ON THE SPACES OF SIEGEL CUSP FORMS OF DEGREE TWO*

By Ryuji Tsushima

0. Introduction. Let $\mathfrak{S}_g$ be the Siegel upper half plane of degree g and $\Gamma_g(l)$ the principal congruence subgroup of the Siegel modular group of level l. Let Γ be a subgroup of finite index of $\Gamma_g(1)$. We denote by $S_k(\Gamma)$ the vector space of Siegel cusp forms of weight k with respect to Γ.

Y. Morita and U. Christian calculated the dimension of $S_k(\Gamma_2(l))$ ($l \geq 3$) by the Selberg's trace formula ([11], [3] and [4]), and T. Yamazaki obtained the same result by the formula of Riemann-Roch-Hirzebruch ([19]). In these works it was essential that the action of $\Gamma_g(l)$ ($l \geq 3$) on $\mathfrak{S}_g$ is fixed point free.

Let Γ be as above. Then it is known that Γ contains $\Gamma_g(l)$ for some $l \geq 3$, if $g \geq 2$ ([2] and [10]). In this paper we restrict ourselves to the case of degree two and study the action of $\Gamma/\Gamma_2(l)$ on the smooth compactification of the quotient space of $\mathfrak{S}_2$ by $\Gamma_2(l)$, and we represent the dimension of $S_k(\Gamma)$ in terms of group theoretical conditions of $\Gamma/\Gamma_2(l)$ as a subgroup of $\Gamma_2(1)/\Gamma_2(l)$ by using holomorphic Lefschetz formula of Atiyah-Singer (Theorem (5.2)). Especially we compute the dimension of $S_k(\Gamma_2(1))$ and $S_k(\Gamma_2(2))$ explicitly.

In [16] the author computed the dimension of $S_k(\Gamma_3(l))$ ($l \geq 3$). If the fixed subvarieties in $\mathfrak{S}_3$ of the elements of finite order of $\Gamma_3(1)$ are classified and their isotropy groups are determined, then we can get similar results in the case of degree three. And we can hope that the structure of the graded ring of Siegel modular forms with respect to $\Gamma_3(1)$ will be determined by this method.

T. Arakawa calculated the dimension of the space of Siegel cusp forms with respect to some arithmetic discrete subgroups of $Sp(2, \mathbf{R})$ whose $\mathbf{Q}$-rank is equal to 1, by the Selberg's trace formula ([21]). H.

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Yamaguchi obtained similar results by the formula of Riemann-Roch-Hirzebruch ([23]). Recently K. Hashimoto succeeded to calculate the dimension of the space of Siegel cusp forms with respect to discrete subgroups of $Sp(2, \mathbf{R})$ with torsion elements by the Selberg's trace formula and computed the dimension of $S_k(\Gamma_2(1))$ and $S_k(\Gamma_0(p))$ explicitly ([22]). His results include the case when $\mathbf{Q}$-rank is equal to 1.

Notations.

$\mathrm{Fix}(g)$: the fixed subvariety of g.
$|G|$: The order of a group G.
$G(\Gamma)$: $(\pm 1)\Gamma/(\pm 1)\Gamma_2(l)$.
$G(l)$: $\Gamma_2(1)/(\pm 1)\Gamma_2(l)$.
$\mathfrak{G}$: $\Gamma_2(1)/(\pm 1)$.
i: $\sqrt{-1}$
ρ: $e^{2\pi i/3}$.
η: $(1 + 2\sqrt{-2})/3$.
ω: $e^{2\pi i/5}$.
σ: $e^{\pi i/6}$.
$C_{\mathfrak{G}}(\tilde{\Phi}), N_{\mathfrak{G}}(\tilde{\Phi}), C_{G(l)}(\Phi)$, $N_{G(l)}(\Phi)$: Definition (2.1).
$C_G(g)$: the centralizer of $g \in G$ in a group G.
$C(\varphi)$: the centralizer in $C_{G(l)}(\Phi)$ of $\varphi \in C_{G(l)}(\Phi)$.
$\bar{\varphi}, \varphi$: $\tilde{\varphi} \bmod(\pm 1)$, $\tilde{\varphi} \bmod(\pm 1)\Gamma_2(l)$, respectively, for $\tilde{\varphi} \in \Gamma_2(1)$.
$\langle g \rangle$: the subgroup generated by g.
$\mathfrak{G}(\mathbf{R})$: $Sp(2, \mathbf{R})/(\pm 1)$.
K_M: the canonical line bundle of a complex manifold M.
$\Delta(\Phi)$: $\Phi - \Phi^0$.
$\Delta_k(\Phi)$: the k-th fundamental symmetric function of irreducible divisors in $\Delta(\Phi)$.
$c_i(M)$: the i-th Chern class of a manifold M.
$\bar{c}_i(\Phi^0)$: the i-th logarithmic Chern class of Φ^0 in Φ (cf. [16], Section 1).
$N_{X/Y}$: the normal bundle of X in Y.
ζ: $e^{2\pi i/l}$.
Π: $\Pi_{p\,|\,l,p\text{: prime}}$

1. Holomorphic Lefschetz Formula. 1. Let X be a compact complex manifold and V a holomorphic vector bundle over X, and let G be a finite group of automorphism of the pair (X, V). For any $g \in G$, let X^g be $\mathrm{Fix}(g)$ and

$$X^g = \sum_\alpha X^g_\alpha$$

the irreducible decomposition of X^g, and let

$$N^g_\alpha = \sum_\theta N^g_\alpha(\theta)$$

denote the normal bundle of X^g_α decomposed according to the eigenvalues $e^{i\theta}$ of g. We set

$$\begin{aligned}\mathfrak{U}^\theta(N^g_\alpha(\theta)) &= \Pi_\beta\left(\frac{1-e^{-x_\beta-i\theta}}{1-e^{-i\theta}}\right)^{-1} \\ &= \Pi_\beta\left(1+\frac{1}{1-e^{i\theta}}x_\beta+\frac{1+e^{i\theta}}{2(1-e^{i\theta})^2}x_\beta^2+\cdots\right),\end{aligned}$$

where the Chern class of $N^g_\alpha(\theta)$ is

$$c(N^g_\alpha(\theta)) = \prod_\beta (1+x_\beta).$$

Let $\mathfrak{T}(X^g_\alpha)$ be the Todd class of X^g_α and $ch(V|X^g_\alpha)(g)$ the Chern character of $V|X^g_\alpha$ with g-action ([1]). Set

$$\tau(g, X^g_\alpha) = \left\{\frac{ch(V|X^g_\alpha)(g)\cdot\Pi_\theta\mathfrak{U}^\theta(N^g_\alpha(\theta))\cdot\mathfrak{T}(X^g_\alpha)}{\det(1-g\,|\,(N^g_\alpha)^*)}\right\}[X^g_\alpha],$$

and

$$\tau(g) = \sum_\alpha \tau(g, X^g_\alpha).$$

Then we have

THEOREM (1.1). (Holomorphic Lefschetz Theorem [1]).

$$\sum_{p\geq 0} (-1)^p \operatorname{Trace}(g \mid H^p(X, \mathcal{O}(V))) = \tau(g).$$

Let $H^p(X, \mathcal{O}(V))^G$ be the invariant subspace of $H^p(X, \mathcal{O}(V))$ by G. Then we have the following

THEOREM (1.2).

$$\sum_{p\geq 0} (-1)^p \dim H^p(X, \mathcal{O}(V))^G = \frac{1}{|G|} \sum_{g\in G} \tau(g).$$

2. We use the same notation as in [16], Section 1.3. Let Γ be as in the Introduction. $\Gamma/\Gamma_2(l)$ acts on the pair $(\tilde{\mathfrak{S}}_2^*(l), k\tilde{L}_2 - \Delta(2))$ as a group of automorphism, and this induces an automorphism of $S_k(\Gamma_2(l))$ defined by

$$(M\cdot f)(M\cdot Z) = f(Z)\det(CZ + D)^k,$$

where

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, \qquad f \in S_k(\Gamma_2(l)) \quad \text{and} \quad Z \in \mathfrak{S}_2.$$

In the case of degree two we have

$$\det(CZ + D) = \det(-CZ - D).$$

Therefore, the action of M on $S_k(\Gamma_2(l))$ coincides with that of $-M$. Hence it suffices to consider the action of

$$G(\Gamma) = (\pm 1)\Gamma/(\pm 1)\Gamma_2(l).$$

We have

$$S_k(\Gamma) = S_k(\Gamma_2(l))^{G(\Gamma)}.$$

Recall the following vanishing theorem (cf. [16], Corollary (7.10)).

THEOREM (1.3). *If $p > 0$ and $k \geq 4$, then*

$$H^p(\tilde{\mathfrak{S}}_2^*(l), \mathcal{O}(k\tilde{L}_2 - \Delta(2))) = 0.$$

Therefore, we derive the following

COROLLARY (1.4). *If $k \geq 4$, then*

$$\dim S_k(\Gamma) = \frac{1}{|G(\Gamma)|} \sum_{g \in G(\Gamma)} \tau(g).$$

Let

$$s : \tilde{\mathfrak{S}}_2^*(l) \rightarrow \bar{\mathfrak{S}}_2^*(l)$$

be as in [16], Section 1.3. Set theoretically, $\bar{\mathfrak{S}}_2^*(l)$ is a disjoint union of $\mathfrak{S}_2^*(l)$, copies of $\mathfrak{S}_1^*(l)$ and copies of $\mathfrak{S}_0^*(l)$. We call a copy of $\mathfrak{S}_1^*(l)$ and a copy of $\mathfrak{S}_0^*(l)$ in $\bar{\mathfrak{S}}_2^*(l)$ a cusp of degree 1 and a cusp of degree 0, respectively. In Section 2, Section 3 and Section 4, we consider fixed subvarieties of elements of $G(l)$ in $\mathfrak{S}_2^*(l)$, in s^{-1} (cusps of degree 1) and in s^{-1} (cusps of degree 0), respectively.

2. Fixed Subvarieties in the Interior of the Siegel Space. 1. Fixed subvarieties in $\mathfrak{S}_2$ of the elements of finite order of $\Gamma_2(1)$ were classified by [17] using [5] and also by [20]. Let $\tilde{\varphi}$ be an element of finite order of $\Gamma_2(1)$, and let $\tilde{\Phi}$ be $\mathrm{Fix}(\tilde{\varphi})$. We denote $\tilde{\varphi} \bmod(\pm 1)$ by $\bar{\varphi}$ and $\tilde{\varphi} \bmod(\pm 1)\Gamma_2(l)$ by φ, and we denote the image of $\tilde{\Phi}$ by the natural projection of $\mathfrak{S}_2$ to $\mathfrak{S}_2^*(l)$ by Φ^0. The closure Φ of Φ^0 in $\tilde{\mathfrak{S}}_2^*(l)$ is an irreducible component of $\mathrm{Fix}(\varphi)$. We denote $\Gamma_2(1)/(\pm 1)$ by $\mathfrak{G}$.

Definition (2.1).

I) i) $C_{\mathfrak{G}}(\tilde{\Phi}) = \{g \in \mathfrak{G} \mid g \cdot x = x \text{ for any } x \in \tilde{\Phi}\}$,
ii) $C_{\mathfrak{G}}^p(\tilde{\Phi}) = \{g \in C_{\mathfrak{G}}(\tilde{\Phi}) \mid g \cdot x = x \text{ implies } x \in \tilde{\Phi}\}$,
iii) $N_{\mathfrak{G}}(\tilde{\Phi}) = \{g \in \mathfrak{G} \mid g \cdot \tilde{\Phi} = \tilde{\Phi}\}$.

Elements of $C_{\mathfrak{G}}^p(\tilde{\Phi})$ are called *proper* elements of $C_{\mathfrak{G}}(\tilde{\Phi})$.

II) Let π be the natural homomorphism of $\mathfrak{G}$ to $G(l)$.

i) $C_{G(l)}(\Phi) = \pi(C_{\mathfrak{G}}(\tilde{\Phi})) = \{g \in G(l) \mid g \cdot x = x \text{ for any } x \in \Phi\}$,
ii) $C_{G(l)}^p(\Phi) = \pi(C_{\mathfrak{G}}^p(\tilde{\Phi})) = \{g \in C_{G(l)}(\Phi) \mid \Phi \text{ is closed in } \mathrm{Fix}(g)\}$,
iii) $N_{G(l)}(\Phi) = \pi(N_{\mathfrak{G}}(\tilde{\Phi})) = \{g \in G(l) \mid g \cdot \Phi = \Phi\}$.

$C_{\mathfrak{G}}(\tilde{\Phi})$ and $C_{G(l)}(\Phi)$ are isomorphic through π.

