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ON SIEGEL MODULAR FORMS OF GENUS TWO (II).[1]

By Jun-ichi Igusa.

Introduction. The main subject we shall discuss in this second paper is the Siegel modular forms of genus two with levels. The method we used in the first paper [5] did not give sufficient information even for level two. Therefore the problem (raised by Grothendieck) whether modular varieties become non-singular or not for higher levels was beyond our reach. With some other applications in mind, we therefore investigated "theta-constants" as modular forms and proved, among others, a fundamental lemma in our recent paper [6]. Using the results in that paper, we shall show that *modular varieties of high levels do not have non-singular coverings even locally around their singular points.* Also we shall determine how $\Gamma_2(1)/\Gamma_2(2) = Sp(2, \mathbf{Z}/2\mathbf{Z})$ acts on the ring of modular forms $A(\Gamma_2(2))$ and obtain the characters of its action on the homogeneous parts $A(\Gamma_2(2))_k$ for $k = 0, 1, 2, \cdots$. In this way, we shall determine $A(\Gamma_2(1))$ reproducing our earlier structure theorem on $A(\Gamma_2(1))^{(2)}$. Furthermore, the *polynomial expressions* of the four basic Eisenstein series of level one by theta-constants and a known identity of this kind (between a certain Eisenstein series of level two and the eighth power of Riemann's theta-constant [1]) will be obtained. We note that this identity was previously obtained using the Siegel main theorem on quadratic forms.

1. Normality of $C[\theta_{\mathfrak{m}}\ \theta_{\mathfrak{n}}]$. We shall use the same notation as in TC except that we have $g = 2$ in this paper (and may omit $g = 2$ for the sake of simplicity). Also we shall sometimes treat characteristics as line vectors instead of column vectors (as in the case $g = 1$). We recall that theta-constants $\theta_{\mathfrak{m}}(\tau)$ are defined by putting $z = 0$ in

$$\theta_{\mathfrak{m}}(\tau, z) = \sum_{p \in Z^2} e[\tfrac{1}{2} \cdot {}^t(p + m'/2)\tau(p + m'/2) + {}^t(p + m'/2)(z + m''/2)].$$

Also we shall denote the function $\tau \to \theta_{\mathfrak{m}}(\tau)$ defined in the Siegel upper-half plane $\mathfrak{S}$ by $\theta_{\mathfrak{m}}$. There are ten even characteristics mod 2 and they are (0000), (0001), (0010), (0011), (0100), (0110), (1000), (1001), (1100), (1111).

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The purpose of this section is to prove the normality of the graded ring $\boldsymbol{C}[\theta_{\mathfrak{m}}]$ generated over $\boldsymbol{C}$ by the ten "theta-constants."

LEMMA 1. *Let K be a field* (of characteristic zero) *and let $K[X,T]$ be a ring of polynomials in ten letters X_{ij} for $1 \leqq i, j \leqq 3$ and T with coefficients in K. Then the following twenty-one quadratic polynomials*

$$\sum_{1 \leqq j \leqq 3} X_{i_1 j} X_{i_2 j} - \delta_{i_1 i_2} T^2, \qquad \sum_{1 \leqq i \leqq 3} X_{i j_1} X_{i j_2} - \delta_{j_1 j_2} T^2$$
$$(X_{i_1 j_1} X_{i_2 j_2} - X_{i_1 j_2} X_{i_2 j_1}) - X_{i_3 j_3} T,$$

in which the permutation $(i_1 i_2 i_3) \to (j_1 j_2 j_3)$ is even, generate a prime ideal in $K[X,T]$. Moreover, if (x,t) is a generic zero over K, the integral domain $K[x,t]$ is normal.

Proof. We observe that we have an identity of the form

$$\begin{aligned}\det(X_{ij}) - T^3 = {} & X_{11}((X_{22}X_{33} - X_{23}X_{32}) - X_{11}T) \\ & + X_{21}((X_{32}X_{13} - X_{33}X_{12}) - X_{21}T) \\ & + X_{31}((X_{12}X_{23} - X_{13}X_{22}) - X_{31}T) \\ & + ((X_{11})^2 + (X_{21})^2 + (X_{31})^2 - T^2)T.\end{aligned}$$

Hence the first part becomes a special case of a known theorem and a proof can be found in Weyl [11, pp. 144-7]. We shall prove the second part. The main point is that every element of $K[x,t]$ can be written uniquely as a linear combination of monomials

$$x_{i_1 j_1} \cdots x_{i_p j_p} t^e \quad i_1 \leqq \cdots \leqq i_p,\ j_1 \geqq \cdots \geqq j_p$$

with coefficients in K in which x_{i1} and x_{3j} appear at most once. In order to make the argument clear, we consider the set of pairs of non-negative integers and introduce a lexicographic order in it. This set is well ordered. Then to each monomial $X_{i_1 j_1} \cdots X_{i_p j_p} T^e$ of $K[X,T]$, we associate the following element $(p, \sum i_\alpha + (3 - j_\alpha))$ of the well-ordered set as its "weight." Now we introduce three operations in $K[X,T]$. The first operation is

$$X_{i_1 j_1} X_{i_2 j_2} \to X_{i_1 j_2} X_{i_2 j_1} + X_{i_3 j_3} T$$

whenever both $i_1 < i_2$ and $j_1 < j_2$. The second and the third operations are

$$X_{i_1 1} X_{i_2 1} \to \delta_{i_1 i_2} T^2 - \sum_{j > 1} X_{i_1 j} X_{i_2 j}$$
$$X_{3 j_1} X_{3 j_2} \to \delta_{j_1 j_2} T^2 - \sum_{i < 3} X_{i j_1} X_{i j_2}.$$

We observe that, by the first operation each monomial is replaced by a sum of two monomials one of the same weight and another of a smaller weight. On the other hand, by the two other operations each monomial is replaced by a sum of two or three monomials of smaller weights. If a polynomial in X_{ij}, T is given, we apply the first operation repeatedly (as long as it can be applied) and we get a new polynomial which is a linear combination of monomials $X_{i_1j_1}\cdots X_{i_pj_p}T^e$ in which $i_1\leqq\cdots\leqq i_p$, $j_1\geqq\cdots\geqq j_p$. The new polynomial may be called "properly ordered." We then apply two other operations and order properly the new polynomial so obtained. Because weights decrease by the second and the third operations, the process necessarily stops after a finite number of times. In this way, we get a properly ordered polynomial each monomial of which contains X_{i1} and X_{3j} at most once. If we replace X_{ij} by x_{ij} and T by t, the three operations reduce to modifying expressions of the element of $K[x,t]$. Thus every element of $K[x,t]$ can be written in the way stated before. We shall prove its uniqueness. Suppose that we have a relation of the form

$$P_0(x_{ij})+P_1(x_{ij})t+\cdots+P_n(x_{ij})t^n=0$$

in which $P_e(x_{ij})t^e$ is a linear combination of the said monomials for $0\leqq e\leqq n$. Also the relation is homogeneous in x_{ij} and t. We can assume that $P_0(x_{ij})$ is different from zero. Let (x_1,x_2,x_3) and (y_1,y_2,y_3) be two independent generic points over K of a quadratic cone defined by $X_1^2+X_2^2+X_3^2=0$. Then $(x_{ij},t)\to(x_iy_j,0)$ is a specialization over K, hence $P_0(x_iy_j)=0$. We shall show that this implies $P_0(x_{ij})=0$. We know that $P_0(X_{ij})$ is a linear combination of $X_{i_1j_1}\cdots X_{i_nj_n}$ with coefficients c_{ij} say. If we introduce three more letters Y_1, Y_2, Y_3, the condition $P_0(x_iy_j)=0$ means that

$$\sum c_{ij}X_{i_1}\cdots X_{i_n}Y_{j_1}\cdots Y_{j_n}$$

is contained in the ideal of $K[X_1,X_2,X_3,Y_1,Y_2,Y_3]$ generated by

$$X_1^2+X_2^2+X_3^2 \text{ and } Y_1^2+Y_2^2+Y_3^2.$$

On the other hand, neither X_3 nor Y_1 appears twice in the products $X_{i_1}\cdots X_{i_n}Y_{j_1}\cdots Y_{j_n}$. Since such monomials are linearly independent over K modulo the ideal in question, we get $c_{ij}=0$ and $P_0(x_{ij})=0$. But this is a contradiction. This completes the proof of the key point.

