An explicit formula for the dimension of spaces of Siegel modular forms of degree two

By Yasuo MORITA*

§ 0. Introduction

0.1. The main result. First we shall summarize the main result of this paper. Let $\mathfrak{H}$ be the Siegel upper-half-plane of degree two: $\mathfrak{H}=\{Z\in M_2(\boldsymbol{C})\mid {}^tZ=Z,\ \mathrm{Im}(Z)>0\}$. Let $Sp(2, \boldsymbol{R})$ be the symplectic group of degree two. Then an element γ of $Sp(2, \boldsymbol{R})/\pm 1$ operates on $\mathfrak{H}$ by

$$\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}: Z \longrightarrow (AZ+B)(CZ+D)^{-1}.$$

Let N be a natural number and let $\Gamma_2(N)$ be the principal congruence subgroup of the Siegel modular group $Sp(2, \boldsymbol{Z})/\pm 1$ of level N:

$$\Gamma_2(N)=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(2, \boldsymbol{Z}) \,\middle|\, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\right\}\Big/\pm 1.$$

Let k be an integer and let $S_k(\Gamma_2(N))$ be the complex vector space of the cusp forms of weight k with respect to $\Gamma_2(N)$. Namely, let $S_k(\Gamma_2(N))$ be the space of holomorphic functions $f(Z)$ on $\mathfrak{H}$ satisfying the following two conditions:

(i) $f((AZ+B)(CZ+D)^{-1})=\det(CZ+D)^k f(Z)$ for all $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_2(N)$;

(ii) $\det\{\mathrm{Im}(Z)\}^{k/2}|f(Z)|$ is bounded on $\mathfrak{H}$.

Now we can state the main result of this paper:

MAIN THEOREM. *Suppose $N\geqq 3$ and $k\geqq 7$. Then we have the following dimension formula:*

$$\dim_{\boldsymbol{C}} S_k(\Gamma_2(N)) = \frac{1}{2^9\cdot 3^3\cdot 5}[\Gamma_2(1):\Gamma_2(N)](2k-2)(2k-3)(2k-4)$$
$$-\frac{1}{2^4\cdot 3^2}N\mu_2(N)[\Gamma_1(1):\Gamma_1(N)](2k-3)+\frac{1}{2^3\cdot 3}\mu_2(N)[\Gamma_1(1):\Gamma_1(N)],$$

where $\mu_2(N)$ is the number of inequivalent 0-dimensional cusps of $\Gamma_2(N)\backslash\mathfrak{H}$ (which

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is equal to the number of inequivalent 1*-dimensional cusps of* $\Gamma_2(N)\backslash\mathfrak{H}$), $\Gamma_1(N)$ *(resp.* $\Gamma_1(1)$) *is the principal congruence subgroup of* $SL(2, \boldsymbol{Z})/\pm 1$ *of level* N *(resp.* 1), *and* [:] *denotes the group index.*

0.2. An outline of the proof. Following the general method discovered by A. Selberg, R. Godement expressed the dimension as an integral of an infinite series over the fundamental domain of $\Gamma_2(N)$ (cf. Theorem 2 of §5). Therefore what we must do is to interchange the integral and the infinite sum and to calculate the resulting integrals explicitly. In the case of a properly discontinuous group operating on a bounded symmetric domain with a compact quotient, this problem was solved by R. P. Langlands. But, in the non-compact case, some more difficulties occur: First, the infinite sum in the integral does not converge absolutely and uniformly so that there are some troubles in interchanging the integral and the infinite sum; secondly, infinitely many conjugacy classes of the discontinuous group have non-zero contribution so that we must calculate certain infinite sums. But, fortunately for us, these difficulties appear even in the one-dimensional case and they were overcome by M. Eichler, A. Selberg, and H. Shimizu (cf. Shimizu [15]) in that case. We shall follow their methods and settle these difficulties. That is, we shall overcome the first difficulty by multiplying certain dumping factors, and the second difficulty by using some special values of Dirichlet series. The proof then follows by careful case-by-case analysis.

We shall prove some preliminary results in §1, §2, §3, and §4. We shall start the proof of our main theorem in §5 and complete it in §7.

0.3. Remarks.

(1) J. Igusa determined the structure of the graded ring of holomorphic automorphic forms of the group $\Gamma_2(4)$ by using theta-constants (cf. Igusa [5]). Thus our results can be regarded as a generalization of Igusa's result.

(2) Recently, T. Yamazaki has calculated the dimension of $S_k(\Gamma_2(N))$ $(N\geq 3)$ by using the Riemann-Roch theorem and Igusa's desingularization of the Satake compactification of $\Gamma_2(N)\backslash\mathfrak{H}$ (cf. Yamazaki [19]).

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[Notation]

We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$, respectively, the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers. For any natural number n and for any commutative ring S with an identity element, $M_n(S)$, $GL(n, S)$, and $SL(n, S)$ denote the ring of all matrices of size n with entries in S, the group of all invertible elements in $M_n(S)$, the group of all elements in $M_n(S)$ whose determinants are one, respectively. Further, $Sp(n, S)$ and $U(n)=U(n, \boldsymbol{R})$ denote the symplectic group of degree n over S and the unitary group of degree n, respectively. For any element A of $M_n(S)$, we denote by $\mathrm{tr}(A)$ and $\det(A)$ the trace of A and the determinant of A, respectively. Sometimes we denote by 0, 1 and $|A|$ the zero matrix of $M_n(S)$, the unit matrix of $M_n(S)$ and $\det(A)$, respectively.

We denote by i the complex number $\sqrt{-1}$. For any element Z of $M_n(\boldsymbol{C})$, we denote by $X=\mathrm{Re}(Z)$ and $Y=\mathrm{Im}(Z)$ the real part of Z and the imaginary part of Z, respectively (i.e., $Z=X+iY$ with X, $Y\in M_n(\boldsymbol{R})$). For any complex number $z=x+iy$, we denote by $\mathrm{abs}(z)=|z|$ and $\bar{z}$ the absolute value $\sqrt{x^2+y^2}$ of z and the complex conjugate $x-iy$ of z, respectively.

Let Z be a hermitian matrix. Then we write $Z>0$ (resp. $Z\geqq 0$) if Z is positive definite (resp. positive semi-definite).

We denote by $\Gamma(s)$ and $\zeta(s)$ the Gamma function and the Riemann zeta function, respectively. We denote by $\mathrm{Max}(a, b, \cdots)$ (resp. $\mathrm{Min}(a, b, \cdots)$) the maximal element (resp. the minimal element) of $a, b, \cdots$. We denote by $0(s)$ a continuous function of s which satisfies $\lim_{s\to 0} 0(s)=0$. We denote by $*$ (resp. const.) an unspecified matrix (resp. an unspecified constant). We usually use $a, b, c, \cdots$ (resp. $A, B, C, \cdots$) to denote complex numbers (resp. elements of $M_n(\boldsymbol{C})$). For example, we write any element of $Sp(2, \boldsymbol{C})$ as

$$\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \text{ or } \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Some more standard notations will be used.

§ 1. A classification of conjugacy classes

1.1. Preparatory lemmas. First we quote a well-known fact:

LEMMA 1. *Let γ be a unipotent element of $Sp(2, Z)$. Then γ is conjugate in $Sp(2, Z)$ to an element δ of the form*

$$\begin{pmatrix} 1 & 0 & b_1 & b_{12} \\ 0 & 1 & b_{12} & b_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \multicolumn{2}{c}{0} \\ a & 1 & & \\ & & 1 & -a \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}.$$

Moreover, if γ is conjugate in $Sp(2, R)$ to an element of the form $\begin{pmatrix} 1 & 0 & b_1 & b_{12} \\ 0 & 1 & b_{12} & b_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}$, then γ is conjugate in $Sp(2, Z)$ to an element δ of the same form.

Now we shall prove the following lemma:

LEMMA 2. *Let N and n be natural numbers such that $N \geq 3$. Let $\Gamma(N)$ be the principal congruence subgroup of the unimodular group $GL(n, Z)$ of level N:*

$$\Gamma(N) = \{\gamma \in GL(n, Z) \mid \gamma - 1 \in NM_n(Z)\}.$$

Let ζ be an eigen-value of γ such that ζ is a root of unity. Then $\zeta = 1$.[1]

PROOF. Suppose that $\zeta \neq 1$. Let m be the smallest positive integer such that $\zeta^m = 1$. Let l be a prime number that divides m. Since $\zeta \neq 1$, there is such a prime number l. Then $\gamma^{m/l}$ is an element of $\Gamma(N)$, and $\zeta^{m/l}$ is an eigen-value of $\gamma^{m/l}$. Since $\zeta^{m/l}$ is a primitive l^{th} root of unity, $\zeta^{m/l} - 1$ generates the prime ideal $\mathfrak{l}$ of $Q(\zeta^{m/l})$ such that $\mathfrak{l}^{l-1} = (l)$. Therefore $\zeta^{m/l} - 1$ is not divisible by the natural number N, which is greater than 2. On the other hand, since $\gamma^{m/l}$ belongs to $\Gamma(N)$, there is an element δ of $M_n(Z)$ such that $\gamma^{m/l} - 1 = N\delta$. Hence we have $\zeta^{m/l} - 1 = N\eta$ with an eigen-value η of δ. Since δ belongs to $M_n(Z)$, η is an algebraic integer in $Q(\zeta^{m/l})$. Hence $\zeta^{m/l} - 1$ is divisible by N, which is a contradiction. Therefore we have proved Lemma 2.

In particular, we have the following

PROPOSITION 1. *Let γ be an element of the principal congruence subgroup $\Gamma_2(N)$ of $Sp(2, Z)$ of level N. Suppose N is not less than 3. Then some power of γ is unipotent if and only if γ is unipotent. In particular, $\Gamma_2(N)$ is torsion free.*

1.2. A classification of conjugacy classes. First we shall classify conjugacy

[1] The author was informed of this lemma by Y. Ihara and G. Shimura

classes by means of fixed points on

$$\bar{D}=\{Z\in M_2(\boldsymbol{C})\mid {}^tZ=Z,\ 1-Z\,{}^t\bar{Z}\geqq 0\},$$

where we make an element of $Sp(2, \boldsymbol{R})$ act on $\bar{D}$ in the following way: Let $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of $Sp(2, \boldsymbol{R})$ and put

$$\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}=\begin{pmatrix} i1 & i1 \\ -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} i1 & i1 \\ -1 & 1 \end{pmatrix}.$$

Then we define the action of γ by

$$Z\longmapsto (A_0Z+B_0)(C_0Z+D_0)^{-1}.$$

We see that any element of $Sp(2, \boldsymbol{R})$ acts on $\bar{D}$ as a topological automorphism. Moreover it is well-known that there are only three $Sp(2, \boldsymbol{R})$ orbits:

$$Sp(2, \boldsymbol{R})\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ Sp(2, \boldsymbol{R})\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } Sp(2, \boldsymbol{R})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\bar{D}$ is contractible and $\gamma\in Sp(2, \boldsymbol{R})$ acts on $\bar{D}$ as a topological automorphism, it follows from the Lefschetz fixed point theorem that γ has a fixed point on $\bar{D}$.

Now let γ be an element of $Sp(2, \boldsymbol{R})$. Then, taking a conjugate element if necessary, we may assume that γ fixes (1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or (2) $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ or (3) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore γ can be written in the following way:

(1) $\gamma=\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with $A+Bi\in U(2)$;

or

(2) $$\gamma=\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ -b & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ & & 1 & d_2 \\ \multicolumn{2}{c}{0} & 0 & d_4 \end{pmatrix} \text{ with } \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\in U(1)$$

and

$$\begin{pmatrix} 1 & 0 & 0 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ & & 1 & d_2 \\ \multicolumn{2}{c}{0} & 0 & d_4 \end{pmatrix}\in Sp(2, \boldsymbol{R});$$

or

(3) $\gamma=\begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}$ with $U\in GL(2, \boldsymbol{R})$, $S\in M_2(\boldsymbol{R})$ and ${}^tS=S$.

Now suppose that γ belongs to the principal congruence subgroup of $Sp(2, \boldsymbol{Z})$ of level $N\geqq 3$. If γ is conjugate in $Sp(2, \boldsymbol{R})$ to $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ ($A+Bi\in U(2)$) (case (1)), γ generates a compact and discrete subgroup in $Sp(2, \boldsymbol{R})$. Hence γ is of finite order, hence $\gamma=1$ by Proposition 1. In the same manner, we see that, if γ is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ -b & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ \multicolumn{2}{c}{\multirow{2}{*}{0}} & 1 & d_2 \\ & & 0 & d_4 \end{pmatrix}$$

(case (2)), $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\in U(1)$ is not equal to $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ only if $a_4=d_4^{-1}\neq\pm 1$. In the same manner, we see that, if γ is conjugate in $Sp(2, \boldsymbol{R})$ to $\begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}$ (case (3)), $\det(U)\neq -1$ and U is not of finite order. (If $\det(U)=-1$, the characteristic polynomial of γ is not congruent modulo N to $(X-1)^4$.)

Now we shall prove the following theorem:

THEOREM 1. *Let $\Gamma_2(N)$ be the principal congruence subgroup of the Siegel modular group $Sp(2, \boldsymbol{Z})/\pm 1$ of level $N\geqq 3$. Let γ be an element of $\Gamma_2(N)$. Then γ is conjugate in $Sp(2, \boldsymbol{R})/\pm 1$ to one of the following representatives:*

(i) $$\begin{pmatrix} \cos\lambda & 0 & \sin\gamma & 0 \\ 0 & a & 0 & 0 \\ -\sin\lambda & 0 & \cos\lambda & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \quad (\sin\lambda\neq 0,\ a\neq\pm 1);$$

(ii) $$\begin{pmatrix} a_1 & 0 & & \\ 0 & a_2 & & \\ & & a_1^{-1} & 0 \\ & & 0 & a_1^{-1} \end{pmatrix} \quad (a_1^2,\ a_2^2,\ a_1a_2\neq 1);$$

(iii) $$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ & & a^{-1} & 0 \\ & & 0 & 1 \end{pmatrix} \quad (a\neq\pm 1);$$

(iv) $$\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & & \\ & & a^{-1} & 0 \\ & & 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \quad (a\neq\pm 1);$$

(v)
$$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \multicolumn{2}{c}{0} \\ a & 1 & & \\ & & 1 & -a \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \quad (a \neq 0,\ b \neq 0);$$

(vi)
$$\begin{pmatrix} 1 & 0 & b_1 & b_{12} \\ 0 & 1 & b_{12} & b_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix};$$

(vii)
$$\begin{pmatrix} \mu & 0 & \multicolumn{2}{c}{0} \\ 0 & \mu & & \\ & & \mu^{-1} & 0 \\ \multicolumn{2}{c}{0} & 0 & \mu^{-1} \end{pmatrix}\begin{pmatrix} \cos\lambda & \sin\lambda & \multicolumn{2}{c}{0} \\ -\sin\lambda & \cos\lambda & & \\ & & \cos\lambda & \sin\lambda \\ \multicolumn{2}{c}{0} & -\sin\lambda & \cos\lambda \end{pmatrix} \quad (\mu \neq \pm 1,\ \sin\lambda \neq 0)\ .$$

PROOF. Let γ be an element of $\Gamma_2(N)$. Then, by the above remark, we may assume that γ is conjugate in $Sp(2, \boldsymbol{R})/\pm 1$ to

(1)
$$\delta = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ -b & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ & & 1 & d_2 \\ \multicolumn{2}{c}{0} & 0 & d_4 \end{pmatrix} \in Sp(2, \boldsymbol{R})/\pm 1$$

with $a_4 = d_4^{-1} \neq \pm 1$, $a^2 + b^2 = 1$ and $b \neq 0$, or

(2)
$$\delta = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}$$

with ${}^tS = S$, $\det(U) \neq -1$.

We contend that, in the first case, δ is conjugate in $Sp(2, \boldsymbol{R})/\pm 1$ to an element of the form

$$\begin{pmatrix} \cos\lambda & 0 & \sin\lambda & 0 \\ 0 & a & 0 & 0 \\ -\sin\lambda & 0 & \cos\lambda & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix}$$

with $\sin\lambda \neq 0$ and $a^2 \neq 1$. Since δ has this property if and only if δ^{-1} has this property, we may assume $a_4^2 > 1$. Let

$$\varepsilon_1=\begin{pmatrix}1 & 0 & & \\ \alpha & 1 & & \\ & & 1 & -\alpha \\ & & 0 & 1\end{pmatrix}\in Sp(2, \boldsymbol{R}) .$$

Then

$$\varepsilon_1\delta\varepsilon_1^{-1}=\begin{pmatrix}a & 0 & b & 0\\ 0 & 1 & 0 & 0\\ -b & 0 & a & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & *\\ a_3+(a-a_4)\alpha & \alpha_4 & * & *\\ & & 1 & -a_3d_4-(a-a_4)\alpha d_4\\ & 0 & 0 & d_4\end{pmatrix} .$$

Therefore, since $(a-a_4)(a+a_4)=a^2-a_4^2<0$, we may assume

$$\delta=\begin{pmatrix}\cos\lambda & 0 & \sin\lambda & 0\\ 0 & 1 & 0 & 0\\ -\sin\lambda & 0 & \cos\lambda & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & & \\ 0 & a & & \\ & & 1 & 0\\ & & 0 & a^{-1}\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & b_{12}\\ 0 & 1 & b_{12} & b_2\\ & & 1 & 0\\ & & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}\cos\lambda & 0 & \sin\lambda & 0\\ 0 & a & 0 & 0\\ -\sin\lambda & 0 & \cos\lambda & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & b_{12}\\ 0 & 1 & b_{12} & b_2\\ & & 1 & 0\\ & & 0 & 1\end{pmatrix} .$$

Let

$$\varepsilon_2=\begin{pmatrix}1 & 0 & 0 & \alpha_2\\ \alpha_1 & 1 & \alpha_2 & \alpha_3\\ & & 1 & -\alpha_1\\ & & 0 & 1\end{pmatrix}\in Sp(2, \boldsymbol{R}) .$$

Then

$$\varepsilon_2\delta\varepsilon_2^{-1}=\begin{pmatrix}\cos\lambda & 0 & \sin\lambda & 0\\ 0 & a & 0 & 0\\ -\sin\lambda & 0 & \cos\lambda & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & (1.4)\\ (2.1) & 1 & (2.3) & (2.4)\\ & & 1 & -(2.1)\\ & & 0 & 1\end{pmatrix}$$

with

$$(2.1)=(a^{-1}\cos\lambda-1)\alpha_1-a^{-1}\sin\lambda\alpha_2 ,$$

$$(2.3)=(1.4)=a^{-1}\sin\lambda\alpha_1+(a^{-1}\cos\lambda-1)\alpha_2+b_{12} ,$$

$$(2.4)=a^{-1}(\alpha_1\cos\lambda-\alpha_2\sin\lambda)(-\alpha_2+b_{12})+(-\alpha_3+b_{12}\alpha_1+b_2)$$
$$+\alpha_1a^{-1}(\alpha_1\sin\lambda+\alpha_2\cos\lambda)+a^{-2}\alpha_3 .$$

Since

$$\begin{vmatrix} a^{-1}\cos\lambda-1 & -a^{-1}\sin\lambda \\ a^{-1}\sin\lambda & a^{-1}\cos\lambda-1 \end{vmatrix} = (a^{-1}\cos\lambda-1)^2+(a^{-1}\sin\lambda)^2 \neq 0$$

by our assumption, we can solve $(2.1)=(2.3)=(1.4)=0$ as a system of linear equations in α_1 and α_2. Moreover, we can solve $(2.4)=(a^{-2}-1)\alpha_3+\cdots=0$ in α_3. Therefore we have proved our contention.

Now let $\delta=\begin{pmatrix}1 & S\\ 0 & 1\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & {}^tU^{-1}\end{pmatrix}$ with ${}^tS=S$, $U\in GL(2,\boldsymbol{R})$ and $\det(U)\neq -1$. Let $P(X)=\det(X1-U)$ be the characteristic polynomial of U. Then the discriminant of the equation $P(X)=0$ is $\operatorname{tr}(U)^2-4\det(U)$. Therefore, if $\det(U)<0$, U has two distinct real characteristic roots. Hence, if $\det(U)<0$, $U=G\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}G^{-1}$ with some $G\in GL(2,\boldsymbol{R})$ and $ab<0$. Therefore, in this case, δ is conjugate in $Sp(2,\boldsymbol{R})$ to

$$\delta'=\begin{pmatrix}1 & 0 & s_1 & s_{12}\\ 0 & 1 & s_{12} & s_2\\ & & 1 & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}\begin{pmatrix}a & 0 & & \\ 0 & b & \multicolumn{2}{c}{0}\\ & & a^{-1} & 0\\ \multicolumn{2}{c}{0} & 0 & b^{-1}\end{pmatrix}.$$

Let $\varepsilon=\begin{pmatrix}1 & 0 & t_1 & t_{12}\\ 0 & 1 & t_{12} & t_2\\ & & 1 & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}\in Sp(2,\boldsymbol{R})$. Then

$$\varepsilon\delta'\varepsilon^{-1}=\begin{pmatrix}1 & 0 & (1.3) & (1.4)\\ 0 & 1 & (2.3) & (2.4)\\ & & 1 & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}\begin{pmatrix}a & 0 & & \\ 0 & b & \multicolumn{2}{c}{0}\\ & & a^{-1} & 0\\ \multicolumn{2}{c}{0} & 0 & b^{-1}\end{pmatrix}$$

with

$$\begin{pmatrix}(1.3) & (1.4)\\ (2.3) & (2.4)\end{pmatrix}=\begin{pmatrix}s_1+(1-a^2)t_1 & s_{12}+(1-ab)t_{12}\\ s_{12}+(1-ab)t_{12} & s_2+(1-b^2)t_2\end{pmatrix}.$$

If a^2, ab, $b^2\neq 1$, we can solve the linear equations $(1.3)=(1.4)=(2.3)=(2.4)=0$ in t_1, t_{12}, $t_2\in\boldsymbol{R}$. Hence δ is conjugate to $\begin{pmatrix}a & 0 & & \\ 0 & b & \multicolumn{2}{c}{0}\\ & & a^{-1} & 0\\ \multicolumn{2}{c}{0} & 0 & b^{-1}\end{pmatrix}$. Moreover, we have proved that, if $P(X)=\det(X1-U)=0$ has two distinct real roots a and b, δ is conjugate in $Sp(2,\boldsymbol{R})$ to

$$\delta'=\begin{pmatrix} a & 0 & & \\ 0 & b & & \\ & & a^{-1} & 0 \\ & & 0 & b^{-1} \end{pmatrix} \cdots \text{ if } a^2,\ ab,\ b^2 \neq 1,\ a \neq b\,,$$

or

$$\delta'=\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & s_2 \\ & & a^{-1} & 0 \\ & & 0 & \pm 1 \end{pmatrix} \cdots \text{ if one root } b \text{ is } \pm 1 \text{ and the other root } a \text{ is not } \pm 1\,,$$

or

$$\delta''=\begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & & \\ & & a^{-1} & 0 \\ & & 0 & a \end{pmatrix} \cdots \text{ if } ab=1 \text{ and } a^2 \neq 1\,.$$

Here, since $P(X)\equiv(X-1)^4 \pmod N$, $\delta''=\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & s_2 \\ & & a^{-1} & 0 \\ & & 0 & 1 \end{pmatrix}$ in the second case. Further we note that $\begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & & \\ & & a^{-1} & 0 \\ & & 0 & a \end{pmatrix}$ commute and that $\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & & \\ & & a^{-1} & 0 \\ & & 0 & a \end{pmatrix}$ and $\begin{pmatrix} a & 0 & & \\ 0 & a & & \\ & & a^{-1} & 0 \\ & & 0 & a^{-1} \end{pmatrix}$ are conjugate.

Now suppose that $\det(X\cdot 1-U)=0$ has only one real root a. Then we see easily that $U=G\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}G^{-1}$ with some $G\in GL(2, \boldsymbol{R})$. Hence δ is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\delta'=\begin{pmatrix} 1 & 0 & s_1 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & & \\ 0 & a & & \\ & & a^{-1} & 0 \\ & & -a^{-2}b & a^{-1} \end{pmatrix}.$$

Let $\varepsilon=\begin{pmatrix} 1 & 0 & t_1 & t_{12} \\ 0 & 1 & t_{12} & t_2 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$ be an element of $Sp(2, \boldsymbol{R})$. Then we have

$$\varepsilon\delta'\varepsilon^{-1}=\begin{pmatrix}1 & 0 & (1.3) & (1.4)\\ 0 & 1 & (2.3) & (2.4)\\ & & 1 & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}\begin{pmatrix}a & b & & \\ 0 & a & \multicolumn{2}{c}{0}\\ & & a^{-1} & 0\\ \multicolumn{2}{c}{0} & -a^{-2}b & a^{-1}\end{pmatrix}$$

with

$$(1.3)=s_1+(1-a^2)t_1-2abt_{12}-b^2t_2\ ,$$

$$(1.4)=(2.3)=s_{12}+(1-a^2)t_{12}-abt_2\ ,$$

$$(2.4)=s_2+(1-a^2)t_2\ .$$

Therefore we see that δ is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\delta''=\begin{pmatrix}a & b & 0 & 0\\ 0 & a & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & -a^{-2}b & a^{-1}\end{pmatrix}\cdots\ \text{if}\ a^2\neq 1\ ,$$

or

$$\delta'''=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & s_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\pm1 & b & 0 & 0\\ 0 & \pm1 & 0 & 0\\ 0 & 0 & \pm1 & 0\\ 0 & 0 & -b & \pm1\end{pmatrix}\cdots\ \text{if}\ a=\pm1\ \text{and}\ b\neq 0\ ,$$

or

$$\delta''''=\begin{pmatrix}\pm1 & 0 & s_1 & s_{12}\\ 0 & \pm1 & s_{12} & s_2\\ 0 & 0 & \pm1 & 0\\ 0 & 0 & 0 & \pm1\end{pmatrix}\cdots\ \text{if}\ a=\pm1\ \text{and}\ b=0\ .$$

Here we have $a\neq -1$ by Proposition 1. Moreover we note that

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}a & b & 0 & 0\\ 0 & a & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & -a^{-2}b & a^{-1}\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}a & 0 & 0 & -b\\ 0 & a^{-1} & -a^{-2}b & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & a\end{pmatrix}=\begin{pmatrix}a & 0 & 0 & 0\\ 0 & a^{-1} & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & a\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & -a^{-1}b\\ 0 & 1 & -a^{-1}b & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and that δ''' is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & & \\ a & 1 & \multicolumn{2}{c}{0} \\ & & 1 & -a \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}.$$

Now suppose that $\det(X\cdot 1-U)=0$ has two imaginary roots. Then we have $U=G\cdot\mu\cdot V\cdot G^{-1}$ with $G\in SL(2, \boldsymbol{R})$, $\mu\in\boldsymbol{R}$, $\lambda\in\boldsymbol{R}$ and $V=\begin{pmatrix}\cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda\end{pmatrix}$. Suppose $\mu=\pm 1$. Then we see $P(X)=(X\mp e^{i\lambda})(X\mp e^{-i\lambda})(X\mp e^{i\lambda})(X\mp e^{-i\lambda})$. Moreover, since γ belongs to $Sp(2, \boldsymbol{Z})$, $P(X)$ is a monic polynomial with integer coefficients. Hence we see that $\pm e^{i\lambda}$ is a unit in an imaginary quadratic field. Therefore $\pm e^{i\lambda}$ is a root of unity and $\pm\begin{pmatrix}\cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda\end{pmatrix}$ is of finite order. Therefore, by Proposition 1, γ is unipotent, which is contradiction. Therefore we have proved that, if $P(X)=0$ has imaginary roots, γ is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\delta=\begin{pmatrix} 1 & 0 & s_1 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 & & \\ 0 & \mu & \multicolumn{2}{c}{0} \\ & & \mu^{-1} & 0 \\ \multicolumn{2}{c}{0} & 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \cos\lambda & \sin\lambda & & \\ -\sin\lambda & \cos\lambda & \multicolumn{2}{c}{0} \\ & & \cos\lambda & \sin\lambda \\ \multicolumn{2}{c}{0} & -\sin\lambda & \cos\lambda \end{pmatrix}$$

with $\mu\neq\pm 1$ and $\sin\lambda\neq 0$. Put $\varepsilon=\begin{pmatrix} 1 & 0 & t_1 & t_{12} \\ 0 & 1 & t_{12} & t_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}$. Then

$$\varepsilon\delta\varepsilon^{-1}=\begin{pmatrix} 1 & 0 & (1.3) & (1.4) \\ 0 & 1 & (2.3) & (2.4) \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 & & \\ 0 & \mu & \multicolumn{2}{c}{0} \\ & & \mu^{-1} & 0 \\ \multicolumn{2}{c}{0} & 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \cos\lambda & \sin\lambda & & \\ -\sin\lambda & \cos\lambda & \multicolumn{2}{c}{0} \\ & & \cos\lambda & \sin\lambda \\ \multicolumn{2}{c}{0} & -\sin\lambda & \cos\lambda \end{pmatrix}$$

with

$$(1.3)=s_1+(1-\mu^2\cos^2\lambda)t_1-2\mu^2\sin\lambda\cos\lambda t_{12}-\mu^2\sin^2\lambda t_2\,,$$

$$(1.4)=(2.3)=s_{12}+\mu^2\sin\lambda\cos\lambda t_1+(1+\mu^2\sin^2\lambda-\mu^2\cos^2\lambda)t_{12}-\mu^2\sin\lambda\cos\lambda t_2\,,$$

$$(2.4)=s_2-\mu^2\sin^2\lambda t_1+2\mu^2\sin\lambda\cos\lambda t_{12}+(1-\mu^2\cos^2\lambda)t_2\,.$$

Since

$$\begin{vmatrix} 1-\mu^2\cos^2\lambda & -2\mu^2\sin\lambda\cos\lambda & -\mu^2\sin^2\lambda \\ \mu^2\sin\lambda\cos\lambda & 1+\mu^2\sin^2\lambda-\mu^2\cos^2\lambda & -\mu^2\sin\lambda\cos\lambda \\ -\mu^2\sin^2\lambda & 2\mu^2\sin\lambda\cos\lambda & 1-\mu^2\cos^2\lambda \end{vmatrix}$$

$$=(1-\mu^2)\{(1+\mu^2\sin^2\lambda-\mu^2\cos^2\lambda)^2+4\mu^4\sin^2\lambda\cos^2\lambda\}\neq 0\,,$$

we can solve the linear equations $(1.3)=(1.4)=(2.3)=(2.4)=0$ in t_1, t_{12}, $t_2 \in \boldsymbol{R}$. Hence δ is conjugate in $Sp(2, \boldsymbol{R})/\pm 1$ to

$$\begin{pmatrix} \mu & 0 & & \\ 0 & \mu & & \\ & & \mu^{-1} & 0 \\ & & 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \cos\lambda & \sin\lambda & & \\ -\sin\lambda & \cos\lambda & & \\ & & \cos\lambda & \sin\lambda \\ & & -\sin\lambda & \cos\lambda \end{pmatrix},$$

which completes the proof of Theorem 1.