THEOREM (2.2). ([17]). *An irreducible component of the fixed subvariety in $\tilde{\mathfrak{S}}_2^*(l)$ of an element of $G(l)$ which intersects $\mathfrak{S}_2^*(l)$ is equivalent* *to one of the following under the action of $G(l)$, and they are not equivalent to each other. $|N_{G(l)}(\Phi)|$ are calculated below. We omit the* (2, 1) *coefficients, since the matrices are symmetric.*

	Φ	$\tilde{\Phi}$	$\lvert C_{G(l)}(\Phi)\rvert$	$\lvert N_{G(l)}(\Phi)\rvert$
1)	Φ_1	$\begin{pmatrix} z_1 & z_2 \\ & z_3 \end{pmatrix}$	1	$(1/2)l^{10}\Pi(1-P^{-2})(1-p^{-4})$
2)	Φ_2	$\begin{pmatrix} z_1 & 0 \\ & z_2 \end{pmatrix}$	2	$l^6\Pi(1-p^{-2})^2$
3)	Φ_3	$\begin{pmatrix} z_1 & 1/2 \\ & z_2 \end{pmatrix}$	2	$\begin{cases} l^6\Pi(1-p^{-2})^2, & \text{if } 2 \nmid l \\ (4/3)l^6\Pi(1-p^{-2})^2, & \text{if } 2 \mid l \end{cases}$
4)	Φ_4	$\begin{pmatrix} z & 0 \\ & z \end{pmatrix}$	4	$2l^3\Pi(1-p^{-2})$
5)	Φ_5	$\begin{pmatrix} z & 1/2 \\ & z \end{pmatrix}$	4	$\begin{cases} 4l^3\Pi(1-p^{-2}), & \text{if } 2 \nmid l \\ (8/3)l^3\Pi(1-p^{-2}), & \text{if } 2 \mid l \end{cases}$
6)	Φ_6	$\begin{pmatrix} z & z/2 \\ & z \end{pmatrix}$	6	$\begin{cases} 6l^3\Pi(1-p^{-2}), & \text{if } 3 \nmid l \\ (9/2)l^3\Pi(1-p^{-2}), & \text{if } 3 \mid l \end{cases}$
7)	Φ_7	$\begin{pmatrix} i & 0 \\ & z \end{pmatrix}$	4	$2l^3\Pi(1-p^{-2})$
8)	Φ_8	$\begin{pmatrix} \rho & 0 \\ & z \end{pmatrix}$	6	$3l^3\Pi(1-p^{-2})$
9)	Φ_9	$\begin{pmatrix} i & 0 \\ & i \end{pmatrix}$	16	16
10)	Φ_{10}	$\begin{pmatrix} \rho & 0 \\ & \rho \end{pmatrix}$	36	36
11)	Φ_{11}	$\begin{pmatrix} \rho & 0 \\ & i \end{pmatrix}$	12	12
12)	Φ_{12}	$\frac{\sqrt{-3}}{3}\begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix}$	12	12

13)	Φ_{13}	$\begin{pmatrix} \eta & (\eta-1)/2 \\ & \eta \end{pmatrix}$	24	24
14)	Φ_{14}	$\begin{pmatrix} \omega & \omega+\omega^3 \\ & -\omega^4 \end{pmatrix}$	5	5

The isotropy groups of these fixed subvarieties were determined by [4]. In the following we do not list the matrices themselves but indicate them by the symbols of [17], and we list only the representatives of conjugacy classes of proper elements of $C_{G(l)}(\Phi)$. We denote by $C(\varphi)$ the centralizer in $C_{G(l)}(\Phi)$ of φ for $\varphi \in C_{G(l)}(\Phi)$ and indicate the eigenvalues $e^{i\theta}$ of the normal bundle of Φ by θ.

THEOREM (2.3). *The representatives of conjugacy classes of proper elements of $C_{G(l)}(\Phi)$ are as follows:*

	φ	$\tilde{\varphi}$ (symbol of [17])	$\lvert C(\varphi)\rvert$	θ	$\det(CZ+D)$ for $z \in \tilde{\Phi}$
1)	φ_1	I ①	1		1
2)	φ_2	II ① a)	2	π	-1
3)	φ_3	II ① b)	2	π	-1
4)	φ_4	II ② c)	4	π, π	1
5)	φ_5	II ② d)	4	π, π	1
6)	φ_6	II ③ c)	3	$2\pi/3, 4\pi/3$	1
7)	$\varphi_7(1)$	III ⑤ a)	4	$\pi, 3\pi/2$	i
	$\varphi_7(2)$	III ⑤ b)	4	$\pi/2, \pi$	$-i$
8)	$\varphi_8(1)$	III ① a)	6	$2\pi/3, 4\pi/3$	ρ^2
	$\varphi_8(2)$	III ① b)	6	$2\pi/3, 4\pi/3$	ρ
	$\varphi_8(3)$	III ③ a)	6	$4\pi/3, 5\pi/3$	$-\rho^2$
	$\varphi_8(4)$	III ③ b)	6	$\pi/3, 2\pi/3$	$-\rho$
9)	$\varphi_9(1)$	II ② a)	16	π, π, π	-1
	$\varphi_9(2)$	IV ① a)	8	$\pi/2, 3\pi/2, 3\pi/2$	$-i$
	$\varphi_9(3)$	IV ① b)	8	$\pi/2, \pi/2, 3\pi/3$	i
10)	$\varphi_{10}(1)$	II ③ a)	36	$4\pi/3, 4\pi/3, 4\pi/3$	ρ
	$\varphi_{10}(2)$	II ③ b)	36	$2\pi/3, 2\pi/3, 2\pi/3$	ρ^2
	$\varphi_{10}(3)$	IV ② c)	18	$2\pi/3, \pi, 4\pi/3$	-1
	$\varphi_{10}(4)$	IV ② e)	12	$\pi/3, 4\pi/3, 4\pi/3$	$-\rho$
	$\varphi_{10}(5)$	IV ② f)	12	$2\pi/3, 2\pi/3, 5\pi/3$	$-\rho^2$
	$\varphi_{10}(6)$	IV ② g)	36	$2\pi/3, 2\pi/3, 5\pi/3$	$-\rho^2$
	$\varphi_{10}(7)$	IV ② h)	36	$\pi/3, 4\pi/3, 4\pi/3$	$-\rho$
	$\varphi_{10}(8)$	IV ③ a)	12	$2\pi/3, 5\pi/3, 5\pi/3$	ρ^2
	$\varphi_{10}(9)$	IV ③ b)	12	$\pi/3, \pi/3, 4\pi/3$	ρ
11)	$\varphi_{11}(1)$	IV ⑥ a)	12	$2\pi/3, 5\pi/6, \pi$	σ^7
	$\varphi_{11}(2)$	IV ⑥ b)	12	$\pi, 7\pi/6, 4\pi/3$	σ^5

	$\varphi_{11}(3)$	IV ⑥ c)	12	$2\pi/3,\ \pi,\ 11\pi/6$	σ
	$\varphi_{11}(4)$	IV ⑥ d)	12	$\pi/6,\ \pi,\ 4\pi/3$	σ^{11}
12)	φ_{12}	IV ② a)	6	$2\pi/3,\ \pi,\ 4\pi/3$	-1
13)	φ_{13}	IV ① c)	4	$\pi/2,\ \pi,\ 3\pi/2$	-1
14)	$\varphi_{14}(1)$	IV ④ a)	5	$2\pi/5, 4\pi/5, 8\pi/5$	ω
	$\varphi_{14}(2)$	IV ④ b)	5	$4\pi/5, 6\pi/5, 8\pi/5$	ω^2
	$\varphi_{14}(3)$	IV ④ c)	5	$2\pi/5, 4\pi/5, 6\pi/5$	ω^3
	$\varphi_{14}(4)$	IV ④ d)	5	$2\pi/5, 6\pi/5, 8\pi/5$	ω^4

2. Lemma (2.4). *Let $g \in \mathfrak{G}$ and $\bar{\varphi} \in C^p_{\mathfrak{G}}(\tilde{\Phi})$. Then g belongs to $N_{\mathfrak{G}}(\tilde{\Phi})$ if and only if $g^{-1}\cdot\bar{\varphi}\cdot g$ belongs to $C_{\mathfrak{G}}(\tilde{\Phi})$. The map:*

$$N_{\mathfrak{G}}(\tilde{\Phi}) \to C_{\mathfrak{G}}(\tilde{\Phi})$$

$$g \mapsto g^{-1}\cdot\bar{\varphi}\cdot g$$

induces an injection of $C_{\mathfrak{G}}(\bar{\varphi})\backslash N_{\mathfrak{G}}(\tilde{\Phi})$ to $C_{\mathfrak{G}}(\tilde{\Phi})$.

Proof. If $g \in N_{\mathfrak{G}}(\tilde{\Phi})$ and $Z \in \tilde{\Phi}$, then $g^{-1}(\bar{\varphi}(g(Z))) = g^{-1}(g(Z)) = Z$. Conversely, if $g^{-1}\cdot\bar{\varphi}\cdot g \in C_{\mathfrak{G}}(\tilde{\Phi})$, then $\bar{\varphi}(g(Z)) = g(Z)$ for $Z \in \tilde{\Phi}$. Since $\bar{\varphi}$ is a proper element of $C_{\mathfrak{G}}(\tilde{\Phi})$, we infer $g(Z) \in \tilde{\Phi}$.

The image of the map in the lemma consists of the elements of $C_{\mathfrak{G}}(\tilde{\Phi})$ which are conjugate to $\bar{\varphi}$ in $N_{\mathfrak{G}}(\tilde{\Phi})$. We can determine this image easily, and as a result the image consists of the elements of $C_{\mathfrak{G}}(\tilde{\Phi})$ which are conjugate to $\bar{\varphi}$ in $C_{\mathfrak{G}}(\tilde{\Phi})$. Now we have the following

Corollary (2.5).

2) $N_{\mathfrak{G}}(\tilde{\Phi}_2) = C_{\mathfrak{G}}(\bar{\varphi}_2)$.
3) $N_{\mathfrak{G}}(\tilde{\Phi}_3) = C_{\mathfrak{G}}(\bar{\varphi}_3)$.
4) $N_{\mathfrak{G}}(\tilde{\Phi}_4) = C_{\mathfrak{G}}(\bar{\varphi}_4)$.
5) $N_{\mathfrak{G}}(\tilde{\Phi}_5) = C_{\mathfrak{G}}(\bar{\varphi}_5)$.
6) $N_{\mathfrak{G}}(\tilde{\Phi}_6) = C_{\mathfrak{G}}(\bar{\varphi}_6) \cup C_{\mathfrak{G}}(\bar{\varphi}_6)\cdot\bar{\delta}$, *where $\bar{\delta}$ is defined below.*
7) $N_{\mathfrak{G}}(\tilde{\Phi}_7) = C_{\mathfrak{G}}(\bar{\varphi}_7(1))$.
8) $N_{\mathfrak{G}}(\tilde{\Phi}_8) = C_{\mathfrak{G}}(\bar{\varphi}_8(1))$.

Let $\tilde{\alpha}_\theta$, $\tilde{\epsilon}$, $\tilde{\delta}$ and $\tilde{\xi}$ be

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{and}$$

$$\begin{pmatrix} 1+\sqrt{3} & -1+\sqrt{3} & 0 & 0 \\ -1+\sqrt{3} & 1+\sqrt{3} & 0 & 0 \\ 0 & 0 & (1+\sqrt{3})/4\sqrt{3} & (1-\sqrt{3})/4\sqrt{3} \\ 0 & 0 & (1-\sqrt{3})/4\sqrt{3} & (1+\sqrt{3})/4\sqrt{3} \end{pmatrix}, \text{ respectively.}$$

THEOREM (2.6). (4), 5) *and* 6) *are due to* [13]).

$$2)\ C_{\mathfrak{G}}(\overline{\varphi}_2) = \langle\overline{\delta}\rangle \times \left\{ \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \in \mathrm{Sp}(2, \mathbf{Z}) \right\} / (\pm 1).$$

$$3)\ C_{\mathfrak{G}}(\overline{\varphi}_3) = \overline{\epsilon}\langle\overline{\delta}\rangle\overline{\epsilon}^{-1} \times \overline{\epsilon} \left\{ \begin{pmatrix} a_1 & 0 & b_1/2 & 0 \\ 0 & a_2 & 0 & b_2/2 \\ 2c_1 & 0 & d_1 & 0 \\ 0 & 2c_2 & 0 & d_2 \end{pmatrix} \in \mathrm{Sp}(2, \mathbf{R}) \,\middle|\, \begin{array}{l} a_i, b_i, c_i, d_i \in \mathbf{Z}, \\ a_i \equiv d_{3-i} \bmod 2, \\ b_i \equiv c_{3-i} \bmod 2, \\ \text{for } i = 1, 2 \end{array} \right\} \overline{\epsilon}^{-1} / (\pm 1).$$

$$4)\ C_{\mathfrak{G}}(\overline{\varphi}_4) = \langle\overline{\delta}\rangle \times \langle\overline{\varphi}_4\rangle \times \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \in \mathrm{Sp}(2, \mathbf{Z}) \right\} / (\pm 1).$$

5) $C_{\mathfrak{G}}(\overline{\varphi}_5) = \overline{\epsilon}\langle\overline{\delta}\rangle\overline{\epsilon}^{-1} \times \overline{\epsilon} H_1 \overline{\epsilon}^{-1}$, where

$$H_1 = \left\{ \begin{pmatrix} a & 0 & b/2 & 0 \\ 0 & a & 0 & b/2 \\ 2c & 0 & d & 0 \\ 0 & 2c & 0 & d \end{pmatrix} \cdot (\tilde{\alpha}_{\pi/4})^{\gamma} \,\middle|\, \begin{array}{l} \gamma = 0, 1, 2, 3, ad - bc = 1, \\ a, b, c, d \in (\sqrt{2})^{-\beta}\mathbf{Z}, \\ a \equiv d, b \equiv c \bmod 2(\sqrt{2})^{-\beta}, \\ \text{where } \beta = (1 + (-1)^{\gamma+1})/2 \end{array} \right\} / (\pm 1).$$

6) $C_{\mathfrak{G}}(\overline{\varphi}_6) = \overline{\xi} H_2 \overline{\xi}^{-1}$, where

$$H_2 = \left\{ \begin{pmatrix} a & 0 & b/4 & 0 \\ 0 & a & 0 & b/4 \\ 12c & 0 & d & 0 \\ 0 & 12c & 0 & d \end{pmatrix} \cdot (\tilde{\alpha}_{\pi/6})^{\gamma} \,\middle|\, \begin{array}{l} \gamma = 0, \ldots, 5, ad - 3bc = 1, \\ a, d \in (\sqrt{3})^{\beta}\mathbf{Z}, \\ b, c \in (\sqrt{3})^{-\beta}\mathbf{Z}, \\ \text{where } \beta = (1 + (-1)^{\gamma+1})/2 \end{array} \right\} / (\pm 1).$$

$$7)\ C_{\mathfrak{G}}(\bar{\varphi}_7(1)) = \left\{ \tilde{\varphi}_7(1)^\gamma \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \middle| \begin{array}{l} \gamma = 0, 1, 2, 3, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \end{array} \right\} / (\pm 1).$$

$$8)\ C_{\mathfrak{G}}(\bar{\varphi}_8(1)) = \left\{ \tilde{\varphi}_8(1)^\gamma \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \middle| \begin{array}{l} \gamma = 0, \ldots, 5, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \end{array} \right\} / (\pm 1).$$

Proof. We prove only 3), 4), 5) and 6). Others are easily proved.