Now, for the proof of the normality of $K[x,t]$, we can assume that K is algebraically closed or at least contains the primitive fourth root of unity i. Then there exists an automorphism of $K[x,t]$ over K permuting any x_{ij} with t (multiplied by a fourth root of unity). In fact, since the permutation of any two lines or columns of the 3×3 matrix (x_{ij}) followed by the change of signs of any line or column gives rise to an automorphism of $K[x,t]$ over K, we have only to show that x_{13}, for instance, and $-t$ are interchangeable. This we see by verifying that $t \to -x_{13}$ and

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \to \begin{pmatrix} ix_{11} & ix_{12} & -t \\ -x_{32} & x_{31} & ix_{23} \\ x_{22} & -x_{21} & ix_{33} \end{pmatrix}$$

give rise to an automorphism of $K[x,t]$ over K. (The meaning of this automorphism will become clear in the next section.) This being remarked, we consider (x,t) as a homogeneous generic point over K of a variety embedded in a projective space. Then its affine open set $t \neq 0$ is the variety of the special orthogonal group, and it is non-singular. Therefore, by the above remark, the projective variety is non-singular. The rest of the proof is similar to our proof of the artihmetic normality of the Grassmann variety [4]. We observe that the conductor of $K[x,t]$ is irrelevant in the sense it contains a power of the maximal ideal of $K[x,t]$ generated by x_{ij} and t. Consequently, the graded ring $K[x,t]$ and its normalization coincide except for homogeneous parts of lower degrees. Suppose that they coincide at degree n but not at degree $n-1$. Let ξ be an element of degree $n-1$ in the normalization which is not contained in $K[x,t]$. Then $t\xi$ is in $K[x,t]_n$, hence we have

$$t\xi = P_0(x_{ij}) + P_1(x_{ij})t + \cdots + P_n(x_{ij})t^n$$

in which $P_e(x_{ij})t^e$ is a linear combination of the previously explained monomials (of degree n) for $0 \leqq e \leqq n$. Since ξ is not contained in $K[x,t]$, we have $P_0(x_{ij}) \neq 0$. Apply the specialization $(x_{ij}, t) \to (x_i y_j, 0)$ over K as before. Then, since ξ is integral over $K[x,t]$, we have $t\xi \to 0$, hence $P_0(x_i y_j) = 0$ and $P_0(x_{ij}) = 0$. But this is a contradiction. q. e. d.

LEMMA 2. *Let R be a* (noetherian) *normal domain in which 2 is a unit and let $a_1, a_2, \cdots, a_n$ be non-units of R such that $Ra_i + Ra_j$ has rank two for $i \neq j$. Suppose further that the quotient ring R/Ra_i has no nilpotent element. Let b_i be a square root of a_i for $i = 1, 2, \cdots, n$. Then $R[b_1, b_2, \cdots, b_n]$ is a normal domain.*

Proof. Put $R_1 = R[b_1]$. If we can show that R_1 and $a_2, \cdots, a_n$ satisfy similar conditions as R and $a_1, a_2, \cdots, a_n$, we can apply an induction on n. Let K be the field of fractions of R. Then $K(b_1)$ is clearly a quadratic extension of K and R_1 is the normalization of R in $K(b_1)$. Moreover 2 is a unit and $a_2, \cdots, a_n$ are non-units of R_1. The "going-down" theorem of Cohen-Seidenberg [cf. 8, pp. 31-32] asserts that $R_1a_i + R_1a_j$ has rank two for $i \neq j$. We shall show that R_1/R_1a_i has no nilpotent element for $i = 2, \cdots, n$. Otherwise there exists an element c of R_1 which itself is not but its square is contained in R_1a_2 say. Write c in the form $p + qb_1$ with p, q in R. Then we have $c^2 = (p^2 + q^2a_1) + 2pqb_1 = (p' + q'b_1)a_2$ with some p', q' in R. This implies $p^2 + q^2a_1 = p'a_2$, $2pq = q'a_2$, hence $p^2 + q^2a_1$ and $2pq$ are in Ra_2. Since Ra_2 is an intersection of prime ideals not containing a_1, this implies that p and q are in Ra_2, hence c is in R_1a_2. But this is a contradiction. The lemma is thus proved.

LEMMA 3. *Let K, x_{ij}, t be as in Lemma 1 and let $(x_{ij})^{\frac{1}{2}}$, $t^{\frac{1}{2}}$ be square roots of x_{ij}, t for $1 \leqq i, j \leqq 3$. Then $K[x^{\frac{1}{2}}, t^{\frac{1}{2}}]$ is a normal domain.*

Proof. We shall show that the ten principal ideals $K[x,t]x_{ij}$, $K[x,t]t$ are prime and distinct. First of all, a part of the proof of Lemma 1 shows that $K[x,t]t$ is the kernel of the homomorphism of $K[x,t]$ associated with the specialization $(x_{ij}, t) \rightarrow (x_iy_j, 0)$ over K. Therefore $K[x,t]t$ is a prime ideal. Since we know that any x_{ij} is interchangeable with t (multiplied by a fourth root of unity), we see that $K[x,t]x_{ij}$ is a prime ideal also. Since under the specialization $(x_{ij}, t) \rightarrow (x_iy_j, 0)$ over K, no x_{ij} is specialized to zero, the prime ideal $K[x,t]t$ is different from any other prime ideal $K[x,t]x_{ij}$. Therefore, no two of the ten prime ideals are same. The rest follows from Lemma 2 by taking $K[x,t]$ as R and the ten elements x_{ij}, t as $a_1, a_2, \cdots, a_n$.