§ 2. Determination of centralizers

2.1. Determination of centralizers. In 2.1, we shall determine the centralizer in $Sp(2, \boldsymbol{R})$ of an element of $Sp(2, \boldsymbol{R})$ which appeared in Theorem 1 as one of the representatives.

PROPOSITION 2. *Let*

$$\delta=\begin{pmatrix} \cos\lambda & 0 & \sin\lambda & 0 \\ 0 & a & 0 & 0 \\ -\sin\lambda & 0 & \cos\lambda & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix}$$

be an element of $Sp(2, \boldsymbol{R})$ *with* $\sin\lambda \neq 0$, $a \neq 1$. *Then the centralizer of* δ *in* $Sp(2, \boldsymbol{R})$ *is the set of all matrices in* $Sp(2, \boldsymbol{R})$ *of the form*

$$\begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \beta & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{pmatrix}$$

PROOF. Put

$$\varepsilon=\begin{pmatrix} i & 0 & i & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$\varepsilon^{-1}\delta\varepsilon=\begin{pmatrix} e^{i\lambda} & 0 & & \\ 0 & a & & \\ & & e^{-i\lambda} & 0 \\ & & 0 & a^{-1} \end{pmatrix},$$

where $e^{\pm i\lambda}$ and $a^{\pm 1}$ are all distinct. Hence the centralizer of $\varepsilon^{-1}\delta\varepsilon$ in $GL(4, \boldsymbol{C})$ is the set of diagonal matrices. Hence the centralizer of δ in $Sp(2, \boldsymbol{R})$ is the set of all matrices of the form

$$\begin{pmatrix} (1/2)(\mu+\nu) & 0 & (1/2i)(\mu-\nu) & 0 \\ 0 & \rho & 0 & 0 \\ (-1/2i)(\mu-\nu) & 0 & (1/2)(\mu+\nu) & 0 \\ 0 & 0 & 0 & \rho' \end{pmatrix} \in Sp(2, \boldsymbol{R})$$

with $\mu, \nu, \rho, \rho' \in \boldsymbol{C}$, which can be easily seen to be equal to the set of all matrices of the form $\begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \beta & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{pmatrix}$. Q.E.D.

In the same manner, we can prove the following four propositions:

PROPOSITION 3. *Let*

$$\delta = \begin{pmatrix} a_1 & 0 & & \\ 0 & a_2 & & 0 \\ & & a_1^{-1} & 0 \\ & 0 & 0 & a_2^{-1} \end{pmatrix}$$

be an element of $Sp(2, \boldsymbol{R})$ *with* $a_1^2, a_2^2, a_1a_2 \neq 1$. *Then the centralizer of* δ *in* $Sp(2, \boldsymbol{R})$ *is the set of all matrices of the following form:*

$$\begin{pmatrix} \alpha & 0 & & \\ 0 & \beta & & 0 \\ & & \alpha^{-1} & 0 \\ & 0 & 0 & \beta^{-1} \end{pmatrix} \cdots \text{ if } a_1 \neq a_2 ,$$

or

$$\begin{pmatrix} V & 0 \\ 0 & {}^tV^{-1} \end{pmatrix} \quad (V \in GL(2, \boldsymbol{R})) \cdots \text{ if } a_1 = a_2 .$$

PROPOSITION 4. *Let*

$$\delta = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ & & a^{-1} & 0 \\ & 0 & 0 & 1 \end{pmatrix}$$

be an element of $Sp(2, \boldsymbol{R})$ *with* $a^2 \neq 1$. *Then the centralizer of* δ *in* $Sp(2, \boldsymbol{R})$ *is* *the set of all matrices of the following form:*

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta \\ & & \alpha^{-1} & 0 \\ & 0 & 0 & 1 \end{pmatrix} \cdots \ \textit{if } b \neq 0 ,$$

or

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \beta_2 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & \beta_3 & 0 & \beta_4 \end{pmatrix} \cdots \ \textit{if } b = 0 .$$

PROPOSITION 5. *Let*

$$\delta = \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & & 0 \\ 0 & a^{-1} & & \\ & & a^{-1} & 0 \\ & 0 & 0 & a \end{pmatrix}$$

be an element of $Sp(2, \boldsymbol{R})$ with $a^2 \neq 1$, $b \neq 0$. Then the centralizer of δ in $Sp(2, \boldsymbol{R})$ is the set of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & \beta \\ 0 & 1 & \beta & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & & 0 \\ 0 & \alpha^{-1} & & \\ & & \alpha^{-1} & 0 \\ & 0 & 0 & \alpha \end{pmatrix} .$$

PROPOSITION 6. *Let*

$$\delta = \begin{pmatrix} \mu & 0 & & 0 \\ 0 & \mu & & \\ & & \mu^{-1} & 0 \\ & 0 & 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \cos\lambda & \sin\lambda & & 0 \\ -\sin\lambda & \cos\lambda & & \\ & & \cos\lambda & \sin\lambda \\ & 0 & -\sin\lambda & \cos\lambda \end{pmatrix}$$

be an element of $Sp(2, \boldsymbol{R})$ with $\mu^2 \neq 1$, $\sin\lambda \neq 0$. Then the centralizer of δ in $Sp(2, \boldsymbol{R})$ is the set of all matrices of the form

$$\begin{pmatrix} \alpha & 0 & & 0 \\ 0 & \alpha & & \\ & & \alpha^{-1} & 0 \\ & 0 & 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta & & 0 \\ -\sin\beta & \cos\beta & & \\ & & \cos\beta & \sin\beta \\ & 0 & -\sin\beta & \cos\beta \end{pmatrix} .$$

Now we shall determine the centralizer in the remaining case.

PROPOSITION 7. *Let δ be*

$$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ a & 1 & \multicolumn{2}{c}{0} \\ & & 1 & -a \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \quad (a, b \in \boldsymbol{R}, \quad a, b \neq 0) .$$

Then the centralizer of δ in $Sp(2, \boldsymbol{R})$ is the set of all matrices of the form

$$\pm\begin{pmatrix} 1 & 0 & \beta_1 & \beta_{12} \\ 0 & 1 & \beta_{12} & \beta_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ \alpha & 1 & \multicolumn{2}{c}{0} \\ & & 1 & -\alpha \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}$$

with $\alpha, \beta_* \in \boldsymbol{R}$, $\beta_1 = a^{-1}b\alpha$ and $\beta_{12} = -\frac{1}{2}b\alpha + \frac{1}{2}a^{-1}b\alpha^2$.

PROOF. Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix}$$

be an element of $Sp(2, \boldsymbol{R})$ and suppose that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\delta = \delta\begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

Then we have

(1) $C\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}C,$

(2) $C\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} + D\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}D,$

(3) $A\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}A + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 0 \end{pmatrix}C,$

and

(4) $A\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}B + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}D.$

By (1), we have

$$\begin{pmatrix} c_1+ac_2 & c_2 \\ c_3+ac_4 & c_4 \end{pmatrix} = \begin{pmatrix} c_1-ac_3 & c_2-ac_4 \\ c_3 & c_4 \end{pmatrix},$$

hence $c_2=-c_3$, $c_4=0$. By (2), we have

$$\begin{pmatrix} bc_1 & -abc_1 \\ -bc_2 & abc_2 \end{pmatrix} + \begin{pmatrix} d_1 & -ad_1+d_2 \\ d_3 & -ad_3+d_4 \end{pmatrix} = \begin{pmatrix} d_1-ad_3 & d_2-ad_4 \\ d_3 & d_4 \end{pmatrix},$$

hence $c_1=c_2=d_3=0$, $d_1=d_4$. Now, since $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \boldsymbol{R})$, we may put

$$A={}^t\begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix}^{-1}, \quad B=\begin{pmatrix} \beta_1 & \beta_{12} \\ \beta_{12} & \beta_2 \end{pmatrix}\begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix}, \quad C=0$$

and

$$D=\begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix}.$$

Then (3) holds automatically, and it follows from (4) that

$$\begin{aligned} &{}^t\begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix}^{-1}\begin{pmatrix} b & -ab \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \beta_1 & \beta_{12} \\ \beta_{12} & \beta_2 \end{pmatrix}\begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_{12} \\ \beta_{12} & \beta_2 \end{pmatrix}\begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix} + \begin{pmatrix} b & -ab \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_1 & d_2 \\ 0 & d_1 \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} &\begin{pmatrix} bd_1^{-1} & -abd_1^{-1} \\ -bd_1^{-2}d_2 & abd_1^{-2}d_2 \end{pmatrix} + \begin{pmatrix} d_1\beta_1 & -ad_1\beta_1+d_2\beta_1+d_1\beta_{12} \\ d_1\beta_{12} & -ad_1\beta_{12}+d_2\beta_{12}+d_1\beta_2 \end{pmatrix} \\ &= \begin{pmatrix} d_1\beta_1 & d_2\beta_1+d_1\beta_{12} \\ ad_1\beta_1+d_1\beta_{12} & ad_2\beta_1+ad_1\beta_{12}+d_2\beta_{13}+d_1\beta_2 \end{pmatrix} + \begin{pmatrix} bd_1 & bd_2-abd_1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore $d_1=\pm 1$, $bd_2=-ad_1\beta_1=2d_1\beta_{12}+d_2\beta_1$, from which we have easily our assertion. Q.E.D.

We shall determine the centralizer of

$$\begin{pmatrix} 1 & 0 & s_1 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}$$

in the next section.

2.2. An application. In 2.2, we shall study the relation between the stabilizers of cusps of $Sp(2, \boldsymbol{Z})$ and the classification of the conjugacy classes of elements of $\Gamma_2(N)$.

By the theory of the compactification, it is well-known that any cusp of $Sp(2, \boldsymbol{Z})$ is equivalent by an element of $Sp(2, \boldsymbol{Z})$ to

$$\left\{\begin{pmatrix} i\infty & * \\ * & i\infty \end{pmatrix}\right\} \text{ or } \left\{\begin{pmatrix} z_1 & * \\ * & i\infty \end{pmatrix} \middle| z_1 \in \boldsymbol{C},\ \operatorname{Im}(z_1) > 0\right\}.$$

Moreover it is well-known that the stabilizer of $\left\{\begin{pmatrix} i\infty & * \\ * & i\infty \end{pmatrix}\right\}$ in $Sp(2, \boldsymbol{Z})$ is

$$\Gamma_\infty^0 = \left\{\begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix} \middle| U \in GL(2, \boldsymbol{Z}),\ S \in M_2(\boldsymbol{Z}),\ {}^tS = S\right\}$$

and that the stabilizer of

$$\left\{\begin{pmatrix} z_1 & * \\ * & i\infty \end{pmatrix} \middle| z_1 \in \boldsymbol{C},\ \operatorname{Im}(z_1) > 0\right\}$$

in $Sp(2, \boldsymbol{Z})$ is

$$\Gamma_\infty^1 = \left\{\begin{pmatrix} * & 0 & * & * \\ * & \pm 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \in Sp(2, \boldsymbol{Z})\right\}.$$

Further it is easy to see that Γ_∞^0 is the semi-direct product of

$$\left\{\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix} \middle| U \in GL(2, \boldsymbol{Z})\right\} \text{ and } \left\{\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \middle| S \in M_2(\boldsymbol{Z}),\ {}^tS = S\right\}$$

with the latter as the normal subgroup and that Γ_∞^1 is the semi-direct product of

$$\left\{\begin{pmatrix} \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_3 & 0 & \alpha_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in SL(2, \boldsymbol{Z})\right\}$$

and

$$\left\{\begin{pmatrix} 1 & 0 & 0 & \beta_{12} \\ 0 & 1 & \beta_{12} & \beta_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ \gamma & \pm 1 & \multicolumn{2}{c}{0} \\ & & 1 & \mp\gamma \\ \multicolumn{2}{c}{0} & 0 & \pm 1 \end{pmatrix} \middle| \beta_{12},\ \beta_2,\ \gamma \in \boldsymbol{Z}\right\}$$

with the latter as the normal subgroup.

PROPOSITION 8. *Let γ be an element of $\Gamma_2(N) \cap \Gamma_\infty^0$ with $N \geqq 3$. Then γ is conjugate in $Sp(2, \boldsymbol{R})$ to one of the following representatives:*

(i) $$\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \text{ with } a^2 \neq 1 ,$$

(ii) $$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } a,\ b \neq 0 ,$$

(iii) $$\begin{pmatrix} 1 & 0 & b_1 & b_{12} \\ 0 & 1 & b_{12} & b_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

PROOF. Since γ belongs to $\Gamma_2(N) \cap \Gamma_\infty^0$, we may write

$$\gamma = \begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}$$

with $U \in SL(2, \boldsymbol{Z})$, $S \in M_2(\boldsymbol{Z})$ and ${}^tS = S$. Therefore the roots of $P(X) = \det(X \cdot 1 - \gamma) = 0$ are a, a^{-1}, a, a^{-1} with $a^2 \neq 1$, or $e^{i\theta}$, $e^{-i\theta}$, $e^{i\theta}$, $e^{-i\theta}$ $(\theta \neq \pm\pi)$, or 1, 1, 1, 1. Hence Proposition 8 follows from Theorem 1.

In the same manner, we have the following

PROPOSITION 9. *Let γ be an element of $\Gamma_2(N) \cap \Gamma_\infty^1$ with $N \geqq 3$. Then γ is conjugate in $Sp(2, \boldsymbol{R})$ to one of the following representatives:*

(i) $$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } a^2 \neq 1 ,$$

(ii) $$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } a,\ b \neq 0 ,$$

(iii) $$\begin{pmatrix} 1 & 0 & b_1 & b_{12} \\ 0 & 1 & b_{12} & b_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Now we shall show an application of the results of 2.1.

PROPOSITION 10. *Let γ_1, γ_2 be elements of Γ_∞^0 that are conjugate in $Sp(2, \boldsymbol{Z})$. Suppose they are conjugate in $Sp(2, \boldsymbol{R})$ to*

$$\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & \multicolumn{2}{c}{0} \\ 0 & a^{-1} & & \\ & & a^{-1} & \\ \multicolumn{2}{c}{0} & 0 & a \end{pmatrix}$$

with $a^2 \neq 1$, $b \neq 0$. Then $\gamma_1 = \varepsilon\gamma_2\varepsilon^{-1}$ with $\varepsilon \in \Gamma_\infty^0$, i.e., they are conjugate in Γ_∞^0.

PROOF. Let ε be an element of $Sp(2, \boldsymbol{Z})$ with $\gamma_1 = \varepsilon\gamma_2\varepsilon^{-1}$. We have proved in the proof of Theorem 1 that there are elements ε_1 and ε_2 of $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in Sp(2, \boldsymbol{R})\right\}$ such that

$$\varepsilon_1\gamma_1\varepsilon_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & b' \\ 0 & 1 & b' & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} a' & 0 & \multicolumn{2}{c}{0} \\ 0 & a'^{-1} & & \\ & & a'^{-1} & 0 \\ \multicolumn{2}{c}{0} & 0 & a' \end{pmatrix}$$

and

$$\varepsilon_2\gamma_2\varepsilon_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & b'' \\ 0 & 1 & b'' & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} a'' & 0 & \multicolumn{2}{c}{0} \\ 0 & a''^{-1} & & \\ & & a''^{-1} & 0 \\ \multicolumn{2}{c}{0} & 0 & a'' \end{pmatrix}.$$

Here we may assume $a' = a'' = a$ since a' and a'' are both roots of $P(X) = \det(X \cdot 1 - \gamma) = (X-a)^2(X-a^{-1})^2 = 0$ and

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ & & 0 & 1 \\ \multicolumn{2}{c}{0} & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \beta \\ 0 & 1 & \beta & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ & & 0 & 1 \\ \multicolumn{2}{c}{0} & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & -\beta & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ & & 0 & 1 \\ \multicolumn{2}{c}{0} & -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 & \multicolumn{2}{c}{0} \\ 0 & \alpha^{-1} & & \\ & & \alpha^{-1} & 0 \\ \multicolumn{2}{c}{0} & 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ & & 0 & 1 \\ \multicolumn{2}{c}{0} & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} & 0 & \multicolumn{2}{c}{0} \\ 0 & \alpha & & \\ & & \alpha & 0 \\ \multicolumn{2}{c}{0} & 0 & \alpha^{-1} \end{pmatrix}.$$

Moreover, since

$$\begin{pmatrix} \mu & 0 & & \\ 0 & 1 & & \\ & & \mu^{-1} & 0 \\ & & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & \beta \\ 0 & 1 & \beta & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}\begin{pmatrix} \mu & 0 & & \\ 0 & 1 & & \\ & & \mu^{-1} & 0 \\ & & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \mu\beta \\ 0 & 1 & \mu\beta & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix},$$

we may assume $b'=b''=1$. Then

$$\varepsilon_1\gamma_1\varepsilon_1^{-1}=\varepsilon_2\gamma_2\varepsilon_2^{-1},\ \gamma_1=\varepsilon\gamma_2\varepsilon^{-1} .$$

Hence $\varepsilon_1\gamma_1\varepsilon_1^{-1}=(\varepsilon_1\varepsilon)\gamma_2(\varepsilon_1\varepsilon)^{-1}=\varepsilon_2\gamma_2\varepsilon_2^{-1}$. Hence, by Proposition 5, there exists

$$\varepsilon_3=\begin{pmatrix} 1 & 0 & 0 & \beta \\ 0 & 1 & \beta & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 & & \\ 0 & \alpha^{-1} & & \\ & & \alpha^{-1} & 0 \\ & & 0 & \alpha \end{pmatrix}$$

such that $\varepsilon_1\varepsilon=\varepsilon_3\varepsilon_2$. Hence $Sp(2, \boldsymbol{Z})\ni\varepsilon=\varepsilon_1^{-1}\varepsilon_3\varepsilon_2\in\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}\subseteq Sp(2, \boldsymbol{R})$. So we have proved $\varepsilon\in\Gamma_\infty^0$. Q.E.D.

In the same manner, we can prove the following two propositions:

PROPOSITION 11. *Let γ_1, γ_2 be elements of Γ_∞^1 which are conjugate in $Sp(2, \boldsymbol{Z})$. Suppose they are conjugate in $Sp(2, \boldsymbol{R})$ to*

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ & & a^{-1} & 1 \\ & & 0 & 0 \end{pmatrix}$$

with $a^2\neq 1$, $b\neq 0$. Then $\gamma_1=\varepsilon\gamma_2\varepsilon^{-1}$ with $\varepsilon\in\Gamma_\infty^1$, i.e., they are conjugate in Γ_∞^1.

PROPOSITION 12. *Let γ_1, γ_2 be elements of Γ_∞^0 (resp. Γ_∞^1). Suppose they are conjugate in $Sp(2, \boldsymbol{R})$ to*

$$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ a & 1 & & \\ & & 1 & -a \\ & & 0 & 1 \end{pmatrix}\quad (a,\ b\neq 0) .$$

Then $\gamma_1=\varepsilon\gamma\varepsilon^{-1}$ with $\varepsilon\in\Gamma_\infty^0$ (resp. $\varepsilon\in\Gamma_\infty^1$), i.e., they are conjugate in Γ_∞^0 (resp. Γ_∞^1).

§ 3. Binary quadratic forms

In this section, we shall summarize some results about binary quadratic forms.

3.1. Equivalences of quadratic forms. Since all results of 3.1 are well-known, we shall omit the proof of them (cf. Gauss [2]). Put

$$G^0=\left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp(2, \boldsymbol{R}) \mid C=0\right\} .$$

Then G^0 is the semi-direct product of

$$\left\{\begin{pmatrix} G & 0 \\ 0 & {}^tG^{-1}\end{pmatrix}\middle| G\in GL(2, \boldsymbol{R})\right\} \text{ and } \left\{\begin{pmatrix} 1 & S \\ 0 & 1\end{pmatrix}\middle| S\in M_2(\boldsymbol{R}),\ {}^tS=S\right\}$$

with the latter group as the normal subgroup. Here, an element

$$\begin{pmatrix} G & 0 \\ 0 & {}^tG^{-1}\end{pmatrix}$$

of the former group acts on the latter group as

$$\begin{pmatrix} G & 0 \\ 0 & {}^tG^{-1}\end{pmatrix}: S \longmapsto GS{}^tG .$$

We say that two symmetric matrices S and S' of size two are *properly equivalent* (resp. *equivalent*) (resp. *$\boldsymbol{R}$-equivalent*) if there is an element G of $SL(2, \boldsymbol{Z})$ (resp. $GL(2,\boldsymbol{Z})$) (resp. $GL(2, \boldsymbol{R})$) such that $S'=GS{}^tG$. We note that, if two integral symmetric matrices S, S' of size two are equivalent, corresponding $\begin{pmatrix} 1 & S \\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix} 1 & S' \\ 0 & 1\end{pmatrix}$ are conjugate.

PROPOSITION 13. *Any non-zero real symmetric matrix of size two is $\boldsymbol{R}$-equivalent to one of the following representatives*:

$$\text{(i)} \quad \begin{pmatrix} \pm 1 & 0 \\ 0 & 0\end{pmatrix}, \qquad \text{(ii)} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \text{(iii)} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}.$$

Moreover the following facts are well-known:

PROPOSITION 14. *Let S be a symmetric matrix of size two with integral entries. Then*

(i) *If* $\det(S)=0$, *S is properly equivalent to* $\begin{pmatrix} 0 & 0 \\ 0 & s_2\end{pmatrix}$ $(s_2\in \boldsymbol{Z})$ *and any two of them are not equivalent.*

(ii) *If* $-\det(S)\in(\boldsymbol{Q}^\times)^2$, *$S$ is properly equivalent to* $\begin{pmatrix} 0 & s_{12} \\ s_{12} & s_2\end{pmatrix}$, *where s_{12} is a positive integer, s_2 is a non-negative integer and* $0\leqq s_2\leqq 2s_{12}-1$. *Any two of them are not properly equivalent.*

(iii) *If* $-\det(S)\neq 0, \notin(\boldsymbol{Q}^\times)^2$, *$S$ is properly equivalent to* $t\begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2\end{pmatrix}$, *where t, s_1, s_{12}, s_2 are integers, $s_1+s_2>0$ and $(s_1, s_{12}, s_2)=1$. Moreover, the number of the proper equivalence classes with fixed t and $s_1s_2-s_{12}^2$ is finite. It is equal to the* *class number of the order of the quadratic field of discriminant* $D(S)$, *where* $D(S)$ *is* $s_{12}^2-s_1s_2$ *if* $s_1\equiv s_2\equiv 0 \mod 2$ *and* $4(s_{12}^2-s_1s_2)$ *otherwise.*

3.2. Units of quadratic forms

PROPOSITION 15. *The centralizer of*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \quad (s_2 \neq 0)$$

in $Sp(2, \boldsymbol{R})$ *is the set of all elements of* $Sp(2, \boldsymbol{R})$ *of the form*

$$\begin{pmatrix} * & 0 & * & * \\ * & \pm 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}.$$

PROOF. Since the set

$$\left\{\begin{pmatrix} * & 0 & * & * \\ * & \pm 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \in Sp(2, \boldsymbol{R})\right\},$$

is generated by the matrices

$$\begin{pmatrix} \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & \pm 1 & 0 & 0 \\ \alpha_3 & 0 & \alpha_4 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \left(\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in SL(2, \boldsymbol{R})\right),$$

$$\begin{pmatrix} 1 & 0 & \multicolumn{2}{c}{0} \\ \beta & 1 & & \\ & & 1 & -\beta \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} (\beta \in \boldsymbol{R}) \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \gamma_1 & \gamma_{12} \\ 0 & 1 & \gamma_{12} & \gamma_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} (\gamma_1, \gamma_{12}, \gamma_2 \in \boldsymbol{R}),$$

we see easily that

$$\left\{\begin{pmatrix} * & 0 & * & * \\ * & \pm 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \in Sp(2, \boldsymbol{R})\right\}$$

is contained in the centralizer. So it is enough to show the opposite inclusion. Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \in Sp(2, \boldsymbol{R})$$

and suppose

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then

$$A = A + \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} C, \; A\begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} + B = B + \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} D, \; C = C, \; C\begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} + D = D .$$

Therefore

$$\begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ s_2 c_3 & s_2 c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} 0 & s_2 c_2 \\ 0 & s_2 c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} 0 & s_2 a_2 \\ 0 & s_2 a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ s_2 d_3 & s_2 d_4 \end{pmatrix}.$$

Therefore $c_2 = c_3 = c_4 = 0$, $a_2 = d_3 = 0$, $a_4 = d_4$. Since $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \boldsymbol{R})$, $a_4 = d_4$ if and only if $a_4 = d_4 = \pm 1$. Hence we have proved our assertion.

LEMMA 3. *Let S be an integral symmetric matrix of size two with* $-\det(S) \neq 0$, $\notin (\boldsymbol{Q}^{\times})^2$. *Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of $Sp(2, \boldsymbol{Q})$ and suppose*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}$$

with some $\begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix} \in Sp(2, \boldsymbol{Q})$. Then $C = 0$.

PROOF. Since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix},$$

we have $C\,{}^tD - D\,{}^tC - CS\,{}^tC = 0$. Since $C\,{}^tD = D\,{}^tC$, we have $CS\,{}^tC = 0$. Since

$-\det(S)\neq 0$, $\notin(\boldsymbol{Q}^\times)^2$, S does not represent 0 in the rational number field. Therefore $C=0$. Q.E.D.

LEMMA 4. *Let S be a symmetric matrix of size two with* $\det(S)\neq 0$. *Suppose* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is an element of* $Sp(2, \boldsymbol{R})$ *such that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}=\begin{pmatrix} 1 & S' \\ 0 & 1 \end{pmatrix}.$$

Then $C=0$.

PROOF. The above condition is equivalent to $A=A+SC$, $AS+B=B+SD$, $C=C$, $CS+D=D$. Therefore we have $SC=0$ and consequently $C=0$. Q.E.D.

Now let S be a symmetric matrix of size two. We call $G\in GL(2, \boldsymbol{Z})$ (resp. $G\in SL(2, \boldsymbol{Z})$) a *unit* (resp. a *proper unit*) of S if G satisfies $GS{}^tG=S$.

PROPOSITION 16. *Let S be an integral symmetric matrix of size two with* $\det(S)\neq 0$. *Then* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(2, \boldsymbol{R})$ *centralizes* $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$ *if and only if $C=0$ and $A={}^tD^{-1}$ satisfies $AS{}^tA=S$. In other words,* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(2, \boldsymbol{Z})$ *centralizes* $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$ *if and only if* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_\infty^0$ *and A is a unit of S.*

This proposition is a direct consequence of Lemma 4.

PROPOSITOIN 17. *Let S be an integral symmetric matrix of size two. If* $-\det(S)\in(\boldsymbol{Q}^\times)^2$, *$S$ has no proper unit other than ± 1. If* $\det(S)$ *is positive, put*

$$S=t\begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix}$$

as in Proposition 14. *Then the number of the proper units of S is equal to the number of the units of the order of the quadratic field of discriminant $D(S)$. If* $\det(S)$ *is negative and* $-\det(S)\notin(\boldsymbol{Q}^\times)^2$, *put*

$$S=t\begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix}$$

as in Proposition 14 *and put* $S={}^tV\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}{}^tV$ *with* $V\in SL(2, \boldsymbol{R})$. *Then G is a proper unit of S if and only if*

$$G=V\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}V^{-1},$$

where ε is a unit of the order of the real quadratic field of discriminant $D(S)$ with positive norm.

Since the results of the proposition are well-known, we omit the proof.