Proof of 3). Let $\mathfrak{G}(\mathbf{R}) = \mathrm{Sp}(2, \mathbf{R})/(\pm 1)$. Since $\tilde{\varphi}_3 = \tilde{\epsilon} \cdot \tilde{\varphi}_2 \cdot \tilde{\epsilon}^{-1}$, we have

$$C_{\mathfrak{G}(\mathbf{R})}(\bar{\varphi}_3) = \bar{\epsilon} \cdot C_{\mathfrak{G}(\mathbf{R})}(\bar{\varphi}_2) \cdot \bar{\epsilon}^{-1}.$$

Therefore it follows

$$C_{\mathfrak{G}}(\bar{\varphi}_3) = \bar{\epsilon} \cdot C_{\mathfrak{G}(\mathbf{R})}(\bar{\varphi}_2) \cdot \bar{\epsilon}^{-1} \cap \mathfrak{G}.$$

The assertion is proved from this fact.

Proof of 4). Let $C_{\mathfrak{G}}(\bar{\varphi}_4)^+ = C_{\Gamma_2(1)}(\tilde{\varphi}_4)/(\pm 1)$. Then since $\tilde{\delta} \cdot \tilde{\varphi}_4 = -\tilde{\varphi}_4 \cdot \tilde{\delta}$, we have

$$C_{\mathfrak{G}}(\bar{\varphi}_4) = \langle \bar{\delta} \rangle \cdot C_{\mathfrak{G}}(\bar{\varphi}_4)^+,$$

which is a semi-direct product of $\langle \bar{\delta} \rangle$ and $C_{\mathfrak{G}}(\bar{\varphi}_4)^+$. $\tilde{M} \in \Gamma_2(1)$ belongs to $C_{\Gamma_2(1)}(\tilde{\varphi}_4)$ if and only if $\tilde{M}$ is written in the following form:

$$\tilde{M} = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ -a_2 & a_1 & -b_2 & b_1 \\ c_1 & c_2 & d_1 & d_2 \\ -c_2 & c_1 & -d_2 & d_1 \end{pmatrix}.$$

Then for some $\theta_1 \in \mathbf{R}$,

$$\tilde{M} = \begin{pmatrix} a_1' & a_2' & b_1' & b_2' \\ -a_2' & a_1' & -b_2' & b_1' \\ c & 0 & d_2' & d_2' \\ 0 & c & -d_2' & d_1' \end{pmatrix} \cdot \tilde{\alpha}_{\theta_1} \quad (c^2 = c_1^2 + c_2^2).$$

If $c \neq 0$, then the condition to be symplectic implies $a_2' = b_2' = d_2' = 0$. If $c = 0$, then $(a_1')^2 + (a_2')^2 \neq 0$, and so for some $\theta_2 \in \mathbf{R}$,

$$\tilde{M} = \begin{pmatrix} a & 0 & b_1'' & b_2'' \\ 0 & a & -b_2'' & b_1'' \\ 0 & 0 & d_1'' & d_2'' \\ 0 & 0 & -d_2'' & d_2'' \end{pmatrix} \cdot \tilde{\alpha}_{\theta_2} \quad (a^2 = (a_1')^2 + (a_2')^2).$$

$a \neq 0$ and the condition to be symplectic imply $b_2'' = d_2'' = 0$. Therefore, in any case, $\tilde{M}$ can be written in the following form:

$$\tilde{M} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \cdot \tilde{\alpha}_{\theta}.$$

The condition $\tilde{M} \in \Gamma_2(1)$ implies that $a, b, c, d \in \mathbf{Z}$ and $\theta = \pi k/2$ $(k \in \mathbf{Z})$. Therefore, we can conclude

$$C_{\mathfrak{G}}(\overline{\varphi}_4)^+ = \langle \overline{\varphi}_4 \rangle \times \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \in \mathrm{Sp}(2, \mathbf{Z}) \right\} / (\pm 1).$$

This implies that the semi-direct product before is also a direct product.

Proof of 5). Since $\tilde{\varphi}_5 = \tilde{\epsilon}\cdot\tilde{\varphi}_4\cdot\tilde{\epsilon}^{-1}$, we have

$$C_{\mathfrak{G}(\mathbf{R})}(\overline{\varphi}_5) = \overline{\epsilon}\cdot C_{\mathfrak{G}(\mathbf{R})}(\overline{\varphi}_4)\cdot\overline{\epsilon}^{-1}.$$

Therefore,

$$C_{\mathfrak{G}}(\overline{\varphi}_5) = \overline{\epsilon}\cdot C_{\mathfrak{G}(\mathbf{R})}(\overline{\varphi}_4)\cdot\overline{\epsilon}^{-1} \cap \mathfrak{G}.$$

The above argument implies that

$$C_{\mathfrak{G}(\mathbf{R})}(\overline{\varphi}_4) = \left\{\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}\cdot\tilde{\alpha}_\theta \in \mathrm{Sp}(2, \mathbf{R})\right\}/(\pm 1).$$

The assertion is proved from this fact.

Proof of 6). The fact that $\mathrm{Trace}(\tilde{\varphi}_6) \neq 0$ implies that

$$C_{\mathfrak{G}}(\overline{\varphi}_6) = C_{\Gamma_2(1)}(\tilde{\varphi}_6)/(\pm 1).$$

$\tilde{M} \in \Gamma_2(1)$ belongs to $C_{\Gamma_2(1)}(\tilde{\varphi}_6)$ if and only if $\tilde{M}$ is written in the following form:

$$\tilde{M} = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ -a_2 & a_1 + a_2 & b_1 - b_2 & b_1 \\ c_1 & c_2 & d_1 & d_2 \\ -c_1 - c_2 & c_1 & -d_2 & d_1 - d_2 \end{pmatrix}.$$

Then

$$\xi^{-1}\cdot\tilde{M}\cdot\xi = \begin{pmatrix} a_1' & a_2' & b_1' & b_2' \\ -a_2' & a_1' & -b_2' & b_1' \\ c_1' & c_2' & d_1' & d_2' \\ -c_2' & c_1' & -d_2' & d_1' \end{pmatrix}$$

Therefore, we infer that $C_{\mathfrak{S}(\mathbf{R})}(\overline{\varphi}_6) = \overline{\xi}\cdot C_{\mathfrak{S}(\mathbf{R})}(\overline{\varphi}_4)\cdot\overline{\xi}^{-1}$. The assertion follows from this fact.

COROLLARY (2.7). *We define $N_{\Gamma_2(1)}(\tilde{\Phi})$ similarly as in Definition* (2.1) *and $N(\tilde{\Phi})(l)$ to be $N_{\Gamma_2(1)}(\tilde{\Phi}) \cap \Gamma_2(l)$. Let $l \geq 3$. Then we have the following.*

$$2)\ N(\tilde{\Phi}_2)(l) = \left\{\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \in \Gamma_2(l)\right\}.$$

$$3)\ N(\tilde{\Phi}_3)(l) = \tilde{\epsilon}\left\{\begin{pmatrix} a_1 & 0 & b_1/2 & 0 \\ 0 & a_2 & 0 & b_2/2 \\ 2c_1 & 0 & d_1 & 0 \\ 0 & 2c_2 & 0 & d_2 \end{pmatrix} \in \mathrm{Sp}(2, \mathbf{R})\ \middle|\ \begin{array}{l} a_i, b_i, c_i, d_i \in \mathbf{Z}, \\ a_i - 1 \equiv b_i \equiv 0 \bmod l, \\ a_i \equiv d_{3-i} \bmod 2l, \\ b_i \equiv c_{3-i} \bmod 2l \end{array}\right\}\tilde{\epsilon}^{-1}.$$

$$4)\ N(\tilde{\Phi}_4)(l) = \left\{\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \in \Gamma_2(l)\right\}.$$

$$5)\ N(\tilde{\Phi}_5)(l) = \tilde{\epsilon}\left\{\begin{pmatrix} a & 0 & b/2 & 0 \\ 0 & a & 0 & b/2 \\ 2c & 0 & d & 0 \\ 0 & 2c & 0 & d \end{pmatrix}\ \middle|\ \begin{array}{l} a, b, c, d \in \mathbf{Z}, \\ ad - bc = 1, \\ a - 1 \equiv b \equiv 0 \bmod l, \\ a - d \equiv b - c \equiv 0 \bmod 2l \end{array}\right\}\tilde{\epsilon}^{-1}.$$

$$6)\ N(\tilde{\Phi}_6)(l) = \tilde{\xi}\left\{\begin{pmatrix} a & 0 & b/4 & 0 \\ 0 & a & 0 & b/4 \\ 12c & 0 & d & 0 \\ 0 & 12c & 0 & d \end{pmatrix}\ \middle|\ \begin{array}{l} a, b, c, d \in \mathbf{Z}, \\ ad - 3bc = 1, \\ a \equiv d \equiv 1 \bmod l, \\ b \equiv c \equiv 0 \bmod l \end{array}\right\}\tilde{\xi}^{-1}.$$

$$7)\text{ and }8)\ N(\tilde{\Phi}_7)(l) = N(\tilde{\Phi}_8)(l) = \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \in \Gamma_2(l)\right\}.$$

The values of $|N_{G(l)}(\Phi)|$ in Theorem (2.2) are obtained from Corollary (2.5), Theorem (2.6) and Corollary (2.7). See the proof of Theorem (2.8) for the details.

3. THEOREM (2.8).

1) $\tau(\varphi_1, \Phi_1) = 2^{-10}3^{-3}5^{-1}((2k-2)(2k-3)(2k-4)l^{10} - 240(2k-3)l^8 + 1440l^7)\Pi(1-p^{-2})(1-p^{-4})$.

2) $\tau(\varphi_2, \Phi_2) = 2^{-7}3^{-2}(-1)^k((k-1)(k-2)l^6 - 6(2k-3)l^5 + 36l^4)\Pi(1-p^{-2})^2$.

3) $\tau(\varphi_3, \Phi_3) =$

$$\begin{cases} 2^{-6}3^{-1}(-1)^k((k-1)(k-2)l^6 - 3(2k-3)l^5 + 12l^4)\Pi(1-p^{-2})^2, & \text{if } 2 \nmid l, \\ 2^{-4}3^{-2}(-1)^k((k-1)(k-2)l^6 - 3(2k-3)l^5 + 12l^4)\Pi(1-p^{-2})^2, & \text{if } 2 \mid l. \end{cases}$$

4) $\tau(\varphi_4, \Phi_4) = 2^{-5}3^{-1}((2k-3)l^3 - 12l^2)\Pi(1-p^{-2})$.

5) $$\tau(\varphi_5, \Phi_5) = \begin{cases} 2^{-5}((2k-3)l^3 - 8l^2)\Pi(1-p^{-2}), & \text{if } 2 \nmid l, \\ 2^{-4}3^{-1}((2k-3)l^3 - 8l^2)\Pi(1-p^{-2}), & \text{if } 2 \mid l. \end{cases}$$

6) $$\tau(\varphi_6, \Phi_6) = \begin{cases} 2^{-1}3^{-2}((2k-3)l^3 - 9l^2)\Pi(1-p^{-2}), & \text{if } 3 \nmid l, \\ 2^{-3}3^{-1}((2k-3)l^3 - 9l^2)\Pi(1-p^{-2}), & \text{if } 3 \mid l. \end{cases}$$

7) $\tau(\varphi_7(1), \Phi_7) = 2^{-6}3^{-1}(i)^k(1+i)((2k-3+i)l^3 - 12l^2)\Pi(1-p^{-2})$.

$\tau(\varphi_7(2), \Phi_7) = 2^{-6}3^{-1}(-i)^k(1-i)((2k-3-i)l^3 - 12l^2)\Pi(1-p^{-2})$.

8) $\tau(\varphi_8(1), \Phi_8) = 2^{-3}3^{-3}(\rho^2)^k((3k-4+\rho^2)l^3 - 18l^2)\Pi(1-p^{-2})$.

$\tau(\varphi_8(2), \Phi_8) = 2^{-3}3^{-3}(\rho)^k((3k-4+\rho)l^3 - 18l^2)\Pi(1-p^{-2})$.