Now the preparation is made to prove the following theorem:

THEOREM 1. *The graded rings $\boldsymbol{C}[(\theta_{\mathfrak{m}})^2]$, $\boldsymbol{C}[\theta_{\mathfrak{m}}]$ and $\boldsymbol{C}[\theta_{\mathfrak{m}}\theta_{\mathfrak{n}}]$ are all normal, hence $A(\Gamma(4,8)) = \boldsymbol{C}[\theta_{\mathfrak{m}}\theta_{\mathfrak{n}}]$.*

Proof. If we use Riemann's theta-formula, which we explained in TC, Section 4, we can verify that

$$t = -(\theta_{0110})^2$$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} (\theta_{1001})^2 & -i(\theta_{1000})^2 & (\theta_{0100})^2 \\ i(\theta_{0001})^2 & (\theta_{0000})^2 & i(\theta_{1100})^2 \\ (\theta_{0010})^2 & -i(\theta_{0011})^2 & (\theta_{1111})^2 \end{pmatrix}$$

define a zero of the ideal introduced in Lemma 1. Since the field $\boldsymbol{C}(\theta_{\mathfrak{m}})$ has dimension four over $\boldsymbol{C}$, we actually have a generic zero over $\boldsymbol{C}$ and the notation is justified. Therefore $\boldsymbol{C}[(\theta_{\mathfrak{m}})^2] = \boldsymbol{C}[x,t]$ is normal by Lemma 1. Hence $\boldsymbol{C}[\theta_{\mathfrak{m}}] = \boldsymbol{C}[x^{\frac{1}{2}}, t^{\frac{1}{2}}]$ is normal by Lemma 3. Since we have $\boldsymbol{C}[\theta_{\mathfrak{m}}\theta_{\mathfrak{n}}] = \boldsymbol{C}[\theta_{\mathfrak{m}}]^{(2)}$, therefore $\boldsymbol{C}[\theta_{\mathfrak{m}}\theta_{\mathfrak{n}}]$ is also normal. Then, TC, Theorem 5 implies that $\boldsymbol{C}[\theta_{\mathfrak{m}}\theta_{\mathfrak{n}}]$

coincides with the ring of modular forms $A(\Gamma(4,8))$ belonging to the congruence group $\Gamma(4,8)$. This completes the proof.

2. Going-down process. Since we know the ring $A(\Gamma(4,8))$, by taking invariant subrings successively, we can determine $A(\Gamma(4))$, $A(\Gamma(2,4))$, $A(\Gamma(2))$ and finally $A(\Gamma(1))$. This procedure is explained in TC, Section 6. The ring $A(\Gamma(4))$ is generated over $\boldsymbol{C}$ by the ten $(\theta_{\mathfrak{m}})^2$, fifteen products $\theta_{\mathfrak{m}_1}\theta_{\mathfrak{m}_2}\,\theta_{\mathfrak{m}_3}\theta_{\mathfrak{m}_4}$ of distinct $\theta_{\mathfrak{m}}$'s satisfying

$$\sum_{1\leqq\alpha\leqq 4} m_{\alpha i}' \equiv \sum_{1\leqq\alpha\leqq 4} m_{\alpha i}'' \equiv 0 \bmod 2,$$

fifteen "complementary" products $\theta_{\mathfrak{m}_1}\theta_{\mathfrak{m}_2}\,\theta_{\mathfrak{m}_3}\theta_{\mathfrak{m}_4}\,\theta_{\mathfrak{m}_5}\theta_{\mathfrak{m}_6}$ and by

$$\theta = \prod_{\mathfrak{m}\text{ even, mod } 2} \theta_{\mathfrak{m}}.$$

The ring $A(\Gamma(2,4))$ is generated over $\boldsymbol{C}$ by $(\theta_{\mathfrak{m}}\theta_{\mathfrak{n}})^2$ and θ, i.e. $A(\Gamma(2,4)) = \boldsymbol{C}[x,t]^{(2)}[\theta]$. In particular, *the modular variety* proj. $A(\Gamma(2,4))$ *is nonsingular.* The ring $A(\Gamma(2))$ is generated over $\boldsymbol{C}$ by the ten $(\theta_{\mathfrak{m}})^4$, the fifteen products $(\theta_{\mathfrak{m}_1}\theta_{\mathfrak{m}_2}\,\theta_{\mathfrak{m}_3}\theta_{\mathfrak{m}_4})^2$, the fifteen complementary products $(\theta_{\mathfrak{m}_1}\theta_{\mathfrak{m}_2}\,\theta_{\mathfrak{m}_3}\theta_{\mathfrak{m}_4}\,\theta_{\mathfrak{m}_5}\theta_{\mathfrak{m}_6})^2$ and by θ. If we cross out from these products (except θ) those which are not of the form $x_{i_1j_1}\cdots x_{i_pj_p}t^e$ where $i_1 \leqq \cdots \leqq i_p$, $j_1 \geqq \cdots \geqq j_p$ and in which x_{i1} and x_{3j} appear at most once, we get t^2, $(x_{12})^2$, $(x_{13})^2$, $(x_{22})^2$, $(x_{23})^2$, $x_{13}x_{22}x_{31}t$ and θ. Since we have

$$2x_{13}x_{22}x_{31}t = (t^2 - (x_{12})^2 - (x_{13})^2 - (x_{22})^2 - (x_{23})^2)t^2 + (x_{12}x_{23})^2 - (x_{13}x_{22})^2,$$

actually the element $x_{13}x_{22}x_{31}t$ is redundant. Thus $A(\Gamma(2))$ is generated over $\boldsymbol{C}$ by θ and by

$$y_0 = t^2, \quad y_1 = (x_{13})^2, \quad y_2 = (x_{22})^2$$
$$y_3 = (x_{12})^2 - t^2, \quad y_4 = (x_{23})^2 - t^2.$$

Furthermore these elements are related as follows

$$(y_0y_1 + y_0y_2 + y_1y_2 - y_3y_4)^2 - 4y_0y_1y_2(y_0 + y_1 + y_2 + y_3 + y_4) = 0$$
$$\theta^2 = \tfrac{1}{4}\cdot(y_0y_1 + y_0y_2 + y_1y_2 - y_3y_4) \cdot (2y_0y_1y_2 + y_0y_1y_3 + y_0y_1y_4 + y_0y_2y_3 + y_0y_2y_4 + 2y_0y_3y_4 + y_1y_2y_3 + y_1y_2y_4 + 2y_1y_3y_4 + 2y_2y_3y_4 + y_3^2y_4 + y_3y_4^2).$$

In particular *the modular variety* proj. $A(\Gamma(2))$ *is a quartic hypersurface* (defined by the above equation) *in the four dimensional projective space.*[2] The going-down process from $A(\Gamma(2))$ to $A(\Gamma(1))$ is more involved. We have to know how $\Gamma(1)/\Gamma(2)$ acts on $A(\Gamma(2))$ and, in particular, how the characters of its action on the homogeneous parts $A(\Gamma(2))_k$ decompose into simple characters for $k=0,1,2,\cdots$.

We first observe that $\Gamma(1)/\Gamma(2)=Sp(2,\boldsymbol{Z}/2\boldsymbol{Z})$ is isomorphic to the symmetric group π_6 of permutations on six letters. The isomorphism can be given as follows. Suppose that M is an element of $\Gamma(1)$ composed of a, b, c, d and $\mathfrak{m}$ an arbitrary characteristic. Then $M\cdot\mathfrak{m}$ was defined as

$$M\cdot\mathfrak{m}=\begin{pmatrix} d & -c \\ -b & a\end{pmatrix}\mathfrak{m}+\begin{pmatrix}(c^td)_0\\(a^tb)_0\end{pmatrix}.$$

If we consider characteristics mod 2, the group $\Gamma(1)/\Gamma(2)$ operates on the set of odd (as well as even) characteristics. There are six odd characteristics mod 2 and they are (0101), (0111), (1010), (1011), (1101), (1110). Since the only element of $Sp(2,\boldsymbol{Z}/2\boldsymbol{Z})$ keeping these characteristics fixed mod 2 is the identity, we have a monomorphism of $Sp(2,\boldsymbol{Z}/2\boldsymbol{Z})$ to π_6. Since the orders of the two groups are same, we have an isomorphism.