PROPOSITION 18. *Let γ be an element of $Sp(2, \mathbf{Z})$ which is conjugate in $Sp(2, \mathbf{Z})$ to*

$$\begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \quad (s_{12} \neq 0) .$$

If γ belongs to $\Gamma_\infty^1 - \Gamma_\infty^0$ or $\Gamma_\infty^1 \cap \Gamma_\infty^0 \cap \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$, there exist

$$\varepsilon_1 \in \Gamma_\infty^1 \cap \left\{\begin{pmatrix} * & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

and $\varepsilon_2 \in \Gamma_\infty^0 \cap \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$ such that

$$\varepsilon_1 \gamma \varepsilon_1^{-1} = \begin{pmatrix} 1 & 0 & b_1 & b_{12} \\ 0 & 1 & b_{12} & b_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \left(b_1 \neq 0 \text{ if } \gamma \notin \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}\right)$$

and

$$(\varepsilon_2 \varepsilon_1) \gamma (\varepsilon_2 \varepsilon_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} .$$

If γ belongs to $\Gamma_\infty^0 \cap \Gamma_\infty^1 - \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$, then there exists $\varepsilon_3 \in \Gamma_\infty^1$ such that

$$\varepsilon_3 \gamma \varepsilon_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} .$$

If γ belongs to $\Gamma_\infty^0 - \Gamma_\infty^1$, there exist $\varepsilon_4 \in \Gamma_\infty^0 \cap \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$ and $\varepsilon_5 \in \Gamma_\infty^1$ such that

$$\varepsilon_4 \gamma \varepsilon_4^{-1} \in \left\{\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & * & * \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & & \\ * & 1 & \multicolumn{2}{c}{0} \\ & & 1 & * \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}\right\}$$

and

$$(\varepsilon_5\varepsilon_4)\gamma(\varepsilon_5\varepsilon_4)^{-1}=\begin{pmatrix}1&0&0&s_{12}\\0&1&s_{12}&s_2\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

PROOF. Let γ be an element of $\Gamma^0_\infty\cap\Gamma^1_\infty-\left\{\begin{pmatrix}1&*\\0&1\end{pmatrix}\right\}$. Then we may write

$$\gamma=\begin{pmatrix}1&0&0&\beta_{12}\\0&1&\beta_{12}&\beta_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha&1&0&0\\0&0&1&-\alpha\\0&0&0&1\end{pmatrix}\quad(\alpha\neq0)\ .$$

(If

$$\gamma=\begin{pmatrix}1&0&\beta_1&\beta_{12}\\0&1&\beta_{12}&\beta_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha&1&0&0\\0&0&1&-\alpha\\0&0&0&1\end{pmatrix}\quad(\alpha\neq0,\ \beta_1\neq0)\ ,$$

γ is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\begin{pmatrix}1&0&\beta_1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha&1&0&0\\0&0&1&-\alpha\\0&0&0&1\end{pmatrix}(\alpha\neq0,\ \beta_1\neq0)\ ,$$

which is not conjugate in $Sp(2, \boldsymbol{R})$ to

$$\begin{pmatrix}1&0&0&\beta_{12}\\0&1&\beta_{12}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

(cf. Proposition 7 and Proposition 16).) Since

$$\begin{pmatrix}1&0&\rho&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&\beta_{12}\\0&1&\beta_{12}&\beta_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha&1&0&0\\0&0&1&-\alpha\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&\rho&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}1&0&0&\beta_{12}-\rho\alpha\\0&1&\beta_{12}-\rho\alpha&\beta_2-\rho\alpha^2\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha&1&0&0\\0&0&1&-\alpha\\0&0&0&1\end{pmatrix}$$

and

$$\begin{pmatrix} 0&0&1&0\\ 0&1&0&0\\ -1&0&0&0\\ 0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&\beta_{12}\\ 0&1&\beta_{12}&\beta_2\\ &0&1&0\\ &&0&1 \end{pmatrix}\begin{pmatrix} 1&0&&0\\ \alpha&1&&\\ &0&1&-\alpha\\ &&0&1 \end{pmatrix}\begin{pmatrix} 0&0&1&0\\ 0&1&0&0\\ -1&0&0&0\\ 0&0&0&1 \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} 1&0&0&-\alpha\\ 0&1&-\alpha&\beta_2-2\alpha\beta_{12}\\ &0&1&0\\ &&0&1 \end{pmatrix}\begin{pmatrix} 1&0&&0\\ \beta_{12}&1&&\\ &0&1&-\beta_{12}\\ &&0&1 \end{pmatrix},$$

we see that there exists $\varepsilon_3'\in\Gamma_\infty^1$ such that

$$\varepsilon_3'\gamma\varepsilon_3'^{-1}=\begin{pmatrix} 1&0&0&s_{12}'\\ 0&1&s_{12}'&s_2'\\ &0&1&0\\ &&0&1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1&0&&0\\ \rho&\pm1&&\\ &0&1&\mp\rho\\ &&0&\pm1 \end{pmatrix}\begin{pmatrix} 1&0&0&s_{12}'\\ 0&1&s_{12}'&s_2'\\ &0&1&0\\ &&0&1 \end{pmatrix}\begin{pmatrix} 1&0&&0\\ \rho&\pm1&&\\ &0&1&\mp\rho\\ &&0&\pm1 \end{pmatrix}^{-1}=\begin{pmatrix} 1&0&0&\pm s_{12}'\\ 0&1&\pm s_{12}'&s_2'\pm2\rho s_{12}'\\ &0&1&0\\ &&0&1 \end{pmatrix},$$

there exists $\varepsilon_3''\in\Gamma_\infty^1$ such that

$$(\varepsilon_3''\varepsilon_3')\gamma(\varepsilon_3''\varepsilon_3')^{-1}=\begin{pmatrix} 1&0&0&s_{12}''\\ 0&1&s_{12}''&s_2''\\ &0&1&0\\ &&0&1 \end{pmatrix}$$

with $s_{12}''>0$, $2s_{12}''>s_2''\geqq0$. Then, by Lemma 4 and Proposition 14, $s_2''=s_2$ and $s_{12}''=s_{12}$, and hence $\varepsilon_3=\varepsilon_3''\varepsilon_3'$ satisfies the above condition.

Since the existence of ε_1, ε_2, ε_4 and ε_5 is easy, we left it to the reader.

PROPOSITION 19. *Let γ be an element of $Sp(2, \boldsymbol{Z})$ that is conjugate in $Sp(2, \boldsymbol{Z})$ to*

$$\begin{pmatrix} 1&0&s_1&s_{12}\\ 0&1&s_{12}&s_2\\ &0&1&0\\ &&0&1 \end{pmatrix}\quad(s_1,\ s_{12},\ s_2\in\boldsymbol{Z}).$$

Suppose $s_{12}^2 - s_1 s_2 \neq 0,\ \notin (Q^\times)^2$. *Then, if* γ *belongs to* Γ_∞^0, γ *has the form*

$$\begin{pmatrix} 1 & 0 & s_1' & s_{12}' \\ 0 & 1 & s_{12}' & s_2' \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \quad (s_1',\ s_{12}',\ s_2' \in Z) .$$

If γ *belongs to* Γ_∞^1, *there exists*

$$\varepsilon = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, Z)$ *such that*

$$\varepsilon\gamma\varepsilon^{-1} = \begin{pmatrix} 1 & 0 & s_1'' & s_{12}'' \\ 0 & 1 & s_{12}'' & s_2'' \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} .$$

PROOF. Since γ is unipotent, the proposition is an easy consequence of Lemma 3.

3.3. Some results on Dirichlet series. Let $\mathfrak{O}$ be the order of the quadratic field of discriminant d and let $h(d)$ be the class number of $\mathfrak{O}$. If $d<0$, let $w(d)$ be the order of the unit group of $\mathfrak{O}$. If $d>0$, let $\varepsilon(d)>1$ be the generator of the unit group of $\mathfrak{O}$ with positive norm. Let $\zeta(s)$ be the Riemann zeta function. Put

$$\xi_1(L, s) = \zeta(2s) \left\{ \sum_{k=1}^{\infty} \frac{h(-4k)}{w(-4k)k^s} + \sum_{\substack{d>0 \\ d\equiv 3 \pmod 4}} \frac{h(-d)}{w(-d)d^s} \right\} ,$$

$$\begin{aligned} \xi_2(L, s) = \zeta(2s) &\left\{ \sum_{k=1}^{\infty} \frac{h(4k)\log\varepsilon(4k)}{k^s} + \sum_{\substack{d>0 \\ d\equiv 1 \pmod 4}} \frac{h(d)\log\varepsilon(d)}{d^s} \right\} \\ &+ \frac{2^{1-2s}-1}{1-2^{-2s}} \log 2\zeta(2s-1) + 2\zeta(2s-1)\zeta'(2s)\zeta(2s)^{-1} - 2\zeta'(2s-1) , \end{aligned}$$

$$\xi_1(\hat{L}, s) = \zeta(2s)4^s \sum_{\substack{d>0 \\ d\equiv 0 \text{ or } 3 \pmod 4}} \frac{h(-d)}{w(-d)d^s}$$

and

$$\begin{aligned} \xi_2(\hat{L}, s) = \zeta(2s)4^s &\sum_{\substack{d>0 \\ d\equiv 0 \text{ or } 1 \pmod 4}} \frac{h(d)\log\varepsilon(d)}{d^s} - 2^{2s}\log 2\zeta(2s-1) \\ &+ 2^{2s}\zeta(2s-1)\zeta'(2s)\zeta(2s)^{-1} - 2^{2s}\zeta'(2s-1) . \end{aligned}$$

These are the zeta functions of a pre-homogeneous vector space and the following results are known (cf. Shintani [17]):

(i) $\xi_1(L, s)$, $\xi_2(L, s)$, $\xi_1(\hat{L}, s)$ and $\xi_2(\hat{L}, s)$ are absolutely convergent for $\mathrm{Re}(s)>3/2$. They are analytically continued to the whole plane as meromorphic functions that are holomorphic except $s=1$ and $3/2$.

(ii) Their Laurent expansions near $s=1$ or $3/2$ are given by

$$\xi_1(L, s)=\frac{\pi}{6}\left(s-\frac{3}{2}\right)^{-1}+\cdots=-\frac{1}{4}(s-1)^{-1}+\cdots,$$

$$\xi_2(L, s)=\frac{\pi^2}{3}\left(s-\frac{3}{2}\right)^{-1}+\cdots=-\frac{1}{2}(s-1)^{-2}-\{2\log 2+\log 2\pi\}(s-1)^{-1}+\cdots.$$

$$\xi_1(\hat{L}, s)=\frac{\pi}{3}\left(s-\frac{3}{2}\right)^{-1}+\cdots=-\frac{1}{2}(s-1)^{-1}+\cdots$$

and

$$\xi_2(\hat{L}, s)=\frac{2}{3}\pi^2\left(s-\frac{3}{2}\right)^{-1}+\cdots=-(s-1)^{-2}-\{2\log 2+\log 2\pi\}(s-1)^{-1}+\cdots.$$

(iii) They satisfy the following functional equations:

$$\xi_1\left(L, \frac{3}{2}-s\right)=\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)\pi^{(1/2)-2s}\{2^{1-2s}\cos \pi s\xi_1(\hat{L}, s)-2^{-1}\zeta(2s-1)\}.$$

$$\xi_2\left(L, \frac{3}{2}-s\right)=\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)\pi^{(1/2)-2s}$$
$$\times\left\{2^{1-2s}(2\pi\xi_1(\hat{L}, s)+\sin \pi s\xi_2(\hat{L}, s))+\sin \pi s\left(\frac{\Gamma'(s)}{\Gamma(s)}-\frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)}\right)\zeta(2s-1)\right\}.$$

§ 4. Estimates of infinite series

4.1. Elementary estimates.

LEMMA 5. *Let a, c be positive real numbers, let b be a real number and let r be a real number greater than $1/2$. Then there exists a positive constant $\kappa=\kappa(r)$ depending only on r and satisfying*

$$\sum_{x=-\infty}^{\infty}\frac{1}{\{a(x+b)^2+c\}^r}\leqq\kappa\cdot\mathrm{Max}\left(\frac{1}{a^{1/2}c^{r-(1/2)}}, \frac{1}{c^r}\right).$$

PROOF. Let $[-b]$ be the largest integer which is not larger than $-b$. Then

$$\frac{1}{\{a(x+b)^2+c\}^r}\begin{cases}\leqq\displaystyle\int_x^{x+1}\frac{dt}{\{a(t+b)^2+c\}^r}\cdots \text{ if } x\leqq[-b]-1,\\ \leqq\dfrac{1}{c^r}\cdots \text{ if } [-b]-1<x\leqq[-b]+2,\end{cases}$$

$$\left| \leqq \int_{x-1}^{x} \frac{dt}{\{a(t+b)^2+c\}^r} \cdots \text{ if } [-b]+2 \leqq x \right. .$$

Hence we have

$$\sum_{x=-\infty}^{\infty} \frac{1}{\{a(x+b)^2+c\}^r} \leqq \frac{3}{c^r} + \int_{-\infty}^{\infty} \frac{dt}{(at^2+c)^r} .$$

Therefore, if we put

$$\kappa = 3 + \int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^r} ,$$

we have our assertion.

Let Y_1, Y_2 be two hermitian matrices. We write $Y_1 \geqq Y_2$ if $Y_1 - Y_2$ is positive semi-definite. We see that, if $Y_1 \geqq Y_2 \geqq 0$, $\det(Y_1) \geqq \det(Y_2)$.

LEMMA 6. *Let Y be a real positive reduced symmetric matrix of size two, i.e.,*

$$Y = \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}$$

with $0 \leqq 2y_{12} \leqq y_1 \leqq y_2$, $0 < y_1$. Then we have

$$2\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \geqq Y = \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix} \geqq \frac{1}{2}\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} > 0 .$$

PROOF. Since

$$S = \begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix} \geqq 0$$

if $s_1 > 0$ and $s_1 s_2 - s_{12}^2 \geqq 0$, our lemma follows immediately.

LEMMA 7. *Let*

$$X = \begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}$$

be two real symmetric matrices. Suppose Y is positive definite. Then we have

$$\operatorname{abs}|X+iY|^2 \geqq \det(Y)^2\{1+\operatorname{tr}(Y^{-1}XY^{-1}X)\} .$$

Moreover, the right-hand side is equal to

$$(y_1y_2 - y_{12}^2)^2 + y_2^2(x_1 - 2x_{12}y_2^{-1}y_{12} + x_2y_2^{-2}y_{12}^2)^2$$
$$+ y_2^{-2}(y_1y_2 - y_{12}^2)(y_1y_2 + y_{12}^2)\left(x_2 - \frac{2x_{12}y_2y_{12}}{y_1y_2 + y_{12}^2}\right)^2 + 2(y_1y_2 + y_{12}^2)^{-1}(y_1y_2 - y_{12}^2)^2 x_{12}^2$$

$$= (y_1y_2-y_{12}^2)^2+y_2^2(x_1-2x_{12}y_2^{-1}y_{12}+x_2y_2^{-2}y_{12}^2)^2$$
$$+2(y_1y_2-y_{12}^2)(x_{12}-y_2^{-1}y_{12}x_2)^2+y_2^{-2}(y_1y_2-y_{12}^2)^2x_2^2$$
$$= (y_1y_2-y_{12}^2)^2+2(y_1y_2+y_{12}^2)\left\{x_{12}-\frac{x_1y_2+x_2y_1}{y_1y_2+y_{12}^2}y_{12}\right\}^2$$
$$+(y_1y_2+y_{12}^2)^{-1}y_2^2(y_1y_2-y_{12}^2)(x_1-y_2^{-2}y_{12}^2x_2)^2+(y_1y_2-y_{12}^2)^2y_2^{-2}x_2^2 .$$

PROOF. Since Y is positive definite, there is a matrix $T\in M_2(\boldsymbol{R})$ satisfying $Y=T^tT$. Since $T^{-1}X^tT^{-1}$ is real symmetric, there is an orthogonal matrix V such that $(V^{-1}T^{-1})X^t(V^{-1}T^{-1})=D$ is a diagonal matrix

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Then

$$\begin{aligned} \text{abs}\{\det(X+iY)\}^2 &= \text{abs}[\det\{(TV)(D+i1)^t(TV)\}]^2 \\ &= \det(TV)^4(1+d_1^2)(1+d_2^2) \\ &\geqq \det(Y)^2(1+d_1^2+d_2^2) \\ &= \det(Y)^2\{1+\text{tr}(D^2)\} \\ &= \det(Y)^2\{1+\text{tr}(V^{-1}T^{-1}X^tT^{-1\,t}V^{-1}V^{-1}T^{-1}X^tT^{-1\,t}V^{-1})\} \\ &= \det(Y)^2\{1+\text{tr}(Y^{-1}XY^{-1}X)\} . \end{aligned}$$

Since

$$Y^{-1}=\frac{1}{(y_1y_2-y_{12}^2)}\begin{pmatrix} y_2 & -y_{12} \\ -y_{12} & y_1 \end{pmatrix},$$

we have

$$XY^{-1}=\frac{1}{(y_1y_2-y_{12}^2)}\begin{pmatrix} x_1y_2-x_{12}y_{12} & -x_1y_{12}+x_{12}y_1 \\ x_{12}y_2-x_2y_{12} & -x_{12}y_{12}+x_2y_1 \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} &(y_1y_2-y_{12}^2)^2\text{tr}(XY^{-1}XY^{-1}) \\ &= (x_1y_2-x_{12}y_{12})^2+2(-x_1y_{12}+x_{12}y_1)(x_{12}y_2-x_2y_{12})+(-x_{12}y_{12}+x_2y_1)^2 \\ &= y_2^2(x_1-2x_{12}y_2^{-1}y_{12}+x_2y_2^{-2}y_{12}^2)^2 \\ &\quad +(y_1{}^2-y_2^{-2}y_{12}^4)x_2^2-4x_{12}y_{12}(y_1-y_2^{-1}y_{12}^2)x_2+x_{12}^2(2y_1y_2-2y_{12}^2) \\ &= y_2^2(x_1-2x_{12}y_2^{-1}y_{12}+x_2y_2^{-2}y_{12}^2)^2 \\ &\quad +y_2^{-2}(y_1y_2-y_{12}^2)(y_1y_2+y_{12}^2)\left(x_2-\frac{2x_{12}y_2y_{12}}{y_1y_2+y_{12}^2}\right)^2 \\ &\quad +2(y_1y_2+y_{12}^2)^{-1}(y_1y_2-y_{12}^2)^2x_{12}^2 . \end{aligned}$$

Hence we have proved the first two assertions. Since the remaining parts can be proved in the same manner, we omit the proof.

Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \left(\text{resp.} \begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}\right)$$

be an element of $Sp(2, \boldsymbol{Z})$ $\left(\text{resp. } Sp(2, \boldsymbol{Z}) \cap \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} = \Gamma_\infty^0\right)$. Then, if we put

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and $Z = X + iY$ is an element of the Siegel upper-half plane of degree two, we have $(A'Z+B')(C'Z+D')^{-1} = U(AZ+B)(CZ+D)^{-1}\,{}^tU + S$. Hence we have

$$Z - (A'\bar{Z}+B')(C'\bar{Z}+D')^{-1} = X - S - \mathrm{Re}\{U(A\bar{Z}+B)(C\bar{Z}+D)^{-1}\,{}^tU\}$$
$$+ i\,[Y + U\,{}^t(C\bar{Z}+D)^{-1}Y(CZ+D)^{-1}\,{}^tU]\,.$$

Now we have the following

PROPOSITION 20. *Let $Z = X + iY$ be an element of the Siegel upper-half-plane of degree two, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ an element of $Sp(2, \boldsymbol{Z})$. Let k be a positive number $\geqq 3$. Suppose*

$$Y = \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix} \geqq \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ with } \mu \leqq 1\,.$$

Then there exists a constant κ_1 depending only on k and satisfying

$$\sum_{\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma_\infty^0 \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}} \mathrm{abs}\,|Z - (A'\bar{Z}+B')(C'\bar{Z}+D')^{-1}|^{-k}$$
$$\leqq \kappa_1 \mu^{-3} \sum_{U \in GL(2, \boldsymbol{Z})} \det[Y + U\,{}^t(C\bar{Z}+D)^{-1}Y(CZ+D)^{-1}\,{}^tU]^{-k+3/2}\,.$$

PROOF. By the preceding remark, it is enough to prove that there exists a positive constant κ_1 depending only on k and satisfying

$$I = \sum_{\substack{S \in M_2(\boldsymbol{Z}) \\ S\,:\,\text{symmetric}}} \mathrm{abs}\,|X_0 + S + iY_0|^{-k} \leqq k_1 \mu^{-3} \det(Y_0)^{-k+3/2}$$

for all $X_0 + iY_0$ of the Siegel upper-half-plane with $Y_0 \geqq \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Here we may assume Y_0 is reduced since we may replace X_0, Y_0, S by $VX_0\,{}^tV$, $VY_0\,{}^tV$, $VS\,{}^tV$ with any $V \in GL(2, \boldsymbol{Z})$. Now put

$$S = \begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix}, \quad X_0 = \begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix}, \quad S + X_0 = \begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}.$$

Then, by Lemma 7, we have

$$I \leqq \sum_{s_1, s_{12}, s_2 \in Z} \left\{ (y_1 y_2 - y_{12}^2)^2 + y_2^2 (t_1 - 2t_{12} y_2^{-1} y_{12} + t_2 y_2^{-2} y_{12}^2)^2 \right.$$
$$+ y_2^{-2} (y_1 y_2 - y_{12}^2)(y_1 y_2 + y_{12}^2) \left(t_2 - \frac{2t_{12} y_2 y_{12}}{y_1 y_2 + y_{12}^2} \right)^2$$
$$\left. + 2(y_1 y_2 + y_{12}^2)^{-1} (y_1 y_2 - y_{12}^2)^2 t_{12}^2 \right\}^{-k/2} .$$

Here, since $0 \leqq 2y_{12} \leqq y_1 \leqq y_2$ and

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \leqq \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix} ,$$

we have

$$2^{-1} y_1 y_2 \leqq y_1 y_2 \pm y_{12}^2 \leqq 2 y_1 y_2 , \quad \mu \leqq y_1 \leqq y_2 .$$

Hence we have

$$(y_1 y_2 - y_{12}^2)^2 \geqq 2^{-2} y_1^2 y_2^2 ,$$
$$y_2{}^2 \geqq 2^{-2} y_2^2 \geqq 2^{-2} \operatorname{Min}(1, y_1^2) y_2^2 ,$$
$$y_2^{-2} (y_1 y_2 - y_{12}^2)(y_1 y_2 + y_{12}^2) \geqq 2^{-2} y_1^2 = 2^{-2} y_1^2 \operatorname{Min}(1, y_2^2) ,$$
$$2(y_1 y_2 + y_{12}^2)^{-1} (y_1 y_2 - y_{12}^2)^2 \geqq 2^{-2} y_1 y_2 \geqq 2^{-2} y_1 y_2 \operatorname{Min}(1, y_1 y_2) .$$

Therefore, by Lemma 5,

$$I \leqq 2^k \sum_{s_1, s_{12}, s_2 \in Z} \left\{ y_1^2 y_2^2 + \operatorname{Min}(1, y_1^2) y_2^2 (t_1 - 2t_{12} y_2^{-1} y_{12} + t_2 y_2^{-2} y_{12}^2)^2 \right.$$
$$\left. + y_1^2 \operatorname{Min}(1, y_2^2) \left(t_2 - \frac{2t_{12} y_2 y_{12}}{y_1 y_2 + y_{12}^2} \right)^2 + y_1 y_2 \operatorname{Min}(1, y_1 y_2) t_{12}^2 \right\}^{-k/2}$$
$$\leqq 2^k \kappa\left(\frac{k}{2}\right) \kappa\left(\frac{k}{2} - \frac{1}{2}\right) \kappa\left(\frac{k}{2} - 1\right) (y_1 y_2)^{-k+3/2} \mu^{-3}$$
$$\leqq 2^k \kappa\left(\frac{k}{2}\right) \kappa\left(\frac{k}{2} - \frac{1}{2}\right) \kappa\left(\frac{k}{2} - 1\right) 2^{k-3/2} (y_1 y_2 - y_{12}^2)^{-k+3/2} \mu^{-3} .$$

Hence we have proved our assertion.

LEMMA 8. *Let the notation be as before. Suppose that Y is reduced. Then we have the following inequality:*

$$2^2 \det \{ Y + U^t (C\bar{Z} + D)^{-1} Y (CZ + D)^{-1}\, {}^t U \}$$
$$\geqq \det(Y) [1 + \operatorname{tr}\{ (T^{-1} U^t (C\bar{Z} + D)^{-1} T)^t \, \overline{(T^{-1} U (C\bar{Z} + D)^{-1} T)} \} ,$$

where

$$T = \begin{pmatrix} y_1^{1/2} & 0 \\ 0 & y_2^{1/2} \end{pmatrix} .$$

PROOF. Since

$$Y \geqq 2^{-1}\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix},$$

we have

$$\begin{aligned} &2^2 \det\{Y + U^t(C\bar{Z}+D)^{-1}Y(CZ+D)^{-1\,t}U\} \\ &\quad \geqq \det\{T^tT + U^t(C\bar{Z}+D)^{-1}T^tT(CZ+D)^{-1\,t}U\} \\ &\quad = \det(Y)\det\{1 + (T^{-1}U^t(C\bar{Z}+D)^{-1}T)^t\overline{(T^{-1}U^t(C\bar{Z}+D)^{-1}T)}\} \\ &\quad \geqq \det(Y)[1 + \operatorname{tr}\{(T^{-1}U^t(C\bar{Z}+D)^{-1}T)^t\overline{(T^{-1}U^t(C\bar{Z}+D)^{-1}T)}\}]. \end{aligned}$$

Q.E.D.

LEMMA 9. *Let the notation be as in Lemma* 8. *Put*

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

Then we have the following equality:

$$\begin{aligned} &\operatorname{tr}\{(T^{-1}UWT)^t\overline{(T^{-1}UWT)}\} \\ &= \left(|\alpha_1|^2 + \frac{y_2}{y_1}|\alpha_2|^2\right)\left(u_1 + \frac{\operatorname{Re}\left(\alpha_1\bar{\alpha}_3 + \frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)}{|\alpha_1|^2 + \frac{y_2}{y_1}|\alpha_2|^2}u_2\right)^2 \\ &\quad + \frac{\operatorname{Im}(\alpha_1\bar{\alpha}_3)^2 + \frac{y_2{}^2}{y_1{}^2}\operatorname{Im}(\alpha_2\bar{\alpha}_4)^2 + \frac{1}{2}\frac{y_2}{y_1}|\alpha_1\alpha_4 - \alpha_2\alpha_3|^2 + \frac{1}{2}\frac{y_2}{y_1}|\alpha_1\bar{\alpha}_4 - \bar{\alpha}_2\alpha_3|^2}{|\alpha_1|^2 + \frac{y_2}{y_1}|\alpha_2|^2}u_2^2 \\ &\quad + \left(\frac{y_1}{y_2}|\alpha_1|^2 + |\alpha_2|^2\right)\left(u_3 + \frac{\operatorname{Re}\left(\frac{y_1}{y_2}\alpha_1\bar{\alpha}_3 + \alpha_2\bar{\alpha}_4\right)}{\frac{y_1}{y_2}|\alpha_1|^2 + |\alpha_2|^2}u_4\right)^2 \\ &\quad + \frac{\frac{y_1^2}{y_2^2}\operatorname{Im}(\alpha_1\bar{\alpha}_3)^2 + \operatorname{Im}(\alpha_2\bar{\alpha}_4)^2 + \frac{1}{2}\frac{y_1}{y_2}|\alpha_1\alpha_4 - \alpha_2\alpha_3|^2 + \frac{1}{2}\frac{y_1}{y_2}|\alpha_1\bar{\alpha}_4 - \bar{\alpha}_2\alpha_3|^2}{\frac{y_1}{y_2}|\alpha_1|^2 + |\alpha_2|^2}u_4^2. \end{aligned}$$

PROOF.

$$\begin{aligned} &\operatorname{tr}\{(T^{-1}UWT)^t\overline{(T^{-1}UWT)}\} = |\alpha_1u_1 + \alpha_3u_2|^2 + \frac{y_2}{y_1}|\alpha_2u_1 + \alpha_4u_2|^2 \\ &\quad + \frac{y_1}{y_2}|\alpha_1u_3 + \alpha_3u_4|^2 + |\alpha_2u_3 + \alpha_4u_4|^2 \\ &= \left(|\alpha_1|^2 + \frac{y_2}{y_1}|\alpha_2|^2\right)u_1{}^2 + 2\operatorname{Re}\left(\alpha_1\bar{\alpha}_3 + \frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)u_1u_2 + \left(|\alpha_3|^2 + \frac{y_2}{y_1}|\alpha_4|^2\right)u_2^2 \end{aligned}$$

$$+\left(\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2\right)u_3{}^2+2\,\mathrm{Re}\left(\frac{y_1}{y_2}\alpha_1\bar{\alpha}_3+\alpha_2\bar{\alpha}_4\right)u_3u_4+\left(\frac{y_1}{y_2}|\alpha_3|^2+|\alpha_4|^2\right)u_4^2$$

$$=\left(|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2\right)\left(u_1+\frac{\mathrm{Re}\left(\alpha_1\bar{\alpha}_3+\frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)}{|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2}u_2\right)^2+\left(|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2\right)^{-1}$$

$$\times\left\{\left(|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2\right)\left(|\alpha_3|^2+\frac{y_2}{y_1}|\alpha_4|^2\right)-\mathrm{Re}\left(\alpha_1\bar{\alpha}_3+\frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)^2\right\}u_2^2$$

$$+\left(\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2\right)\left(u_3+\frac{\mathrm{Re}\left(\frac{y_1}{y_2}\alpha_1\bar{\alpha}_3+\alpha_2\bar{\alpha}_4\right)}{\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2}u_4\right)^2+\left(\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2\right)^{-1}$$

$$\times\left\{\left(\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2\right)\left(\frac{y_1}{y_2}|\alpha_3|^2+|\alpha_4|^2\right)-\mathrm{Re}\left(\frac{y_1}{y_2}\alpha_1\bar{\alpha}_3+\alpha_2\alpha_4\right)^2\right\}u_2^2\,.$$

Here we have

$$\left(|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2\right)\left(|\alpha_3|^2+\frac{y_2}{y_1}|\alpha_4|^2\right)-\mathrm{Re}\left(\alpha_1\bar{\alpha}_3+\frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)^2$$

$$=\{|\alpha_1\alpha_3|^2-\mathrm{Re}\,(\alpha_1\bar{\alpha}_3)^2\}+\frac{y_2{}^2}{y_1{}^2}\{|\alpha_2\alpha_4|^2-\mathrm{Re}\,(\alpha_2\bar{\alpha}_4)^2\}$$

$$+\frac{y_2}{y_1}\left\{|\alpha_1\alpha_4|^2+|\alpha_2\alpha_3|^2-\frac{1}{2}(\alpha_1\bar{\alpha}_3+\bar{\alpha}_1\alpha_3)(\alpha_2\bar{\alpha}_4+\bar{\alpha}_2\alpha_4)\right\}$$

$$=\mathrm{Im}\,(\alpha_1\bar{\alpha}_3)^2+\frac{y_2{}^2}{y_1{}^2}\mathrm{Im}\,(\alpha_2\bar{\alpha}_4)^2+\frac{1}{2}\frac{y_2}{y_1}|\alpha_1\alpha_4-\alpha_2\alpha_3|^2+\frac{1}{2}\frac{y_2}{y_1}|\alpha_1\bar{\alpha}_4-\bar{\alpha}_2\alpha_3|^2\,.$$

Hence, by symmetry, we have our assertion.