$\tau(\varphi_8(3), \Phi_8) = 2^{-3}3^{-2}(-\rho^2)^k(1+2\rho)((k-2-\rho^2)l^3 - 6l^2)\Pi(1-p^{-2})$.

$\tau(\varphi_8(4), \Phi_8) = 2^{-3}3^{-2}(-\rho)^k(1+2\rho^2)((k-2-\rho)l^3 - 6l^2)\Pi(1-p^{-2})$.

9) $\tau(\varphi_9(1), \Phi_9) = 2^{-3}(-1)^k$.

$\tau(\varphi_9(2), \Phi_9) = 2^{-2}(-i)^k(1+i)$.

$\tau(\varphi_9(3), \Phi_9) = 2^{-2}(i)^k(1-i)$.

10) $\tau(\varphi_{10}(1), \Phi_{10}) = 3^{-2}(\rho)^k(1+2\rho)$.

$\tau(\varphi_{10}(2), \Phi_{10}) = 3^{-2}(\rho^2)^k(1+2\rho^2)$.

$\tau(\varphi_{10}(3), \Phi_{10}) = 2^{-1}3^{-1}(-1)^k.$

$\tau(\varphi_{10}(4), \Phi_{10}) = \tau(\varphi_{10}(7), \Phi_{10}) = 3^{-1}(-\rho)^k.$

$\tau(\varphi_{10}(5), \Phi_{10}) = \tau(\varphi_{10}(6), \Phi_{10}) = 3^{-1}(-\rho^2)^k.$

$\tau(\varphi_{10}(8), \Phi_{10}) = 3^{-1}(\rho^2)^k(1 + 2\rho).$

$\tau(\varphi_{10}(9), \Phi_{10}) = 3^{-1}(\rho)^k(1 + 2\rho^2).$

11) $\tau(\varphi_{11}(1), \Phi_{11}) = 2^{-1}3^{-1}(\sigma^7)^k(1 - \rho)(1 - \sigma^7)^{-1}.$

$\tau(\varphi_{11}(2), \Phi_{11}) = 2^{-1}3^{-1}(\sigma^5)^k(1 - \rho^2)(1 - \sigma^5)^{-1}.$

$\tau(\varphi_{11}(3), \Phi_{11}) = 2^{-1}3^{-1}(\sigma)^k(1 - \rho)(1 - \sigma)^{-1}.$

$\tau(\varphi_{11}(4), \Phi_{11}) = 2^{-1}3^{-1}(\sigma^{11})^k(1 - \rho^2)(1 - \sigma^{11})^{-1}.$

12) $\tau(\varphi_{12}, \Phi_{12}) = 2^{-1}3^{-1}(-1)^k.$

13) $\tau(\varphi_{13}, \Phi_{13}) = 2^{-2}(-1)^k.$

14) $\tau(\varphi_{14}(1), \Phi_{14}) = 5^{-1}(\omega)^k(1 - \omega^2).$

$\tau(\varphi_{14}(2), \Phi_{14}) = 5^{-1}(\omega^2)^k(1 - \omega^4).$

$\tau(\varphi_{14}(3), \Phi_{14}) = 5^{-1}(\omega^3)^k(1 - \omega).$

$\tau(\varphi_{14}(4), \Phi_{14}) = 5^{-1}(\omega^4)^k(1 - \omega^3).$

Proof. 1) follows from the fact that $\tau(\varphi_1, \Phi_1) = \dim S_k(\Gamma_2(l))$. 9), ..., 14) are easy. We prove only 3), 6) and 7). 2), 4), 5) and 8) are similarly proved. We always denote a divisor, the line bundle determined by the divisor and its first Chern class by the same symbol.

Proof of 3). Let

$$g = \tilde{\epsilon}\begin{pmatrix} a_1 & 0 & b_1/2 & 0 \\ 0 & a_2 & 0 & b_2/2 \\ 2c_1 & 0 & d_1 & 0 \\ 0 & 2c_2 & 0 & d_2 \end{pmatrix}\tilde{\epsilon}^{-1} \in N(\tilde{\Phi}_3)(l).$$

Then g acts on $\tilde{\Phi}_3$ as

$$g\cdot\begin{pmatrix} z_1 & 1/2 \\ & z_2 \end{pmatrix} = \begin{pmatrix} \dfrac{a_1z_1+b_1/2}{2c_1z_1+d_1} & \dfrac{1}{2} \\ & \dfrac{a_2z_2+b_2/2}{2c_2z_2+d_2} \end{pmatrix}.$$

Therefore, if we define $\psi:\mathfrak{S}_1 \times \mathfrak{S}_1 \to \tilde{\Phi}_3$ by

$$\psi(z_1, z_2) = \begin{pmatrix} z_1/2 & 1/2 \\ & z_2/2 \end{pmatrix},$$

then we have

$$(\psi^{-1}\cdot g\cdot\psi)(z_1, z_2) = \left(\frac{a_1z_1+b_1}{c_1z_1+d_1}, \frac{a_2z_2+b_2}{c_2z_2+d_2}\right).$$

We put

$$g^{\psi} = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right),$$

and

$$N(\tilde{\Phi}_3)(l)^{\psi} = \{g^{\psi} \mid g \in N(\tilde{\Phi}_3)(l)\}.$$

Then $N(\tilde{\Phi}_3)(l)^{\psi}$ is a subgroup of $SL(2, \mathbf{Z}) \times SL(2, \mathbf{Z})$ which contains $\Gamma_1(2l) \times \Gamma_1(2l)$ as a normal subgroup of index six (resp., eight), if l is odd (resp., even). We prove only the case when l is odd. The case when l is even is similarly proved.

Assume that l is odd. We denote the elements of $SL(2, \mathbf{Z}/2l\mathbf{Z})$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2l, \quad \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \bmod 2l, \quad \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \bmod 2l,$$

$$\begin{pmatrix} 1+l & l \\ l & 1+l \end{pmatrix} \bmod 2l, \quad \begin{pmatrix} 1+l & l \\ l & 1 \end{pmatrix} \bmod 2l$$

and

$$\begin{pmatrix} 1 & l \\ l & 1+l \end{pmatrix} \bmod 2l,$$

by $A_1, \ldots, A_6$, respectively. Then

$$F = N(\tilde{\Phi}_3)(l)^{\psi}/\Gamma_1(2l) \times \Gamma_1(2l)$$

consists of (A_1, A_1), (A_2, A_3), (A_3, A_2), (A_4, A_4), (A_5, A_6) and (A_6, A_5).

Let $P_{1,0}$ be the subgroup of $SL(2, \mathbf{Z})$ defined by

$$P_{1,0} = \left\{ \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix} \middle| \, n \in \mathbf{Z} \right\}.$$

Then the cusps of $\tilde{\mathfrak{S}}_1^*(2l)$ correspond bijectively to

$$\Gamma_1(2l)\backslash\Gamma_1(1)/P_{1,0}$$

by the correspondence:

$$\Gamma_1(2l)\begin{pmatrix} a & b \\ c & d \end{pmatrix} P_{1,0} \mapsto \Gamma_1(2l)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty.$$

Let M be the subset of $\mathbf{Z}/2l\mathbf{Z} \times \mathbf{Z}/2l\mathbf{Z}$ consisting of elements of order $2l$. Then we have another bijection:

$$\Gamma_1(2l)\backslash\Gamma_2(1)/P_{1,0} \to M/(\pm 1),$$

$$\Gamma_1(2l)\begin{pmatrix} a & b \\ c & d \end{pmatrix} P_{1,0} \mapsto \pm(a \bmod 2l, c \bmod 2l).$$

Let U, V and W be the sets of cusps of $\tilde{\mathfrak{S}}_1^*(2l)$ corresponding to

$$\{(a, b) \in M/(\pm 1) \,|\, a \not\equiv 0,\ b \equiv 0 \bmod 2\},$$

$$\{(a, b) \in M/(\pm 1) \,|\, a \equiv 0,\ b \not\equiv 0 \bmod 2\},$$

$$\{(a, b) \in M/(\pm 1) \,|\, a \not\equiv 0,\ b \not\equiv 0 \bmod 2\},$$

respectively, by the above bijection. U, V and W consist of $(1/2)l^2\Pi(1 - p^{-2})$ cusps, respectively. A_2, A_3 and A_4 fix any $u \in U$, any $v \in V$ and any $w \in W$, respectively, and do not fix other cusps, and A_5 and A_6 fix no cusps.

Let S denote $\tilde{\mathfrak{S}}_1^*(2l) \times \tilde{\mathfrak{S}}_1^*(2l)$, and for a cusp p of $\tilde{\mathfrak{S}}_1^*(2l)$ let C_p and D_p denote the divisor $\{p\} \times \tilde{\mathfrak{S}}_1^*(2l)$ and $\tilde{\mathfrak{S}}_1^*(2l) \times \{p\}$ on S, respectively. Then (A_2, A_3) transforms C_u and D_v into themselves and fixes $u \times v$ for any $u \in U$ and $v \in V$. Similarly, (A_3, A_2) (resp., (A_4, A_4)) transforms C_v and D_u (resp., C_{w_1} and D_{w_2}) into themselves and fixes $v \times u$ (resp., $w_1 \times w_2$) for any $v \in V$ and $u \in U$ (resp., $w_1, w_2 \in W$).

Let ψ be as above. Then ψ induces a morphism of $\mathfrak{S}_1^*(2l) \times \mathfrak{S}_1^*(2l)$ to Φ_3. This morphism induces a rational map of S to Φ_3 which is not holomorphic at $U \times V$, $V \times U$ and $W \times W$. Let $\tilde{S}$ be the blowing up of S at $U \times V$, $V \times U$ and $W \times W$. Then it is easily seen that ψ induces a morphism of $\tilde{S}$ to Φ_3 and $\bar{S} = F\backslash\tilde{S}$ is isomorphic to Φ_3. Let p_1 and p_2 be the projection of S to the first and the second factors, and let Q, π, $\bar{\psi}$ and i be as in the following diagram:

$$\begin{tikzcd}
S \arrow[r, hook] & \mathfrak{S}_1^*(2l) \times \mathfrak{S}_1^*(2l) \arrow[r] & \Phi_3 \arrow[dd, equal] & \\
\tilde{S} \arrow[u, "Q"] \arrow[d, "\pi"'] & & & \\
\bar{S} \arrow[rr, "\sim", "\bar{\psi}"'] & & \Phi_3 \arrow[r, hook, "i"'] & \tilde{\mathfrak{S}}_2^*(l).
\end{tikzcd}$$

Let $\tilde{C}_p$ and $\tilde{D}_p$ denote the proper transforms of C_p and D_p by Q, respectively, and let $\bar{C}_p$ and $\bar{D}_p$ denote $\pi(\tilde{C}_p)$ and $\pi(\tilde{D}_p)$, respectively. Let $\tilde{E}_{p\times q}$ be the exceptional curve on $\tilde{S}$ over $p \times q \in S$, and let $\bar{E}_{p\times q}$ be $\pi(\tilde{E}_{p\times q})$.

Since

$$(C_p)^2[S] = (D_p)^2[S] = 0,$$

we have

$$(\tilde{C}_p)^2[\tilde{S}] = (\tilde{D}_p)^2[\tilde{S}] = -(1/2)l^2\Pi(1 - p^{-2}).$$

Since $\pi^*(\bar{C}_p) = \tilde{C}_p + \tilde{C}_{p'} + \tilde{C}_{p''}$, where $\tilde{C}_p$, $\tilde{C}_{p'}$ and $\tilde{C}_{p''}$ are disjoint, we have

$$(1)\qquad (\bar{C}_p)^2[\bar{S}] = (\bar{C}_p)^2[(1/6)\pi_*(\tilde{S})] = (\pi^*(\bar{C}_p))^2[(1/6)\tilde{S}]$$
$$= -(1/4)l^2\Pi(1 - p^{-2}).$$

Similarly we have

$$(2)\qquad (\bar{D})^2[\bar{S}] = -(1/4)l^2\Pi(1 - p^{-2}).$$

Since π ramifies at $\tilde{E}_{p\times q}$ with ramification index 2, and $\pi^*(\bar{E}_{p\times q}) = 2(\tilde{E}_{p\times q} + \tilde{E}_{p'\times q'} + \tilde{E}_{p''\times q''})$, we have

$$(3)\qquad (\bar{E}_{p\times q})^2[\bar{S}] = (\pi^*(\bar{E}_{p\times q}))^2[(1/6)\tilde{S}] = -2.$$

Further we have

$$(4)\qquad ((i\cdot\bar{\psi})^*(\tilde{L}_2))^2[\bar{S}] = ((i\cdot\bar{\psi}\cdot\pi)^*(\tilde{L}_2))^2[(1/6)\tilde{S}]$$
$$= (Q^*(p_1^*(\tilde{L}_1) + p_2^*(\tilde{L}_1)))^2[(1/6)\tilde{S}]$$
$$= (1/6)(p_1^*(\tilde{L}_1) + p_2^*(\tilde{L}_1))^2[S]$$
$$= (1/48)l^6\Pi(1 - p^{-2})^2,$$

$$(5)\qquad ((i\cdot\bar{\psi})^*(\tilde{L}_2)\cdot\bar{E}_{p\times q})[\bar{S}]$$
$$= ((i\cdot\bar{\psi}\cdot\pi)^*(\tilde{L}_2)\cdot(\tilde{E}_{p\times q} + \tilde{E}_{p'\times q'} + \tilde{E}_{p''\times q''}))[(1/3)\tilde{S}]$$
$$= (Q^*(p_1^*(\tilde{L}_1) + p_2^*(\tilde{L}_1))\cdot Q^*(0 + 0 + 0))[(1/3)\tilde{S}] = 0,$$

and

$$(6)\qquad ((i\cdot\bar{\psi})^*(\tilde{L}_2)\cdot\bar{C}_p)[\bar{S}]$$
$$= ((i\cdot\bar{\psi}\cdot\pi)^*(\tilde{L}_2)\cdot(\tilde{C}_p + \tilde{C}_{p'} + \tilde{C}_{p''}))[(1/6)\tilde{S}]$$
$$= (p_2^*(\tilde{L}_1)\cdot(C_p + C_{p'} + C_{p''}))[(1/6)S] = (1/8)l^3\Pi(1 - p^{-2}).$$