This being remarked, we shall determine the character of the representation of $Sp(2,\boldsymbol{Z}/2\boldsymbol{Z})$ on $A(\Gamma(2))_k$. We observe that we have

$$\dim_{\boldsymbol{C}} A(\Gamma(2))_k=\binom{k/2+4}{4}-\binom{k/2}{4}$$
$$=1/12\cdot(k^3+3k^2+14k+12)$$

for $k=0,2,4,\cdots$ and

$$\dim_{\boldsymbol{C}} A(\Gamma(2))_k=\dim_{\boldsymbol{C}} A(\Gamma(2))_{k-5}$$

for $k=1,3,5,\cdots$. In particular the dimensions are 1, 0, 5, 0, 15, 1, $\cdots$ for $k=0$, 1, 2, 3, 4, 5, $\cdots$. We shall determine how $Sp(2,\boldsymbol{Z}/2\boldsymbol{Z})$ operates on $A(\Gamma(2))_2$. This can be done using TC, Theorem 6. We do it as follows. We call the six odd characteristics mod 2 simply $1,2,\cdots,6$. We know in general that the symmetric group π_g on g letters is generated by (12) and $(12\cdots g)$. The element M of $\Gamma(1)$ which gives rise to (12) satisfies

$$M\equiv\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\bmod 2.$$

[2] If S is a graded integral domain (of finite type over $\boldsymbol{C}$), we have proj. $S=$proj. $S^{(d)}$ for every positive integer d. We are using this fact for $d=2$ in the above two cases.

Also the element M of $\Gamma(1)$ which gives rise to (123456) satisfies

$$M \equiv \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix} \bmod 2.$$

We shall denote the elements of $\Gamma(1)$ standing on the right sides by M_{12} and $M_{1\cdots 6}$ respectively. We have seen in TC, Section 2 that the transformation law of theta-functions implies

$$\theta_{M\cdot\mathfrak{m}}(M\cdot\tau) = \text{æ}(M)e(\phi_{\mathfrak{m}}(M))\det(c\tau+d)^{\frac{1}{2}}\theta_{\mathfrak{m}}(\tau)$$

for all M in $\Gamma(1)$ in which

$$\phi_{\mathfrak{m}}(M) = -\tfrac{1}{8}\cdot({}^tm'{}^tbdm' + {}^tm''{}^tacm'' - 2{}^tm'{}^tbcm'' - 2^t(a^tb)_0(dm'-cm'')).$$

We can determine $\text{æ}(M_{12})$ and $\text{æ}(M_{1\cdots 6})$ as in TC, Section 3. In particular we have

$$\text{æ}(M_{12})^2 = 1, \qquad \text{æ}(M_{1\cdots 6})^2 = e(\tfrac{1}{4}).$$

Now, in order to obtain a concrete matricial representation of $Sp(2,\mathbf{Z}/2\mathbf{Z})$ acting on the five dimensional complex vector space $A(\Gamma(2))_2$, we arrange $y_0, y_1, \cdots, y_4$ in a column vector $\mathfrak{y}$. Applying the formula we just copied from TC, we then get

$$M^{-1}\cdot\mathfrak{y} = \rho(M)\mathfrak{y}$$

with

$$\rho(M_{12}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(M_{1\cdots 6}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We also have

$$M_{12}^{-1}\cdot\theta = M_{1\cdots 6}^{-1}\cdot\theta = -\theta.$$

We shall combine these results with the known theory of representations of symmetric groups [3, 11].

We know in general that conjugacy classes of π_g correspond to partitions of g. In fact every element of π_g has a cycle expression and it determines a partition of g. Two elements of π_g are conjugate if and only if they correspond to the same partition of g. On the other hand, simple characters of π_g also correspond to partitions of g. In fact each partition of g determines a primitive idempotent of the group ring of π_g over $\boldsymbol{Q}$ and it gives a simple character of π_g and conversely. Now the number 6 has the following eleven partitions

$$\begin{aligned}6&=5+1=4+2=4+1+1=3+3=3+2+1\\&=3+1+1+1=2+2+2=2+2+1+1\\&=2+1+1+1+1=1+1+1+1+1+1.\end{aligned}$$

We arrange eleven conjugacy classes and simple characters of π_6 in this order (one from top to bottom another from left to right) and write them as follows:

1	−1	0	1	0	0	−1	0	0	1	−1	120
1	0	−1	0	0	1	0	0	−1	0	1	144
1	−1	1	0	−1	0	0	−1	1	−1	1	90
1	1	−1	0	−1	0	0	1	1	−1	−1	90
1	−1	0	1	2	−2	1	2	0	−1	1	40
1	0	0	−1	1	0	1	−1	0	0	−1	120
1	2	0	1	−1	−2	1	−1	0	2	1	40
1	−1	3	−2	−3	0	2	3	−3	1	−1	15
1	1	1	−2	1	0	−2	1	1	1	1	45
1	3	3	2	1	0	−2	−1	−3	−3	−1	15
1	5	9	10	5	16	10	5	9	5	1	1.

The last column which is separated from the main part indicates the numbers of elements in the conjugacy classes.

The table shows that the representation ρ is irreducible and its character is the character χ_{222} corresponding to the partition $6=2+2+2$. Using the representation ρ, we operate $Sp(2,\boldsymbol{Z}/2\boldsymbol{Z})$ on the graded ring $\boldsymbol{C}[Y]$ of polynomials in five letters $Y_0, Y_1, \cdots, Y_4$ each of degree two with coefficients in $\boldsymbol{C}$. If we denote by Ψ_k the character of the representation on $\boldsymbol{C}[Y]_k$, for every M in $Sp(2,\boldsymbol{Z}/2\boldsymbol{Z})$, we have the following identity

$$\sum_{0\leq k<\infty}\Psi_{2k}(M)t^k=1/\det(1_5-\rho(M)t)$$

of formal power-series in t. We then observe that $A(\Gamma(2))^{(2)}$ is the quotient ring of $\boldsymbol{C}[Y]$ by a principal ideal generated by

$$(Y_0Y_1+Y_0Y_2+Y_1Y_2-Y_3Y_4)^2-4Y_0Y_1Y_2(Y_0+Y_1+\cdots+Y_4).$$

This polynomial (in fact both

$$Y_0Y_1+Y_0Y_2+Y_1Y_2-Y_3Y_4 \text{ and } Y_0Y_1Y_2(Y_0+Y_1+\cdots+Y_4))$$

are invariant by the operations of $Sp(2,\mathbf{Z}/2\mathbf{Z})$. Therefore, if X_k is the character of the action of $Sp(2,\mathbf{Z}/2\mathbf{Z})$ on $A(\Gamma(2))_k$, we have

$$\sum_{0\leqq k<\infty} X_{2k}t^k = (1-t^4)\sum_{0\leqq k<\infty} \Psi_{2k}t^k.$$

Moreover, if k is odd, we have $A(\Gamma(2))_k = A(\Gamma(2))_{k-5}\cdot\theta$ and the element θ has the property $M\cdot\theta=\epsilon(M)\theta$, in which $\epsilon(M)=\pm 1$ according as M corresponds to even or odd permutation of π_6. We note that ϵ is the simple character corresponding to the partition $6=1+1+1+1+1+1$. In this way, we get the following definitive result:

THEOREM 2. *The characters* X_k *of the representations of* $Sp(2,\mathbf{Z}/2\mathbf{Z})$ *on* $A(\Gamma(2))_k$ *are given by*

$$\sum_{0\leqq k<\infty} X_k(M)t^k = (1+\epsilon(M)t^5)(1-t^8)/\det(1_5-\rho(M)t^2).$$

A complete table of $\det(1_5-\rho(M)t)$ can be obtained very easily and it is as follows:

$(1-t)^2(1+t)(1+t+t^2)$	(123456)
$1-t^5$	(12345)
$(1-t)(1+t)^2(1+t^2)$	(1234)(56)
$(1-t)^2(1+t)(1+t^2)$	(1234)
$(1-t)^3(1+t+t^2)$	(123)(456)
$(1+t)(1-t+t^2)(1+t+t^2)$	(123)(45)
$(1-t)(1+t+t^2)^2$	(123)
$(1-t)^4(1+t)$	(12)(34)(56)
$(1-t)^3(1+t)^2$	(12)(34)
$(1-t)^2(1+t)^3$	(12)
$(1-t)^5$	(1).