Let $\mathfrak{H}$ be the Siegel upper-half-plane of degree two: $\{Z\in M_2(\boldsymbol{C})\,|\,{}^tZ=Z,\ \mathrm{Im}\,(Z)>0\}$. Let μ be a positive number smaller than 1. We define a subset $\mathfrak{H}(\mu)$ of $\mathfrak{H}$ by

$$\mathfrak{H}(\mu)=\left\{Z\in\mathfrak{H}\,|\,Y\geqq\begin{pmatrix}\mu&0\\0&\mu\end{pmatrix},\ \ y_2\geqq y_1\geqq 2y_{12}\geqq 0\right\}\,.$$

We note here that $\mathfrak{H}\left(\frac{\sqrt{3}}{4}\right)$ contains a fundamental domain of the Siegel modular group of degree two.

PROPOSITION 21. *Let k be a positive number $\geqq 7$ and let Z be an element of $\mathfrak{H}(\mu)$. Then there exists a positive constant κ_2 depending only on k and satisfying the following inequality:*

$$\sum_{\left\{\substack{\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in Sp(2,Z)\\ |C|\neq 0}\right.}\frac{\det(Y)^{k-3}}{\mathrm{abs}\,|Z-(A\bar{Z}+B)(C\bar{Z}+D)^{-1}|^k\,\mathrm{abs}\,|C\bar{Z}+D|^k}$$

$$\leqq \kappa_2 \mu^{-6} \frac{1}{\det(Y)^3} \sum_{\substack{(C, D) \\ |C| \neq 0}} \frac{1}{\operatorname{abs}|CZ+D|^{k-3}},$$

where (C, D) *runs over all non-associative mutually prime symmetric pairs with* $\det(C) \neq 0$. *We note that the last series converges by the result of H. Braun.*

PROOF. By Proposition 20, we have

$$I = \sum_{\substack{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2, Z) \\ |C| \neq 0}} \frac{|Y|^{k-3}}{\operatorname{abs}|Z-(A\bar{Z}+B)(C\bar{Z}+D)^{-1}|^k \operatorname{abs}|C\bar{Z}+D|^k}$$

$$\leqq \kappa_1 \mu^{-3} \sum_{\substack{(C, D) \\ |C| \neq 0}} \sum_{U} \frac{|Y|^{k-3} \operatorname{abs}|C\bar{Z}+D|^{-k}}{|Y+U^t(C\bar{Z}+D)^{-1}Y(CZ+D)^{-1}\,{}^tU|^{k-3/2}}.$$

Here we may assume $C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_4 \end{pmatrix}$, $c_1 > 0$, $c_4 > c_2 \geqq 0$ since any C with $|C| \neq 0$ can be transformed to such a representative by left multiplication of a certain unimodular matrix. Let $*$ be the main involution of $M_2(\boldsymbol{C})$ i.e., $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$. Then we have

$$\begin{aligned} {}^t(C\bar{Z}+D)^{-1}Y(CZ+D)^{-1} &= \frac{1}{\operatorname{abs}|CZ+D|^2}\,{}^t(C\bar{Z}+D)^* Y(CZ+D)^* \\ &= \frac{1}{\operatorname{abs}|CZ+D|^2}\{{}^t(CX+D)^* Y(CX+D)^* + {}^t(CY)^* Y(CY)^*\} \\ &\geqq \frac{1}{\operatorname{abs}|CZ+D|^2}\,{}^t(CY)^* Y(CY)^*. \end{aligned}$$

Hence, as in the proof of Lemma 8, we have

$$I \leqq \kappa_1 \mu^{-3} \sum_{\substack{(C, D) \\ |C| \neq 0}} \sum_{U} \frac{|Y|^{k-3} \operatorname{abs}|CZ+D|^{-k}}{\operatorname{abs}|Y+\operatorname{abs}|CZ+D|^{-2} U^t(CY)^* Y(CY)^{*t} U|^{k-3/2}}$$

$$\leqq 2^{2k-3} \kappa_1 \mu^{-3} \sum_{\substack{(C, D) \\ |C| \neq 0}} \sum_{U} \frac{(y_1 y_2)^{-3/2} \operatorname{abs}|CZ+D|^{-k}}{[1+\operatorname{abs}|CZ+D|^{-2} \operatorname{tr}\{(T^{-1}U^t(CY)^*T)^t(T^{-1}U^t(CY)^*T)\}]^{k-3/2}}.$$

Here, by Lemma 9, we have

$$\operatorname{tr}\{(T^{-1}U^t(CY)^*T)^t(\overline{T^{-1}U(CY)^*T})\}$$

$$\geqq \left(|\alpha_1|^2 + \frac{y_2}{y_1}|\alpha_2|^2\right)\left(u_1 + \frac{\operatorname{Re}\left(\alpha_1\bar{\alpha}_3 + \frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)}{|\alpha_1|^2 + \frac{y_2}{y_1}|\alpha_2|^2} u_2\right)^2$$

$$+\frac{1}{2}\left(|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2\right)^{-1}\frac{y_2}{y_1}|\alpha_1\alpha_4-\alpha_2\alpha_3|^2u_2^2$$

$$+\left(\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2\right)\left(u_3+\frac{\mathrm{Re}\left(\frac{y_1}{y_2}\alpha_1\bar{\alpha}_3+\alpha_2\bar{\alpha}_4\right)}{\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2}u_4\right)^2$$

$$+\frac{1}{2}\left(\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2\right)^{-1}\frac{y_1}{y_2}|\alpha_1\alpha_4-\alpha_2\alpha_4|^2u_4^2\,,$$

where

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}={}^t(CY)^*=\begin{pmatrix}c_4y_2 & -c_4y_{12}\\ -c_1y_{12}-c_2y_2 & c_1y_1+c_2y_{12}\end{pmatrix}.$$

Hence,

$$\geqq c_4{}^2y_2{}^2\left(u_1+\frac{\mathrm{Re}\left(\alpha_1\bar{\alpha}_3+\frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)}{|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2}u_2\right)^2+2^{-3}c_4^{-2}y_2^{-2}\frac{y_2}{y_1}c_1^2c_4^2y_1^2y_2^2u_2^2$$

$$+c_4{}^2y_1y_2\left(u_3+\frac{\mathrm{Re}\left(\frac{y_1}{y_2}\alpha_1\bar{\alpha}_3+\alpha_2\bar{\alpha}_4\right)}{\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2}u_4\right)^2+2^{-3}c_4^{-2}y_1^{-1}y_2^{-1}\frac{y_1}{y_2}c_1^2c_4^2y_1^2y_2^2u_4^2$$

$$\geqq y_2{}^2\left(u_1+\frac{\mathrm{Re}\left(\alpha_1\bar{\alpha}_3+\frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)}{|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2}u_2\right)^2+2^{-3}y_1y_2u_2^2$$

$$+y_1y_2\left(u_3+\frac{\mathrm{Re}\left(\frac{y_1}{y_2}\alpha_1\bar{\alpha}_3+\alpha_2\bar{\alpha}_4\right)}{\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2}u_4\right)^2+2^{-3}y_1^2u_4^2\,.$$

Here we note that, since $u_1u_4-u_2u_3=\pm1$, u_3 is determined by u_1, u_4, u_2 if $u_2\neq0$. Moreover, if $u_2=0$, then $u_1=\pm1$ and $u_4=\pm1$. Hence we have

$$I\leqq 2^{2k-3}\kappa_1\mu^{-3}\sum_{(C,D)}(y_1y_2)^{-3/2}\,\mathrm{abs}\,|CZ+D|^{k-3}$$

$$\times\left[2\sum_{u_1,u_2,u_4\in Z}\left\{\mathrm{abs}\,|CZ+D|^2+y_2^2\left(u_1+\frac{\mathrm{Re}\left(\alpha_1\bar{\alpha}_3+\frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)}{|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2}u_2\right)^2\right.\right.$$

$$\left.+2^{-3}y_1y_2u_2^2+2^{-3}y_1^2u_4^2\right\}^{-k+3/2}$$

$$+4\sum_{u_3\in Z}\left\{\text{abs}\,|CZ+D|^2+y_1y_2\left(u_3\pm\frac{\text{Re}\left(\dfrac{y_1}{y_2}\alpha_1\bar{\alpha}_3+\alpha_2\bar{\alpha}_4\right)}{\dfrac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2}\right)^2\right\}^{-k+3/2}\Bigg].$$

Since $\text{abs}|CZ+D|^2\geqq|CY|^2\geqq|Y|^2\geqq 2^{-2}y_1^2y_2^2$, we have

$$I\leqq 2^{5k-15/2}\kappa_1\mu^{-3}\sum_{\{\substack{(C,D)\\|C|\neq 0}}(y_1y_2)^{-3/2}\,\text{abs}|CZ+D|^{k-3}$$
$$\times[2\mu^{-3}\kappa(k-3/2)\kappa(k-2)\kappa(k-5/2)\,\text{abs}|CZ+D|^{-2k+6}y_1^{-3/2}y_2^{-3/2}$$
$$+2^2\mu^{-1}\kappa(k-3/2)\,\text{abs}|CZ+D|^{-2k+4}y_1^{-1/2}y_2^{-1/2}]\,.$$

Hence there exists a constant κ_2 depending only on k such that

$$I\leqq\kappa_2\mu^{-6}2^{-3}(y_1y_2)^{-3}\sum_{\{\substack{(C,D)\\|C|\neq 0}}\text{abs}|CZ+D|^{-k+3}$$

$$\leqq\kappa_2\mu^{-6}\frac{1}{|Y|^3}\sum_{\{\substack{(C,D)\\|C|\neq 0}}\frac{1}{\text{abs}|CZ+D|^{k-3}}\,. \qquad \text{Q.E.D.}$$

PROPOSITION 22. *Let k be a positive number $\geqq 7$ and let Z be an element of $\mathfrak{H}(\mu)$. Then there exists a constant κ_3 depending only on k and satisfying the following inequality:*

$$\sum_{\left\{\substack{\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(2,Z)\\ \det(C)=0\\ C\neq 0}\right.}\frac{\det(Y)^{k-3}}{\text{abs}|Z-(A\bar{Z}+B)(C\bar{Z}+D)^{-1}|^k\,\text{abs}|C\bar{Z}+D|^k}$$

$$\leqq\kappa_3\mu^{-4}\frac{1}{y_1^4y_2}\sum_{\left\{\substack{(C,D)\\ \det(C)=0\\ C\neq 0}\right.}\frac{1}{\text{abs}|CZ+D|^{k-3}}\,,$$

where (C, D) runs over all non-associative mutually prime symmetric pairs with $\det(C)=0$ and $C\neq 0$. Here we may restrict the above sum to any set consisting of left Γ_∞^0-cosets with the condition $|C|=0$, $C\neq 0$.

PROOF. Since $|C|=0$ and $C\neq 0$, there are two unimodular matrices U_1 and U_2 satisfying

$$C=U_1\begin{pmatrix}c_1&0\\0&0\end{pmatrix}{}^tU_2\quad\text{and}\quad D=U_1\begin{pmatrix}d_1&0\\0&1\end{pmatrix}U_2^{-1}\,.$$

Therefore we may assume

$$C=\begin{pmatrix}c_1&0\\0&0\end{pmatrix}{}^tU_2\,,\quad D=\begin{pmatrix}d_1&0\\0&1\end{pmatrix}U_2^{-1}\,.$$

Put

$$U_2=\begin{pmatrix}\beta_1 & \beta_2\\ \beta_3 & \beta_4\end{pmatrix}.$$

Then we have

$${}^t(C\bar{Z}+D)^{-1}=\frac{1}{|CZ+D|}\begin{pmatrix}\pm\beta_1 & \pm\beta_3\\ -c_1\beta_1\bar{z}_{12}-c_1\beta_3\bar{z}_2\pm d_1\beta_2 & c_1\beta_1\bar{z}_1+c_1\beta_3\bar{z}_{12}\pm d_1\beta_4\end{pmatrix}.$$

Hence, by Proposition 20, Lemma 8 and Lemma 9, we have

$$I=\sum_{\left\{\begin{smallmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\\ |C|=0\\ C\neq 0\end{smallmatrix}\right.}\frac{\det(Y)^{k-3}}{\mathrm{abs}|Z-(A\bar{Z}+B)(C\bar{Z}+D)^{-1}|^k\,\mathrm{abs}|C\bar{Z}+D|^k}$$

$$\leqq\kappa_1\mu^{-3}\sum_{\left\{\begin{smallmatrix}(C,D)\\ |C|=0\\ C\neq 0\end{smallmatrix}\right.}\sum_U\frac{|Y|^{k-3}\,\mathrm{abs}\,|C\bar{Z}+D|^{-k}}{|Y+U^t(C\bar{Z}+D)^{-1}Y(CZ+D)^{-1}\,{}^tU|^{k-3/2}}$$

$$\leqq 2^{2k-3}\kappa_1\mu^{-3}\sum_{\left\{\begin{smallmatrix}(C,D)\\ |C|=0\\ C\neq 0\end{smallmatrix}\right.}\sum_U\frac{|Y|^{-3/2}\,\mathrm{abs}\,|C\bar{Z}+D|^{-k}}{\{1+\mathrm{tr}(T^{-1}U^t(C\bar{Z}+D)^{-1}T)^t(T^{-1}U^t(C\bar{Z}+D)^{-1}T)\}^{k-3/2}}$$

$$\leqq 2^{2k-3}\kappa_1\mu^{-3}\sum_{\left\{\begin{smallmatrix}(C,D)\\ |C|=0\\ C\neq 0\end{smallmatrix}\right.}\sum_U |Y|^{-3/2}\,\mathrm{abs}|C\bar{Z}+D|^{-k}$$

$$\times\left[1+\left(|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2\right)\left(u_1+\frac{\mathrm{Re}\left(\alpha_1\bar{\alpha}_3+\frac{y_2}{y_1}\alpha_2\bar{\alpha}_4\right)}{|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2}u_2\right)^2\right.$$

$$+\left(|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2\right)^{-1}\left(\mathrm{Im}(\alpha_1\bar{\alpha}_3)^2+\frac{y_2^2}{y_1^2}\mathrm{Im}(\alpha_2\bar{\alpha}_4)^2+\frac{1}{2}\frac{y_2}{y_1}\mathrm{Im}(\alpha_1\alpha_4-\alpha_2\alpha_3)^2\right)u_2^2$$

$$+\left(\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2\right)\left(u_3+\frac{\mathrm{Re}\left(\frac{y_1}{y_2}\alpha_1\bar{\alpha}_3+\alpha_2\bar{\alpha}_4\right)}{\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2}u_4\right)^2$$

$$\left.+\left(\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2\right)^{-1}\left(\frac{y_1^2}{y_2^2}\mathrm{Im}(\alpha_1\bar{\alpha}_3)^2+\mathrm{Im}(\alpha_2\bar{\alpha}_4)^2+\frac{1}{2}\frac{y_1}{y_2}\mathrm{Im}(\alpha_1\alpha_4-\alpha_2\alpha_3)^2\right)u_4^2\right]^{-k+3/2},$$

where

$$\alpha_1=\frac{\pm 1}{|C\bar{Z}+D|}\beta_1,\quad \alpha_2=\frac{\pm 1}{|C\bar{Z}+D|}\beta_3,$$

$$\alpha_3=\frac{1}{|C\bar{Z}+D|}(-c_1\beta_1\bar{z}_{12}-c_1\beta_3\bar{z}_2\pm d_1\beta_2)$$

and

$$\alpha_4=\frac{1}{|C\bar{Z}+D|}(c_1\beta_1\bar{z}_1+c_1\beta_3\bar{z}_{12}\pm d_1\beta_4)\ .$$

If $\beta_1=0$, then β_2, $\beta_3=\pm1$. Therefore $\alpha_1=0$,

$$\alpha_2=\frac{\pm1}{|C\bar{Z}+D|}\ ,\quad \alpha_3=\frac{1}{|C\bar{Z}+D|}(\pm c_1\bar{z}_2\pm d_1\beta_2)$$

and

$$\alpha_4=\frac{1}{|C\bar{Z}+D|}(\pm c_1\bar{z}_{12}\pm d_1\beta_4)\ .$$

Hence we have

$$\begin{aligned}I\leqq\ &2^{2k-3}\kappa_1\mu^{-3}\sum_{\left\{\substack{(C,D)\\ |C|=0\\ C\neq0}\right.}\sum_{U}|Y|^{-3/2}\operatorname{abs}|C\bar{Z}+D|^{k-3}\\ &\times\Bigl[\operatorname{abs}|C\bar{Z}+D|^2+\frac{y_2}{y_1}(u_1+|\alpha_2|^{-2}\operatorname{Re}(\alpha_2\bar{\alpha}_4)u_2)^2+\Bigl(\frac{y_2}{y_1}c_1^2y_{12}^2+\frac{1}{2}c_1^2y_2^2\Bigr)u_2^2\\ &\quad+(u_3+|\alpha_2|^{-2}\operatorname{Re}(\alpha_2\bar{\alpha}_4)u_4)^2+\Bigl(c_1^2y_{12}^2+\frac{1}{2}\frac{y_1}{y_2}c_1^2y_2^2\Bigr)u_4^2\Bigr]^{-k+3/2}\\ \leqq\ &2^{3k-6}\kappa_1\mu^{-3}\sum_{\left\{\substack{(C,D)\\ |C|=0\\ C\neq0}\right.}\operatorname{abs}|C\bar{Z}+D|^{k-3}(y_1y_2)^{-3/2}\\ &\times[2\mu^{-1}\kappa(k-3/2)\kappa(k-2)\kappa(k-5/2)y_2^{-2}\operatorname{abs}|C\bar{Z}+D|^{-2k+6}\\ &\quad+\mu^{-1}\kappa(k-3/2)\operatorname{abs}|C\bar{Z}+D|^{-2k+4}]\ .\end{aligned}$$

Since $\operatorname{Im}|C\bar{Z}+D|^2=c_2^1y_2^2$, $\mu\leqq y_1\leqq y_2$, we have our assertion in this case.

Next assume $\beta_3=0$. Then we have

$$\begin{aligned}I\leqq\ &2^{2k-3}\kappa_1\mu^{-3}\sum_{\left\{\substack{(C,D)\\ |C|=0\\ C\neq0}\right.}\sum_{U}|Y|^{-3/2}\operatorname{abs}|C\bar{Z}+D|^{k-3}\\ &\times\Bigl[\operatorname{abs}|C\bar{Z}+D|^2+(u_1+|\alpha_1|^{-2}\operatorname{Re}(\alpha_1\bar{\alpha}_3)u_2)^2+\frac{1}{2}c_1^2y_1y_2u_2^2\\ &\quad+\frac{y_1}{y_2}(u_3+|\alpha_1|^{-2}\operatorname{Re}(\alpha_1\bar{\alpha}_3)u_4)^2+\frac{1}{2}c_1^2y_1^2u_4^2\Bigr]^{-k+3/2}\\ \leqq\ &2^{3k-6}\kappa_1\mu^{-3}\sum_{\left\{\substack{(C,D)\\ |C|=0\\ C\neq0}\right.}\operatorname{abs}|C\bar{Z}+D|^{k-3}(y_1y_2)^{-3/2}\\ &\times[2\mu^{-1}\kappa(k-3/2)\kappa(k-2)\kappa(k-5/2)y_1^{-2}\operatorname{abs}|C\bar{Z}+D|^{-2k+6}\\ &\quad+\mu^{-1}\kappa(k-3/2)y_1^{-1/2}y_2^{1/2}\operatorname{abs}|C\bar{Z}+D|^{-2k+4}]\ .\end{aligned}$$

Therefore we have our assertion also in this case. Here we note that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\infty^1$ if and only if $\beta_3=0$ and $u_2=0$.

Now assume $\beta_1,\ \beta_3 \neq 0$. Then we have

$$|\alpha_1|^2+\frac{y_2}{y_1}|\alpha_2|^2=\left(\beta_1{}^2+\frac{y_2}{y_1}\beta_3^2\right)\frac{1}{\operatorname{abs}|C\bar{Z}+D|^2},$$

$$\begin{aligned}\operatorname{abs}|CZ+D|^2&\left\{\operatorname{Im}(\alpha_1\bar{\alpha}_3)^2+\frac{y_2^2}{y_1^2}\operatorname{Im}(\alpha_2\bar{\alpha}_4)^2\right.\\ &=c_1^2\beta_1^2(\beta_1 y_{12}+\beta_3 y_2)^2+\frac{y_2^2}{y_1^2}c_1^2\beta_3^2(\beta_1 y_1+\beta_3 y_{12})^2\\ &\geqq \beta_1^2(|\beta_1|y_{12}-|\beta_3|y_2)^2+\beta_3^2\frac{y_2^2}{y_1^2}(|\beta_1|y_1-|\beta_3|y_{12})^2,\end{aligned}$$

$$\frac{y_1}{y_2}|\alpha_1|^2+|\alpha_2|^2=\left(\frac{y_1}{y_2}\beta_1^2+\beta_3^2\right)\frac{1}{\operatorname{abs}|C\bar{Z}+D|^2},$$

$$\begin{aligned}\operatorname{abs}|CZ+D|^2&\left\{\frac{y_1^2}{y_2^2}\operatorname{Im}(\alpha_1\bar{\alpha}_3)^2+\operatorname{Im}(\alpha_2\bar{\alpha}_4)^2\right\}\\ &\geqq \beta_1^2\frac{y_1^2}{y_2^2}(|\beta_1|y_{12}-|\beta_3|y_2)^2+\beta_3^2(|\beta_1|y_1-|\beta_3|y_{12})^2.\end{aligned}$$

Therefore, if $|\beta_1|\geqq|\beta_3|\geqq 1$ (resp. $|\beta_3|\geqq|\beta_1|\geqq 1$), we have

$$\begin{aligned}I\leqq\ &2^{2k-3}\kappa_1\mu^{-3}\sum_{\substack{(C,D)\\ |C|=0\\ C\neq 0}}\sum_U (y_1y_2)^{-3/2}\operatorname{abs}|C\bar{Z}+D|^{k-3}\\ &\times\Big[\operatorname{abs}|C\bar{Z}+D|^2+\beta_3^2\left(1+\frac{y_2}{y_1}\right)u_1^2+\beta_3^2\left(1+\frac{y_2}{y_1}\right)^{-1}2^{-2}y_2^2u_2^2\\ &\quad+\beta_3^2\left(\frac{y_1}{y_2}+1\right)u_3^2+\beta_3^2\left(\frac{y_1}{y_2}+1\right)^{-1}2^{-2}y_1^2u_4^2\Big]^{k-3/2}\\ &\Big(\text{resp. }\times\Big[\operatorname{abs}|C\bar{Z}+D|^2+\beta_1^2\left(1+\frac{y_2}{y_1}\right)u_1^2+\beta_1^2\left(1+\frac{y_2}{y_1}\right)^{-1}2^{-2}y_2u_2^2\\ &\quad+\beta_1^2\left(\frac{y_1}{y_2}+1\right)u_3^2+\beta_1^2\left(\frac{y_1}{y_2}+1\right)^{-1}2^{-2}y_1^2u_4^2\Big]^{-k+3/2}\Big)\\ \leqq\ &2^{5k-15/2}\kappa_1\mu^{-3}\sum_{\substack{(C,D)\\ |C|=0\\ C\neq 0}}\sum_U (y_1y_2)^{-3/2}\operatorname{abs}|C\bar{Z}+D|^{k-3}\\ &\times[\operatorname{abs}|C\bar{Z}+D|^2+u_1^2+y_1y_2u_2^2+u_3^2+y_1^2u_4^2]^{-k+3/2}.\end{aligned}$$

Since $\mu\leqq y_1\leqq y_2$, $u_1u_4-u_2u_3=\pm 1$ and

$$\operatorname{Im}|C\bar{Z}+D|^2=c_1^2(\beta_1^2y_1+2\beta_1\beta_3y_{12}+\beta_3^2y_2)^2\geqq c_1^2\left(\beta_3^2\frac{3}{4}y_2\right)^2\geqq\left(\frac{3}{4}\right)^2y_2^2,$$

our assertion follows from Lemma 5. Q.E.D.

PROPOSITION 23. *Let k be a positive number $\geqq 3$ and let Z be an element of $\mathfrak{H}(\mu)$. Then there exist constants κ_4 and κ_5, depending only on k and satisfying the following two inequalities:*

(i) $$\sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty^0 - \Gamma_\infty^1} \frac{\det(Y)^{k-3}}{\operatorname{abs}|Z-(A\bar{Z}+B)(C\bar{Z}+D)^{-1}|^k \operatorname{abs}|C\bar{Z}+D|^k} \leqq \kappa_4 \mu^{-3} \frac{1}{(y_1 y_2)^{3/2}};$$

(ii) $$\sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty^0 \cap \Gamma_\infty^1} \frac{\det(Y)^{k-3}}{\operatorname{abs}|Z-(A\bar{Z}+B)(C\bar{Z}+D)^{-1}|^k \operatorname{abs}|C\bar{Z}+D|^k} \leqq \kappa_5 \mu^{-3} \frac{1}{y_1^2 y_2}.$$

PROOF. We see that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to Γ_∞^0 if and only if $C=0$. Moreover, if we write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} U & S{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}$$

with

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in GL(2, \boldsymbol{Z})$$

and an integral symmetric matrix S of size two,

$$\begin{pmatrix} U & S{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}$$

belongs to Γ_∞^1 if and only if $u_2=0$. Now, by Proposition 20, we have

$$I = \sum_S \frac{\det(Y)^{k-3}}{\operatorname{abs}|Z-(U\bar{Z}+S{}^tU^{-1}){}^tU|^k \operatorname{abs}|{}^tU^{-1}|^k} \leqq \kappa_1 \mu^{-3} \frac{\det(Y)^{k-3}}{\det\{Y+UY{}^tU\}^{k-3/2}}.$$

Hence, by Lemma 8 and Lemma 9, we have

$$I \leqq 2^{2k-3/2} \kappa_1 \mu^{-3} \frac{1}{(y_1 y_2)^{3/2}} \frac{1}{\left(1+u_1^2+\frac{y_2}{y_1}u_2^2+\frac{y_1}{y_2}u_3^2+u_4^2\right)^{k-3/2}}.$$

If

$$\begin{pmatrix} U & S{}^tU^{-1} \\ 0 & U \end{pmatrix}$$

belongs to $\Gamma_\infty^0 - \Gamma_\infty^1$, $u_2 \neq 0$. Therefore, since $u_1u_4 - u_2u_3 = \pm 1$, we have

$$\sum_{\left\{\begin{smallmatrix} \left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right) \in GL(2, Z) \\ u_2 \neq 0 \end{smallmatrix}\right.} \frac{1}{\left(1+u_1^2+\frac{y_2}{y_1}u_2^2+\frac{y_1}{y_2}u_3^2+u_4^2\right)^{k-3/2}}$$

$$\leqq 2 \sum_{u_1, u_2, u_4 \in Z} \frac{1}{\left(1+u_1^2+u_2^2+u_4^2\right)^{k-3/2}} \leqq 2\kappa\left(k-\frac{3}{2}\right)\kappa(k-2)\kappa\left(k-\frac{5}{2}\right).$$

So we have proved the first assertion. If $\begin{pmatrix} U & S^t U^{-1} \\ 0 & U \end{pmatrix}$ belongs to $\Gamma_\infty^0 \cap \Gamma_\infty^1$, we have

$$U=\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}=\begin{pmatrix} \pm 1 & 0 \\ u_3 & \pm 1 \end{pmatrix}.$$

Therefore we have

$$\sum_{\left\{\begin{smallmatrix} \left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right) \in GL(2, Z) \\ u_2=0 \end{smallmatrix}\right.} \frac{1}{\left(1+u_1^2+\frac{y_2}{y_1} u_2^2+\frac{y_1}{y_2} u_3^2+u_4^2\right)^{k-3/2}}$$

$$\leqq 2^2 \sum_{u_3 \in Z} \frac{1}{\left(1+\frac{y_1}{y_2} u_3^2\right)^{k-3/2}} \leqq 2^2 \kappa(k-3/2)\left(\frac{y_2}{y_1}\right)^{1/2}.$$

Hence we have proved the second assertion.

4.2. The Poisson summation formula. First we quote a lemma from Siegel [13].

LEMMA 10. *Let Z be an element of the Siegel upper-half-plane of degree two. Let k be a positive number $\geqq 3$. Then we have*

$$\sum_T \det(T)^{k-3/2} \exp\{2\pi i \operatorname{tr}(TZ)\}=(4\pi)^{1/2}(2\pi i)^{-2k}\Gamma(k)\Gamma(k-1/2) \sum_S \det(Z+S)^{-k},$$

where T runs over all half-integral positive symmetric matrices of size two and S runs over all integral symmetric matrices of size two.

PROPOSITION 24. *Let k (resp. N) be a positive number $\geqq 3$ (resp. $\geqq 1$) and let Z be an element of $\mathfrak{H}(\mu)$. Then there exists a constant κ_6 depending only on k and satisfying the following inequality:*

$$\sum_{U \in GL(2, Z)} \Big| \sum_{\left\{\begin{smallmatrix} S \in M_2(Z) \\ {}^tS=S \end{smallmatrix}\right.} \det(Y)^k \det\{Z-U\bar{Z}\,{}^tU-NS\}^{-k}\Big| \leqq \kappa_6 \mu^{-4} N^4.$$

PROOF. By Lemma 10, we have

$$I=\sum_{U \in GL(2, Z)} \Big| \sum_{\left\{\begin{smallmatrix} S \in M_2(Z) \\ {}^tS=S \end{smallmatrix}\right.} \det(Y)^k \det\{N^{-1}Z-N^{-1}U\bar{Z}\,{}^tU-S\}^{-k}\Big| \cdot N^{-2k}$$

$$\leqq \frac{2^{k-1}\pi^{2k-1/2}}{\Gamma(k)\Gamma(k-1/2)} \det(Y)^k N^{-2k} \sum_U \sum_T \det(T)^{k-3/2} \exp\{-2\pi N^{-1} \operatorname{tr} T(Y+UY\,{}^tU)\}.$$

Since $0 \leqq 2y_{12} \leqq y_1 \leqq y_2$ and $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \leqq Y$, we have

$$I \leqq \frac{2^{2k-1}\pi^{2k-1/2}}{\Gamma(k)\Gamma(k-1/2)} \det(Y)^k N^{-2k} \sum_T |T|^{k-3/2} \exp\left\{-\pi N^{-1} \operatorname{tr} T\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}\right\}$$

$$\times \sum_{U} \exp\{-\pi\mu N^{-1}\operatorname{tr}(TU\,{}^tU)\}\ .$$

Put

$$T=\begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2 \end{pmatrix} \text{ and } U=\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}.$$

Then we have

$$\begin{aligned}\operatorname{tr}(TU\,{}^tU) &= (u_1^2+u_2^2)t_1+2(u_1u_3+u_2u_4)t_{12}+(u_3^2+u_4^2)t_2 \\ &= t_1(u_1+t_1^{-1}t_{12}u_3)^2+t_1^{-1}(t_1t_2-t_{12}^2)u_3^2 \\ &\quad +t_2^{-1}(t_1t_2-t_{12}^2)u_2^2+t_2(u_4+t_2^{-1}t_{12}^2u_2)^2\ .\end{aligned}$$

Now we need the following lemma:

LEMMA 11. *Let a be a positive number and let b, c be real numbers. Then we have*

$$\sum_{u=-\infty}^{\infty} \exp\{-a(u+b)^2-c\} \leqq \operatorname{Max}\left(6,\, 2\int_{-\infty}^{\infty}\exp(-u^2)\,du\right)\exp(-c)\operatorname{Max}\left(1,\, \frac{1}{\sqrt{a}}\right).$$

Since the proof of this lemma is quite easy, we omit it and continue the proof of Proposition 24.