Similarly we have

$$((i\cdot\bar{\psi})^*(\tilde{L}_2)\cdot\bar{D}_p)[\bar{S}] = (1/8)l^3\Pi(1-p^{-2}). \tag{7}$$

Further, by the proportionality ([16], Theorem (1.3)) we have

$$\begin{aligned}\bar{c}_2(\Phi_3^0)[\Phi_3] &= (1/2)(\bar{c}_1(\Phi_3^0))^2[\Phi_3] \\ &= 2(i^*(\tilde{L}_2))^2[\Phi_3] = (1/24)l^6\Pi(1-p^{-2})^2.\end{aligned} \tag{8}$$

Let

$$\bar{C} = \sum_{p\in U}\bar{C}_p, \qquad \bar{D} = \sum_{p\in U}\bar{D}_p \quad\text{and}\quad \bar{E} = \sum_{p\times q\in U\times V}\bar{E}_{p\times q}.$$

Then we have

$$\begin{aligned}\bar{\psi}^*(\Delta(\Phi_3)) &= \bar{C} + \bar{D} + \bar{E}, \\ (i\cdot\bar{\psi})^*(\Delta(2)) &= \bar{C} + \bar{D} + 2\bar{E},\end{aligned} \tag{9}$$

and

$$(i\cdot\bar{\psi})^*(2\tilde{L}_2) = K_{\bar{S}} + i^*(\Delta(\Phi_3)).$$

Therefore, it follows that

$$\bar{\psi}^*(N_3(\pi)) = -(i\cdot\bar{\psi})^*(K_{\mathfrak{S}_2^*(l)}) + K_{\bar{S}} = -(i\cdot\bar{\psi})^*(\tilde{L}_2) + \bar{E}, \tag{10}$$

where $N_3(\pi)$ is the normal bundle of Φ_3. Further, by [16], Proposition (1.2), we have

$$c_1(\bar{S}) = i^*(\bar{c}_1(\Phi_3^0) + \Delta(\Phi_3)) = -2(i\cdot\bar{\psi})^*(\tilde{L}_2) + \bar{C} + \bar{D} + \bar{E}, \tag{11}$$

and

$$\begin{aligned}c_2(\bar{S}) &= i^*(\bar{c}_2(\Phi_3^0) + \bar{c}_1(\Phi_3^0)\cdot\Delta(\Phi_3) + \Delta_2(\Phi_3)) \\ &= i^*(\bar{c}_2(\Phi_3^0)) - 2(i\cdot\bar{\psi})^*(\tilde{L}_2)\cdot(\bar{C} + \bar{D} + \bar{E}) \\ &\quad + (\bar{D}\cdot\bar{E} + \bar{C}\cdot\bar{E} + \bar{C}\cdot\bar{D}).\end{aligned} \tag{12}$$

By (1)$\sim$(12) we can compute the intersection numbers which appear in $\tau(\varphi_3, \Phi_3)$ and obtain the assertion.

Proof of 6). Let

$$g = \tilde{\xi}\begin{pmatrix} a & 0 & b/4 & 0 \\ 0 & a & 0 & b/4 \\ 12c & 0 & d & 0 \\ 0 & 12c & 0 & d \end{pmatrix}\tilde{\xi}^{-1} \in N(\tilde{\Phi}_6)(l).$$

Then g acts on Φ_6 as

$$g\cdot\begin{pmatrix} z & z/2 \\ & z \end{pmatrix} = \begin{pmatrix} \dfrac{az+2b}{3cz/2+d} & \left(\dfrac{az+2b}{3cz/2+d}\right)/2 \\ & \dfrac{az+2b}{3cz/2+d} \end{pmatrix}.$$

Therefore, if we define $\psi : \mathfrak{S}_1 \to \tilde{\Phi}_6$ by

$$\psi(z) = \begin{pmatrix} 2z & z \\ & 2z \end{pmatrix},$$

then we have

$$(\psi^{-1}\cdot g\cdot\psi)(z) = \frac{az+b}{3cz+d}.$$

We put

$$g^{\psi} = \begin{pmatrix} a & b \\ 3c & d \end{pmatrix}$$

and

$$N(\tilde{\Phi}_6)(l)^{\psi} = \{g^{\psi} \mid g \in N(\tilde{\Phi}_6)(l)\}.$$

Then $N(\tilde{\Phi}_6)(l)^{\psi}$ is a subgroup of $SL(2, \mathbf{Z})$ which contains $\Gamma_1(3l)$ as a normal subgroup of index six (resp., nine) if $3 \nmid l$ (resp., $3 \mid l$). Let

$$F = N(\tilde{\Phi}_6)(l)^{\psi}/\Gamma_1(3l),$$

$$R = \tilde{\mathfrak{S}}_1^*(3l),$$

$$\bar{R} = F\backslash R,$$

and let π be the natural projection of R to $\bar{R}$, i the inclusion of Φ_6 to $\tilde{\mathfrak{S}}_2^*(l)$, and $\bar{\psi}$ the isomorphism of $\bar{R}$ to Φ_6. Then it is easily seen that

$$(i\cdot\bar{\psi}\cdot\pi)^*(\tilde{L}_2) = 2\tilde{L}_1, \tag{13}$$

and

$$i^*(\tilde{L}_2) = K_{\Phi_6} + \Delta(\Phi_6). \tag{14}$$

The boundary of Φ_6 intersects the boundary of $\tilde{\mathfrak{S}}_2^*(l)$ at the points which are equivalent to

$$\begin{pmatrix} \infty & -\infty \\ & \infty \end{pmatrix}$$

under the action of $G(l)$ (see Section 4). Therefore, we have

$$i^*(\Delta(2)) = 3\Delta(\Phi_6). \tag{15}$$

Let $N_6(2\pi/3)$ and $N_6(4\pi/3)$ be the subbundles of the normal bundle of Φ_6 corresponding to the eigenvalues ρ and ρ^2, respectively. Then it is easily seen that

$$\begin{aligned} N_6(2\pi/3) &= N_6(4\pi/3) \\ &= (-i^*(K_{\tilde{\mathfrak{S}}_2^*(l)}) + K_{\Phi_6})/2 \\ &= (-i^*(K_{\tilde{\mathfrak{S}}_2^*(l)} + \Delta(2)) + K_{\Phi_6} + 3\Delta(\Phi_6))/2 \\ &= -i^*(\tilde{L}_2) + \Delta(\Phi_6). \end{aligned} \tag{16}$$

It is easily proved as before that Φ_6 has $l^2\Pi(1 - p^{-2})$ (resp., $(3/4)l^2\Pi(1 - p^{-2})$) cusps if $3 \nmid l$ (resp., $3 \mid l$). Therefore, by (13)~(16) we can compute $\tau(\varphi_6, \Phi_6)$.

Proof of 7). We compute only $\tau(\varphi_7(1), \Phi_7)$. $\tau(\varphi_7(2), \Phi_7)$ is similarly computed. Φ_7 is isomorphic to $\tilde{\mathfrak{S}}_1^*(l)$. Let ψ be the isomorphism of $\tilde{\mathfrak{S}}_1^*(l)$ to Φ_7, i the inclusion of Φ_7 to $\tilde{\mathfrak{S}}_2^*(l)$, and j the inclusion of Φ_7 to Φ_2.

Let

$$\begin{pmatrix} z_1 & z_2 \\ & z_3 \end{pmatrix}$$

be the coordinate system of $\mathfrak{S}_2$. Then we have

$$d\varphi_7(1)\left(\left(\frac{\partial}{\partial z_1}\right)_p\right) = -\left(\frac{\partial}{\partial z_1}\right)_p,$$

and

$$d\varphi_7(1)\left(\left(\frac{\partial}{\partial z_2}\right)_p\right) = -i\left(\frac{\partial}{\partial z_2}\right)_p,$$

where $p \in \tilde{\Phi}_7$. Therefore, if we denote by $N_7(\pi)$ and $N_7(3\pi/2)$ the subbundles of the normal bundle of Φ_7 corresponding to the eigenvalues -1 and $-i$, respectively, then we have

$$N_7(\pi) = N_{\Phi_7/\Phi_2} = 0, \tag{17}$$

and

$$N_7(3\pi/2) = j^*(N_{\Phi_2/\tilde{\mathfrak{S}}_2^*(l)}) = -i^*(\tilde{L}_2). \tag{18}$$

Further, we have

$$(i\cdot\psi)^*(\tilde{L}_2) = \tilde{L}_1, \tag{19}$$

$$i^*(\Delta(2)) = \Delta(\Phi_7), \tag{20}$$

and

$$c_1(\Phi_7) = -2i^*(\tilde{L}_2) + \Delta(\Phi_7).$$

By (17) ~ (21) we can compute $\tau(\varphi_7(1), \Phi_7)$.

3. Fixed Subvarieites over Cusps of Degree One. 1. Let $\bar{\mathfrak{S}}_1^*(l)$ be (the closure of) one of the cusps of degree one in $\bar{\mathfrak{S}}_2^*(l)$. Let

$$\begin{pmatrix} z_1 & z_2 \\ & z_3 \end{pmatrix}$$

be the system of coordinates of $\mathfrak{S}_2$. Cusps of degree 1 in $\bar{\mathfrak{S}}_2^*(l)$ are equivalent to each other under the action of $G(l)$. We assume $\bar{\mathfrak{S}}_1^*(l)$ is defined by $\operatorname{Im} z_3 = \infty$. We denote by $P_{2,1}$ the subgroup of $\Gamma_2(1)$ consisting of elements which transform the rational boundary component of $\mathfrak{S}_2$ defined by $\operatorname{Im} z_3 = \infty$ to itself and $P_{2,1} \cap \Gamma_2(l)$ by $P_{2,1}(l)$. An element M of $P_{2,1}$ is uniquely decomposed as

$$M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & n \\ m & 1 & n & 0 \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}), \qquad u = \pm 1 \quad \text{and} \quad m, n, r \in \mathbf{Z}.$$

If M is in $P_{2,1}(l)$, then this decomposition is done in $P_{2,1}(l)$. M acts on $\mathfrak{S}_2$ as

$$M \cdot \begin{pmatrix} z_1 & z_2 \\ & z_3 \end{pmatrix} = \begin{pmatrix} \dfrac{az_1 + b}{cz_1 + d} & \dfrac{z_2 + mz_2 + n}{(cz_1 + d)u} \\ & z_3 + mz_2 + r + \dfrac{(mz_1 + z_2 + n)(dm - cz_2 - cn)}{cz_1 + d} \end{pmatrix}.$$

$s^{-1}(\bar{\mathfrak{S}}_1^*(l))$ has a structure of an elliptic surface. See [8] and [12] for the details. Let $p \in \mathfrak{S}_1^*(l)$, then $s^{-1}(p)$ has the structure of elliptic curve determined by the lattice $l(\tilde{p}\mathbf{Z} + \mathbf{Z})$, where $\tilde{p} \in \mathfrak{S}_1$ is a representative of p. Therefore, a point of $s^{-1}(\mathfrak{S}_1^*(l))$ is determined by p and $q \in \mathbf{C}/l(\tilde{p}\mathbf{Z} + \mathbf{Z})$. Let $\tilde{q} \in \mathbf{C}$ be a representative of q. We denote the point determined by p and q by

$$\begin{pmatrix} \tilde{p} & \tilde{q} \\ & \infty \end{pmatrix}$$

symbolically. If p is a cusp of $\bar{\mathfrak{S}}_1^*(l)$, then $s^{-1}(p)$ has a structure of an l-gon of rational curves.

Let M be as before. Then the element $(\pm 1)MP_{2,1}(l)$ of $P_{2,1}/(\pm 1)P_{2,1}(l)$ acts on $s^{-1}(\bar{\mathfrak{S}}_1^*(l))$ as

$$(\pm 1)MP_{2,1}(l)\cdot\begin{pmatrix} \tilde{p} & \tilde{q} \\ & \infty \end{pmatrix} = \begin{pmatrix} \dfrac{a\tilde{p} + b}{c\tilde{p} + d} & \dfrac{\tilde{q} + m\tilde{p} + n}{(c\tilde{p} + d)u} \\ & \infty \end{pmatrix},$$

and acts on the normal bundle of $s^{-1}(\bar{\mathfrak{S}}_1^*(l))$ as a multiplication by

$$\exp\left(2\pi i\left(mz_2 + r + \frac{(mz_1 + z_2 + n)(dm - cz_2 - cn)}{cz_1 + d}\right)\Big/ l\right).$$

Therefore, to classify the fixed subvarieties in $s^{-1}(\bar{\mathfrak{S}}_1^*(l))$, it suffices to consider the cases when

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \text{ or } \pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the case when l is odd, we denote by Φ_{17}^0

$$\bigcup_{z \in \mathfrak{S}_1}\left\{\begin{pmatrix} z & 1/2 \\ & \infty \end{pmatrix}, \begin{pmatrix} z & (lz+1)/2 \\ & \infty \end{pmatrix}, \begin{pmatrix} z & (lz+l+1)/2 \\ & \infty \end{pmatrix}\right\},$$

and in the case when l is even, we denote by Φ_{17}^0

$$\bigcup_{z\in\mathfrak{S}_1}\left\{\begin{pmatrix} z & 1/2 \\ & \infty \end{pmatrix}, \begin{pmatrix} z & (l+1)/2 \\ & \infty \end{pmatrix}, \begin{pmatrix} z & (lz+1)/2 \\ & \infty \end{pmatrix}, \begin{pmatrix} z & (lz+l+1)/2 \\ & \infty \end{pmatrix}\right\}.$$

Let Φ_{17} be the closure of Φ_{17}^0 in $s^{-1}(\bar{\mathfrak{S}}_1^*(l))$. Φ_{17} is an irreducible curve and has a structure of a covering space over $\bar{\mathfrak{S}}_1^*(l)$ of degree three (resp., four) if l is odd (resp., even) by s.