The symbols standing on the right are representatives of conjugacy classes and the ordering is the same as before. We note that this table can be obtained knowing only that we have $\mathrm{trace}(\rho)=\chi_{222}$ and knowing the values of χ_{222}.

3. Going-down process (continued). We recall that, if G is an arbitrary finite group and X a character of G, the multiplicity m_χ of a simple character χ in X can be determined as

$$m_\chi = 1/\mathrm{ord}(G)\cdot\sum_{s\in G} X(s)\chi(s^{-1}).$$

Also, if G is a symmetric group, we have $\chi(s^{-1})=\chi(s)$ for every s. Using the results in the previous section, therefore, we can decompose the character X_k into simple characters. In particular $\dim_{\boldsymbol{C}} A(\Gamma(1))_k$ can be calculated as the multiplicity of the principal character in X_k and we get

$$\begin{aligned}\sum_{0\leqq k<\infty} \dim_{\boldsymbol{C}} A(\Gamma(1))_k t^k &\\ &=(1+t^{35})/(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})\\ &=1+t^4+t^6+t^8+2t^{10}+3t^{12}+\cdots.\end{aligned}$$

The calculation (which we are not writing down here) shows also that, if we denote by $\Gamma(1)_e$ the subgroup of $\Gamma(1)$ of index two defined by $\epsilon(M)=1$, we get

$$\begin{aligned}\sum_{0\leqq k<\infty} \dim_{\boldsymbol{C}} A(\Gamma(1)_e)_k t^k &\\ &=(1+t^{30})/(1-t^4)(1-t^5)(1-t^6)(1-t^{12}).\end{aligned}$$

At any rate, since we know

$$\begin{aligned}\sum_{0\leqq k<\infty} \dim_{\boldsymbol{C}} A(\Gamma_1(1))_k t^k &=1/(1-t^4)(1-t^6)\\ &=1+t^4+t^6+t^8+t^{10}+2t^{12}+\cdots,\end{aligned}$$

the ring $A(\Gamma(1))$ is generated over $\boldsymbol{C}$ by two modular forms of weights four, six and three cusp forms of weights ten, twelve, thirty-five and all are unique up to constant factors. We also remarked previously [5, Appendix] that the definition and elementary properties of Eisenstein series plus this information imply that, if we denote by ψ_k the Eisenstein series of weights k for $k=4$, $6,\cdots$, the modular forms of weights four, six are constant multiples of ψ_4, ψ_6 and the cusp forms of weights ten and twelve are constant multiples of $\psi_4\psi_6-\psi_{10}$ and $3^2 7^2(\psi_4)^3+2\cdot 5^3(\psi_6)^2-691\psi_{12}$. We shall construct these modular forms explicitly using theta-constants.

We shall denote $(\theta_m)^4$ corresponding to ten even characteristics mod 2 in the order of Section 1 by $(1),(2),\cdots,(10)$. Then a general observation in TC, Appendix shows that

$$\tfrac{1}{4}\cdot\sum_{1\leqq i\leqq 10}(i)^2=\tfrac{1}{6}\cdot\sum_{i<j}e(i,j)(i)(j)$$

is an element of $A(\Gamma(1))_4$ which is mapped to 1 by the square of the Siegel operator. Hence this is ψ_4. In order to obtain the element of degree six, we apply a similar method, i.e. the symmetrization of products of theta-constants. We observe that the symmetrization of $(1)^3$ and $(1)^2(2)$ are both zero. Therefore we shall consider the symmetrization of $(1)(2)(3)$. We observe that $\Gamma(1)$ does not operate transitively on the set of all triples of even characteristics (mod 2). In fact

$$e(i,j,k)=e(i,j)e(j,k)e(k,i)$$

is invariant under the operations of $\Gamma(1)$, and it is 1 for $(1,2,3)$ and is -1 for $(1,2,5)$. They are known as "syzygous" and "azygous" triples. At any rate, we can verify that $\Gamma(1)_e$, hence a fortiori $\Gamma(1)$, operates transitively on the sets of syzygous and azygous triples of even characteristics. We therefore apply symmetrizations with respect to $\Gamma(1)_e$ to $((1)(2)(3))^m$ and to $((1)(2)(5))^m$, in general, and put

$$(\text{syzy})_m=\tfrac{1}{4}\cdot\sum_{\text{syzygous}}(\pm(i)(j)(k))^m$$

$$(\text{azy})_m=\tfrac{1}{8}\cdot\sum_{\text{azygous}}(\pm(i)(j)(k))^m$$

for $m=1,2,3,\cdots$. The summation in $(\text{syzy})_1$, for instance, is extended over sixty syzygous triples of which thirty have plus-sign and the remaining thirty have minus-sign. The summation in $(\text{azy})_1$ is similar. We observe that $(\text{syzy})_m$ for every m and $(\text{azy})_m$ for even m are elements of $A(\Gamma(1))_{6m}$ while $(\text{azy})_m$ for odd m belongs to $A(\Gamma(1)_e)_{6m}$ and not to $A(\Gamma(1))$ unless it is zero. Furthermore the square of the Siegel operator maps $(\text{syzy})_m$ to 1, hence $(\text{syzy})_1$ is just a different expression for ψ_6. As for the cusp form of weight ten, we have one such form, i.e. θ^2. Moreover the Fourier expansion of θ itself starts as follows

$$\theta\begin{pmatrix} w_1 & \epsilon \\ \epsilon & w_2 \end{pmatrix}=-2^7 i\cdot e(w_1/2)e(w_2/2)(\pi\epsilon)+\cdots.$$

There is no ready-made cusp form of weight twelve. We therefore symmetrize $(1)^6$ and try to determine a linear combination of the symmetrized element and $(\psi_4)^3$, $(\psi_6)^2$ to get a cusp form. In the calculation, it is understood that we use the expressions of ψ_4, ψ_6 by theta-constants. This simple consideration works and we get a cusp form together with the following Fourier expansion

$$\Big(3^2\sum_{1\leqq i\leqq 10}(i)^6-2^2 11(\psi_4)^3+2^3(\psi_6)^2\Big)\begin{pmatrix} w_1 & \epsilon \\ \epsilon & w_2 \end{pmatrix}$$
$$=2^{15}3^4 11 e(w_1)e(w_2)+\cdots.$$

Finally we shall obtain the cusp form of weight thirty-five. The formula for $\dim_{\mathbf{C}} A(\Gamma(1)_e)_k$ shows that it is a product of θ and an element of $A(\Gamma(1)_e)_{30}$ not in $A(\Gamma(1))$. We just have one such element, i.e. $(\text{azy})_5$ provided it is different from zero. We can verify easily that $(\mathrm{azy})_m$ are different from zero except for $m=1,3$. Furthermore $(\mathrm{azy})_5$ has the following Fourier expansion

$$(\mathrm{azy})_5\begin{pmatrix} w_1 & \epsilon \\ \epsilon & w_2 \end{pmatrix} = -2^{29}5^3 e(3w_1/2)e(3w_2/2)(e(w_1)-e(w_2))+\cdots.$$