Let T be any half-integral positive symmetric matrix of size two. Then we have

$$\operatorname{tr}(TU\,{}^tU) \geqq (u_1+t_1^{-1}t_{12}u_3)^2+\frac{1}{4}t_1^{-1}u_3^2+\frac{1}{4}t_1^{-1}u_2^2+(u_4+t_1^{-1}t_{12}u_2)^2\ .$$

Therefore it follows from the above lemma that

$$\begin{aligned} I \leqq\ & \frac{2^{k+1}\pi^{2k-1/2}}{\Gamma(k)\Gamma(k-1/2)}\operatorname{Max}\left(6,\, 2\int_{-\infty}^{\infty}\exp(-u^2)\,du\right)^4 \mu^{-2}N^{-2k+2}\det(Y)^k \\ & \times \sum_{T} |T|^{k-3/2}(t_1t_2)^{1/2}\exp\{-\pi N^{-1}t_1y_1-\pi N^{-1}t_2y_2\}\ .\end{aligned}$$

Since t_{12} is half-integral and $t_1t_2-t_{12}^2>0$,

$$\begin{aligned} I \leqq\ & \frac{2^{k+2}\pi^{2k-1/2}}{\Gamma(k)\Gamma(k-1/2)}\operatorname{Max}\left(6,\, 2\int_{-\infty}^{\infty}\exp(-u^2)\,du\right)^4 \mu^{-2}N^{-2k+2} \\ & \times \sum_{t_1,\,t_2\in N}(t_1y_1t_2y_2)^k\exp(-2N^{-1}t_1y_1-2N^{-1}t_2y_2) \\ \leqq\ & \frac{2^{k+2}\pi^{2k-1/2}}{\Gamma(k)\Gamma(k-1/2)}\operatorname{Max}\left(6,\, 2\int_{-\infty}^{\infty}\exp(-u^2)\,du\right)^4 \mu^{-2}N^2 \\ & \times \operatorname*{Max}_{0\leqq t<\infty}\{t^k\exp(-t)\}^2\Big\{\sum_{t\in N}\exp(-\mu N^{-1}t)\Big\}^2\ .\end{aligned}$$

Since

$$\sum_{t\in N}\exp(-\mu N^{-1}t)\leqq\int_0^\infty\exp(-\mu N^{-1}t)dt=\mu^{-1}N\int_0^\infty\exp(-t)dt\,,$$

we have our assertion.

PROPOSITION 25. *Let k (resp. N) be a positive number $\gneqq 3$ (resp. $\geqq 1$) and let Z be an element of $\mathfrak{H}(\mu)$. Then there exists a positive constant κ_7 depending only on k and satisfying the following inequality:*

$$\sum_{-\infty<\nu,\,s_{12}<+\infty}\left|\sum_{-\infty<s_2<+\infty}\det(Y)^k\det\left\{Z-\begin{pmatrix}1&0\\\nu&1\end{pmatrix}Z\,{}^t\!\begin{pmatrix}1&0\\\nu&1\end{pmatrix}-N\begin{pmatrix}0&s_{12}\\s_{12}&s_2\end{pmatrix}\right\}^{-k}\right|$$
$$\leqq\kappa_7\mu^{-2}N^2\,.$$

Proof. Put

$$Z=\begin{pmatrix}z_1&z_{12}\\z_{12}&z_2\end{pmatrix},\quad X=\begin{pmatrix}x_1&x_{12}\\x_{12}&x_2\end{pmatrix},\quad Y=\begin{pmatrix}y_1&y_{12}\\y_{12}&y_2\end{pmatrix}.$$

Then the above sum is equal to

$$\begin{aligned}I=&\sum_{-\infty<\nu,\,s_{12}<\infty}\left|\sum_{-\infty<s_2<\infty}\frac{(y_1y_2-y_{12}^2)^k}{\{2iy_1(2iy_2-Ns_2-2\nu\bar z_{12}-\nu^2\bar z_1)-(2iy_{12}-Ns_{12}-\nu\bar z_1)^2\}^k}\right|\\
=&\sum_{\nu,\,s_{12}}\frac{(y_1y_2-y_{12}^2)^k}{2^ky_1^k}\left|\sum_{s_2}\frac{1}{\{-Ns_2+2iy_2-2\nu\bar z_{12}-\nu^2\bar z_1-(2iy_1)^{-1}(2iy_{12}-Ns_{12}-\nu\bar z_1)^2\}^k}\right|\\
=&\sum_{\nu,\,s_{12}}\frac{(y_1y_2-y_{12}^2)^k}{2^ky_1^k}\\
&\times\left|\sum_{n=1}^{\infty}\frac{(2\pi i)^kn^{k-1}N^{-k}}{(k-1)!}\exp[2\pi N^{-1}ni\{2iy_2-2\nu\bar z_{12}-\nu^2\bar z_1-(2iy_1)^{-1}(2iy_{12}-Ns_{12}-\nu\bar z_1)^2\}]\right|\\
\leqq&\sum_{\nu,\,s_{12},\,n}\frac{(y_1y_2-y_{12}^2)^k}{2^ky_1^k}n^{k-1}\frac{(2\pi)^k}{(k-1)!}N^{-k}\\
&\times\exp\left[-4\pi N^{-1}n\left\{y_2-y_1^{-1}y_{12}^2+\frac14\nu^2y_1+\frac14y_1^{-1}(Ns_{12}+\nu x_1)^2\right\}\right].\end{aligned}$$

Hence, by Lemma 11, we have

$$\begin{aligned}I\leqq&\frac{2^{-k}(2\pi)^k(y_1y_2-y_{12}^2)^k\mu^{-1/2}}{(k-1)!\,y_1^{k-1/2}}N^{-k}\,\mathrm{Max}\left(6,\,2\int_{-\infty}^{\infty}\exp(-u^2)du\right)\\
&\times\sum_{\nu,\,n}n^{k-1}\exp[-4\pi N^{-1}n(y_2-y_1^{-1}y_{12}^2)-\pi\nu^2N^{-1}ny_1]\\
\leqq&\frac{2^{-k+1}(2\pi)^k(y_2-y_1^{-1}y_{12}^2)^{k+1/2}\mu^{-1}}{(k-1)!}N^{-k+1/2}\,\mathrm{Max}\left(6,\,2\int_{-\infty}^{\infty}\exp(-u^2)du\right)^2\\
&\times\sum_n n^{k-1}\exp[-4\pi(y_2-y_1^{-1}y_{12}^2)N^{-1}n]\\
\leqq&\frac{2^{-k+1}(2\pi)^k\mu^{-1}N}{(k-1)!}\,\mathrm{Max}\left(6,\,2\int_{-\infty}^{\infty}\exp(-u^2)du\right)^2\end{aligned}$$

$$\times \sum_n \{N^{-1}n(y_2-y_1^{-1}y_{12}^2)\}^{k+1/2}\exp\{-N^{-1}n(y_2-y_1^{-1}y_{12}^2)\}\exp\{-N^{-1}\mu n\}$$

$$\leqq \frac{2^{-k+1}(2\pi)^k \operatorname{Max}\left(6, 2\int_{-\infty}^{\infty}\exp(-u^2)\,du\right)^2}{(k-1)!}$$

$$\times \mu^{-1} \operatorname*{Max}_{0\leqq t<\infty} \{t^{k+1/2}\exp(-t)\} \frac{N}{1-\exp(-N^{-1}\mu)}$$

$$\leqq \frac{1}{(k-1)!} 2^2\pi^k \operatorname{Max}\left(6, 2\int_{-\infty}^{\infty}\exp(-u^2)\,du\right)^2 \operatorname*{Max}_{0\leqq t<\infty} \{t^{k+1/2}\exp(-t)\}\mu^{2-}N^2 . \qquad \text{Q.E.D.}$$

§ 5. The Selberg trace formula

5.1. Some results of Godement. Let $\mathfrak{H}$ be the Siegel upper-half-plane of degree two. We write an element

$$Z=\begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}$$

of $\mathfrak{H}$ as $X+iY$ with two real matrices

$$X=\begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix} \quad \text{and} \quad Y=\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix} .$$

Let k (resp. N) be a positive integer $\gneqq 4$ (resp. $N\geqq 1$). Let $\Gamma_2(N)$ be the principal congruence subgroup of the Siegel modular group of degree two of level N:

$$\Gamma_2(N)=\left\{\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(2, Z)\,\middle|\,\begin{pmatrix} A & B \\ C & D \end{pmatrix}\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(\mathrm{mod}\ N)\right\}\Big/\pm 1 .$$

We call f a cusp form of $\Gamma_2(N)$ of weight k if f satisfies the following conditions:

(i) f is a holomorphic function on $\mathfrak{H}$;

(ii) $f((AZ+B)(CZ+D)^{-1})=\det(CZ+D)^k f(Z)$ for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in \Gamma_2(N)$;

(iii) $\det(\mathrm{Im}(Z))^{k/2}|f(Z)|$ is bounded on $\mathfrak{H}$.

Owing to Satake, it is known that (iii) is equivalent to the condition that $f(Z)$ belongs to the kernel of the Φ-operator of Siegel. Hence, if we denote by $S_k(\Gamma_2(N))$ the complex vector space of all cusp forms of $\Gamma_2(N)$ of degree two, $S_k(\Gamma_2(N))$ is a finite dimensional complex vector space.

Now it is well-known that $S_k(\Gamma_2(N))$ has a structure of a finite dimensional Hilbert space with the Peterson metric. R. Godement studied the kernel function of this Hilbert space and proved the following theorem (cf. [11]):

THEOREM 2. *Let the notation be as above. Let $F(N)$ be a fundamental domain of $\Gamma_2(N)$. Put*

$$K(Z, Z')=\sum_{\gamma=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_2(N)}\frac{\det\{\mathrm{Im}(Z')\}^k}{\det\left(\dfrac{Z-\gamma\bar{Z}'}{2i}\right)^k\det(C\bar{Z}'+D)^k},$$

$$a(k)=2^{-8}\pi^{-3}(2k-2)(2k-3)(2k-4),$$

$$dXdY=dx_1dx_{12}dx_2dy_1dy_{12}dy_2.$$

Then we have

(i) $f(Z)=a(k)\int_{F(N)}K(Z, Z')\dfrac{dX'dY'}{\det(Y')^3}$ *for any* $f\in S_k(\Gamma_2(N))$,

(ii) $|K(Z, Z)|$ *is bounded on* $F(N)$,

(iii) $\dim S_k(\Gamma_2(N))=a(k)\int_{F(N)}K(Z, Z)\dfrac{dX\,dY}{\det(Y)^3}$.

5.2. A reformulation. Now let $\begin{pmatrix} A & B \\ C & D\end{pmatrix}=\gamma$ be an element of $Sp(2, R)$ and put

$$H_\gamma(Z)=\frac{\det\{\mathrm{Im}(Z)\}^k}{\det\left(\dfrac{Z-\gamma\bar{Z}}{2i}\right)^k\det(C\bar{Z}+D)^k}.$$

Then we see that $H_{\delta^{-1}\gamma\delta}(Z)=H_\gamma(\delta Z)$ for any $\delta\in Sp(2, \boldsymbol{R})$. Since $\dfrac{dxdy}{\det(Y)^3}$ is invariant by the operation of an element of $\gamma\in Sp(2, \boldsymbol{R})$, we have

$$\dim S_k(\Gamma(N))=a(k)\sum_{\delta\in\Gamma_2(N)\backslash\Gamma_2(1)}\int_{F(1)}\sum_{\gamma\in\Gamma_2(N)}H_{\delta^{-1}\gamma\delta}(Z)\frac{dXdY}{\det(Y)^3}.$$

Here we note that $\delta^{-1}\gamma\delta$ belongs to $\Gamma_2(N)$ since $\Gamma_2(N)$ is a normal subgroup of $\Gamma_2(1)$. Put $F=F(1)$ and $dZ=\dfrac{dXdY}{\det(Y)^3}$. It is well-known that we can take the following set F as a fundamental domain of $Sp(2, \boldsymbol{Z})/\pm 1$: F is the set consisting of all elements $Z=X+iY\in\mathfrak{H}$ satisfying (i) $-1/2\leqq x_1, x_{12}, x_2\leqq 1/2$, (ii) $0\leqq 2y_{12}\leqq y_1\leqq y_2$ (i.e., Y is reduced) and (iii) $\mathrm{abs}|CZ+D|\geqq 1$ for any $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ of $Sp(2, \boldsymbol{Z})/\pm 1$. (We note that (iii) implies in particular $y_1\geqq\dfrac{\sqrt{3}}{2}$, $y_2\geqq\dfrac{\sqrt{3}}{2}$ and $\det(Y)\geqq 1$.) Moreover, it is well-known that F has a one-dimensional cusp

$$F_1=\left\{\begin{pmatrix} z_1 & * \\ * & i\infty\end{pmatrix}\middle|\begin{matrix} z_1\in\boldsymbol{C},\ \mathrm{Im}(z_1)>0 \\ |z_1|\geqq 1,\ -1/2\leqq\mathrm{Re}(z_1)\leqq 1/2\end{matrix}\right\}$$

and a zero-dimensional cusp $F_0=\left\{\begin{pmatrix} i\infty & * \\ * & i\infty\end{pmatrix}\right\}$, where we identify $\begin{pmatrix} z_1 & * \\ * & i\infty\end{pmatrix}$ and $\begin{pmatrix} z_1' & * \\ * & i\infty\end{pmatrix}$ $(z_1, z_1'\in\boldsymbol{C}\cup\{i\infty\})$ if and only if $z_1=z_1'$, and that their stabilizer in

$Sp(2, \boldsymbol{Z})$ is

$$\Gamma^1_\infty = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in Sp(2, \boldsymbol{Z}) \right\}$$

and

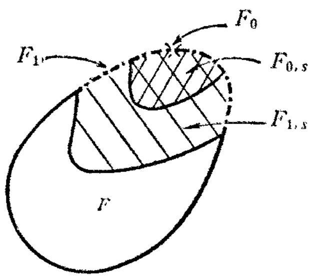
$$\Gamma^0_\infty = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in Sp(2, \boldsymbol{Z}) \right\}$$

respectively. For any positive number $s \leqq 1$, put

$$F_{1,s} = \left\{ Z = X + iY \in F \mid y_2 - y_1^{-1} y_{12}^2 \geqq \exp\left(\frac{1}{s^2}\right) \right\}$$

and

$$F_{0,s} = \left\{ Z = X + iY \in F \mid y_1 \geqq \exp\left(\frac{1}{s}\right),\ \ y_2 - y_1^{-1} y_{12}^2 \geqq \exp\left(\frac{1}{s^2}\right) \right\}.$$



Then we see that $F_{0,s}$ is contained in $F_{1,s}$, $F_{1,s}$ (resp. $F_{0,s}$) is a neighborhood of F_1 (resp. F_0) and $F_{1,s}$ (resp. $F_{0,s}$) decreases as s decreases to zero and has F_1 (resp. F_0) as its limit.

THEOREM 3. *Let the notation be as above. Suppose $k \geqq 7$. Then we have the follwoing formula*:

$$\begin{aligned} &\dim S_k(\Gamma_2(N)) \\ &= a(k) \sum_{\delta \in \Gamma_2(N) \backslash \Gamma_2(1)} \Bigg[\sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \notin \Gamma^1_\infty \cup \Gamma^0_\infty}} \int_F H_{\gamma'}(Z) dZ \\ &\quad + \lim_{s \to +0} \sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^1_\infty}} \left\{ \int_{F - F_{1,s}} H_{\gamma'}(Z) dZ + \int_{F_{1,s}} H_{\gamma'}(Z) \frac{dZ}{(y_2 - y_1^{-1} y_{12}^2)^s} \right\} \\ &\quad + \lim_{s \to +0} \sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta\gamma\delta \in \Gamma^0_\infty - \Gamma^1_\infty}} \left\{ \int_{F - F_{0,s}} H_{\gamma'}(Z) dZ + \int_{F_{0,s}} H_{\gamma'}(Z) \frac{dZ}{(y_1 y_2 - y_{12}^2)^s} \right\} \Bigg]. \end{aligned}$$

We need a lemma to prove this theorem.

LEMMA 12. *Let s be a positive number $\leqq 1$. Then there exists a constant κ_8 satisfying the follwoing conditions*:

$$\int_{F_{1,s}} \frac{1}{y_1^2 y_2} \frac{dX dY}{(y_2 - y_1^{-1} y_{12}^2)^s} \leqq \kappa_8 \frac{1}{s^2} \exp\left(\frac{-1}{2s}\right),$$

$$\int_{F_{0,s}} \frac{1}{y_1^{3/2}y_2^{3/2}} \frac{dXdY}{(y_1y_2-y_{12}^2)^s} \leqq \kappa_8 \frac{1}{s^2} \exp\left(\frac{-1}{2s}\right),$$

$$\int_{F-F_{1,s}} \frac{dXdY}{y_1^2y_2} \leqq \kappa_8 \frac{1}{s^4},$$

$$\int_{F_{1,s}-F_{0,s}} \frac{dXdY}{y_1^{3/2}y_2^{3/2}} \leqq \kappa_8 \exp\left(\frac{s-1}{2s^2}\right)$$

and

$$\int_{F-F_{0,s}} \frac{dXdY}{y_1^{3/2}y_2^{3/2}} \leqq \kappa_8 \frac{1}{s^4}.$$

PROOF. Since $0\leqq 2y_{12}\leqq y_1\leqq y_2$, we have $y_1y_2-y_{12}^2\geqq \frac{3}{4}y_1y_2$ and $y_2-y_1^{-1}y_{12}^2\geqq \frac{3}{4}y_2$. Since F is contained in

$$\left\{ X+iY\in \mathfrak{H} \,\middle|\, \begin{array}{c} -\frac{1}{2}\leqq x_1, x_{12}, x_2\leqq \frac{1}{2}, 0\leqq 2y_{12}\leqq y_1\leqq y_2 \\ \frac{\sqrt{3}}{2}\leqq y_1,\ \frac{\sqrt{3}}{2}\leqq y_2 \end{array} \right\}$$

we have

$$\begin{aligned} \int_{F_{1,s}} \frac{dXdY}{y_1^2y_2(y_2-y_1^{-1}y_{12}^2)^s} &\leqq \left(\frac{4}{3}\right)^s \int_{\left\{\substack{0\leqq 2y_{12}\leqq y_1\leqq y_2 \\ \exp(1/s^2)\leqq y_2 \\ \sqrt{3}/2\leqq y_1}\right.} \frac{dy_1dy_{12}dy_2}{y_1^{2+s/2}y_2^{1+s/2}} \\ &= \left(\frac{4}{3}\right)^s \frac{1}{2}\int_{\exp(1/s^2)}^{\infty} \int_{\sqrt{3}/2}^{y_2} \frac{dy_1dy_2}{y_1^{1+s/2}y_2^{1+s/2}} \\ &\leqq \left(\frac{4}{3}\right)^s \left(\frac{2}{\sqrt{3}}\right)^{s/2} \frac{1}{s}\int_{\exp(1/s^2)}^{\infty} \frac{dy_2}{y_2^{1+s/2}} \\ &\leqq \left(\frac{32}{9\sqrt{3}}\right)^{s/2} \frac{2}{s^2}\exp\left(\frac{-1}{2s}\right). \end{aligned}$$

In the same manner, we have

$$\int_{F_{0,s}} \frac{dXdY}{(y_1y_2)^{3/2}(y_1y_2-y_{12}^2)^s} \leqq \left(\frac{16}{3\sqrt{3}}\right)^s \frac{1}{s^2}\exp\left(\frac{-1}{s}\right),$$

$$\int_{F-F_{1,s}} \frac{dXdY}{y_1^2y_2} \leqq \frac{1}{2}\int_{\sqrt{3}/2}^{\exp(1/s^2)}\int_{\sqrt{3}/2}^{\exp(1/s^2)} \frac{dy_1dy_2}{y_1y_2} \leqq \frac{1}{2}\left(\frac{1}{s^2}-\log\frac{\sqrt{3}}{2}\right)^2,$$

$$\int_{F_{1,s}-F_{0,s}} \frac{dXdY}{(y_1y_2)^{3/2}} \leqq 8\left\{\exp\left(\frac{1}{2s}\right)-\left(\frac{\sqrt{3}}{2}\right)^{1/2}\right\}\exp\left(\frac{-1}{2s^2}\right)$$

and

$$\int_{F-F_{0,s}} \frac{dXdY}{(y_1y_2)^{3/2}} \leqq \frac{1}{2}\left(\frac{1}{s^2}-\log\frac{\sqrt{3}}{2}\right)^2.$$

Since $\frac{2}{\sqrt{3}}<e$, $-\log\frac{\sqrt{3}}{2}\leqq 1\leqq\frac{1}{s^4}$. Hence, putting $\kappa_3=\mathrm{Max}\left(\frac{64}{9\sqrt{3}}, \frac{16}{3\sqrt{3}}, 8\right)$, we have Lemma 12.

Now we start the proof of Theorem 3. By Proposition 21, we have

$$\sum_{\left\{\substack{\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_2(1)\\ |C|\neq 0}\right.} |H_\gamma(Z)|\leqq\kappa_2\cdot 2^{2k}\cdot\left(\frac{\sqrt{3}}{4}\right)^{-6}\sum_{\left\{\substack{(C,D)\\ |C|\neq 0}\right.}\frac{1}{\mathrm{abs}|CZ+D|^{k-3}}.$$

Since $\sum_{(C,D)}\frac{1}{\mathrm{abs}|CZ+D|^{k-3}}$ is bounded on F ($k\gtrless 6$) (cf. e.g. Maass [8]), we have

$$\int_F\sum_{\left\{\substack{\gamma\in\Gamma_2(N)\\ \gamma'=\delta^{-1}\gamma\delta=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\\ |C|\neq 0}\right.}|H_{\gamma'}(Z)|dZ<\mathrm{const.}\int_F dZ<\infty.$$

By Proposition 22, we have

$$\sum_{\left\{\substack{\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_2(1)\\ |C|=0\\ C\neq 0}\right.}|H_\gamma(Z)|\leqq\kappa_3\cdot 2^{2k}\cdot\left(\frac{\sqrt{3}}{4}\right)^{-6}\sum_{\left\{\substack{(C,D)\\ |C|=0\\ C\neq 0}\right.}\frac{y_1^{-1}y_2^2}{\mathrm{abs}|CZ+D|^{k-3}}.$$

Here we may assume

$$C=\begin{pmatrix}c_1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1 & u_3\\ u_2 & u_4\end{pmatrix},\quad D=\pm\begin{pmatrix}d_1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_4 & -u_2\\ -u_3 & u_1\end{pmatrix}$$

with some $\begin{pmatrix}u_1 & u_2\\ u_3 & u_4\end{pmatrix}\in GL(2, \boldsymbol{Z})$, $c_1, d_1\in\boldsymbol{Z}$, $c_1\neq 0$. Then $\mathrm{abs}|CZ+D|=|c_1\{u_1^2z_1+2u_1u_3z_{12}+u_3^2z_2\}+d_1|$. Hence, if $u_3\neq 0$ (i.e., $\begin{pmatrix}A & B\\ C & D\end{pmatrix}\notin\Gamma_\infty^0\cdot\Gamma_\infty^1$),

$$\mathrm{abs}|CZ+D|\geqq|c_1|(u_1^2y_1+2u_1u_3y_{12}+u_3^2y_2)\geqq|c_1|u_3^2(y_2-y_1^{-1}y_{12}^2)\geqq y_2-y_1^{-1}y_{12}^2.$$

Hence we have

$$\int_F\sum_{\left\{\substack{\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\notin\Gamma_\infty^0\cdot\Gamma_\infty^1\\ |C|=0}\right.}|H_\gamma(Z)|dZ$$

$$\leqq\kappa_3 2^{2k}\left(\frac{\sqrt{3}}{4}\right)^{-6}\mathrm{Max}\left\{\sum_{(C,D)}\left\{\frac{1}{\mathrm{abs}|CZ+D|^{k-7/2}}\right\}\right\}\int_F\frac{dXdY}{y_1^4y_2^{1+1/2}}<\infty.$$

Moreover, we see easily from the proof of Proposition 22 that the above result remains true even if we replace the condition $\gamma\notin\Gamma_\infty^0\cdot\Gamma_\infty^1$ by $\gamma\notin\Gamma_\infty^1$. Hence it follows from (ii) of Theorem 2 that

$$\int_F\sum_{\left\{\substack{\gamma\in\Gamma_2(N)\\ \gamma'=\delta^{-1}\gamma\delta\notin\Gamma_\infty^1\cup\Gamma_\infty^0}\right.}|H_{\gamma'}(Z)|dZ<\infty$$

and

$$\int_F |\sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^1_\infty \cup \Gamma^0_\infty}} H_{\gamma'}(Z)|dZ < \infty.$$

Furthermore, by Proposition 24 and Proposition 25, we have

$$\int_F |\sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^1_\infty}} H_{\gamma'}(Z)|dZ < \infty$$

and

$$\int_F |\sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^0_\infty - \Gamma^1_\infty}} H_{\gamma'}(Z)|dZ < \infty.$$

Hence we have

$$\begin{aligned}\dim S_k(\Gamma_2(N)) = a(k) \sum_{\delta \in \Gamma_2(N)\backslash\Gamma_2(1)} \Bigg[& \sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \notin \Gamma^1_\infty \cup \Gamma^0_\infty}} \int_F H_{\gamma'}(Z)dZ \\ & + \lim_{s \to +0} \Bigg\{ \int_{F - F_{1,s}} \sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^1_\infty}} H_{\gamma'}(Z)dZ \\ & + \int_{F_{1,s}} \sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^1_\infty}} H_{\gamma'}(Z) \frac{dZ}{(y_2 - y_1^{-1}y_{12}^2)^s} \Bigg\} \\ & + \lim_{s \to +0} \Bigg\{ \int_{F - F_{0,s}} \sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^0_\infty - \Gamma^1_\infty}} H_{\gamma'}(Z)dZ \\ & + \int_{F_{0,s}} \sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^\infty_0 - \Gamma^1_\infty}} H_{\gamma'}(Z) \frac{dZ}{(y_1y_2 - y_{12}^2)^s} \Bigg\} \Bigg].\end{aligned}$$

Since, by Proposition 22 and Proposition 23, we have

$$\sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^1_\infty}} |H_{\gamma'}(Z)| < \mathrm{Max}(\kappa_3, \kappa_5) 2^{2k} \left(\frac{\sqrt{3}}{4}\right)^{-6} \sum_{(C,D)} \frac{y_1 y_2^2}{\mathrm{abs}|CZ+D|^{k-3}}$$

and

$$\sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \in \Gamma^0_\infty - \Gamma^1_\infty}} |H_{\gamma'}(Z)| < \kappa_4 2^{2k} \left(\frac{\sqrt{3}}{4}\right)^{-3} (y_1 y_2)^{3/2},$$

it follows from Lemma 12 that all the above series are summable. Hence we may interchange the integrals and the infinite sums. Q.E.D.

REMARK. We see from Lemma 12 that

$$\lim_{s \to 0} \int_{F_{1,s} \cup F_{0,s}} \sum_{\substack{\gamma \in \Gamma_2(N) \\ \gamma' = \delta^{-1}\gamma\delta \notin \Gamma^1_\infty \supset \Gamma^0_\infty}} |H_{\gamma'}(Z)|dZ \longrightarrow 0,$$

$$\lim_{s\to 0}\int_{F_{1,s}} \sum_{\substack{\gamma\in\Gamma_2(N)\\ \gamma'=\delta^{-1}\gamma\delta\in\Gamma^1_\infty}} |H_{\gamma'}(Z)| \frac{dZ}{(y_2-y_1^{-1}y_{12}^2)^s} \longrightarrow 0,$$

$$\lim_{s\to 0}\int_{F_{1,s}-F_{0,s}} \sum_{\substack{\gamma\in\Gamma_2(N)\\ \gamma'=\delta^{-1}\gamma\delta\in\Gamma^0_\infty-\Gamma^1_\infty}} |H_{\gamma'}(Z)|\,dZ \longrightarrow 0$$

and

$$\lim_{s\to 0}\int_{F_{0,s}} \sum_{\substack{\gamma\in\Gamma_2(N)\\ \gamma'=\delta^{-1}\gamma\delta\in\Gamma^0_\infty-\Gamma^1_\infty}} |H_{\gamma'}(Z)| \frac{dZ}{(y_1y_2-y_{12}^2)^s} \longrightarrow 0.$$

Hence, in fact, the contribution of $F_{1,s}\supseteq F_{0,s}$ tends to zero as s tends to $+0$. The author was informed of this fact by H. Shimizu.

§6. Vanishing of some integrals

In this section we shall prove that some integrals in Theorem 3 are zero.