Φ_{17} does not ramify over $\mathfrak{S}_1^*(l)$. In the case when l is odd, Φ_{17} has two branches over a cusp of $\bar{\mathfrak{S}}_1^*(l)$, and one of them does not ramify, and the other ramifies with ramification index two. In the case when l is even, we can classify the cusps of $\bar{\mathfrak{S}}_1^*(l)$ into U, V and W as in the proof of Theorem (2.8), 3). Note that l here corresponds to $2l$ before. If $u \in U$, then Φ_{17} has four branches over u, and they do not ramify. If $v \in V$ and $w \in W$, then Φ_{17} has two branches over v and w, and they ramify with ramification index two. In both cases a branch which does not ramify intersects a side of an l-gon of rational curves, and a branch which ramifies intersects a vertex of an l-gon.

For a fixed subvariety Φ in $s^{-1}(\bar{\mathfrak{S}}_1^*(l))$ of an element of $G(l)$, we define $C_{G(l)}(\Phi)$, $C^p_{G(l)}(\Phi)$ and $N_{G(l)}(\Phi)$ as Definition (2.1). Now we have the following

THEOREM (3.1). *Fixed subvarieties in $s^{-1}(\bar{\mathfrak{S}}_1^*(l))$ of elements of $G(l)$ which intersect $s^{-1}(\mathfrak{S}_1^*(l))$ are classified as follows. Φ means the closure of Φ^0 in $s^{-1}(\bar{\mathfrak{S}}_1^*(l))$.*

	Φ	Φ^0	$\lvert C_{G(l)}(\Phi)\rvert$	$\lvert N_{G(l)}(\Phi)\rvert$
15)	Φ_{15}	$\left\{\begin{pmatrix} z_1 & z_2 \\ & \infty \end{pmatrix} \middle\vert \begin{matrix} z_1 \in \mathfrak{S}_1 \\ z_2 \in \mathbf{C} \end{matrix}\right\}$	l	$l^6\Pi(1-p^{-2})$
16)	Φ_{16}	$\left\{\begin{pmatrix} z & 0 \\ & \infty \end{pmatrix} \middle\vert z \in \mathfrak{S}_1\right\}$	$2l$	$l^4\Pi(1-p^{-2})$
17)	Φ_{17}		$2l$	$\begin{cases} l^4\Pi(1-p^{-2}), \text{ if } 2 \nmid l \\ (4/3)l^4\Pi(1-p^{-2}), \text{ if } 2 \mid l \end{cases}$
18)	Φ_{18}	$\begin{pmatrix} i & 0 \\ & \infty \end{pmatrix}$	$4l$	$4l$
19)	Φ_{19}	$\begin{pmatrix} i & (i+1)/2 \\ & \infty \end{pmatrix}$	$4l$	$4l$

20)	Φ_{20}	$\begin{pmatrix} \rho & 0 \\ & \infty \end{pmatrix}$	$6l$	$6l$
21)	Φ_{21}	$\begin{pmatrix} \rho & (\rho+2)/3 \\ & \infty \end{pmatrix}$	$3l$	$3l$

$|C_{G(l)}(\Phi)|$ and $|N_{G(l)}(\Phi)|$ in this theorem are easily determined. In the following theorem we list only the proper elements of $C_{G(l)}(\Phi)$. We indicate the eigenvalues $e^{i\theta}$ of the normal bundle of Φ by θ.

THEOREM (3.2). *The proper elements φ of $C_{G(l)}(\Phi)$ are as follows:*

	φ	Representative of φ in $\Gamma_2(1)$	The condition φ to be proper	θ	$\det(CZ+D)$ for $Z \in \Phi$
15)	$\varphi_{15}(r)$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	$2\pi r/l$	1
16)	$\varphi_{16}(r)$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	π, $2\pi r/l$	-1
17)	$\varphi_{17}(r)$	$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & r \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	π, $2\pi r/l$	-1
18)	$\varphi_{18}(1, r)$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & r \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	$\pi/2$, π, $2\pi r/l$	$-i$
	$\varphi_{18}(2, r)$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & r \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	$\pi, 3\pi/2, 2\pi r/l$	i

19)	$\varphi_{19}(1,r)$	$\begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & r \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$		$\pi/2,\ \pi,$ $\pi(2r-1)/l$	$-i$
	$\varphi_{19}(2,r)$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & r \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$		$\pi,\ 3\pi/2,$ $\pi(2r+1)/l$	i
20)	$\varphi_{20}(1,r)$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & r \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	$2\pi/3,\ 4\pi/3,$ $2\pi r/l$	ρ^2
	$\varphi_{20}(2,r)$	$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & r \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	$2\pi/3,\ 4\pi/3,$ $2\pi r/l$	ρ
	$\varphi_{20}(3,r)$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & r \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	$4\pi/3,\ 5\pi/3,$ $2\pi r/l$	$-\rho^2$
	$\varphi_{20}(4,r)$	$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & r \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$r \not\equiv 0 \bmod l$	$\pi/3,\ 2\pi/3,$ $2\pi r/l$	$-\rho$
21)	$\varphi_{21}(1,r)$	$\begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & -1 & r \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$		$2\pi/3,\ 4\pi/3,$ $2\pi(3r-1)/3l$	ρ^2

$\varphi_{21}(2,r)$	$\begin{pmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & r \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$2\pi/3,\ 4\pi/3,$ $2\pi(3r+1)/3l$	ρ

2. THEOREM (3.3).

15) $$\tau(\varphi_{15}(r), \Phi_{15}) = 2^{-3}3^{-1}\left(\frac{9-(2k-3)l}{(1-\zeta^r)} + \frac{(2k-3)l-6}{(1-\zeta^r)^2} - \frac{4}{(1-\zeta^r)^3}\right)l^3\Pi(1-p^{-2}).$$

16) $$\tau(\varphi_{16}(r), \Phi_{16}) = 2^{-5}3^{-1}(-1)^k\left(\frac{12-(2k-3)l}{1-\zeta^r}\right)l^2\Pi(1-p^{-2}).$$

17) $$\tau(\varphi_{17}(r), \Phi_{17}) = \begin{cases} 2^{-5}(-1)^k\left(\dfrac{8-(2k-3)l}{(1-\zeta^r)} + \dfrac{4}{(1-\zeta^r)^2}\right)l^2\Pi(1-p^{-2}), & \text{if } 2 \nmid l, \\ 2^{-3}3^{-1)^{k}}\left(\dfrac{8-(2k-3)l}{(1-\zeta^r)} + \dfrac{4}{(1-\zeta^r)^2}\right)l^2\Pi(1-p^{-2}), & \text{if } 2 \mid l. \end{cases}$$

18) $$\tau(\varphi_{18}(1,r), \Phi_{18}) = 2^{-2}(-i)^k(1-i)(\zeta^r-1)^{-1}.$$

$$\tau(\varphi_{18}(2,r), \Phi_{18}) = 2^{-2}(i)^k(1+i)(\zeta^r-1)^{-1}.$$

19) $$\tau(\varphi_{19}(1,r), \Phi_{19}) = 2^{-2}(-i)^k(1-i)(\exp(\pi i(2r-1)/l)-1)^{-1}.$$

$$\tau(\varphi_{19}(2,r), \Phi_{19}) = 2^{-2}(i)^k(1+i)(\exp(\pi i(2r+1)/l)-1)^{-1}.$$

20) $$\tau(\varphi_{20}(1,r), \Phi_{20}) = 3^{-1}(\rho^2)^k(\zeta^r-1)^{-1}.$$

$$\tau(\varphi_{20}(2,r), \Phi_{20}) = 3^{-1}(\rho)^k(\zeta^r-1)^{-1}.$$

$$\tau(\varphi_{20}(3,r), \Phi_{20}) = 3^{-1}(-\rho^2)^k(1+2\rho)(\zeta^r-1)^{-1}.$$

$$\tau(\varphi_{20}(4,r), \Phi_{20}) = 3^{-1}(-\rho)^k(1+2\rho^2)(\zeta^r-1)^{-1}.$$

21) $$\tau(\varphi_{21}(1,r), \Phi_{21}) = 3^{-1}(\rho^2)^k(\exp(2\pi i(3r-1)/3l)-1)^{-1}.$$

$$\tau(\varphi_{21}(2,r), \Phi_{21}) = 3^{-1}(\rho)^k(\exp(2\pi i(3r+1)/3l)-1)^{-1}.$$

Proof. We prove only 15) and 17). 16) is similarly proved, and the others are easily proved.

Proof of 15). Let i be the inclusion of Φ_{15} to $\tilde{\mathfrak{S}}_2^*(l)$. Then we have

$$i^*(\tilde{L}_2) = (s \mid \Phi_{15})^*(\tilde{L}_1),$$

$$i^*(\Delta(2)) = i^*(\Phi_{15}) + \Delta(\Phi_{15}), \tag{21}$$

and

$$\Delta(\Phi_{15}) = (s \mid \Phi_{15})^*(\Delta(1)).$$

Therefore, we have

$$(i^*(\tilde{L}_2))^2[\Phi_{15}] = (i^*(\tilde{L}_2)\cdot\Delta(\Phi_{15}))[\Phi_{15}] = (\Delta(\Phi_{15}))^2[\Phi_{15}] = 0. \tag{22}$$

By [16], Remark (3.5), Lemma (4.) and Lemma (6.8), we have

$$(i^*(\Phi_{15})\cdot\Delta(\Phi_{15}))[\Phi_{15}] = -l^3\Pi(1-p^{-2}), \tag{23}$$

$$(i^*(\tilde{L}_2\cdot\Phi_{15}))[\Phi_{15}] = -(1/12)l^4\Pi(1-p^{-2}), \tag{24}$$

and

$$(i^*(\Phi_{15}))^2[\Phi_{15}] = (1/6)l^3\Pi(1-p^{-2}), \tag{25}$$

It is easily seen that

$$(\Delta_2(\Phi_{15}))[\Phi_{15}] = (1/2)l^3\Pi(1-p^{-2}). \tag{26}$$

Further, by [16], Proposition (1.2), Lemma (5.1) and Remark (5.4), we have

$$c_1(\Phi_{15}) = \bar{c}_1(\Phi_{15}^0) + \Delta(\Phi_{15}) = (s \mid \Phi_{15})^*(-3\tilde{L}_1 + \Delta(1)), \tag{27}$$

$$c_2(\Phi_{15}) = \bar{c}_2(\Phi_{15}^0) + \bar{c}_1(\Phi_{15}^0)\cdot\Delta(\Phi_{15}) + \Delta_2(\Phi_{15}) = \Delta_2(\Phi_{15}), \tag{28}$$

and

$$N_{\Phi_{15}/\tilde{\mathfrak{S}}_2^*(l)} = i^*(\Phi_{15}). \tag{29}$$

Since $\varphi_{15}(r)$ acts on the normal bundle of Φ_{15} as a multiplication by ζ^r, we have

$$(30) \qquad ch((k\tilde{L}_2 - \Delta(2)) \mid \Phi_{15})(\varphi_{15}(r)) = \zeta^{-r} ch((k\tilde{L}_2 - \Delta(2)) \mid \Phi_{15}).$$

The assertion is proved by (21) ~ (30).

Proof of 17). We prove only the case when l is odd. Let i be the inclusion of Φ_{17} to $\tilde{\mathfrak{S}}_2^*(l)$, j the inclusion of Φ_{17} to Φ_{15}, and $N_{17}(\pi)$ and $N_{17}(2\pi r/l)$ the subbundles of the normal bundle of Φ_{17} corresponding to the eigenvalue -1 and ζ^r, respectively. Then we have

$$(31) \qquad i^*(\tilde{L}_2) = (s \mid \Phi_{17})^*(\tilde{L}_1),$$

$$(32) \qquad c_1(\Phi_{17}) = -2i^*(\tilde{L}_2) + \Delta(\Phi_{17}),$$

$$(33) \qquad i^*(\Delta(2)) = i^*(\Phi_{15}) + j^*(\Delta(\Phi_{15})),$$

$$(34) \qquad \begin{aligned} N_{17}(\pi) &= -j^*(K_{\Phi_{15}}) + K_{\Phi_{17}} \\ &= -i^*(\tilde{L}_2) + j^*(\Delta(\Phi_{15})) - \Delta(\Phi_{17}), \end{aligned}$$

and

$$(35) \qquad N_{17}(2\pi r/l) = i^*(\Phi_{17}).$$

By (1), (2) in the proof of Theorem (2.8) 3), we have

$$(36) \qquad i^*(\Phi_{15})[\Phi_{17}] = -(1/4)l^2\Pi(1 - p^{-2}),$$

and it is easily seen that

$$(37) \qquad (\Delta(\Phi_{17}))[\Phi_{17}] = l^2\Pi(1 - p^{-2}),$$

and

$$(38) \qquad (j^*(\Delta(\Phi_{15})))[\Phi_{17}] = (3/2)l^2\Pi(1 - p^{-2}).$$

The assertion is proved by (31)-(38).

