | 0 | 0 |
| (1, 2, 0) | −20 | 0 | 1 | 4 | 108 |
| (1, 2, 1) | −16 | 0 | 0 | 2 | −16 |
| (2, 2, 0) | −448 | 4 | 276 | 984 | 3904 |
| (2, 2, 1) | 912 | 2 | 88 | 670 | −624 |
| (2, 2, 2) | −96 | 0 | 6 | 180 | 288 |

So, we obtain the following equalities:

$$(S_1+1)(YTR)=(Y^2Z-XYK)/2,$$

$(S_1+1)(ZTR)=(YZ^2-XZK)/2,$

$(S_1+1)(UTR)=28YZ^2-28XZK+3K^2-(Y^2Z-XYK-X^2YZ+X^3K)/256,$

$(S_1+1)(YZU)=160YZ^2-(Y^2Z-X^2YZ)/32,$

$(S_1+1)(X^2TR)=(X^2YZ-X^3K)/2.$

Then, the following table of the Fourier coefficients is obtained by routine calculations:

| | $(2-S_1)YTR$ | $(2-S_1)ZTR$ | $(2-S_1)UTR$ | $(2-S_1)YZU$ | $(2-S_1)X^2TR$ |
|---|---|---|---|---|---|
| (1, 1, 0) | 1 | 0 | 0 | 0 | 1 |
| (1, 2, 0) | −10 | 0 | −1 | −4 | 54 |
| (1, 2, 1) | −8 | 0 | 0 | −2 | −8 |
| (2, 2, 0) | 256 | 2 | 120 | 48 | 1280 |
| (2, 2, 1) | 156 | 1 | 56 | 23 | 156 |
| (3, 3, 0) | 22104 | 108 | 4464 | −1152 | 141912 |
| (3, 6, 0) | 9828 | 3528 | 145818 | −497412 | 2809188 |
| (3, 6, 3) | 1297728 | 576 | −61056 | −154800 | −3347136 |
| (6, 6, 0) | 203839488 | 44208 | −2655936 | −4247424 | 358594560 |
| (6, 6, 3) | 82486692 | 29304 | −437472 | −3526551 | 26176932 |
| (3, 1, 2) | 38 | 0 | −1 | 2 | −90 |
| (4, 1, 1) | 412 | 0 | −32 | −25 | 2460 |

Noting that the matrix of the coefficients at (1, 1, 0), (1, 2, 0), (1, 2, 1), (2, 2, 0), (2, 2, 1) of $(2-S_1)YTR$ etc. is non singular, we get the representation matrix of $T(3)$ on $V_-$. That is, we have

$$T(3)(2-S_1)^t(YTR,\ ZTR,\ UTR,\ YZU,\ X^2TR)$$

$$=T'(2-S_1)^t(YTR,\ ZTR,\ UTR,\ YZU,\ X^2TR),$$

where

$$T'=\begin{pmatrix} 17496 & 202899456 & -1474560 & -737280 & 4608 \\ 103.5 & 109656 & -1440 & -720 & 4.5 \\ 2376 & -3907584 & 10584 & 12672 & 2088 \\ -5040 & -17547264 & 193536 & 82008 & 3888 \\ 138240 & 0 & 2211840 & 1105920 & 3672 \end{pmatrix}.$$

By routine calculations, we see that the eigen values and the eigen forms of $T'$ are given as follows:

$\lambda(3)=107352$ for

$$(2-S_1)(3YTR+48UTR+24UYZ+X^2TR),$$

$\lambda(3)=307800$ for

$$(2-S_1)(-27YTR-89088ZTR+960UTR+480UYZ+11X^2TR),$$

$\lambda(3)=-88488$ for the space spanned by

$$(2-S_1)(4YTR+64UTR+32UYZ-3X^2TR) \quad \text{and}$$

$$(2-S_1)(4096ZTR+64UTR+32UYZ-3X^2TR),$$

$\lambda(3)=-14760$ for

$$(2-S_1)(3YTR-17408ZTR+320UTR-160UYZ+X^2TR).$$

Then, it is easy to see that the eigen spaces and the eigen values of the action of $T(3)$ of $S_{12}(B(2))$ are given as in the theorem. Next, we see that the eigen spaces of $T(3)$ are also the common eigen spaces of all the Hecke operators $T(n)(2\nmid n)$. Here, we shall prove the assertion only for the eigen space with $\lambda(3)=-88488$, since the proofs for the other cases are virtually the same. As $27136A-496B+11C-7D$ (resp. $27136A'-496B'+11C'-7D'$) is the unique eigen form of $S_{12}(\Gamma_0''(2))$ (resp. $S_{12}(\Gamma_0'(2))$) up to constant such that $\lambda(3)=-88488$, each of them is a common eigen cusp form. But, we have

$$(S_1+1)(27136A-496B+11C-7D)\neq 0 \quad \text{and}$$

$$(S_1+1)(27136A'-496B'+11C'-7D')\neq 0,$$

since the Fourier coefficient at (1, 1, 0) of the former is 16. So, we need only show that the (two dimensional) eigen space $V$ of $T(3)$ in $S_{12}(\Gamma_0(2))$ corresponding to $\lambda(3)=-88488$ is also a common eigen space. The space $V$ is invariant by the action of $U(2)$ and $\rho$, but it is easy to see that $U(2)$ and $\rho$ do not commute on $V$ with each other. So, the assertion is proved. q.e.d.

## § 4. Concluding remarks

We give some remarks on the automorphic forms belonging to some parahoric subgroups which are not minimal. Pairs of examples $(f_6, F_8)$ and $(f_9^{(1)}, F_{12}^{(2)})$ suggest that there exists a good correspondence between $S_k(K(p))$ and $\mathfrak{M}_{k-3}(U_2(D))$. In fact, there exists an interesting relation between the dimensions of $S_k(K(p))$ and $\mathfrak{M}_{k-3}(U_2(D))$. (The dimensions of $\mathfrak{M}_{k-3}(U_2(D))$ have been given in [9] (II), and $S_k(K(p))$ has been obtained explicitly by the author, using Hashimoto [8] which gave a general but not explicit formula of dimensions of Siegel cusp forms of genus two.) This will be given in the forthcoming paper, together with a precise conjecture. As for $S_k(\Gamma_0(p))$, we note the following: Let $V$ be the eigen space in $S_{12}(\Gamma_0(2))$ with $\lambda(3)=-88488$ as in the proof of Th. 3.4. As we have seen, any element of $V$ is not a common eigen form of $U(2)$ and $\rho$. So, judging from the definition of the new forms of genus one case (cf. Atkin and Lehner [3]), it is natural to call the elements of $V$ "old form from $\Gamma_0'(2)$ and $\Gamma_0''(2)$", *despite that they satisfy the Ramanujan Conjecture at* 3. It is plausible that the "image" of $\mathfrak{M}_\nu(U_1)$ is contained in a space consisting of a sort of new forms in $S_{\nu+3}(\Gamma_0(p))$ if $\nu$ is odd.

## References


[1] Andrianov, A. N., Euler products corresponding to Siegel modular forms of genus 2, Russian Math. Surveys **29** (1974), 45-116.

[2] Andrianov, A. N., Modular descent and the Saito-Kurokawa conjecture, Invent. Math. **53** (1979), 267-280.

[3] Atkin, A. O. and Lehner, Hecke operators on $\Gamma_0(m)$, Math. Ann. **185** (1970), 134-160.

[4] Eichler, M., Über die Darstellbarkeit von Modulformen durch Theta Reihen, J. Reine Angew. Math. **195** (1956), 159-171.

[5] Eichler, M., Quadratische formen und modulformen, Acta Arith. **4** (1958), 217-239.

[6] Evdokimov, S. A., Euler products for congruence subgroups of the Siegel groups of genus 2, Math. USSR-Sb. **28** (1976), 431-458.