6.1. The absolutely convergent case

THEOREM 4. *Let the notation be as in Theorem 3 in the last section. Suppose $k\geqq 7$. Then we have*

$$\int_F \sum |H_\gamma(Z)|\,dZ<\infty,$$

where γ runs over all elements of $\Gamma_2(N)$ that are conjugate in $Sp(2,\mathbf{R})$ to

(i)
$$\begin{pmatrix} \cos\lambda & 0 & \sin\lambda & 0\\ 0 & a & 0 & 0\\ -\sin\lambda & 0 & \cos\lambda & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix} \quad (\sin\lambda\neq 0,\ a\neq\pm 1)$$

or

(ii)
$$\begin{pmatrix} a_1 & 0 & & 0\\ 0 & a_2 & & \\ & 0 & a_1^{-1} & 0\\ & & 0 & a_2^{-1}\end{pmatrix}\neq 1$$

or

(iii)
$$\begin{pmatrix} \mu & 0 & & 0\\ 0 & \mu & & \\ 0 & & \mu^{-1} & 0\\ & & 0 & \mu^{-1}\end{pmatrix}\begin{pmatrix} \cos\lambda & \sin\lambda & & 0\\ -\sin\lambda & \cos\lambda & & \\ & 0 & \cos\lambda & \sin\lambda\\ & & -\sin\lambda & \cos\lambda\end{pmatrix} \quad (\mu^2\neq 1,\ \sin\lambda\neq 0).$$

We shall show one application of this theorem before proving it. So let us assume Theorem 4. Then, since $H_{\varepsilon^{-1}\gamma\varepsilon}(Z)=H_\gamma(\varepsilon Z)(\gamma,\varepsilon\in Sp(2,\boldsymbol{R}))$, we see that the contribution to the dimension formula in Theorem 3 of the above types of conjugacy classes is equal to

$$a(k)\sum_{\delta\in\Gamma_2(N)\backslash\Gamma_2(1)}\int_F\sum_{\gamma:\text{ as above}}H_{\delta^{-1}\gamma\delta}(Z)dZ=a(k)\sum_{\delta\in\Gamma_2(N)\backslash\Gamma_2(1)}\sum_{\gamma'}\int_{F_{\delta^{-1}\gamma'\delta}}H_{\delta^{-1}\gamma'\delta}(Z)dZ,$$

where γ' runs over all $Sp(2,\boldsymbol{Z})$ conjugacy classes in $\Gamma_2(N)$ whose $Sp(2,\boldsymbol{R})$ conjugacy classes are as in the above theorem and $F_{\delta^{-1}\gamma'\delta}$ is a fundamental domain of the centralizer $\delta^{-1}\gamma'\delta$ in $\Gamma_2(1)=Sp(2,\boldsymbol{Z})/\pm1$.

Now we can prove the following theorem:

THEOREM 5. *Let the notation and assumption be as in Theorem 3. Then the contribution to the dimension formula in Theorem 3 of the elements of $\Gamma_2(N)$ which are conjugate in $Sp(2,\boldsymbol{R})$ to*

(i)
$$\begin{pmatrix}\cos\lambda & 0 & \sin\lambda & 0\\ 0 & a & 0 & 0\\ -\sin\lambda & 0 & \cos\lambda & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix}\quad(\sin\lambda\neq0,\ a\neq\pm1)$$

or

(ii)
$$\begin{pmatrix}a_1 & 0 & & \\ 0 & a_2 & & 0\\ & & a_1^{-1} & 0\\ 0 & & 0 & a_2^{-1}\end{pmatrix}\neq1$$

or

(iii)
$$\begin{pmatrix}\mu & 0 & & \\ 0 & \mu & & 0\\ & & \mu^{-1} & 0\\ 0 & & 0 & \mu^{-1}\end{pmatrix}\begin{pmatrix}\cos\lambda & \sin\lambda & & \\ -\sin\lambda & \text{con}\,\lambda & & 0\\ & & \cos\lambda & \sin\lambda\\ 0 & & -\sin\lambda & \cos\lambda\end{pmatrix}\quad(\mu^2\neq1,\ \sin\lambda\neq0)$$

is zero. Hence we may disregard in the dimension formula in Theorem 3 terms such that γ' is conjugate in $Sp(2,\boldsymbol{R})$ to one of the above types of elements.

PROOF OF THEOREM 5. By the preceding remark, it is enough to show that

$$\int_{F_\gamma}H_\gamma(Z)dZ=0,$$

if γ is the above type. Hence it is a special case of the conjecture of A. Selberg.

Let C_γ (resp. $C_\gamma^{\boldsymbol{R}}$) be the centralizer of γ in $Sp(2,\boldsymbol{Z})/\pm1$ (resp. $Sp(2,\boldsymbol{R})/\pm1$).

Then we see that the above integral is equal to

$$\int_{C_\gamma \backslash C_\gamma^R} dZ^1 \times \int_{C_\gamma^R \backslash \mathfrak{H}} H_\gamma(Z) dZ^2,$$

where dZ^1 (resp. dZ^2) is the restriction of dZ on C_γ^R (resp. the induced measure on $C_\gamma^R \backslash \mathfrak{H}$). Hence it is enough to prove

$$\int_{C_\gamma^R \backslash \mathfrak{H}} H_\gamma(Z) dZ^2 = 0$$

and so we may replace γ by any conjugate element. Let γ be

$$\begin{pmatrix} \cos\lambda & 0 & \sin\lambda & 0 \\ 0 & a & 0 & 0 \\ -\sin\lambda & 0 & \cos\lambda & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix}$$

($\sin\lambda \neq 0$, $a \neq \pm 1$). Then it follows from Proposition 2 that

$$C_\gamma^R \backslash \mathfrak{H} = \left\{ \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \in M_2(\boldsymbol{C}) \,\middle|\, \begin{matrix} y_1 \geqq 1, & y_1 y_2 - y_{12}^2 > 0 \\ x_1 = 0, & y_2 = 1 \end{matrix} \right\},$$

$$\begin{aligned} H_\gamma(Z) = \det\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}^k \Bigg\{ & \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \begin{pmatrix} -\sin\lambda \bar{z}_1 + \cos\lambda & -\sin\lambda \bar{z}_{12} \\ 0 & a^{-1} \end{pmatrix} \\ & + \begin{pmatrix} \cos\lambda \bar{z}_1 + \sin\lambda_1 & \cos\lambda \bar{z}_{12} \\ a\bar{z}_{12} & a\bar{z}_2 \end{pmatrix} \Bigg\}^{-k} \\ = -k^{\text{th}} & \text{ power of a non-constant linear form in } x_2. \end{aligned}$$

Therefore, rewriting the above integral as a multiple integral, we see that

$$\int_{C_\gamma^R \backslash \mathfrak{H}} H_\gamma(Z) dZ^2 = 0.$$

Let γ be

$$\begin{pmatrix} a_1 & 0 & & \\ 0 & a_2 & & 0 \\ & & a_1^{-1} & 0 \\ & 0 & 0 & a_2^{-1} \end{pmatrix}$$

($a_1^2 \neq 1$). Then it follows from Proposition 3 and Proposition 4 that

$$C_\gamma^R \backslash \mathfrak{H} \cong \left\{ \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \in M_2(\boldsymbol{C}) \,\middle|\, \begin{matrix} y_1 = y_2 = 1 \\ y_1 y_2 - y_{12}^2 > 0 \end{matrix} \right\} \ldots \text{if } a_2^2 \neq 1,\ a_1 \neq a_2^{\pm 1},$$

or

$$\cong \left\{ \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \in M_2(\boldsymbol{C}) \,\middle|\, \begin{matrix} y_1 y_2 - y_{12}^2 > 0 \\ z_2 = 1 \end{matrix} \right\} \ldots \text{if } a_2 = 1,$$

or

$$\cong\left\{\begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}\in M_2(\boldsymbol{C})\,\middle|\,\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\ldots \text{if } a_1=a_2$$

and, by Theorem 1, that these three cover all cases.

Since

$$H_\gamma(Z)=\det\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}^k\left\{\begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}\begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_2^{-1} \end{pmatrix}-\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} \bar z_1 & \bar z_{12} \\ \bar z_{12} & \bar z_2 \end{pmatrix}\right\}^{-k}$$
$$=-k^{\text{th}} \text{ power of a non-constant linear form in } x_{1'},$$

we see that

$$\int_{C_\gamma^R\backslash\mathfrak{H}} H_\gamma(Z)dZ^2=0.$$

Let γ be

$$\begin{pmatrix} \mu & 0 & & \\ 0 & \mu & & \\ & & \mu^{-1} & 0 \\ & & 0 & \mu^{-1} \end{pmatrix}\begin{pmatrix} \cos\lambda & \sin\lambda & & \\ -\sin\lambda & \cos\lambda & & \\ & & \cos\lambda & \sin\lambda \\ & & -\sin\lambda & \cos\lambda \end{pmatrix}$$

($\mu^2\neq 1$, $\sin\lambda\neq 0$). Then it follows from Proposition 6 that

$$C_\gamma^R\backslash\mathfrak{H}\cong\left\{\begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}\in M_2(\boldsymbol{C})\,\middle|\,\begin{matrix} y_1>0,\ y_1y_2-y_{12}^2>0 \\ \text{and } (\sharp) \end{matrix}\right\},$$

$$H_\gamma(Z)=\det\left[\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}^k\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}^{-k}\right.$$
$$\left.\times\left\{\begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}-\begin{pmatrix} \mu^2 & 0 \\ 0 & \mu^2 \end{pmatrix}\begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}\begin{pmatrix} \bar z_1 & \bar z_{12} \\ \bar z_{12} & \bar z_2 \end{pmatrix}\begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix}\right\}^{-k}\right],$$

where ($\sharp$) means that $\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}$ belongs to a fundamental domain of

$$\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}\longrightarrow\begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$
$$(\alpha\in\boldsymbol{R}^\times,\ \beta\in\boldsymbol{R}).$$

Now put

$$\begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix}-\begin{pmatrix} \mu^2 & 0 \\ 0 & \mu^2 \end{pmatrix}\begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}\begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix}\begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix}=\begin{pmatrix} x_1' & x_1' \\ x_{12}' & x_2' \end{pmatrix}.$$

Then we have

$$x_1'=(1-\mu^2\cos^2\lambda)x_1-2\mu^2\sin\lambda\cos\lambda\, x_{12}-\mu^2\sin^2\lambda x_2,$$
$$x_{12}'=\mu^2\sin\lambda\cos\lambda x_1+(1-\mu^2\cos^2\lambda+\mu^2\sin^2\lambda)x_{12}-\mu^2\sin\lambda\cos\lambda x_2,$$
$$x_2'=-\mu^2\sin^2\lambda x_1+2\mu^2\sin\lambda\cos\lambda x_{12}+(1-\mu^2\cos^2\lambda)x_2$$

and

$$\begin{vmatrix} 1-\mu^2\cos\lambda, & -2\mu^2\sin\lambda\cos\lambda, & -\mu^2\sin\lambda \\ \mu^2\sin\lambda\cos\lambda, & 1-\mu^2\cos^2\lambda+\mu^2\sin^2\lambda, & -\mu^2\sin\lambda\cos\lambda \\ -\mu^2\sin^2\lambda, & 2\mu^2\sin\lambda\cos\lambda, & 1-\mu^2\cos^2\lambda \end{vmatrix}$$
$$=(1-\mu^2)\{(1-\mu^2\cos^2\lambda+\mu^2\sin^2\lambda)^2+4\mu^4\sin^2\lambda\cos^2\lambda\}\neq 0.$$

Hence, replacing the variables x_1, x_{12}, x_2 by x_1', x_{12}', x', we see that our integral vanishes also in this case. Q.E.D.

Now we shall prove Theorem 4.

By the proof of Proposition 8 and Proposition 9, we see that, if γ is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\text{(i)}\ \begin{pmatrix} \cos\lambda & 0 & \sin\lambda & 0 \\ 0 & a & 0 & 0 \\ -\sin\lambda & 0 & \cos\lambda & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} (\sin\lambda\neq 0,\ a\neq\pm 1) \text{ or } \quad \text{(ii)}\ \begin{pmatrix} a_1 & 0 & & \\ 0 & a_2 & & 0 \\ & 0 & a_1^{-1} & 0 \\ & & 0 & a_2^{-1} \end{pmatrix} \neq 1 \text{ or}$$

$$\text{(iii)}\quad \begin{pmatrix} \mu & 0 & & \\ 0 & \mu & & 0 \\ & 0 & \mu^{-1} & 0 \\ & & 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \cos\lambda & \sin\lambda & & \\ -\sin\lambda & \cos\lambda & & 0 \\ & 0 & \cos\lambda & \sin\lambda \\ & & -\sin\lambda & \cos\lambda \end{pmatrix} (\mu^2\neq 1,\ \sin\lambda\neq 0),$$

γ is contained in Γ_∞^0 (resp. Γ_∞^1) only if γ is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & & 0 \\ & 0 & a^{-1} & 0 \\ & & 0 & a \end{pmatrix} (a^2\neq 1) \left(\text{resp.} \begin{pmatrix} a & 0 & & \\ 0 & 1 & & 0 \\ & 0 & a^{-1} & 0 \\ & & 0 & 1 \end{pmatrix} (a^2\neq 1) \right).$$

Therefore it is enough to check these two cases since we have already proved in the proof of Theorem 3 that the integral

$$\int_F \sum_{\gamma\in\Gamma-\Gamma_\infty^0-\Gamma_\infty^1} |H_\gamma(Z)|\, dZ$$

is finite. Now put

$$\gamma=\begin{pmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}\in\Gamma_\infty^0, \quad U=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } S=\begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix}$$

and suppose that γ is conjugate in $Sp(2, \boldsymbol{R})$ to $\begin{pmatrix} \lambda & 0 & & \\ 0 & \lambda^{-1} & & 0 \\ & 0 & \lambda^{-1} & 0 \\ & & 0 & \lambda \end{pmatrix} (\lambda\neq\pm 1)$. Put

$$V=\begin{pmatrix} \dfrac{c}{\sqrt{(a+d)^2-4}} & \dfrac{-a+d-\sqrt{(a+d)^2-4}}{2\sqrt{(a+d)^2-4}} \\ 1 & \dfrac{-a+d+\sqrt{(a+d)^2-4}}{2c} \end{pmatrix} \in SL(2, \boldsymbol{R}).$$

Then we have

$$\begin{pmatrix} V & 0 \\ 0 & {}^tV^{-1} \end{pmatrix}\begin{pmatrix} U & S^tU^{-1} \\ 0 & {}^tU^{-1} \end{pmatrix}\begin{pmatrix} V^{-1} & 0 \\ 0 & {}^tV \end{pmatrix}=\begin{pmatrix} \begin{matrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{matrix} & VS^tU^{-1}\,{}^tV \\ 0 & \begin{matrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{matrix} \end{pmatrix}.$$

Put $V=\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$, $T=\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$, $t_1=(\lambda^2-1)^{-1}(v_1^2s_1+2v_1v_2s_{12}+v_2^2s_2)$ and $t_2=(\lambda^{-2}-1)^{-1}(v_3^2s_1+2v_3v_4s_{12}+v_4^2s_2)$. Then we have

$$\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \begin{matrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{matrix} & VS^tU^{-1}\,{}^tV \\ 0 & \begin{matrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{matrix} \end{pmatrix}\begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & (\sharp) \\ 0 & 1 & (\sharp) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \begin{matrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{matrix} & 0 \\ 0 & \begin{matrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{matrix} \end{pmatrix},$$

where $(\sharp)=v_1v_3s_1+(v_2v_3+v_1v_4)s_{12}+v_2v_4s_2$. Since this is conjugate in $GL(4, \boldsymbol{C})$ to

$$\begin{pmatrix} \begin{matrix} \lambda & (\sharp) \\ 0 & \lambda \end{matrix} & 0 \\ 0 & \begin{matrix} \lambda^{-1} & (\sharp) \\ 0 & \lambda^{-1} \end{matrix} \end{pmatrix},$$

it is semi-simple if and only if

$$\begin{aligned}(\sharp)&=v_1v_3s_1+(v_2v_3+v_1v_4)s_{12}+v_2v_4s_2\\ &=\frac{1}{\sqrt{(a+d)^2-4}}\{cs_1-(a-d)s_{12}+bs_2\}=0.\end{aligned}$$

Since $\lambda\neq\pm1$, $b\neq0$. Hence s_2 is determined by a, b, c, d, s_1 and s_{12}. Hence we have

$$I=\sum_{\left\{\substack{\gamma\in\Gamma_\infty^0 \\ \gamma\sim\begin{pmatrix} \begin{smallmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix}\lambda^{-1} & 0 \\ 0 & \lambda\end{smallmatrix}\end{pmatrix} (\lambda\neq\pm1)}\right.} |H_\gamma(Z)| \leq \sum_{\left\{\substack{U=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \\ b\neq0}\right.} \sum_{s_1, s_{12}} |H_\gamma(Z)|,$$

where

$$Z=\begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}$$

belongs to F,

$$\gamma=\begin{pmatrix}1 & S\\ 0 & 1\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & {}^tU^{-1}\end{pmatrix},\quad S=\begin{pmatrix}s_1 & s_{12}\\ s_{12} & s_2\end{pmatrix},$$

$s_2=-b^{-1}\{cs_1-(a-d)s_{12}\}$. Therefore, by Lemma 7, we have

$$I\leqq \sum_{\substack{U=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\\ b\neq 0\\ s_1,\, s_{12}}} \frac{(y_1y_2-y_{12}^2)^k\,(y_1'y_2'-y_{12}'^2)^{-k}}{\{1+2(y_1'y_2'-y_{12}'^2)^{-1}(x_{12}'+y_1'^{-1}y_{12}'x_1')^2+y_1'^{-2}x_1'^2\}^{k/2}}$$

where

$$\begin{pmatrix}y_1' & y_{12}'\\ y_{12}' & y_2'\end{pmatrix}=\begin{pmatrix}y_1 & y_{12}\\ y_{12} & y_2\end{pmatrix}+\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}y_1 & y_{12}\\ y_{12} & y_2\end{pmatrix}\begin{pmatrix}a & c\\ b & d\end{pmatrix}$$

and

$$\begin{pmatrix}x_1' & x_{12}'\\ x_{12}' & x_2'\end{pmatrix}=-\begin{pmatrix}s_1 & s_{12}\\ s_{12} & s_2\end{pmatrix}+\begin{pmatrix}x_1 & x_{12}\\ x_{12} & x_2\end{pmatrix}-\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x_1 & x_{12}\\ x_{12} & x_2\end{pmatrix}\begin{pmatrix}a & c\\ b & d\end{pmatrix}.$$

Since

$$\begin{pmatrix}y_1' & y_{12}'\\ y_{12}' & y_2'\end{pmatrix}\geqq\begin{pmatrix}y_1 & y_{12}\\ y_{12} & y_2\end{pmatrix},\quad y_1'y_2'-y_{12}'^2\geqq y_1y_2-y_{12}^2\geqq 1\,.$$

Moreover, $y_1'\geqq y_1\geqq \dfrac{\sqrt{3}}{2}$. Hence

$$I\leqq \text{const.}\sum_{\substack{U=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\\ b\neq 0}}\frac{(y_1y_2-y_{12}^2)^k y_1'}{(y_1'y_2'-y_{12}'^2)^{k-1/2}}$$

by Lemma 5. Since $2\begin{pmatrix}y_1 & 0\\ 0 & y_2\end{pmatrix}\geqq\begin{pmatrix}y_1 & y_{12}\\ y_{12} & y_2\end{pmatrix}\geqq\dfrac{1}{2}\begin{pmatrix}y_1 & 0\\ 0 & y_2\end{pmatrix}$, we see that

$$\begin{aligned}2\begin{pmatrix}y_1 & 0\\ 0 & y_2\end{pmatrix}+2\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}y_1 & 0\\ 0 & y_2\end{pmatrix}\begin{pmatrix}a & c\\ b & d\end{pmatrix}&\geqq\begin{pmatrix}y_1' & y_{12}'\\ y_{12}' & y_2'\end{pmatrix}\\ &\geqq\frac{1}{2}\begin{pmatrix}y_1 & 0\\ 0 & y_2\end{pmatrix}+\frac{1}{2}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}y_1 & 0\\ 0 & y_2\end{pmatrix}\begin{pmatrix}a & c\\ b & d\end{pmatrix}.\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{y_1'}{y_1'y_2'-y_{12}'^2}&\leqq 8\,\frac{(1+a^2)y_1+b^2y_2}{\{(1+a^2)y_1+b^2y_2\}\{c^2y_1+(1+d^2)y_2\}-\{acy_1+bdy_2\}^2}\\ &=8\,\frac{(1+a^2)y_1+b^2y_2}{c_1^2y_1^2+\{1+a^2+d^2+(ad-bc)^2\}y_1y_2+b^2y_2^2}\\ &\leqq 8\,\frac{1}{y_2}\,.\end{aligned}$$

Therefore, by Lemma 5, Lemma 8 and Lemma 9, we have

$$I\leq \text{const.}\sum_{\left\{\substack{U=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\\ b\neq 0}\right.}\frac{(y_1y_2-y_{12}^2)^k}{y_2(y_1'y_2'-y_{12}'^2)^{k-3/2}}$$

$$\leq \text{const.}\sum_{\left\{\substack{U=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\\ b\neq 0}\right.}\frac{(y_1y_2-y_{12}^2)^{3/2}}{y_2\left\{1+a^2+\frac{y_2}{y_1}b^2+\frac{y_1}{y_2}c^2+d^2\right\}^{k-3/2}}$$

$$\leq \text{const.}\sum_{a,b,d}\frac{(y_1y_2-y_{12}^2)^{3/2}}{y_2\{1+a^2+b^2+d^2\}^{k-3/2}}$$

$$\leq \text{const.}\frac{(y_1y_2-y_{12}^2)^{3/2}}{y_2}\ .$$

Since

$$\int_F y_2^{-1}(y_1y_2-y_{12}^2)^{3/2}dZ\leq\int_{\left\{\substack{0\leq 2y_{12}\leq y_1\leq y_2\\ \frac{\sqrt{3}}{2}\leq y_1}\right.} y_2^{-1}(y_1y_2-y_{12}^2)^{-3/2}dy_1dy_{12}dy_2<\infty\ ,$$

we have proved our assertion in the first case. Next put

$$\gamma=\begin{pmatrix} 1 & 0 & 0 & s_{12}\\ 0 & 1 & s_{12} & s_2\\ & & 1 & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ a_3 & 1 & \multicolumn{2}{c}{0}\\ & & 1 & -a_3\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 & b & 0\\ 0 & 1 & 0 & 0\\ c & 0 & d & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\in\Gamma_\infty^1$$

and suppose that γ conjugate in $Sp(2, R)$ to

$$\begin{pmatrix} \lambda & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0}\\ & & \lambda^{-1} & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}\quad(\lambda^2\neq 1).$$

Then we can prove in the same manner as before that s_2 is determined by s_{12}, a_3, a, b, c and d and that b, $c\neq 0$. Hence we have

$$I=\sum_{\left\{\substack{\gamma\in\Gamma_\infty^1\\ \gamma\sim\begin{pmatrix}\lambda & 0 & & \\ 0 & 1 & & \\ & & \lambda^{-1} & 0\\ & & 0 & 1\end{pmatrix}(\lambda^2\neq 1)}\right.}|H_\gamma(Z)|\leq\sum_{\left\{\substack{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\\ c\neq 0}\right.}\sum_{a_3,s_{12}}|H_\gamma(Z)|$$

$$\leq\sum_{\left\{\substack{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\\ c\neq 0}\right.}\sum_{a_3,s_{12}}\frac{(y_1y_2-y_{12}^2)^k(y_1'y_2'-y_{12}'^2)^{-k}|cz_1+d|^{-k}}{\{1+2(y_1'y_2'-y_{12}'^2)^{-1}(x_{12}'+y_1'^{-1}y_{12}'x_1')^2+y_1'^{-2}x_1'^2\}^{k/2}}$$

$$\leqq \text{const.} \sum_{\left\{\begin{smallmatrix}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \\ c\neq 0\end{smallmatrix}\right.} \sum_{a_3} \frac{(y_1y_2-y_{12}^2)^k|cz_1+d|^{-k}}{(y_1'y_2'-y_{12}'^2)^{k-1/2}(1+y_1'^{-2}x_1'^2)^{k/2-1/2}}$$

$$\leqq \text{const.} \sum_{\left\{\begin{smallmatrix}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \\ c\neq 0\end{smallmatrix}\right.} \frac{(y_1y_2-y_{12}^2)^k|cz_1+d|^{-k+1}}{(y_1y_2)^{k-1/2}\left(\dfrac{y_1}{y_2}\right)^{1/2}(1+y_1'^{-2}x_1'^2)^{k/2-1/2}}$$

$$= \text{const.} \sum_{\left\{\begin{smallmatrix}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \\ c\neq 0\end{smallmatrix}\right.} \frac{y_2y_1^{k-1}(1+|cz_1+d|^{-2})^{k-1}}{|z_1-(a\bar{z}_1+b)(c\bar{z}_1+d)^{-1}|^{k-1}|cz_1+d|^{k-1}}$$

$$\leqq \text{const.} \left[\sum_{\left\{\begin{smallmatrix}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in SL(2,Z) \\ c\neq 0\end{smallmatrix}\right.} \frac{y_1^{k-1}}{|z_1-(a\bar{z}_1+b)(c\bar{z}_1+d)^{-1}|^{k-1}|cz_1+d|^{k-1}}\right]\cdot y_2 .$$

Since the sum in the bracket is bounded (which is a result of Shimizu), we have

$$I \leqq \text{const.}\ y_2 .$$

Since

$$\int_F y_2 dZ \leqq \int_{\left\{\begin{smallmatrix} 0\leqq 2y_{12}\leqq y_1\leqq y_2 \\ \frac{\sqrt{3}}{2}\leqq y_1\end{smallmatrix}\right.} y_2(y_1y_2-y_{12}^2)^{-3}dy_1dy_{12}dy_2 < \infty ,$$

we have checked our assertion in the second case. Consequently, we have completed the proof of Theorem 4, and hence the proof of Theorem 5 also.

6.2. Another case.

THEOREM 6. *Let the notation and assumption be as in Theorem 3. Then the contribution to the dimension formula in Theorem 3 of the elements of $\Gamma_2(N)$ which are conjugate in $Sp(2, \boldsymbol{R})$ to*

(i) $$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ a & 1 & \multicolumn{2}{c}{0} \\ & & 1 & -a \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} \quad (a, b\neq 0) \quad \textit{or}$$

(ii) $$\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & \multicolumn{2}{c}{0} \\ & & a^{-1} & 0 \\ \multicolumn{2}{c}{0} & 0 & a \end{pmatrix} \quad (a^2\neq 1, b\neq 0) \quad \textit{or}$$

(iii) $$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ & & a^{-1} & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix} (a^2\neq 1, b\neq 0)$$

is zero. Hence we may disregard in the dimension formula in Theorem 3 the terms such that γ' is conjugate in $Sp(2, \boldsymbol{R})$ to one of the above types of elements.

We need one more lemma to prove this theorem:

LEMMA 13. *Let s be a positive number $\leqq 2^{-1}$ and put*

$$B_{1,s}=\left\{Z=\begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \in M_2(\boldsymbol{C}) \,\middle|\, y_1>0,\ y_2-y_1^{-1}y_{12}^2 \geqq \exp\left(\frac{1}{s^2}\right)\right\}.$$

Then $B_{1,s}$ contains $\bigcup_{\varepsilon \in \Gamma_\infty^1} \varepsilon F_{1,s}$ and is contained in $\bigcup_{\varepsilon \in \Gamma_{2}(1)} \varepsilon F_{1,s}$.

PROOF OF LEMMA 13. Let $Z \in F_{1,s}$ and $\varepsilon \in \Gamma_\infty^1$. Put $\varepsilon=\begin{pmatrix} a & 0 & b & * \\ * & * & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & * \end{pmatrix}$. Then we have $\det\{\mathrm{Im}\,(\varepsilon(Z))\}=\dfrac{1}{|cz_1+d|^2}\det\{\mathrm{Im}\,(Z)\}$ and $\mathrm{Im}\,\{(az_1+b)(cz_1+d)^{-1}\}=\dfrac{1}{|cz_1+d|^2}\mathrm{Im}\,(z_1)$. Hence their quotient $y_2-y_1^{-1}y_{12}^2=\dfrac{(y_1y_2-y_{12}^2)}{y_1}$ is invariant by the operation of the elements of Γ_∞^1. Therefore $B_{1,s} \supseteq \bigcup_{\varepsilon \in \Gamma_\infty^1} \varepsilon F_{1,s}$.

Next, suppose that $W=\begin{pmatrix} w_1 & w_{12} \\ w_{12} & w_2 \end{pmatrix}=\begin{pmatrix} u_1 & u_{12} \\ u_{12} & u_2 \end{pmatrix}+i\begin{pmatrix} v_1 & v_{12} \\ v_{12} & v_2 \end{pmatrix}$ $(w_1, w_{12}, w_2 \in \boldsymbol{C}, u_1, u_{12}, u_2, v_1, v_{12}, v_2 \in \boldsymbol{R})$ belongs to $B_{1,s}$. Then there is an element $\delta_1=\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ of $Sp(2, \boldsymbol{Z})$ such that

$$(aw_1+b)(cw_1+d)^{-1} \in \left\{z_1 \,|\, y_1>0,\ -\frac{1}{2}<x_1 \leqq \frac{1}{2},\ |z_1| \geqq 1\right\}.$$

Since $(aw_1+b)(cw_1+d)^{-1}$ is the $(1, 1)$ component of $\delta_1 W=W'=U'+iV'$, we have $|w_1'| \geqq 1\left(\text{hence } v_1' \geqq \dfrac{\sqrt{3}}{2}\right)$, $v_2'-v_1'^{-1}v_{12}'^2=v_2-v_1^{-1}v_{12}^2 \geqq \exp\left(\dfrac{1}{s^2}\right)>1$ and $\det\{\mathrm{Im}(W')\}= v_1'(v_2'-v_1'^{-1}v_{12}'^2) \geqq \dfrac{\sqrt{3}}{2}\exp\left(\dfrac{1}{s^2}\right)>1$. Hence $\mathrm{abs}\,\{\det(CW'+D)\} \geqq 1$ for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \boldsymbol{Z})$, since $\mathrm{abs}\,\{\det\,(CW'+D)\} \geqq |V'| \geqq 1$ if $|C| \neq 0$ $\left(\text{resp.} \geqq |c_1w_1'+d_1| \geqq 1 \text{ if } C=\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix},\ c_1 \neq 0,\ D=\begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix}\right)$ $\left(\text{resp.} \geqq v_2'-v_1'^{-1}v_{12}^2 \geqq 1 \text{ if } |C|=0,\ C \neq 0,\ C \neq \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}\right)$.