4. Fixed Subvarieties over Cusps of Degree Zero. 1. Let p be one of the cusps of degree 0 in $\bar{\mathfrak{S}}_2^*(l)$. Let

$$\begin{pmatrix} z_1 & z_2 \\ & z_3 \end{pmatrix}$$

be the coordinate system of $\mathfrak{S}_2$. Cusps of degree 0 in $\bar{\mathfrak{S}}_2^*(l)$ are equivalent to each other under the action of $G(l)$. Therefore, we assume that p is defined by $\operatorname{Im} z_1 = \operatorname{Im} z_3 = \infty$. $s^{-1}(p)$ is a reducible rational variety composed of $(1/4)l^3\Pi(1 - p^{-2})$ projective lines meeting three at each one of the $(1/6)l^3\Pi(1 - p^{-2})$ vertexes ([8]). These projective lines are equivalent to each other, and these vertexes are equivalent to each other, under the action of $G(l)$. Therefore, it suffices to consider a single projective line and a single vertex to classify the fixed subvarieties in $s^{-1}(p)$.

We denote by $P_{2,0}$ the subgroup of $\Gamma_2(1)$ consisting of elements which fix the rational boundary component of $\mathfrak{S}_2$ defined by $\operatorname{Im} z_1 = \operatorname{Im} z_3 = \infty$ and $P_{2,0} \cap \Gamma_2(l)$ by $P_{2,0}(l)$. An element M of $P_{2,0}$ is written as

$$M = \begin{pmatrix} U & R\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix},$$

where $U \in GL(2, \mathbf{Z})$, and R is a symmetric matrix with integral coefficients. Let $Q_{2,0}$ be the subgroup of $P_{2,0}$ consisting of elements such that $U = 1_2$, and let $Q_{2,0} \cap \Gamma_2(l) = Q_{2,0}(l)$. Let e be the map:

$$\mathfrak{S}_2 \to Q_{2,0}(l)\backslash\mathfrak{S}_2$$

$$\begin{pmatrix} z_1 & z_2 \\ & z_3 \end{pmatrix} \mapsto \begin{pmatrix} Z_1 & Z_2 \\ & Z_3 \end{pmatrix} = \begin{pmatrix} \exp(2\pi i z_1/l) & \exp(2\pi i z_2/l) \\ & \exp(2\pi i z_3/l) \end{pmatrix}.$$

Let $\mathfrak{X}_2$ be the torus embedding corresponding to the Delony-Voronoi decomposition of degree two ([12]). $\mathfrak{X}_2$ has $Q_{2,0}(l)\backslash\mathfrak{S}_2$ as an open subset, and the action of $P_{2,0}/Q_{2,0}(l)$ extends to $\mathfrak{X}_2$. $P_{2,0}(l)/(\pm 1)Q_{2,0}(l)$ acts on $\mathfrak{X}_2$ without fixed points. Let π be the natural projection of $\mathfrak{X}_2$ to $P_{2,0}(l)\backslash\mathfrak{X}_2$. The neighborhood of $s^{-1}(p)$ is constructed as $P_{2,0}(l)\backslash\mathfrak{X}_2$.

Let Σ be the central cone in the Delony-Voronoi decomposition ([8]

and [12]). $\mathfrak{X}_\Sigma$ = Spec $\mathbf{C}[Z_1Z_2, Z_2^{-1}, Z_2Z_3]$ has a structure of an affine open subset of $\mathfrak{X}_2$ ([9]). The neighborhood of the affine line defined by $Z_1Z_2 = Z_2Z_3 = 0$ is mapped isomorphically into $P_{2,0}\backslash\mathfrak{X}_2$ by π. Therefore, it suffices to consider this affine line instead of the projective line in $s^{-1}(p)$ and the origin $(Z_1Z_2, Z_2^{-1}, Z_2Z_3) = (0, 0, 0)$ instead of the vertex. We denote the point $(Z_1Z_2, Z_2^{-1}, Z_2Z_3) = (0, \exp(-2\pi i\, z_2/l), 0)$ on the affine line by

$$\begin{pmatrix} \infty & z_2 \\ & \infty \end{pmatrix}$$

symbolically. Therefore, the origin is written as

$$\begin{pmatrix} \infty & -\infty \\ & \infty \end{pmatrix}$$

symbolically.

Let

$$M = \begin{pmatrix} U & R\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix} \in P_{2,0},$$

where

$$U \in GL(2, \mathbf{Z}) \quad \text{and} \quad R = \begin{pmatrix} r & s \\ s & t \end{pmatrix}.$$

We denote the element $(\pm 1)MQ_{2,0}(l)$ of $P_{2,0}/(\pm 1)Q_{2,0}(l)$ by

$$U(r, s, t).$$

M acts on $Z \in \mathfrak{S}_2$ as

$$Z \mapsto UZ\,{}^tU + R,$$

and $(\pm 1)MQ_{2,0}(l)$ acts on $\mathfrak{X}_\Sigma$ by this action through π.

Let U_1, U_2, U_3 and U_4 be

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively, and let $V_1, V_2, \ldots, V_6$ be

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix},$$

respectively. Then by [8] the elements of $P_{2,0}/(\pm 1)Q_{2,0}(l)$ which transform the affine line $Z_1Z_2 = Z_2Z_3 = 0$ to itself are

$$U_i(r, s, t) \quad (i = 1, 2, 3, 4),$$

and the elements of $P_{2,0}/(\pm 1)Q_{2,0}(l)$ which fix the origin $Z_1Z_2 = Z_2^{-1} = Z_2Z_3 = 0$ are

$$V_i(r, s, t) \quad (i = 1, 2, \ldots, 6).$$

By this result we can determine the fixed subvarieties in $s^{-1}(p)$ and their isotropy groups. Let Φ_{22}^0 be

$$\pi\left\{\begin{pmatrix} \infty & z \\ & \infty \end{pmatrix} \middle| z \in \mathbf{C}\right\}$$

and Φ_{22} the closure of Φ_{22}^0 in $s^{-1}(p)$.

THEOREM (4.1). *The fixed subvarieties in $s^{-1}(p)$ are classified as follows.*

	Φ	$\lvert C_{G(l)}(\Phi)\rvert$	$\lvert N_{G(l)}(\Phi)\rvert$
22)	Φ_{22}	$2l^2$	$4l^3$
23)	$\Phi_{23} = \pi\left\{\begin{pmatrix} \infty & 0 \\ & \infty \end{pmatrix}\right\}$	$4l^2$	$4l^2$

24)	$\Phi_{24} = \pi\left\{\begin{pmatrix} \infty & 1/2 \\ & \infty \end{pmatrix}\right\}$	$4l^2$	$4l^2$
25)	$\Phi_{25} = \pi\left\{\begin{pmatrix} \infty & -\infty \\ & \infty \end{pmatrix}\right\}$	$6l^3$	$6l^3$

In the following theorem we list only the proper elements of $C_{G(l)}(\Phi)$. We indicate the eigenvalues $e^{i\theta}$ by $e^{i\theta}$.

THEOREM (4.2). *The proper elements φ of $C_{G(l)}(\Phi)$ are as follows.*

	φ	Representative of φ in $P_{2,0}/(\pm 1)Q_{2,0}(l)$	The condition φ to be proper	$e^{i\theta}$
22)	$\varphi_{22}(1, r, t)$	$U_1(r, 0, t)$	$rt \not\equiv 0 \bmod l$	ζ^r, ζ^t
	$\varphi_{22}(3, r, t)$	$U_3(r, 0, t)$	$r + t \not\equiv 0 \bmod l$	$\pm\zeta^{(r+t)/2}$
23)	$\varphi_{23}(2, r, t)$	$U_2(r, 0, t)$	$r + t \not\equiv 0 \bmod l$	$\pm\zeta^{(r+t)/2}$
	$\varphi_{23}(4, r, t)$	$U_4(r, 0, t)$	$rt \not\equiv 0 \bmod l$	ζ^r, ζ^t
24)	$\varphi_{24}(2, r, t)$	$U_2(r, -1, t)$	$r + t \not\equiv 0 \bmod l$	$\pm\zeta^{(r+t)/2}$
	$\varphi_{24}(4, r, t)$	$U_4(r, -1, t)$	$rt \not\equiv 0 \bmod l$	ζ^r, ζ^t
25)	$\varphi_{25}(1, r, s, t)$	$V_1(r, s, t)$	$(r + s)(s + t)s \not\equiv 0 \bmod l$	$\zeta^{r+s}, \zeta^{s+t}, \zeta^{-s}$
	$\varphi_{25}(2, r, s, t)$	$V_2(r, s, t)$	$r + s + t \not\equiv 0 \bmod l$	$\rho^{\alpha}\zeta^{(r+s+t)/3}\ (\alpha = 0, 1, 2)$
	$\varphi_{25}(4, r, s, t)$	$V_4(r, s, t)$	$(r + 2s + t)s \not\equiv 0 \bmod l$	$\zeta^{-s}, \pm\zeta^{(r+2s+t)/2}$

$\det(CZ + D)$ for $Z \in \Phi$ is equal to $\det(D)$, since $C = 0$.

Remark (4.3). $C_{G(l)}(\Phi_{25})$ has other proper elements whose representatives are $V_i(r, s, t)$ $(i = 3, 5, 6)$. We omitted them in the above theorem and the following theorem, since they are conjugate to $\varphi_{25}(i, r, s, t)$ $(i = 2$ or $4)$ in $C_{G(l)}(\Phi_{25})$. It suffices to double (resp. treble) the contribution of $\varphi_{25}(2, r, s, t)$ (resp., $\varphi_{25}(4, r, s, t)$) in the dimension formula (Theorem (5.2), (49)) instead of them.

2. Theorem (4.4).

22) $$\tau(\varphi_{22}(1, r, t), \Phi_{22}) = \frac{1}{(\zeta^r - 1)(\zeta^t - 1)}\left\{\frac{2}{(\zeta^r - 1)} + \frac{2}{(\zeta^t - 1)} + 3\right\}.$$

$$\tau(\varphi_{22}(2, r, t), \Phi_{22}) = \frac{(-1)^k}{(\zeta^{r+t} - 1)}\left\{\frac{4}{(\zeta^{r+t} - 1)} + 3\right\}.$$

23) and 24) $$\tau(\varphi_{23}(2, r, t), \Phi_{23}) = \tau(\varphi_{24}(2, r, t), \Phi_{24}) = 2^{-1}(\zeta^{r+t} - 1)^{-1}.$$

$$\tau(\varphi_{23}(4, r, t), \Phi_{23}) = \tau(\varphi_{24}(4, r, t), \Phi_{24}) = 2^{-1}(-1)^k(\zeta^r - 1)^{-1}(\zeta^t - 1)^{-1}.$$

25) $$\tau(\varphi_{25}(1, r, s, t), \Phi_{25}) = (\zeta^{r+s} - 1)^{-1}(\zeta^{s+t} - 1)^{-1}(\zeta^{-s} - 1)^{-1}.$$

$$\tau(\varphi_{25}(2, r, s, t), \Phi_{25}) = (\zeta^{r+s+t} - 1)^{-1}.$$

$$\tau(\varphi_{25}(4, r, s, t), \Phi_{25}) = (-1)^k(\zeta^{r+2s+t} - 1)^{-1}(\zeta^{-s} - 1)^{-1}.$$

Proof. We prove only 22). Others are easily proved.

Proof of 22). We compute only $\tau(\varphi_{22}(1, r, t), \Phi_{22})$. Let i be the inclusion of Φ_{22} to $\tilde{\mathfrak{S}}_2^*(l)$, and $N_{22}(2\pi/l)$ and $N_{22}(2\pi t/l)$ the subbundles of the normal bundle of Φ_{22} corresponding to the eigenvalues ζ^r and ζ^t, respectively. By [16], Theorem (2.4) and Remark (3.5), we have

$$i^*(\tilde{L}_2)[\Phi_{22}] = 0, \tag{39}$$

$$N_{22}(2\pi r/l)[\Phi_{22}] = -2, \tag{40}$$

$$N_{22}(2\pi t/l)[\Phi_{22}] = -2, \tag{41}$$

and

$$i^*(\Delta(2))[\Phi_{22}] = -2. \tag{42}$$

Further, we have

$$c_1(\Phi_{22})[\Phi_{22}] = 2, \tag{43}$$

and

$$ch((k\tilde{L}_2 - \Delta(2))\,|\,\Phi_{22})(\varphi_{22}(1, r, t)) = \zeta^{-r-t} ch((k\tilde{L}_2 - \Delta(2))\,|\,\Phi_{22}). \tag{44}$$

The assertion is proved by (39)-(44).