We shall summarize our results in the following way:

THEOREM 3. *The graded ring $A(\Gamma(1))$ is generated by ψ_4, ψ_6 and three cusp forms χ_{10}, χ_{12}, χ_{35} of weights ten, twelve, thirty-five with Fourier expansions of the form*

$$\chi_{10}\begin{pmatrix} w_1 & \epsilon \\ \epsilon & w_2 \end{pmatrix} = e(w_1)e(w_2)(\pi\epsilon)^2+\cdots$$

$$\chi_{12}\begin{pmatrix} w_1 & \epsilon \\ \epsilon & w_2 \end{pmatrix} = e(w_1)e(w_2)+\cdots$$

$$\chi_{35}\begin{pmatrix} w_1 & \epsilon \\ \epsilon & w_2 \end{pmatrix} = e(2w_1)e(2w_2)(e(w_1)-e(w_2))(\pi\epsilon)+\cdots.$$

They are all expressed by theta-constants. Moreover the ideal of cusp forms is generated in $A(\Gamma(1))$ by χ_{10}, χ_{12}, χ_{35}.

Only the last statement needs a proof and it is as follows. Let J be the ideal of $A(\Gamma(1))$ generated by χ_{10}, χ_{12}, χ_{35}. Since a cusp form of odd weight is in $A(\Gamma(1))\chi_{35}$, it is in J. Therefore we have only to show that cusp forms of even weights can be written in the form $\psi\chi_{10}-\psi'\chi_{12}$ with ψ, ψ' in $A(\Gamma(1))^{(2)}$. Since $A(\Gamma(1))^{(2)}$ is just a ring of polynomials in the four independent variables ψ_4, ψ_6, χ_{10}, χ_{12} over $\boldsymbol{C}$, we have $\psi\chi_{10}-\psi'\chi_{12}=0$ if and only if $\psi=\psi''\chi_{12}$, $\psi'=\psi''\chi_{10}$ with ψ'' in $A(\Gamma(1))^{(2)}$. Therefore, for every even positive integer k, the dimension of $(A(\Gamma(1))/J)_k$ is given by

$$\begin{aligned}\mathrm{coeff}_{t^k}\,((1-t^{10}-t^{12}+t^{22})/(1-t^4)(1-t^6)(1-t^{10})(1-t^{12}))\\ =\mathrm{coeff}_{t^k}\,(1/(1-t^4)(1-t^6)),\end{aligned}$$

and this is the dimension of $(A(\Gamma_1(1))_k$. Hence J_k is precisely the kernel of the Siegel operator $A(\Gamma(1))_k\to A(\Gamma_1(1))_k$, and the theorem is proved.

We note that $(\chi_{35})^2$ is in $A(\Gamma(1))^{(2)}$, hence it is a polynomial in ψ_4, ψ_6, χ_{10}, χ_{12} with coefficients in $\boldsymbol{C}$. This modular form is connected with the "skew-invariant" of binary sextics [cf. 5]. Also we make the following statement:

COROLLARY. *The graded ring $A(\Gamma(1)_e)$ is generated by ψ_4, ψ_6 and the three cusp forms θ, χ_{12}, $(\mathrm{azy})_5$ of weights five, twelve, thirty.*

Finally, we shall consider the ring of modular forms $A(\Gamma(1,2))$. Since

$\Gamma(1,2)$ is the stabilizer of $(1)=(\theta_{0000})^4$, it is a subgroup of $\Gamma(1)$ of index ten. Moreover the image of $\Gamma(1,2)$ in π_6 can be determined. It is the subgroup of π_6 generated by permutations of 1, 4, 6 (among themselves), permutations of 2, 3, 5 and by (12)(34)(56). In particular, the numbers of elements in the image corresponding to various partitions of 6 in the previous section are in that order 12, 0, 18, 0, 4, 12, 4, 6, 9, 6, 1. Using this fact and the results of the previous section, we get

$$\sum_{0\leqq k<\infty} \dim_{C} A(\Gamma(1,2))_k t^k$$
$$= P(t)/(1-t^2)(1-t^4)(1-t^6)(1-t^{12})$$
$$= 1+2t^4+2t^6+4t^8+5t^{10}+9t^{12}+\cdots,$$

in which $P(t)=(1-t^2+t^4+t^8)+t^{10}(1+t^4-t^6+t^8)$. In particular the dimension of $A(\Gamma(1,2))_4$ is two. We know two linearly independent elements of $A(\Gamma(1,2))_4$ already, and they are $(1)^2=(\theta_{0000})^8$ and ψ_4. On the other hand, if we denote the Eisenstein series of weight four belonging to $\Gamma(1,2)$ simply by ψ_4', this is also an element of $A(\Gamma(1,2))_4$. Hence we get a relation of the form

$$\psi_4'=p(\theta_{0000})^8+q\psi_4$$

with some p, q in $\boldsymbol{C}$. If we symmetrize this relation with respect to $\Gamma(1)$, we get $4p+10q=4$. On the other hand, if we apply the square of the Siegel operator to the above relation, we get $p+q=1$. The two equations have a unique solution $(p,q)=(1,0)$. Going back to the definitions of ψ_4' and θ_{0000}, we therefore get the *Braun identity*

$$\sum_{c\,{}^t d \text{ even}} \det(c\tau+d)^{-4}=\Big(\sum_{p\in Z^2} e(\tfrac{1}{2}\cdot {}^t p\tau p)\Big)^8.$$

We can give a more systematic treatment using Fourier expansions of Eisenstein series and also we can discuss relations of Eisenstein series of higher weights.

4. Singularities of modular varieties. We recall that, if Γ is a subgroup of $Sp(2,\boldsymbol{R})$ containing $\Gamma(2,4)$ as a subgroup of finite index, the corresponding modular variety proj. $A(\Gamma)$ has proj. $A(\Gamma(2,4))$ as a (global) non-singular covering. Therefore, the singularities of proj. $A(\Gamma)$, in particular those of proj. $A(\Gamma(1))$ and of proj. $A(\Gamma(2))$, are not too bad. We shall show, however, that the singularities of proj. $A(\Gamma(n))$ are quite bad for $n\geqq 3$. Let ∞ be the 0-dimensional boundary component of $\mathfrak{S}$ defined by

$$\infty=\lim_{\eta\to\infty} i\eta 1_2.$$

The values of $\theta_{\mathfrak{m}}$ at ∞, which are the images of $\theta_{\mathfrak{m}}$ by the square of the Siegel operator, are simply 1 for $m' = (00)$ and zero otherwise. At any rate, the point ∞ has a unique image point in every modular variety.

THEOREM 4. *There is no regular local ring which is integral over the analytic local ring of the modular variety* proj. $A(\Gamma(4))$ *at the image point of* ∞.

Before we start proving this theorem, we recall the footnote 2 saying in particular that, in the consideration of modular varieties we can restrict modular forms to those of even weights. Therefore we shall talk about rings of modular forms without adding the superscript (2). We need some lemmas.