Therefore there is an element $\delta_2=\begin{pmatrix} U & S{}^tU^{-1} \\ 0 & {}^tU^{-1}\end{pmatrix}\in Sp(2, \boldsymbol{Z})$ such that $\delta_2 W'\in F$. Put $Z=\delta_2 W'$, $U^{-1}=\begin{pmatrix}\alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4\end{pmatrix}$. Then $V'=U^{-1}Y{}^tU^{-1}$ and $v_1'=\alpha_1^2 y_1+2\alpha_1\alpha_2 y_{12}+\alpha_2^2 y_2$. Since $0\leqq 2y_{12}\leqq y_1\leqq y_2$, $\alpha_1, \alpha_2\in \boldsymbol{Z}$ and $(\alpha_1, \alpha_2)\neq(0, 0)$, we have

$$v_1'\geqq \alpha_1^2 y_1-2|\alpha_1||\alpha_2|y_1+\alpha_2^2 y_1=(|\alpha_1|-|\alpha_2|)^2 y_1\geqq y_1$$

if $|\alpha_1|\neq|\alpha_2|$ and

$$v_1'=\alpha_1^2(y_1\pm 2y_{12}+y_2)\geqq \alpha_1^2 y_2\geqq \alpha_1^2 y_1\geqq y_1$$

if $|\alpha_1|=|\alpha_2|$. Therefore $v_1'\geqq y_1$ in any case, hence we have

$$\begin{aligned} y_2-y_1^{-1}y_{12}^2 &= \frac{y_1y_2-y_{12}^2}{y_1} \\ &= \frac{v_1'v_2'-v_{12}'^2}{y_1}\geqq v_1'-v_1'^{-1}v_{12}'^2\geqq \exp\left(\frac{1}{s^2}\right). \end{aligned}$$

Therefore we have completed the proof of Lemma 13.

Now we start the proof of Theorem 6. First put

$$B_{0,s}=\bigcup_{\varepsilon\in\Gamma_\infty^0}\varepsilon F_{0,s}.$$

Then we see that there exists a set $B'_{0,s}\supseteqq R^3$ such that $Z=X+iY\in B_{0,s}$ if and only if $Y=\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2\end{pmatrix}\in B'_{0,s}$. Let γ be an element of $\Gamma_2(N)$ that is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ a & 1 & & \\ & & 1 & -a \\ & & 0 & 1\end{pmatrix}\quad (a, b\neq 0).$$

Since we are going to show that the contribution of $Sp(2, \boldsymbol{Z})$ conjugacy classes of elements of $\Gamma_2(N)$ which are conjugate in $Sp(2, \boldsymbol{R})$ to any representative of the above form is zero, we may replace γ by any element that is conjugate in $Sp(2, \boldsymbol{Z})$ to γ. So, by Lemma 1, we may assume that $\gamma=\begin{pmatrix} 1 & 0 & b_1 & b_{12} \\ 0 & 1 & b_{12} & b_2 \\ & & 1 & 0 \\ & & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ a & 1 & & \\ & & 1 & -a \\ & & 0 & 1\end{pmatrix}$ $\in\Gamma_\infty^0\cap\Gamma_\infty^1$. Here $\varepsilon\gamma\varepsilon^{-1}$ $(\varepsilon\in Sp(2, \boldsymbol{Z}))$ belongs to Γ_∞^0 (resp. Γ_∞^1) if and only if ε belongs to Γ_∞^0 (resp. Γ_∞^1) by Proposition 12. Moreover, the centralizer C_γ of γ in $Sp(2, \boldsymbol{Z})/\pm 1$ is contained in $\Gamma_\infty^0 \cap \Gamma_\infty^1$. Hence $B_{1,\varepsilon}$ and $B_{0,\varepsilon}$ are stable by the operation of an element of C_γ. Let $F_{\gamma,\varepsilon}$ be the fundamental domain of C_γ in $\mathfrak{H}-B_{1,\varepsilon}-B_{0,\varepsilon}$. Then, by the remark after the proof of Theorem 3 and by Lemma 12, the contribution of $Sp(2, \boldsymbol{Z})/\pm 1$ conjugacy classes containing some γ of the above type is equal to $\sharp\{\Gamma_2(N)\backslash\Gamma_2(1)\}$ times

$$I=\lim_{\varepsilon\to+0}\sum_{\substack{\gamma:\text{above type}\\ \text{non-coojugate}}}\int_{F_{\gamma,\varepsilon}}H_\gamma(Z)dZ\,,$$

which we shall prove to be zero. Let $\delta_\gamma=\begin{pmatrix}1&0&t_1&t_{12}\\0&1&t_{12}&t_2\\ & &1&0\\ \multicolumn{2}{c}{0}&0&1\end{pmatrix}$ be an element of $Sp(2, \boldsymbol{R})$ satisfying $\delta_\gamma\gamma\delta_\gamma^{-1}=\begin{pmatrix}1&0&b_1&0\\0&1&0&0\\ & &1&0\\ \multicolumn{2}{c}{0}&0&1\end{pmatrix}\begin{pmatrix}1&0& & \\a&1&\multicolumn{2}{c}{0}\\ & &1&-a\\ \multicolumn{2}{c}{0}&0&1\end{pmatrix}$, whose existence was proved in the proof of Theorem 1. Then we have

$$I=\lim_{\varepsilon\to+0}\sum_\gamma\int_{F'_{\gamma,\varepsilon}}H_{\delta_\gamma\gamma\delta_\gamma^{-1}}(Z)dZ\,,$$

where $F'_{\gamma,\varepsilon}=\delta_\gamma F_{\gamma,\varepsilon}$ is a fundamental domain of $\delta_\gamma C_\gamma\delta_\gamma^{-1}$ in $\mathfrak{H}-\delta_\gamma B_{0,\varepsilon}-B_{1,\varepsilon}$. By Proposition 7, there is a set $F''_{\gamma,\varepsilon}\subset\boldsymbol{R}^4$ such that Z belongs to $F'_{\gamma,\varepsilon}$ if and only if (y_1, y_{12}, y_2, x_2) belongs to $F''_{\gamma,\varepsilon}$. Since

$$\begin{aligned}H_{\delta_\gamma\gamma\delta_\gamma^{-1}}(Z)&=\det\begin{pmatrix}y_1&y_{12}\\y_{12}&y_2\end{pmatrix}^k\\&\times\left\{\left(y_1-\frac{1}{2i}b_1\right)\left(y_2-\frac{1}{2i}ax_{12}+\frac{1}{2}ay_{12}-\frac{1}{2i}a^2x_1+\frac{1}{2}a^2y_1\right)-\left(y_{12}-\frac{1}{2i}ax_1+\frac{1}{2}ay_1\right)^2\right\}^{-k}\\&=-k^{\text{th}}\text{ power of a non-constant linear form in }x_{12},\end{aligned}$$

we have

$$\begin{aligned}I&=\lim_{\varepsilon\to0}\sum_\gamma\int_{F''_{\gamma,\varepsilon}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_{\delta_\gamma\gamma\delta_\gamma^{-1}}(Z)dx_{12}dx_1dx_2\frac{dY}{|Y|^3}\\&=0\,.\end{aligned}$$

Hence we have checked our assertion in the first case.

Next let γ be an element of $\Gamma_2(N)$ that is conjugate in $Sp(2, \boldsymbol{R})$ to

$$\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & & 0 \\ & & a^{-1} & 0 \\ & 0 & 0 & a \end{pmatrix} (a^2\neq 1,\ b\neq 0) \left(\text{resp.} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ & & a^{-1} & 0 \\ & 0 & 0 & 1 \end{pmatrix} (a^2\neq 1,\ b\neq 0)\right).$$

If γ is not conjugate in $Sp(2, \boldsymbol{Z})/\pm 1$ to any element of Γ_∞^0 (resp. Γ_∞^1), we can prove the assertion more simply, so we omit the proof in such a case and assume that γ is conjugate in $Sp(2, \boldsymbol{Z})/\pm 1$ to an element of Γ_∞^0 (resp. Γ_∞^1). Now suppose that γ itself is contained in Γ_∞^0 (resp. Γ_∞^1). Then $\varepsilon\gamma\varepsilon^{-1}$ ($\varepsilon \in Sp(2, \boldsymbol{Z})$) belongs to Γ_∞^0 (resp. Γ_∞^1) if and only if ε belongs to Γ_∞^0 (resp. Γ_∞^1) by Proposition 10 (resp. Proposition 11). Moreover, the centralizer C_γ of γ in $Sp(2, \boldsymbol{Z})/\pm 1$ is contained in Γ_∞^0 (resp. Γ_∞^1). Hence $B_{0,\varepsilon}$ (resp. $B_{1,\varepsilon}$) is stable by the operation of an element of C_γ. Let $F_{\gamma,\varepsilon}$ be the fundamental domain of C_γ in $\mathfrak{H}-B_{0,\varepsilon}$ (resp. $\mathfrak{H}-B_{1,\varepsilon}$). Then we see that the contribution of $Sp(2, \boldsymbol{Z})/\pm 1$ conjugacy classes containing some γ of the above type is equal to $\sharp\{\Gamma_2(N)\backslash\Gamma_2(1)\}$ times

$$I = \lim_{\varepsilon\to +0} \sum_{\substack{\gamma:\text{above type}\\ \text{non-conjugate}}} \int_{F_{\gamma,\varepsilon}} H_\gamma(Z)dZ\,.$$

By the proof of Theorem 1, there is an element

$$\delta_\gamma \in \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in Sp(2, \boldsymbol{R})\right\} \left(\text{resp. } \delta_\gamma \left\{\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in Sp(2, \boldsymbol{R})\right\}\right)$$

such that

$$\delta_\gamma\gamma\delta_\gamma^{-1} = \begin{pmatrix} 1 & 0 & 0 & b' \\ 0 & 1 & b' & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & & 0 \\ & & a^{-1} & 0 \\ & 0 & 0 & a \end{pmatrix} \quad (a^2\neq 1,\ b'\neq 0)$$

$$\left(\text{resp.} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & b' \\ & & a^{-1} & 0 \\ & 0 & 0 & 1 \end{pmatrix} \quad (a^2\neq 1,\ b'\neq 0)\right).$$

Therefore

$$I = \lim_{\varepsilon\to +0} \sum_{\gamma} \int_{F_{\gamma',\varepsilon}} H_{\delta_\gamma\gamma\delta_\gamma^{-1}}(Z)dZ\,,$$

where $F'_{\gamma,\epsilon}$ is a fundamental domain of $\delta_\gamma C_\gamma \delta_\gamma^{-1}$ in $\mathfrak{H}-\delta_\gamma B_{0,\epsilon}$ (resp. $\mathfrak{H}-B_{1,\epsilon}$). By Proposition 5 (resp. Proposition 4), there is a set $F''_{\gamma,\epsilon}\subseteq \boldsymbol{R}^4$ such that Z belongs to $F'_{\gamma,\epsilon}$ if and only if $(y_1, y_{12}, y_2, x_{12})$ (resp. (y_1, y_{12}, y_2, x_2)) belongs to $F''_{\gamma,\epsilon}$. Since $H_{\delta_\gamma\gamma\delta_\gamma^{-1}}(Z)$ is the $-k^{\text{th}}$ power of a non-constant linear form in x_1, we see that

$$\int_{F_{\gamma'}} H_{\delta_\gamma\gamma\delta_\gamma^{-1}}(Z)dZ=0 .$$

Therefore we have completed the proof of Theorem 6.

§7. An explicit formula

We have proved in section 6 that, if γ is an element of

$$\Gamma_2(N)=\left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp(2, \boldsymbol{Z})\,\middle|\,\begin{pmatrix} A & B \\ C & D\end{pmatrix}\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\bmod N\right\}\Big/\pm 1$$

(N may be any natural number) and γ is conjugate in $Sp(2, \boldsymbol{R})$ to

(i) $$\begin{pmatrix} \cos\lambda & 0 & \sin\lambda & 0 \\ 0 & a & 0 & 0 \\ -\sin\lambda & 0 & \cos\lambda & 0 \\ 0 & 0 & 0 & a^{-1}\end{pmatrix} \quad (\sin\lambda\neq 0,\ a\neq 1) ,$$

(ii) $$\begin{pmatrix} a_1 & 0 & & \\ 0 & a_2 & & 0 \\ & & a_1^{-1} & 0 \\ 0 & & 0 & a_2^{-1}\end{pmatrix}(a_1^2, a_2^2, a_1a_2\neq 1) ,$$

(iii) $$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ & & a^{-1} & 0 \\ 0 & & 0 & 1\end{pmatrix} \quad (a^2\neq 1) ,$$

(iv) $$\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b & 0 \\ & & 1 & 0 \\ 0 & & 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 & & \\ 0 & a^{-1} & & 0 \\ & & a^{-1} & 0 \\ 0 & & 0 & a\end{pmatrix} \quad (a^2\neq 1) ,$$

(v) $$\begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ 0 & & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ a & 1 & & 0 \\ & & 1 & -a \\ 0 & & 0 & 1\end{pmatrix} \quad (a, b\neq 0) \quad \text{or}$$

(vi) $$\begin{pmatrix} \mu & 0 & & \\ 0 & \mu & & 0 \\ & & \mu^{-1} & 0 \\ 0 & & 0 & \mu^{-1}\end{pmatrix}\begin{pmatrix} \cos\lambda & \sin\lambda & & \\ -\sin\lambda & \cos\lambda & & 0 \\ & & \cos\lambda & \sin\lambda \\ 0 & & -\sin\lambda & \cos\lambda\end{pmatrix} \quad (\mu^2\neq 1,\ \sin\lambda\neq 0) ,$$

γ has no contribution to the dimension formula.

Now let us assume that N is not less than 3. Then, in view of Theorem 1 and Lemma 1, it is enough to calculate the contribution of elements γ of $\Gamma_2(N)$ such that γ is conjugate in $Sp(2, \boldsymbol{Z})$ to $\delta=\begin{pmatrix}1 & S\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & s_1 & s_{12}\\ 0 & 1 & s_{12} & s_2\\ & & 1 & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}$. (We do not need this assumption (i.e. $N\geqq 3$) hereafter.)

7.1. The contribution of the identity element.

THEOREM 7. *Let the notation and assumption be as in Theorem* 3. *Then the contribution of the identity element to the dimension formula in Theorem* 3 *is*

$$a(k)[\Gamma_2(1) : \Gamma_2(N)]\int_F dZ=\frac{1}{2^9\cdot 3^3\cdot 5}[\Gamma_2(1) : \Gamma_2(N)](2k-2)(2k-3)(2k-4)\ .$$

PROOF. Since $H_{\delta^{-1}1_{\delta}}(Z)=1$ and $\int_F dZ=\dfrac{\pi^3}{270}$ (cf. Siegel [14]), we immediately have our assertion.

7.2. The contribution of degenerate quadratic forms.

THEOREM 8. *Let the notation and assumption be as in Theorem* 3. *Then we have*

(i) $\sum_{\gamma}\int_F |H_\gamma(Z)|dZ<\infty$,

where γ runs over all elements of $\Gamma_2(1)$ that are conjugate in $Sp(2, \boldsymbol{Z})$ to $\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & d\\ & & 1 & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}$ $(d\in \boldsymbol{Z},\ d\neq 0)$;

(ii) *the contribution to the dimension formula of elements of $\Gamma_2(N)$ which are conjugate in $Sp(2, \boldsymbol{Z})$ to* $\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & d\\ & & 1 & 0\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}$ $(d\in \boldsymbol{Z},\ d\neq 0)$ *is*

$$a(k)[\Gamma_2(1):\ \Gamma_2(N)]\sum_{\substack{-\infty<d<\infty\\ d\neq 0}}\int_{C_\delta} H_\delta(Z)dz$$
$$=-a(k)[\Gamma_2(1):\ \Gamma_2(N)]\frac{2^2\pi}{3(k-1)(k-2)}\sum_{d=1}^{\infty}\frac{1}{(Nd)^2}$$

$$=-\frac{1}{2^4 3^2}\mu_2(N)[\Gamma_1(1):\Gamma_1(N)]N(2k-3),$$

where $\delta=\begin{pmatrix}1&0&0&0\\0&1&0&Nd\\&&1&0\\0&&0&1\end{pmatrix}$, C_δ *is a fundamental domain of the centralizer of* δ *in* $Sp(2,\boldsymbol{Z})$ *and* $\mu_2(N)$ *is the number of inequivalent* 1-*dimensional cusps of* $\Gamma_2(N)\backslash\mathfrak{H}$.

PROOF. Since $H_{\varepsilon^{-1}\gamma\varepsilon}(Z)=H_\gamma(\varepsilon Z)\,(\varepsilon\in Sp(2,\boldsymbol{R}))$, it follows from Proposition 14 and Proposition 15 that

$$\begin{aligned}I_1&=\sum_\gamma\int_F|H_\gamma(Z)|dZ\\&=\sum_d\int_{C_\delta}|H_\gamma(Z)|dZ\\&=\sum_d\int_{C_\delta}\frac{(y_1y_2-y_{12}^2)^{k-3}}{|y_1(y_2-(1/2i)d)-y_{12}^2|^k}dx_1dx_{12}dx_2dy_1dy_{12}dy_2,\end{aligned}$$

where $C_\delta=\{Z\in\mathfrak{H}\,|\,-1/2\leqq x_1,x_{12},x_2\leqq 1/2,\ 0\leqq 2y_{12}\leqq y_1,\ |z_1|\geqq 1\}$. Therefore, we have

$$\begin{aligned}I_1&=\sum_d\int_{\left\{\substack{0\leqq 2y_{12}\leqq y_1\\ 0\leqq u\\ -1/2\leqq x_1\leqq 1/2\\ |z_1|\geqq 1}\right.}\frac{u^{k-3}y_1^{-1}}{(u^2+(d_1^2/4)y_1^2)^{k/2}}dx_1dudy_{12}dy_1\\&=\frac{1}{2}\sum_d\int_{\left\{\substack{0\leqq u\\ -1/2\leqq x_1\leqq 1/2\\ |z_1|\geqq 1}\right.}\frac{u^{k-3}}{(u^2+(d_1^2/4)y_1^2)^{k/2}}dudx_1dy_1\\&=2\sum_d\frac{1}{d^2}\int_{\left\{\substack{-1/2\leqq x_1\leqq 1/2\\ |z_1|\geqq 1}\right.}\frac{dx_1dy_1}{y_1^2}\int_0^\infty\frac{u^{k-3}du}{(u^2+1)^{k/2}}<\infty.\end{aligned}$$

So we have proved the first assertion.

By the assertion (i), the contribution of elements of $\Gamma_2(N)$ which are conjugate in $Sp(2,\boldsymbol{Z})$ to $\begin{pmatrix}1&0&0&0\\0&1&0&d\\&&1&0\\0&&0&1\end{pmatrix}$ $(d\in\boldsymbol{N}, d\neq 0)$ is equal to

$$I_2=a(k)\sum_{\varepsilon\in\Gamma_2(N)\backslash\Gamma_2(1)}\sum_\gamma\int_F H_{\varepsilon^{-1}\gamma\varepsilon}(Z)dZ.$$

Here, by Proposition 14 and Proposition 15, we have

$$I_2=a(k)[\Gamma_2(1):\Gamma_2(N)]\sum_{\left\{\substack{-\infty<d<\infty\\ d\neq 0}\right.}\int_{C_\delta}H_\delta(Z)dZ.$$

Here we have

$$\int_{C_\delta} H_\delta(Z)dZ=\int_{C_\delta}\frac{(y_1y_2-y_{12}^2)^{k-3}}{(y_1y_2-y_{12}^2-(Nd/2i)y_1)^k}dx_1dx_{12}dx_2dy_1dy_{12}dy_2$$

$$=\int_{\left\{\begin{smallmatrix}0\le 2y_{12}\le y_1\\ 0\le u\\ -1/2\le x_1\le 1/2\\ |z_1|\ge 1\end{smallmatrix}\right.}\frac{u^{k-3}y_1^{-1}}{(u+(Nd/2)iy_1)^k}dy_{12}dudx_1dy_1$$

$$=\frac{1}{2}\int_{\left\{\begin{smallmatrix}0\le u\\ -1/2\le x_1\le 1/2\\ |z_1|\ge 1\end{smallmatrix}\right.}\frac{u^{k-3}}{(u+(Nd/2)iy_1)^k}dudx_1dy_1$$

$$=\frac{-2}{(Nd)^2}\int_0^\infty\frac{u^{k-3}du}{(u+1)^k}\int_{\left\{\begin{smallmatrix}-1/2\le x_1\le 1/2\\ |z_1|\ge 1\end{smallmatrix}\right.}\frac{dx_1dy_1}{y_1^2}$$

$$=\frac{-2}{3(k-1)(k-2)}\frac{1}{N^2d^2}.$$

Hence we have

$$I_2=-a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{2^2\pi}{3(k-1)(k-2)}\sum_{d=1}^{\infty}\frac{1}{(Nd)^2}$$

$$=-\frac{1}{2^5\cdot 3^2}(2k-3)[\Gamma_2(1):\Gamma_2(N)]N^{-2}.$$

Since

$$[\Gamma_2(1):\Gamma_2(N)]=\frac{1}{2}N^{10}\prod_{\left\{\begin{smallmatrix}p:\ \text{prime}\\ p|N\end{smallmatrix}\right.}(1-p^{-2})(1-p^{-4}),$$

$$[\Gamma_1(1):\Gamma_1(N)]=\frac{1}{2}N^3\prod_{\left\{\begin{smallmatrix}p:\ \text{prime}\\ p|N\end{smallmatrix}\right.}(1-p^{-2})$$

and

$$\mu_2(N)=\frac{1}{2}N^4\prod_{\left\{\begin{smallmatrix}p:\ \text{primo}\\ p|N\end{smallmatrix}\right.}(1-p^{-4}),$$

we have our assertion.

7.3 The contribution of non-degenerate quadratic forms. In 7.3 we shall calculate the contribution of non-degenerate quadratic forms. For this purpose we need a few more preparatory lemmas.

LEMMA 14. *Let k be a positive number $\geqq 7$. Then we have*

$$\int_F\sum_\gamma H_\gamma(Z)|dZ<\infty,$$

where γ runs over all elements of $\Gamma_\infty^1-\Gamma_\infty^0$ that are conjugate in $Sp(2,\boldsymbol{R})$ to

$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \in Sp(2, \boldsymbol{Z})$ *with* $\det(S) \neq 0$.

PROOF. Let γ be as above. In view of assertion (i) of Theorem 8, we may assume that γ is conjugate in $Sp(2, \boldsymbol{R})$ to $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \in Sp(2, \boldsymbol{Z})$ with $\det(S) \neq 0$. Hence, by Proposition 14, Proposition 18 and Proposition 19, there exist $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z})$ and $\begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix} \in M_2(\boldsymbol{Z})$ such that

$$\gamma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & s_1 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 & -b & 0 \\ 0 & 1 & 0 & 0 \\ -c & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore we may write

$$\begin{aligned} \gamma &= \begin{pmatrix} 1-acs_1 & 0 & a^2 s_1 & as_{12} \\ -cs_{12} & 1 & as_{12} & s_2 \\ -c^2 s_1 & 0 & 1+acs_1 & cs_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & as_{12} \\ 0 & 1 & as_{12} & s_2 - acs_{12}^2 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & & 0 \\ -cs_{12} & 1 & & \\ & 0 & 1 & cs_{12} \\ & & 0 & 1 \end{pmatrix} \begin{pmatrix} 1-acs_1 & 0 & a^2 s_1 & 0 \\ 0 & 1 & 0 & 0 \\ -c^2 s_1 & 0 & 1+acs_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence by Lemma 5 and Lemma 7, we have,

$$\begin{aligned} I &= \sum_{\gamma} |H_\gamma(Z)| \\ &\leq \sum_{a,c,s_1,s_{12},s_2} |H_\gamma(Z)| \\ &\leq \text{const.} \sum_{a,c,s_1,s_{12}} \frac{|Y|^k \operatorname{abs}|CZ+D|^{-k} y_1'^{-1}(1+y_1'^{-2}x_1'^2)^{-1/2(k-1)}}{\left| Y + \begin{pmatrix} 1 & 0 \\ -cs_{12} & 1 \end{pmatrix}^t (\overline{CZ+D})^{-1} Y (CZ+D)^{-1} \begin{pmatrix} 1 & -cs_{12} \\ 0 & 1 \end{pmatrix} \right|^{k-1}}, \end{aligned}$$

where $C = \begin{pmatrix} -c^2 s_1 & 0 \\ 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1+acs_1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$x_1' + iy_1' = (x_1 + iy_1) - \frac{(1-acs_1)(x_1 - iy_1) + a^2 s_1}{-c^2 s_1 (x_1 - iy_1) + (1+acs_1)}.$$

Therefore

$$I \leq \text{const.} \sum_{a,c,s_1} \frac{y_1^{-1/2} y_{22}^{1/2} y_1'^{k-1} y_2}{(x_1'^2 + y_1'^2)^{1/2(k-1)} \operatorname{abs}|CZ+D|^{k-1}}$$

$$\leq \text{const.} \sum_{\left\{\substack{\varepsilon=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2,Z) \\ c' \neq 0}\right.} \frac{y_1^{-1/2} y_2^{3/2} y_1'^{k-1}}{|z_1-\varepsilon \bar{z}_1|^{k-1} |c' z_1 + d'|^{k-1}}$$

$$\leq \text{const.}\, y_1^{-1/2} y_2^{3/2},$$

here we have used the fact $y_1' = y_1\{1+(cz_1+d)^{-2}\} \leq 2y_1$. Therefore we have

$$\int_F I\,dZ \leq \text{const.} \int_F y_1^{-1/2} y_2^{3/2} dZ < \infty. \qquad \text{Q.E.D.}$$

LEMMA 15. *Let k be a positive number ≥ 7. Then we have*

$$\int_F \sum_\gamma |H_\gamma(Z)| dZ < \infty,$$

where γ runs over all elements of $\Gamma_\infty^0 - \Gamma_\infty^1$ that are conjugate in $Sp(2, \boldsymbol{Z})$ to
$$\begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} \in Sp(2, \boldsymbol{Z}).$$

Since the proof of this lemma can be done in the same manner as the proof of Lemma 14, we omit it.

LEMMA 16. *Let k be a positive number ≥ 7. Then we have*

$$\int_F \sum_\gamma |H_\gamma(Z)| dZ < \infty,$$

where $\gamma = \begin{pmatrix} 1 & 0 & s_1 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix}$ *and s_1, s_{12}, s_2 run over all integers such that* $-(s_1 s_2 - s_{12}^2)$ *belongs to* $(\boldsymbol{Q}^\times)^2$.

PROOF. First we consider the following Diophantine equation:

$$(\#) \qquad -(s_1 s_2 - s_{12}^2) = n^2.$$

Since this equation is equivalent to $-s_1 s_2 = (n - s_{12})(n + s_{12})$, we can solve (#) as follows: Let s_1, s_2 be any integers. Take any pair of integers t_1, t_2 satisfying (i) $t_1 t_2 = -s_1 s_2$ and (ii) $t_1 \equiv t_2 \pmod 2$. Then $n = (t_1 + t_2)/2$ and $s_{12} = (-t_1 + t_2)/2$ give all the solutions of (#). In particular, the number of solutions of (#) for given s_1 and s_2 is not greater than twice the number of positive divisors of $|s_1 s_2|$, which is not greater than $c(\varepsilon)|s_1 s_2|^\varepsilon$, where ε is any positive number and $c(\varepsilon)$ is a positive constant depending only on ε.

By this remark, we have the following estimates:

$$
\begin{aligned}
I &= \sum_{\gamma\,:\,\text{as above}} |H_\gamma(Z)| \\
&= \sum_{s_1, s_{12}, s_2} \frac{(y_1y_2-y_{12}^2)^k}{\operatorname{abs}\left|\begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix} - \frac{1}{2i}\begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix}\right|^k} \\
&\leq \sum_{|s_1|\geqq|s_2|} \frac{2^k c(\varepsilon)|s_1s_2|^\varepsilon}{\left\{4+\dfrac{y_2^2}{y_1^2y_2^2-y_{12}^4}(s_1-y_2^{-2}y_{12}^2s_2)^2+y_2^{-2}s_2^2\right\}^{k/2}} \\
&+ \sum_{|s_1|<|s_2|} \frac{2^k c(\varepsilon)|s_1s_2|^\varepsilon}{\left\{4+\dfrac{y_1^2}{y_1^2y_2^2-y_{12}^4}(s_2-y_1^{-2}y_{12}^2s_1)^2+y_1^{-2}s_1^2\right\}^{k/2}}.
\end{aligned}
$$

Since $0\leqq 2y_{12}\leqq y_1\leqq y_2$ and $(\sqrt{3}/2)\leqq y_1$, we have

$$
\frac{y_2^2}{y_1^2y_2^2-y_{12}^4}\geqq y_1^{-2}, \qquad \frac{y_1^2}{y_1^2y_2^2-y_{12}^4}\geqq y_2^{-2},
$$

$$
\frac{4}{3}\geqq y_1^{-2}, \qquad \frac{4}{3}\geqq y_2^{-2},
$$

$$
(s_1-y_2^{-2}y_{12}^2s_2)^2\geqq\left(|s_1|-\frac{1}{4}|s_2|\right)^2\geqq\frac{1}{4}s_1^2 \qquad (|s_1|\geqq|s_2|)
$$

and

$$
(s_2-y_1^{-2}y_{12}^2s_1)^2\geqq\left(|s_2|-\frac{1}{4}|s_1|\right)^2\geqq\frac{1}{4}s_2^2 \qquad (|s_1|<|s_2|).
$$

Hence we have

$$
\begin{aligned}
I &\leqq 2^k c(\varepsilon) \sum_{|s_1|\geqq|s_2|} \frac{|s_1s_2|^\varepsilon}{\{4+(1/4)y_1^{-2}s_1^2+y_2^{-2}s_2^2\}^{k/2}} \\
&+2^k c(\varepsilon) \sum_{|s_1|<|s_2|} \frac{|s_1s_2|^\varepsilon}{\{4+(1/4)y_2^{-2}s_2^2+y_1^{-2}s_1^2\}^{k/2}} \\
&\leqq 2^{2k} c(\varepsilon) \sum_{s_1, s_2\in Z} \frac{(y_1y_2)^\varepsilon}{\{1+y_1^{-2}s_1^2+y_2^{-2}s_2^2\}^{1/2(k-\varepsilon)}} \\
&\leqq \text{const. } c(\varepsilon)(y_1y_2)^{1+\varepsilon}.
\end{aligned}
$$

Therefore we have

$$
\int_F \sum_\gamma |H_\gamma(Z)|dZ \leqq \text{const. } c\left(\frac{1}{4}\right)\int_F (y_1y_2)^{5/4}dZ<\infty. \qquad \text{Q.E.D.}
$$

Now we state the main results in 7.3:

THEOREM 9. *Let the notation and assumption be as in Theorem* 3. *Then we have the following results*:

(i) *The contribution to the dimension formula of elements of* $\Gamma_2(N)$ *which are conjugate in* $Sp(2, \boldsymbol{Z})$ *to* $\delta=\begin{pmatrix}1 & NS\\ 0 & 1\end{pmatrix}$, $S=\begin{pmatrix}s_1 & s_{12}\\ s_{12} & s_2\end{pmatrix}(-\det(S)\neq 0, \notin (\boldsymbol{Q}^{\times})^2)$ *is equal to*

$$a(k)[\Gamma_2(1):\ \Gamma_2(N)]\lim_{s\to+0}\sum_{\delta}\int_{C_\delta} H_\delta(Z)\frac{dZ}{(y_1y_2-y_{12}^2)^s},$$

where δ *runs over all non-conjugate elements of the above form and* C_δ *is a fundamental domain of the centralizer of* δ *in* $Sp(2, \boldsymbol{Z})$.