5. The Dimension Formula. Let $g \in G(l)$. We compute the trace of the action of g on $S_k(\Gamma_2(l))$. Fix(g) has $M \cdot \Phi_\alpha$ ($M \in G(l)$, $\alpha = 1, \ldots, 25$) as an irreducible component, if and only if

$$M^{-1} \cdot g \cdot M = \varphi,$$

for some $\varphi \in C^p_{G(l)}(\Phi_\alpha)$. If $M' \in C_{G(l)}(\varphi)$, then we have

$$(MM')^{-1} \cdot g \cdot (MM') = \varphi.$$

Therefore, g also fixes $MM' \cdot \Phi_\alpha$. The number of irreducible components of Fix(g) on which g acts as φ is

$$\frac{|C_{G(l)}(\varphi)|}{|C_{G(l)}(\varphi) \cap N_{G(l)}(\Phi_\alpha)|} = \frac{|C_{G(l)}(\varphi)|}{|N_{G(l)}(\Phi_\alpha)|} \cdot \frac{|N_{G(l)}(\Phi_\alpha)|}{|C_{G(l)}(\varphi) \cap N_{G(l)}(\Phi_\alpha)|} \tag{45}$$

The map

$$N_{G(l)}(\Phi_\alpha) \to C_{G(l)}(\Phi_\alpha)$$
$$g \mapsto g^{-1} \cdot \varphi \cdot g$$

induces an injection of $(C_{G(l)}(\varphi) \cap N_{G(l)}(\Phi_\alpha)) \backslash N_{G(l)}(\Phi_\alpha)$ to $C_{G(l)}(\Phi_\alpha)$ (Lemma (2.4)). The image of this map consists of elements of $C_{G(l)}(\Phi_\alpha)$ which are conjugate to φ in $N_{G(l)}(\Phi_\alpha)$. But as a result we can see that the image of this map consists of elements of $C_{G(l)}(\Phi_\alpha)$ which are conjugate to φ in $C_{G(l)}(\Phi_\alpha)$. Therefore, the value in (45) is equal to

$$\frac{|C_{G(l)}(\varphi)|}{|N_{G(l)}(\Phi_\alpha)|} \cdot \frac{|C_{G(l)}(\Phi_\alpha)|}{|C(\varphi)|},$$

where $C(\varphi)$ is the centralizer of φ in $C_{G(l)}(\Phi_\alpha)$. Let $Cl(C^p_{G(l)}(\Phi_\alpha))$ be the set of representatives of conjugacy classes of $C_{G(l)}(\Phi_\alpha)$ contained in $C^p_{G(l)}(\Phi_\alpha)$. We denote the conjugacy relation in $G(l)$ by $\sim$. We proved the following

THEOREM (5.1). *If $k \geq 4$, then*

$\operatorname{Trace}(g \mid S_k(\Gamma_2(l))) = \tau(g)$

$$= \sum_{\alpha=1}^{25} \sum_{\substack{\varphi \in Cl(C^p_{G(l)}(\Phi_\alpha)) \\ \varphi \sim g}} \frac{|C_{G(l)}(\varphi)|}{|N_{G(l)}(\Phi_\alpha)|} \cdot \frac{|C_{G(l)}(\Phi_\alpha)|}{|C(\varphi)|} \tau(\varphi, \Phi_\alpha) \tag{46}$$

$$= \sum_{\alpha=1}^{25} \sum_{\substack{\varphi \in C^p_{G(l)}(\Phi_\alpha) \\ \varphi \sim g}} \frac{|C_{G(l)}(\varphi)|}{|N_{G(l)}(\Phi_\alpha)|} \tau(\varphi, \Phi_\alpha). \tag{47}$$

Let $\Gamma \supset \Gamma_2(l)$ be as in the Introduction. We compute the dimension of $S_k(\Gamma)$. Let $g_1, \ldots, g_h$ be the representatives of conjugacy classes of $G(\Gamma)$. Then we have if $k \geq 4$, then

$$\dim S_k(\Gamma) = \frac{1}{|G(\Gamma)|} \sum_{g \in G(\Gamma)} \tau(g)$$

$$= \sum_{i=1}^{h} \frac{1}{|C_{G(\Gamma)}(g_i)|} \tau(g_i).$$

Therefore we have the following

THEOREM (5.2). *If $k \geq 4$, then*

$\dim S_k(\Gamma)$

(48)

$$= \sum_{i=1}^{h} \frac{1}{|C_{G(\Gamma)}(g_i)|} \sum_{\alpha=1}^{25} \sum_{\substack{\varphi \in Cl(C^p_{G(l)}(\Phi_\alpha)) \\ \varphi \sim g_i}} \frac{|C_{G(l)}(\varphi)| \cdot |C_{G(l)}(\Phi_\alpha)|}{|N_{G(l)}(\Phi_\alpha)| \cdot |C(\varphi)|} \tau(\varphi, \Phi_\alpha)$$

(49)

$$= \sum_{i=1}^{h} \frac{1}{|C_{G(\Gamma)}(g_i)|} \sum_{\alpha=1}^{25} \sum_{\substack{\varphi \in C^p_{G(l)}(\Phi_\alpha) \\ \varphi \sim g_i}} \frac{|C_{G(l)}(\varphi)|}{|N_{G(l)}(\Phi_\alpha)|} \tau(\varphi, \Phi_\alpha).$$

The conjugacy classes of $\Gamma_2(1)/\Gamma_2(p)$ (p = prime) are classified by [18]. If $l = \Pi p$ is square free, then we have

$$\Gamma_2(1)/\Gamma_2(l) \simeq \Pi(\Gamma_2(1)/\Gamma_2(p)).$$

Therefore, $|C_{G(l)}(\varphi)|$ is determined if l is square free.

Now we can compute $\dim S_k(\Gamma_2(1))$ and $\dim S_k(\Gamma_2(2))$.

LEMMA (5.3).

i) $\Sigma_{r=1}^{l-1}(1-\zeta^r)^{-1} = (1/2)(l-1)$.
ii) $\Sigma_{r=1}^{l-1}(1-\zeta^r)^{-2} = -(1/12)(l-1)(l-5)$.
iii) $\Sigma_{r=1}^{l-1}(1-\zeta^r)^{-3} = -(1/8)(l-1)(l-3)$.

Example (5.4). By the above theorem we have if $k \geq 4$, then

$$\dim S_k(\Gamma_2(1)) = \sum_{\alpha=1}^{25} \sum_{\varphi \in C_{G(l)}^{p}(\Phi_\alpha)} \frac{\tau(\varphi, \Phi_\alpha)}{|N_{G(l)}(\Phi_\alpha)|}$$

By Lemma (5.3) and a rather complicated computation, we can see that this is equal to the coefficient of t^k in

$$\frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} - \frac{1}{(1-t^4)(1-t^6)}.$$

Let $A_k(\Gamma)$ be the vector space of Siegel modular forms of weight k with respect to Γ. Then by the surjectivity of Φ-operator ([15]), we have if $k \geq 5$, then $\dim A_k(\Gamma_2(1))$ is equal to the coefficient of t^k in

$$\frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

This coincides with the result of [7], and we can determine the structure of the graded ring:

$$\bigoplus_{k=0}^{\infty} A_k(\Gamma_2(1))$$

by this result ([7]).

Example (5.5). Let Γ be $\Gamma_2(2)$ and $l\,(\geq 3)$ an odd integer. We study the action of

$$G(\Gamma) = \Gamma_2(2)/(\pm 1)\Gamma_2(2l)$$

on $S_k(\Gamma_2(2l))$. By the isomorphism

$$f: G(2l) = \Gamma_2(1)/(\pm 1)\Gamma_2(2l) \xrightarrow{\sim} \Gamma_2(1)/\Gamma_2(2) \oplus \Gamma_2(1)/(\pm 1)\Gamma_2(l)$$

$G(\Gamma)$ is isomorphic to

$$\{1\} \oplus \Gamma_2(1)/(\pm 1)\Gamma_2(l).$$

If $\varphi \in G(2l)$, then we denote $f(\varphi)$ by (φ mod 2, φ mod l). φ is conjugate to an element of $G(\Gamma)$ if and only if φ mod $2 = 1$, and in such a case we have

$$C_{G(2l)}(\varphi) \simeq \Gamma_2(1)/\Gamma_2(2) \oplus C_{G(l)}(\varphi \bmod l),$$

and

$$C_{G(\Gamma)}(\varphi) \simeq \{1\} \oplus C_{G(l)}(\varphi \bmod l).$$

Therefore, it follows that

$$\frac{|C_{G(2l)}(\varphi)|}{|C_{G(\Gamma)}(\varphi)|} = |\Gamma_2(1)/\Gamma_2(2)| = 720.$$

Hence we have if $k \geq 4$, then

$$\dim S_k(\Gamma_2(2)) = 720 \sum_{\alpha=1}^{25} \sum_{\substack{\varphi \in C^p_{G(2l)}(\Phi_\alpha) \\ \varphi \bmod 2 = 1}} \frac{\tau(\varphi, \Phi_\alpha)}{|N_{G(2l)}(\Phi_\alpha)|}.$$

This is easily computed and equal to

$$(1/24)(2k^3 - 9k^2 - 17k + 84) + (5/8)(-1)^k(k^2 - 9k + 20).$$

It is known that $\bar{\mathfrak{S}}_2^*(2)$ has fifteen cusps of degree 1 and fifteen cusps of degree 0 ([14]), and if k (≥ 4) is even, then

$$\dim S_k(\Gamma_1(2)) = (1/2)(k - 4).$$

Therefore, by [15] if k (≥ 6) is even, then

$$\begin{aligned}\dim A_k(\Gamma_2(2)) &\doteq \dim S_k(\Gamma_2(2)) + 15 \dim S_k(\Gamma_1(2)) + 15 \\ &= (1/12)(k^3 + 3k^2 + 14k + 12).\end{aligned}$$

If k (≥ 5) is odd, then

$$\begin{aligned}\dim A_k(\Gamma_2(2)) &= \dim S_k(\Gamma_2(2)) \\ &= (1/12)(k^3 - 12k^2 + 59k - 108).\end{aligned}$$

These results coincide with that of [7].

Remark (5.6). Let χ be a homomorphism of Γ to $\mathbf{C}^*$ such that $\operatorname{Ker}(\chi) = \Gamma'$ is a subgroup of finite index of Γ. We denote by $S_k(\Gamma, \chi)$ the vector space of Siegel cusp forms of weight k and character χ with respect to Γ. Let l (≥ 3) be an integer such that $\Gamma' \supset \Gamma_2(l)$. We can obtain the similar result about $\dim S_k(\Gamma, \chi)$ by studying the action of $\Gamma/\Gamma_2(l)$ on $(\tilde{\mathfrak{S}}_2^*(l), \mathcal{O}(k\tilde{L}_2 - \Delta(2)))$.

Remark (5.7). Let $\Gamma_0(l)$ be the subgroup of $\Gamma_2(l)$ defined by

$$\Gamma_0(l) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| C \equiv 0 \bmod l \right\},$$

and let χ be a character of $(\mathbf{Z}/l\mathbf{Z})^*$. For an element

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of $\Gamma_0(l)$, we define $\chi(M)$ by

$$\chi(M) = \chi(\det(A) \bmod l).$$

Then χ is a character of $\Gamma_0(l)$ whose kernel contains $\Gamma_2(l)$. If l is square free, we can represent $\dim S_k(\Gamma_0(l), \chi)$ explicitly.

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CORRECTIONS TO [16]

P. 941, line 3: "$\bar{c}_1{}^{-2}(3)$" should read "$\bar{c}_1(3)^2$".
P. 949, line 3: The (3,3) coefficient of the third matrix should be 0.
P. 949, line 10: The (3,3) coefficient of the second matrix should be -1.
P. 949, line 1 from bottom: "α_2, β_3, α_2" should read "α_2, β_3, α_3".
P. 950, line 2: "α_1, γ_2, α_1" should read "α_1, γ_2, α_3".
P. 962, line 4 and 8: "$\mathcal{O}_{D_i}(D_i)$" should read "$j_{i*}\mathcal{O}_{D_i}(D_i)$",
P. 962, line 5 and 9: "$\mathcal{O}_{D_i'}(D_i')$" should read "$j_{i*}'\mathcal{O}_{D_i'}(D_i')$",
P. 962, line 10: "$ch(\mathcal{O}_{\bar{D}}(\bar{D}))_{|\bar{D}}$" should read "$ch(j_*\mathcal{O}_{\bar{D}}(\bar{D}))_{|\bar{D}} + ch(\mathcal{O}_{\bar{D}})$",
P. 962, line 12: The right hand side should be added "$ch(\mathcal{O}_{\bar{D}})$", where $j_i: D_i \to \bar{X}$, $j_i': D_i' \to \bar{D}$ and $j: \bar{D} \to \bar{X}$ mean the inclusions, and $|\bar{D}$ means the pullback by j. There exists the relation:

$$ch(\mathcal{O}_{\bar{D}}(\bar{D})) = ch(j_*\mathcal{O}_{\bar{D}}(\bar{D}))_{|\bar{D}} + ch(\mathcal{O}_{\bar{D}}).$$

These errors do not affect the statement of Lemma (5.1).
P. 967, line 7: "$((p_0 + m_0'/2)z + m_0/2)$" should read "$((p_3 + m_3'/2)z + m_0''/2)$".
P. 973, line 2: "$7(12)^{-1}$" should read "$7(18)^{-1}$".
P. 973, line 18: "$7(12)^{-1}$" should read "$7(18)^{-1}$".
P. 975, line 1 from bottom: "p^4" should read "p^{-4}".
P. 976, line 7: "$4\bar{c}_1\Delta_1\Delta_2{}^2$" should read "$3\bar{c}_1\Delta_1\Delta_2{}^2$".