LEMMA 4. *If we put*

$$u_1 = (\theta_{0100}/\theta_{0000})^2, \qquad u_2 = (\theta_{1000}/\theta_{0000})^2$$
$$u_3 = (\theta_{1100}/\theta_{0000})^2,$$

these three functions form a set of local parameters of proj. $A(\Gamma(2,4))$ *at the image point of* ∞.

Proof. We shall use the well-known Jacobian criterion. If we put $y_{00} = t/x_{22}$ and $y_{ij} = x_{ij}/x_{22}$ for (ij) different from (12), (13), (22), (23), the nine functions u_1, u_2, u_3, y_{ij} satisfy the following six identities

$$\begin{aligned}
(y_{21})^2 + 1 - (u_3)^2 - (y_{00})^2 &= 0\\
y_{11}y_{21} - iu_2 + iu_1u_3 &= 0\\
(y_{11})^2 + (y_{21})^2 + (y_{31})^2 - (y_{00})^2 &= 0\\
-(u_2)^2 + 1 + (y_{32})^2 - (y_{00})^2 &= 0\\
-iu_1u_2 + iu_3 + y_{32}y_{33} &= 0\\
-iu_2y_{33} - u_1y_{32} + y_{00}y_{21} &= 0.
\end{aligned}$$

If we evaluate the Jacobian of this system with respect to the six y_{ij} at the image point of ∞, we get

$$\pm 2^3(y_{21})^3 y_{31}(y_{32})^2(\infty) = \pm 2^3 i,$$

and this is different from zero. Therefore u_1, u_2, u_3 form a set of local parameters of proj. $A(\Gamma(2,4))$ at the image point of ∞.

Actually the same proof shows that the three functions in the lemma form a set of local co-ordinates of proj. $A(\Gamma(2,4))$ at the image point of

$$\lim_{\eta\to\infty}\begin{pmatrix} w & 0 \\ 0 & i\eta \end{pmatrix}$$

for every w in the upper-half plane. In fact, the Jacobian in question will just become $\pm 2^3 i(\theta_{01}(w)/\theta_{00}(w))^6$.

LEMMA 5. *Let R be a* (noetherian) *normal domain and $a_1, a_2, \cdots, a_n$ be elements of R such that $Ra_i + Ra_j$ has rank two for $i \neq j$. Let b be a square root of the product $a_1 a_2 \cdots a_n$ and let S be a unique factorization domain which is integral over $R[b]$ and in which every unit has a square root. Then S contains the square roots of $a_1, a_2, \cdots, a_n$.*

Proof. Suppose that a square root b_1 of a_1, say, is not contained in S. Then in the decomposition of a_1 into irreducible elements in S, at least one of them, say P, appears odd times. Since the product $a_1 a_2 \cdots a_n$ is a square in S, another element, say a_2, also contains P odd times. Now the intersection $\mathfrak{p}$ of PS with R is a prime ideal of R, and it contains both a_1 and a_2. Therefore $\mathfrak{p}$ contains a minimal prime ideal $\mathfrak{q}$ of a_1, and $\mathfrak{q}$ is strictly smaller than $\mathfrak{p}$ by assumption. Hence the going-down theorem of Cohen-Seidenberg asserts that PS contains a prime ideal having $\mathfrak{q}$ as its intersection with R. But PS has rank one, and this is a contradiction.

Now, we shall start proving the theorem. We note that proj. $A(\Gamma(4))$ is an abelian covering of proj. $A(\Gamma(2,4))$ with the Galois group isomorphic to $\Gamma(2,4)/\Gamma(4)(\pm 1_4)$. Since this is a vector space over $\boldsymbol{Z}/2\boldsymbol{Z}$ of dimension five, we need five independent radicals, and the following elements of $A(\Gamma(4))$

$$\theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}, \qquad \theta_{0000}\theta_{0010}\theta_{0100}\theta_{0110}$$
$$\theta_{0000}\theta_{0001}\theta_{1000}\theta_{1001}, \qquad \theta_{0000}\theta_{0011}\theta_{1100}\theta_{1111}$$
$$\theta_{0000}\theta_{0100}\theta_{1000}\theta_{1100}$$

divided by $(\theta_{0000})^4$ are such radicals (as we see by TC, Theorem 3). We call them in this order $x_1, x_2, \cdots, x_5$. Then, up to units in the local ring of proj. $A(\Gamma(2,4))$ at the image point of ∞, we have

$$(x_1)^2 \sim 1, \qquad (x_2)^2 \sim u_1(u_2u_3 - u_1)$$
$$(x_3)^2 \sim u_2(u_3u_1 - u_2), \qquad (x_4)^2 \sim u_3(u_1u_2 - u_3)$$
$$(x_5)^2 \sim u_1u_2u_3.$$

Suppose therefore that a regular local ring S is integral over the analytic local ring of proj. $A(\Gamma(4))$ at the image point of ∞. Then S is isomorphic to a ring of convergent power-series in three variables over $\boldsymbol{C}$. In particular, it is a unique factorization domain in which every unit has a square root. Moreover Lemma 4 shows that the analytic local ring of proj. $A(\Gamma(2,4))$ at the image point of ∞ is the ring of convergent power-series in u_1, u_2, u_3

with coefficients in $\boldsymbol{C}$. Therefore we can apply Lemma 5 taking this regular local ring as R. We see, in this way, that S contains the square roots of u_1, u_2, u_3, $u_2u_3-u_1$, $u_3u_1-u_2$, $u_1u_2-u_3$. Namely, if we denote the square roots of u_1, u_2, u_3 by v_1, v_2, v_3, the ring S contains not only v_1, v_2, v_3 but also the square roots of $(v_1)^2-(v_2v_3)^2$, $(v_2)^2-(v_3v_1)^2$, $(v_3)^2-(v_1v_2)^2$. We apply Lemma 5 again to the same S and to the ring of convergent power-series in v_1, v_2, v_3 with coefficients in $\boldsymbol{C}$, and we see that

$$(v_1 \pm v_2v_3)^{\frac{1}{2}}, \qquad (v_2 \pm v_3v_1)^{\frac{1}{2}}, \qquad (v_3 \pm v_1v_2)^{\frac{1}{2}}$$

are all contained in S. We shall show that, if we put

$$\begin{aligned} u_1' &= 2(v_1+v_2v_3)/(1+u_1+u_2+u_3) \\ u_2' &= 2(v_2+v_3v_1)/(1+u_1+u_2+u_3) \\ u_3' &= 2(v_3+v_1v_2)/(1+u_1+u_2+u_3) \end{aligned}$$

and define v_1', v_2', v_3', u_1'', u_2'', u_3'', v_1'', v_2'', v_3'', $\cdots$ similarly, these two sequences are both contained in S.