[7] Hashimoto, K., On Brandt matrices associated with the positive definite quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 227-245.

[8] Hashimoto, K., The dimension of the space of cusp forms of Siegel upper half plane of degree two (I), J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1983), 403-488, (II), to appear.

[9] Hashimoto, K. and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 549-601; (II) ibid. **28** (1982), 695-699; (III) ibid. **30** (1983), 393-401.

[10] Ibukiyama, T., On symplectic Euler factors of genus two, Proc. Japan Acad. **57**, Ser. A (1981), 271-275.

[11] Ibukiyama, T., On the graded rings of Siegel modular forms of genus two belonging to certain level two congruence subgroups, preprint.

[12] Igusa, J., On Siegel modular forms of genus two, Amer. J. Math. **84** (1962), 175-200; II, ibid. **86** (1964), 392-412.

[13] Ihara, Y., On certain arithmetical Dirichlet series, J. Math. Soc. Japan **16** (1964), 214-225.

[14] Iwahori, N. and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke ring of $\mathfrak{p}$-adic Chevalley groups, Publ. Math. IHES **25** (1965), 5-48.

[15] Jacquet, H. and R. P. Langlands, Automorphic forms on $GL(2)$, Lecture Notes in Math. 260, Springer, Berlin-Heidelberg-New York, 1972.

[16] Kojima, H., Siegel modular forms of degree two, to appear.

[17] Kurokawa, N., Examples of eigen values of Hecke operators on Siegel cusp forms of degree two, Invent. Math. **49** (1978), 149-165.

[18] Langlands, R. P., Problems in the theory of automorphic forms, Lectures in Modern Analysis and Applications, Lecture Notes in Math. 170, Springer, Berlin-Heidelberg-New York, 1970, 18-61.

[19] Langlands, R. P., Automorphic representations, Shimura varieties, and motives, Proc. Sympos. Pure Math. Vol. 33, Part 2, 1979, 205-246.

[20] Maass, H., Über eine Spezialshar von Modulformen zweiten Grades, Invent. Math.

52 (1969), 95-104; II, ibid. 53 (1979), 249-253; III, ibid. 255-265.
[21] Matsuda, I., Dirichlet series corresponding to Siegel modular forms of degree 2, level $N$, Sci. Papers College Gen. Ed. Univ. Tokyo 28 (1978), 21-49.
[22] Niwa, S., Modular forms of half integral weight and the integral of certain theta-functions, Nagoya Math. J. **56** (1974), 147-161.
[23] Niwa, S., On Shimura's trace formula, Nagoya Math. J. **66** (1976), 183-202.
[24] Oda, T., On modular forms associated with indefinite quadratic forms of signature $(2, n-2)$, Math. Ann. **231** (1977), 97-144.
[25] Resnikoff, H.L. and R.L. Saldana, Some properties of Eisenstein series of degree two, J. Reine Angew. Math. **265** (1974), 90-109.
[26] Shimizu, H., On zeta functions of quaternion algebras, Ann. of Math. **81** (1965), 166-193.
[27] Shimura, G., On modular forms of half integral weight, Ann. of Math. **97** (1973), 440-481.
[28] Shintani, T., On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J. **58** (1975), 83-126.
[29] Tits, J., Reductive groups over local fields, Proc. Sympos. Pure Math. Vol. 33, part 1, AMS, Providence, 1979, 29-69.
[30] Weyl, H., Classical groups, Princeton Univ. Press (1939).
[31] Yoshida, H., On an explicit construction of Siegel modular forms of genus 2, Proc. Japan Acad. **55**, Ser. A (1979), 297-300.
[32] Yoshida, H., Siegel's modular forms and the arithmetic of quadratic forms, Invent. Math. **60** (1980), 193-248.
[33] Zagier, D., Sur la conjecture de Saito-Kurokawa (d, après H. Maass), in Séminaire Delange-Pisot-Poitou 1979-1980, Progress in Math. 12, Birkhäuser-Verlag, 371-394.

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