(ii) *If* S *is indefinite and* $-\det(S)\neq 0, \notin (\boldsymbol{Q}^{\times})^2$,

$$\int_{C_\delta} H_\delta(Z)\frac{dZ}{(y_1y_2-y_{12}^2)^s}=0.$$

(iii) *If* S *is definite*,

$$\int_{C_\delta} H_\delta(Z)\frac{dZ}{(y_1y_2-y_{12}^2)^s}$$
$$=\left\{\frac{2^5\pi}{(2k-2)(2k-3)(2k-4)}+o(s)\right\}\exp\left\{\pm\frac{\pi}{2}i(-3-2s)\right\}\frac{1}{w(S)|\det(NS)|^{3/2+s}},$$

where the sign is + (*resp.* −) *if* S *is positive* (*resp. negative*) *and* $w(S)$ *is the number of units of* S. *Hence their contribution is equal to*

$$a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{2^5\pi^3}{(2k-2)(2k-3)(2k-4)}\frac{1}{N^3}\lim_{s\to 0}s\xi_1\left(L,\frac{3}{2}+s\right)$$
$$=\frac{1}{2^2\cdot 3}\mu_2(N)[\Gamma_1(1):\Gamma_1(N)],$$

where $\xi_1(L,(3/2)+s)$ *is the Dirichlet series defined in* §3, $\mu_2(N)$ *is the number of inequivalent zero dimensional cusps of* $\Gamma_2(N)\backslash\mathfrak{H}$ *and* $\Gamma_1(N)$ *is the principal congruence subgroup of* $SL(2, \boldsymbol{Z})/\pm 1$ *of level* N.

(iv) *The contribution to the dimension formula of elements of* $\Gamma_2(N)$ *which are conjugate in* $Sp(2, \boldsymbol{Z})$ *to* $\delta=\begin{pmatrix}1 & NS\\ 0 & 1\end{pmatrix}$, $S=\begin{pmatrix}0 & s_{12}\\ s_{12} & s_2\end{pmatrix}$ *is equal to*

$$a(k)[\Gamma_2(1):\ \Gamma_2(N)]\sum_{s_{12}=1}^{\infty}\sum_{0\leq s_2\leq 2s_{12}-1}\lim_{s\to+0}\int_{\begin{cases}0\leq v_1\leq v_2\\ 0\leq v_1v_2-v_{12}^2\\ -1/2\leq x_1,x_{12},x_2\leq 1/2\end{cases}} H_\gamma(Z)\frac{dZ}{(y_2-y_1^{-1}y_{12}^2)^s}$$
$$=a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{-2^5\pi}{(2k-2)(2k-3)(2k-4)}\frac{1}{N^3}\sum_{s_{12}=1}^{\infty}\frac{1}{s_{12}^2}$$

$$=\frac{-1}{2^3\cdot 3}\mu_2(N)[\Gamma_1(1):\Gamma_1(N)],$$

where $\gamma=\begin{pmatrix}1&0&0&s_{12}\\0&1&s_{12}&0\\ & &1&0\\ \multicolumn{2}{c}{0}&0&1\end{pmatrix}$.

PROOF. By Proposition 19, Proposition 24, Lemma 13 and Lemma 16, we see that the contribution of elements γ of $\Gamma_2(N)$ which are conjugate in $Sp(2,\boldsymbol{Z})$ to $\delta=\begin{pmatrix}1&NS\\0&1\end{pmatrix}(-\det(S)\neq 0,\notin(\boldsymbol{Q}^\times)^2)$ is equal to

$$I=a(k)[\Gamma_2(1):\Gamma_2(N)]\left[\sum_{\gamma\notin\left\{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\right\}}\int_F H_\gamma(Z)dZ+\lim_{s\to+0}\sum_{\gamma\in\left\{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\right\}}\int_F H_\gamma(Z)\frac{dZ}{(y_1y_2-y_{12}^2)^s}\right].$$

Now, since we have

$$\sum_\delta\int_{C_\delta}|H_\delta(Z)|\frac{dZ}{(y_1y_2-y_{12}^2)^s}\leq \text{const.}\left\{\xi_1\left(L,\frac{3}{2}+s\right)+\xi_2\left(L,\frac{3}{2}+s\right)\right\}<\infty,$$

for any $s>0$, we easily have the first assertion (i).

Next let $\delta=\begin{pmatrix}1&NS\\0&1\end{pmatrix}$ and suppose that S is indefinite and $-\det(S)\neq 0,\notin(\boldsymbol{Q}^\times)^2$. Let t, V, α be as in Proposition 17, let $w=w(S)$ be the index of the group of proper units of S in the group of units of S, and let $G=V\begin{pmatrix}\varepsilon&0\\0&\varepsilon^{-1}\end{pmatrix}V^{-1}$ be the generator of the group of proper units of S modulo ± 1. Then we have

$$\begin{aligned}\int_{C_\delta}H_\delta(Z)\frac{dZ}{(y_1y_2-y_{12}^2)^s}&=\frac{1}{w}\int_{\left\{\substack{\varepsilon^{-1}\leq y_1\leq\varepsilon\\0\leq y_1y_2-y_{12}^2}\right.}\frac{(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_{12}dy_2}{\{y_1y_2-(y_{12}+(i/2)t\alpha)^2\}^k}\\&=\frac{1}{w}\int_{\left\{\substack{\varepsilon^{-1}\leq y_1\leq\varepsilon\\0\leq u}\right.}\frac{u^{k-3-s}y_1^{-1}dy_1dy_{12}du}{(u-it\alpha y_{12}+(t^2\alpha^2/4))^k}\\&=\frac{1}{w}\int_0^\infty\int_{\varepsilon^{-1}}^{\varepsilon}\int_{-\infty}^\infty\frac{u^{k-3-s}y_1^{-1}}{(u-it\alpha y_{12}+(t^2\alpha^2/4))^k}dy_{12}dy_1du\\&=0,\end{aligned}$$

so that we have proved (ii).

Next let $\delta=\begin{pmatrix}1&NS\\0&1\end{pmatrix}$ and suppose that S is definite. Let V be an element of $SL(2,\boldsymbol{Z})$ such that $VS^tV=\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}$ with some $\alpha,\beta\in\boldsymbol{R}^\times$. Then we have, by

Proposition 16,

$$\int_{C_\delta} H_\delta(Z)\frac{dZ}{(y_1y_2-y_{12}^2)^s}$$

$$=\frac{1}{w(S)}\int_{\left\{\substack{0\le y_1\\ 0\le y_1y_2-y_{12}^2}\right.}\frac{(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_{12}dy_2}{\{(y_1+(i/2)\alpha)(y_2+(i/2)\beta)-y_{12}^2\}^k}$$

$$=\frac{2^{3+2s}}{w(S)|\det(NS)|^{3/2+s}}\int_{\left\{\substack{0\le y_1\\ 0\le y_1y_2-y_{12}^2}\right.}\frac{(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_{12}dy_2}{\{y_1y_2-y_{12}^2\pm i(y_1+y_2)-1\}^k}$$

$$=\frac{2^{3+2s}}{w(S)|\det(NS)|^{3/2+s}}\int_{\left\{\substack{0\le y_1\\ 0\le u}\right.}\frac{u^{k-3-s}y_1^{k-1}dudy_1dy_{12}}{\{(u\pm iy_1-1)y_1\pm i(u+y_{12}^2)\}^k}$$

$$=\frac{2^{3+2s}}{w(S)|\det(NS)|^{3/2+s}}\frac{\Gamma(k-2-s)\Gamma(2+s)}{\Gamma(k)}\exp\left\{\pm\frac{\pi}{2}i(-2-s)\right\}$$

$$\times\int_{0\le y_1} y_1^{k-1}(y_1\pm i)^{-k+2+s}(y_1^2\pm iy_1+y_{12}^2)^{-2-s}dy_1dy_{12}$$

$$=\frac{2^{3+2s}}{w(S)|\det(NS)|^{3/2+s}}\frac{\Gamma(k-2-s)\Gamma(2+s)}{\Gamma(k)}\exp\left\{\pm\frac{\pi}{2}i(-2-s)\right\}$$

$$\times\int_{\left\{\substack{0\le y_1\\ 0\le v}\right.} y_1^{k-1}(y_1\pm i)^{-k+2+s}(y_1^2\pm iy_1+v)^{-2-s}v^{-1/2}dvdy_1$$

$$=\frac{2^{3+2s}}{w(S)|\det(NS)|^{3/2+s}}\frac{\Gamma(k-2-s)\Gamma(1/2)\Gamma(3/2+s)}{\Gamma(k)}\exp\left\{\pm\frac{\pi}{2}i(-2-s)\right\}$$

$$\times\int_{0\le y_1} y_1^{k-5/2-s}(y_1\pm i)^{-k+1/2}dy_1$$

$$=\frac{\Gamma(k-2-s)\Gamma(1/2)\Gamma(3/2+s)\Gamma(k-3/2-s)\Gamma(1+s)}{\Gamma(k)\Gamma(k-1/2)}\cdot 2^{3+2s}$$

$$\times\frac{\exp\{\pm(\pi/2)i(-3-2s)\}}{w(S)|\det(NS)|^{3/2+s}}$$

$$=\left\{\frac{2^5\pi}{(2k-2)(2k-3)(2k-4)}+o(s)\right\}\frac{\exp\{\pm(\pi/2)i(-3-2s)\}}{w(S)|\det(NS)|^{3/2+s}}.$$

Therefore their contribution is equal to

$$a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{2^6\pi}{(2k-2)(2k-3)(2k-4)}\frac{1}{N^3}$$

$$\times\lim_{s\to 0}\{2\pi s+o(s^2)\}\sum_S\frac{1}{w(S)\det(S)^{3/2+s}},$$

where S runs over all inequivalent integral positive symmetric matrices of size two. By the definition of $\xi_1(L,s)$ and Proposition 14, this is equal to

$$a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{2^6\pi^2}{(2k-2)(2k-3)(2k-4)}\frac{1}{N^3}\lim_{s\to+0}s\xi_1\left(L,\frac{3}{2}+s\right)$$
$$=\frac{1}{2^2\cdot 3}\mu_2(N)[\Gamma_1(1):\ \Gamma_1(N)],$$

where we have used the fact that $\lim_{s\to 0} s\xi_1(L,(3/2)+s)=\pi/6$,

$$[\Gamma_2(1):\ \Gamma_2(N)]=\frac{1}{2}N^{10}\prod_{\substack{p\mid N\\ p:\ \text{prime}}}(1-p^{-2})(1-p^{-4}),\ \mu_2(N)=\frac{1}{2}N^4\prod_{\substack{p\mid N\\ p:\ \text{prime}}}(1-p^{-4})$$

and

$$[\Gamma_1(1):\ \Gamma_1(N)]=\frac{1}{2}N^3\prod_{\substack{p\mid N\\ p:\ \text{prime}}}(1-p^{-2}).$$

Next we shall prove (iv). By Proposition 18, the proof of Theorem 3, Lemma 14, Lemma 15 and Lemma 16, we have

$$\sum_{\gamma}\int_F |H_\gamma(Z)|dZ<\infty,$$

where γ runs over all elements of $\Gamma_2'(N)$ which are conjugate in $Sp(2,\boldsymbol{Z})$ to some $\begin{pmatrix}1&0&0&s_{12}\\0&1&s_{12}&s_2\\0&0&1&0\\0&0&0&1\end{pmatrix}$ and which is not contained in

$$\Delta'=\left\{\begin{pmatrix}1&0&0&s'_{12}\\0&1&s'_{12}&s'_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\a_3&1&0&0\\0&0&1&-a_3\\0&0&0&1\end{pmatrix}\ a_3,s'_{12},s'_2\in\boldsymbol{Z},a_3\neq 0\right\}.$$

Moreover, since $\begin{pmatrix}1&0&0&0\\0&1&0&s_2\\0&0&1&0\\0&0&0&1\end{pmatrix}$ is contained in the center of Γ^1_∞, it follows from Proposition 18 and Proposition 25 that

$$\sum_{s_{12}=1}^{\infty}\int_F\left|\sum_{\gamma\in\Delta(s_{12})}H_\gamma(Z)\right|dZ<\infty,$$

where $\Delta(s_{12})$ is the subset consisting of all elements of Δ' which are conjugate in $Sp(2,\boldsymbol{Z})$ to some $\begin{pmatrix}1&0&0&s_{12}\\0&1&s_{12}&s_2\\0&0&1&0\\0&0&0&1\end{pmatrix}$ with a certain fixed s_{12}. Let $c(s_{12})$ be a positive constant, which we shall determine later as a function of s_{12}. Then, by the above remark, the contribution of elements γ of $\Gamma_2(N)$ which are conjugate in $Sp(2, \boldsymbol{Z})$ to $\begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with a certain fixed s_{12} is equal to

$$a(k)[\Gamma_2(1) : \Gamma_2(N)]\Big[\sum_{\gamma \notin \Delta(s_{12})} \int_F H_\gamma(Z)dZ + \sum_{\gamma \in \Delta(s_{12})} \int_{F \cap \{y_2 - y_1^{-1}y_{12}^2 < c(s_{12})\}} H_\gamma(Z)dZ$$
$$+ \lim_{s \to +0} \sum_{\gamma \in \Delta(s_{12})} \int_{F \cap \{y_2 - y_1^{-1}y_{12}^2 \geq c(s_{12})\}} H_\gamma(Z) \frac{dZ}{(y_2 - y_1^{-1}y_{12}^2)^s}\Big].$$

Now we are going to determine the normalizer of $\Delta(s_{12})$ in $Sp(2, \boldsymbol{Z})$. So let γ_1, γ_2 be elements of $\Delta(s_{12})$ and suppose that $\delta \in Sp(2, \boldsymbol{Z})$ satisfies $\gamma_1 = \delta^{-1}\gamma_2\delta$. Then, by Proposition 18, there are $\varepsilon_1, \varepsilon_2 \in \Gamma_\infty^1$ satisfying $\varepsilon_1\gamma_1\varepsilon_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\varepsilon_2\gamma_2\varepsilon_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & s_2 \\ 0 & 1 & s_{12} & s_2' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Moreover, it is easy to see that there are

$$\varepsilon_3, \varepsilon_4 \in \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a_3 \in \boldsymbol{R} \right\}$$

satisfying $(\varepsilon_3\varepsilon_1)\gamma_1(\varepsilon_3\varepsilon_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $(\varepsilon_4\varepsilon_2)\gamma_2(\varepsilon_4\varepsilon_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. It follows that $\varepsilon_4\varepsilon_2\delta(\varepsilon_3\varepsilon_1)^{-1}$ centralize $\begin{pmatrix} 1 & 0 & 0 & s_{21} \\ 0 & 1 & s_{12} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Therefore we may write

$$\varepsilon_4\varepsilon_2\delta(\varepsilon_3\varepsilon_1)^{-1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & a & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} \\ 0 & 0 & a & 0 \end{pmatrix}.$$

In the first case, we see that δ belongs to Γ_∞^1. Since $\varepsilon_2\delta\varepsilon_1^{-1}$ normalize

$$\left\{\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & * & * \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \in Sp(2, \boldsymbol{Z})\right\}$$

(whose normalizer is contained in Γ_∞^0), since $SL(2, \boldsymbol{Z})$ is a subgroup of $GL(2, \boldsymbol{Z})$ of index two and since $GL(2, \boldsymbol{Z})$ acts on the set

$$\left\{\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \in Sp(2, \boldsymbol{Z}) \;\middle|\; -\det(S) \in (\boldsymbol{Q}^\times)^2\right\}$$

in the obvious manner, we see that Γ_∞^1 is a subgroup of the normalizer of

$$\Delta(s_{12}) \cup \left\{\gamma = \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \;\middle|\; s_2 \in \boldsymbol{Z}\right\}$$

in $Sp(2, \boldsymbol{Z})$ of index two.

Now let γ be an element of $\Delta(s_{12})$ and suppose that γ is conjugate in $Sp(2, \boldsymbol{Z})$ to $\begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$. So let $\varepsilon\gamma\varepsilon^{-1} = \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$ with $\varepsilon \in Sp(2, \boldsymbol{Z})$. Then, by the above remark, there are $\varepsilon_5 \in \Gamma_\infty^1$, $\notin \Gamma_\infty^0$, $\varepsilon_6, \varepsilon_7 \in \left\{\begin{pmatrix} 1 & 0 & & 0 \\ a_3 & 1 & & \\ & 0 & 1 & -a_3 \\ & & 0 & 1 \end{pmatrix} \;\middle|\; a_3 \in \boldsymbol{R}\right\}$ and $\varepsilon_8 = 1$ or $\begin{pmatrix} 0 & a & & 0 \\ a^{-1} & 0 & & \\ & 0 & 0 & a^{-1} \\ & & a & 0 \end{pmatrix} \in Sp(2, \boldsymbol{R})$ satisfying $\varepsilon_6\varepsilon\gamma(\varepsilon_6\varepsilon)^{-1} = \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$ and $\varepsilon_6\varepsilon = \varepsilon_8\varepsilon_7\varepsilon_5$. Further, we see that there is a certain ε with $\varepsilon_8 \neq 1$ and such ε is unique up to the elements of Γ_∞, if we fix a conjugacy class.

Let $Z = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}$ be an element of F and suppose $y_2 - y_1^{-1}y_{12}^2 \geqq c(s_{12})$. Let $\gamma \in \Delta(s_{12})$ and $\varepsilon, \varepsilon_5, \varepsilon_6, \varepsilon_7$ and ε_8 be as above. Then we see that $\varepsilon_6\varepsilon Z = Z' = \begin{pmatrix} z_1' & z_{12}' \\ z_{12}' & z_2' \end{pmatrix}$ satisfies

$$y_2'-y_1'^{-1}y_{12}'^2\geqq c(s_{12}) \qquad \text{and} \qquad y_1'\leqq 1\cdots \quad \text{if} \quad \varepsilon_8=1$$

or

$$y_1'-y_2'^{-1}y_{12}'^2\geqq a^2c(s_{12}) \qquad \text{and} \qquad y_2'\leqq a^{-2}\cdots \quad \text{if} \quad \varepsilon_8=\begin{pmatrix} 0 & a & & \\ a^{-1} & 0 & & \\ & & 0 & a^{-1} \\ & & a & 0 \end{pmatrix}.$$

In particular, it is contained in $\{y_1\leqq a^2y_2\}$ or $\{y_1\geqq a^2y_2\}$ (which are fundamental domains of ε_8), if $c(s_{12})$ is greater than a^{-2}. So let us put $c(s_{12})=\underset{\gamma\in\varDelta(s_{12})}{\mathrm{Max}}(|a|^2,|a|^{-2})$ (we note that $\varDelta(s_{12})$ has only finite intersections with the set of $Sp(2,\boldsymbol{Z})$ conjugacy classes in $\Gamma_2(N)$). Here we have

$$\sum_{s_2=0}^{2s_{12}-1}\int_{\left\{\substack{0\leqq y_1\leqq a^2y_2\\ 0\leqq y_1y_2-y_{12}^2}\right.}\frac{y_1^{k-3}(y_2-y_1^{-1}y_{12}^2)^{k-3-s}}{|y_1y_2-(y_{12}-(1/2i)s_{12})^2|^k}dy_1dy_{12}dy_2$$

$$=\sum_{s_2=0}^{2s_{12}-1}\frac{1}{s_{12}^{3+s}}\int_{\left\{\substack{0\leqq y_1\leqq a^2y_2\\ 0\leqq y_1y_2-y_{12}^2}\right.}\frac{y_1^{k-3}(y_2-y_1^{-1}y_{12}^2)^{k-3-s}}{\{(y_1y_2-y_{12}^2+1/4)^2+y_{12}^2\}^{k/2}}dy_1dy_{12}dy_2$$

$$=\left(\sum_{s_2=0}^{2s_{12}-1a^s}a_s\right)\frac{1}{s_{12}^{3+s}}\int_{\left\{\substack{0\leqq y_1\leqq y_2\\ 0\leqq y_1y_2-y_{12}^2}\right.}\frac{y_1^{k-3}(y_2-y_1^{-1}y_{12}^2)^{k-3-s}}{[\{y_1(y_2-y_1^{-1}y_{12}^2)+1/4\}^2+y_{12}^2]^{k/2}}dy_1dy_{12}dy_2,$$

$$\sum_{s_2=0}^{2s_{12}-1}\int_{\left\{\substack{0\leqq a^2y_2\leqq y_1\\ 0\leqq y_1y_2-y_{12}^2}\right.}\frac{a^{2s}(y_1-y_2^{-1}y_{12}^2)^{k-3-s}y_2^{k-3}}{|y_1y_2-(y_{12}-(1/2i)s_{12})^2|^k}dy_1dy_{12}dy_2$$

$$=\left(\sum_{2s=0}^{2s_{12}-1a^s}a_s\right)\frac{1}{s_{12}^{3+s}}\int_{\left\{\substack{0\leqq y_2\leqq y_1\\ 0\leqq y_1y_2-y_{12}^2}\right.}\frac{(y_1-y_2^{-1}y_{12}^2)^{k-3-s}y_2^{k-3}}{[\{(y_1-y_2^{-1}y_{12}^2)y_2+1/4\}^2+y_{12}^2]^{k/2}}dy_1dy_{12}dy_2$$

and

$$\int_{\left\{\substack{0\leqq y_1\leqq y_2\\ 0\leqq y_1y_2-y_{12}^2}\right.}\frac{y^{-3}(y_2-y_1^{-1}y_{12}^2)^{k-3-s}}{[\{y_1(y_2-y_1^{-1}y_{12}^2)+1/4\}^2+y_{12}^2]^{k/2}}dy_1dy_{12}dy_2$$

$$=\int_{\left\{\substack{0\leqq u\\ 0\leqq y_1{}^2\leqq u+y_{12}^2}\right.}\frac{y_1^{-1+s}u^{k-3-s}}{\{(u+1/4)^2+y_{12}^2\}^{k/2}}dy_1d_{12}du$$

$$=\frac{1}{s}\int_{0\leqq u}\frac{(u+y_{12}^2)^{s/2}u^{k-3-s}}{\{(u+1/4)^2+y_{12}^2\}^{k/2}}dy_{12}du$$

$$\leqq\frac{1}{s}\int_{0\leqq u}\frac{u^{k-3-s}}{\{(u+1/4)^2+y_{12}^2\}^{k/2-s/2}}dy_{12}du$$

$$=\frac{1}{s}\int_{-\infty}^{\infty}\frac{dy_{12}}{(y_{12}^2+1)^{k/2-s/2}}\int_0^{\infty}\frac{u^{k-3-s}}{(u+1/4)^{k-1-s}}du<\infty.$$

Therefore, the contribution of elements of $\Gamma_2(N)$ which are conjugate in $Sp(2, \boldsymbol{Z})$ to $\begin{pmatrix} 1 & 0 & 0 & Ns_{12} \\ 0 & 1 & Ns_{12} & Ns_2 \\ & & 1 & 0 \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}$ (s_{12} is a fixed number) is equal to

$$\frac{1}{2}a(k)[\Gamma_2(1):\Gamma_2(N)]$$

$$\times\lim_{s\to+0}\left[\sum_{s_2=0}^{2s_{12}-1}\int_{\left\{\substack{0\leq y_1\leq a^2y_2\\ 0\leq y_1y_2-y_{12}^2}\right.}\frac{y_1^{k-3}(y_2-y_1^{-1}y_{12}^2)^{k-3-s}}{\{y_1y_2-(y_{12}-(1/2i)Ns_{12})^2\}^k}dy_1dy_{12}dy_2\right.$$

$$\left.+\sum_{s_2=0}^{2s_1-1}\int_{\left\{\substack{0\leq a^2y_2\leq y_1\\ 0\leq y_1y_2-y_{12}^2}\right.}\frac{a^{2s}(y_1-y_2^{-1}y_{12}^2)^{k-3-s}y_2^{k-3}}{\{y_1y_2-(y_{12}-(1/2i)Ns_{12})^2\}^k}dy_1dy_{12}dy_2\right]$$

$$=\lim_{s\to+0}a(k)[\Gamma_2(1):\Gamma_2(N)]\left(\sum_{s_2=0}^{2s_{12}-1}a^s\right)\cdot\frac{1}{N^{3+s}s_{12}^{3+s}}$$

$$\times\int_{\left\{\substack{0\leq y_1\leq y_2\\ 0\leq y_1y_2-y_{12}^2}\right.}\frac{y_1^{k-3}(y_2-y_1^{-1}y_{12}^2)^{k-3-s}}{\{y_1y_2-(y_{12}-1/2i)^2\}^k}dy_1dy_{12}dy_2$$

$$=a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{2}{s_{12}^2}\frac{1}{N^3}$$

$$\times\lim_{s\to+0}\int_{\left\{\substack{0\leq u\\ 0\leq y_1{}^2\leq u+y_{12}^2}\right.}\frac{y_1^{-1+s}u^{k-3-s}}{(u+1/4-iy_{12})^k}dy_1dy_{12}du$$

$$=a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{2}{s_{12}^2}\frac{1}{N^3}$$

$$\times\lim_{s\to+0}\frac{1}{s}\int_{0\leq u}\frac{(u+1/4-iy_{12})^k}{(u+y_{12}^2)^{s/2}u^{k-3-s}}dy_{12}du$$

$$=a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{2^5}{s_{12}^2}\frac{1}{N^3}$$

$$\times\lim_{s\to+0}\frac{1}{s}\int_{0\leq v}\frac{(v^2+y_{12}^2)^{s/2}v^{2k-5-2s}}{(v^2+1+2iy_{12})^k}dy_{12}dv$$

$$=a(k)[\Gamma_2(1):\Gamma_2(N)]\frac{2^5}{s_{12}^2}\frac{1}{N^3}$$

$$\times\lim_{s\to+0}\frac{1}{s}\frac{\pi^{1/2}\Gamma(k-2-s)\Gamma\left(\frac{3}{2}+\frac{s}{2}\right)\Gamma\left(k-\frac{3}{2}-\frac{s}{2}\right)\Gamma(1)}{2\Gamma\left(k-\frac{1}{2}-\frac{s}{2}\right)\Gamma(k)\Gamma\left(-\frac{s}{2}\right)},$$

here we have used the following formula (cf. Siegel [12]):

$$\int_0^\infty\int_{-\infty}^\infty \frac{x^{2\lambda-1}(x^2+w^2)^{\nu-1/2}}{(x^2+1-2iw)^{\lambda+\mu+\nu}}dwdx=\frac{\pi^{1/2}\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\nu)\Gamma(\mu-\nu)}{2\Gamma(\lambda+\mu)\Gamma(\lambda+\mu+\nu)\Gamma(1/2-\nu)}$$

$(\mathrm{Re}\,(\lambda)>0, \mathrm{Re}\,(\mu)>0, \mathrm{Re}\,(\mu)>\mathrm{Re}\,(\nu)>-\mathrm{Re}\,(\lambda))$.

Since

$$\lim_{s\to+0}\frac{1}{s}\frac{1}{\Gamma(-s/2)}=\lim_{s\to+0}\frac{1}{-2\Gamma((-s/2)+1)}=-\frac{1}{2},$$

the above formula is equal to

$$a(k)[\Gamma_2(1):\ \Gamma_2(N)]\frac{-2^3}{s_{12}^2}\frac{1}{N^3}\frac{\pi^{1/2}\Gamma(k-2)\Gamma(3/2)\Gamma(k-3/2)}{\Gamma(k-1/2)\Gamma(k)}$$

$$=a(k)[\Gamma_2(1):\ \Gamma_2(N)]\frac{-2^5\pi}{(2k-2)(2k-3)(2k-4)}\frac{1}{N^3}\frac{1}{s_{12}^2}.$$

Since

$$\sum_{s_{12}=1}^\infty\frac{1}{s_1^2}=\frac{\pi^2}{6},$$

we have completed the proof of Theorem 9.

REMARK. T. Shintani has succeeded in simplifying the proof in this section. He has, in fact, calculated the contribution of the conjugate classes of translations under the condition that the discrete subgroup Γ is the principal congruence subgroup of the Siegel modular group of degree n, using his results on the zeta functions of pre-homogeneous vector spaces (cf. [18]).

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Institute for Advanced Study
Princeton, N. J.
U.S.A.
and
Department of Mathematics
Faculty of Science
University of Tokyo
113 Japan