We recall that, in TC, Section 4, we proved the following identity

$$\theta_{\mathfrak{m}}(\tau/2)^2 = \sum_{n' \bmod 2} (-1)^{{}^t m'' n'} \theta\binom{m'+n'}{0}(\tau)\theta\binom{n'}{0}(\tau).$$

In fact this was obtained by putting $z=0$ in the "second principal transformation" of degree two. At any rate, written explicitly, we have the following ten identities:

$$\begin{aligned}
\theta_{0000}(\tau/2)^2 &= \theta_{0000}(\tau)^2+\theta_{0100}(\tau)^2+\theta_{1000}(\tau)^2+\theta_{1100}(\tau)^2 \\
\theta_{0001}(\tau/2)^2 &= \theta_{0000}(\tau)^2-\theta_{0100}(\tau)^2+\theta_{1000}(\tau)^2-\theta_{1100}(\tau)^2 \\
\theta_{0010}(\tau/2)^2 &= \theta_{0000}(\tau)^2+\theta_{0100}(\tau)^2-\theta_{1000}(\tau)^2-\theta_{1100}(\tau)^2 \\
\theta_{0011}(\tau/2)^2 &= \theta_{0000}(\tau)^2-\theta_{0100}(\tau)^2-\theta_{1000}(\tau)^2+\theta_{1100}(\tau)^2 \\
\theta_{0100}(\tau/2)^2 &= 2(\theta_{0000}\theta_{0100}+\theta_{1000}\theta_{1100})(\tau) \\
\theta_{0110}(\tau/2)^2 &= 2(\theta_{0000}\theta_{0100}-\theta_{1000}\theta_{1100})(\tau) \\
\theta_{1000}(\tau/2)^2 &= 2(\theta_{0000}\theta_{1000}+\theta_{0100}\theta_{1100})(\tau) \\
\theta_{1001}(\tau/2)^2 &= 2(\theta_{0000}\theta_{1000}-\theta_{0100}\theta_{1100})(\tau) \\
\theta_{1100}(\tau/2)^2 &= 2(\theta_{0000}\theta_{1100}+\theta_{0100}\theta_{1000})(\tau) \\
\theta_{1111}(\tau/2)^2 &= 2(\theta_{0000}\theta_{1100}-\theta_{0100}\theta_{1000})(\tau).
\end{aligned}$$

Therefore, if we recall that we have

$$v_1 = \theta_{0100}/\theta_{0000}, \qquad v_2 = \theta_{1000}/\theta_{0000}, \qquad v_3 = \theta_{1100}/\theta_{0000},$$

the ten functions $\tau \rightarrow \theta_{\mathfrak{m}}(\tau/2)/\theta_{0000}(\tau)$ are all contained in S. Since $\tau \rightarrow \theta_{0000}(\tau/2)/\theta_{0000}(\tau)$ is one of the four units among them, the modified functions $\tau \rightarrow (\theta_{\mathfrak{m}}/\theta_{0000})(\tau/2)$ are also contained in S. We now observe that our u_1', u_2', u_3' and u_1, u_2, u_3 are related simply as

$$u_i'(\tau) = u_i(\tau/2)$$

for $i = 1, 2, 3$. It is thus clear that the entire argument can be repeated starting from u_1', u_2', u_3' instead of u_1, u_2, u_3. In fact, we have only to replace τ by $\tau/2$. Therefore, by an induction on n we see that the two sequences $u_i^{(n)}$, $v_i^{(n)}$ defined in an obvious way for $n = 1, 2, \cdots$ are contained in S. However, since we have the following relation

$$u_i = (u_i^{(n)})^{2^n}/2^{2(2^n-1)} + \cdots$$

for $n = 1, 2, \cdots$, we see that u_1, u_2, u_3 are contained in the intersection of powers of the ideal of non-units of S. Hence we get $u_1 = u_2 = u_3 = 0$, but this is a contradiction. Theorem 4 is thus proved.

It is not without some interest to see how the iteration process looks like in the elliptic case. Clearly we have to take $u = (\theta_{10}/\theta_{00})^2$, $v = \theta_{10}/\theta_{00}$. As for u' defined by $u'(w) = u(w/2)$, the formula we copied from TC tells that we have

$$u' = 2v/(1+u).$$

This is the quotient of the geometric mean of 1, u by their arithmetic mean. In other words, the process is the one which is familiar in the Gauss theory of "arithmetic-geometric means."

Also it is of some interest to extract a purely algebraic statement from the above proof:

COROLLARY 1. *Let K be an algebraically closed field of characteristic different from 2 and let R be a local ring which is integral over the ring of formal power-series in three variables u_1, u_2, u_3 over K and which contains a square root of the product $u_1u_2u_3(u_1 - u_2u_3)(u_2 - u_3u_1)(u_3 - u_1u_2)$. Then there is no regular local ring S which is integral over R.*

In fact S is isomorphic to a ring of formal power-series in three variables over K and, by Lemma 5, it will contain the square roots v_1, v_2, v_3 of u_1, u_2, u_3 and $(v_1 \pm v_2v_3)^{\frac{1}{2}}$, $(v_2 \pm v_3v_1)^{\frac{1}{2}}$, $(v_3 \pm v_1v_2)^{\frac{1}{2}}$. We define u_1', u_2', u_3' as before. If we can show that S contains the square root of the product

$$u_1'u_2'u_3'(u_1' - u_2'u_3')(u_2' - u_3'u_1')(u_3' - u_1'u_2'),$$

the same induction as before can be applied. Clearly S contains the square roots of u_1', u_2', u_3'. By the obvious symmetry, we have only to show that S contains the square roots of $u_1'-u_2'u_3'$. This is a consequence of the following relation

$$u_1'-u_2'u_3'=2(v_1-v_2v_3)(1+u_1-u_2-u_3)\cdot(1+u_1+u_2+u_3)^{-2},$$

which can be verified easily. Incidentally, in the complex case (hence in general by the universality) this reflects the following relation of theta-constants

$$(\theta_{0100}\theta_{0110})(\tau/2)=2(\theta_{0010}\theta_{0110})(\tau).$$

We note also that, in case K has a valuation like $\boldsymbol{C}$, we can talk about rings of convergent power-series instead of the rings of formal power-series, and get a similar result. This result can be used as a lemma (rather than a corollary) to prove Theorem 4 and, in this way, we can simplify the previous proof.

We shall now derive a more general statement from Theorem 4. The group Γ is assumed, as always, to be commensurable with $\Gamma(1)$:

COROLLARY 2. *If* Γ *has no element of finite order* (other than $\pm$ the identity), *there is no regular local ring which is integral over the analytic local ring of* proj. $A(\Gamma)$ *at the image point of* ∞.

We know that the intersection Γ' of Γ and $\Gamma(4)$ has finite indices in Γ and $\Gamma(4)$. Therefore proj. $A(\Gamma')$ is a (finite) covering of both proj. $A(\Gamma)$ and proj. $A(\Gamma(4))$. Suppose that there exists a regular local ring S which is integral over the analytic local ring R of proj. $A(\Gamma)$ at the image point of ∞. Let R' be the analytic local ring of proj. $A(\Gamma')$ at the image point of ∞. We shall denote by U, V, V' the (irreducible) local analytic varieties associated with S, R, R'. Then U and V, V' can be considered as "sheets" of $\boldsymbol{C}^3$ at the origin and of proj. $A(\Gamma)$, proj. $A(\Gamma')$ at the image points of ∞. Also we have covering morphisms $f\colon U\to V$ and $f'\colon V'\to V$. We shall show that there exists a covering morphism $h\colon U\to V'$ satisfying $f=f'h$. Let Y, Y' be the complements in V, V' of the quotient varieties $\Gamma\backslash\mathfrak{S}$, $\Gamma'\backslash\mathfrak{S}$, and let X be the inverse image in U. Then X, Y, Y' are analytic subsets of U, V, V' of co-dimension two and f' is "unramified" over $V-Y$. We take a point p of U not in X and pick a point q' of V' lying over $f(p)$. Then q' is not in Y' and since $U-X$ is simply connected, if we make an analytic continuation throughout $U-X$, the correspondence $p\to q'$ defines a unique (analytic) morphism of $U-X$ to $V'-Y'$. This morphism can be completed to a morphism h of U to V'. Clearly h satisfies $f=f'h$ and it is a covering morphism. On the other hand, since proj. $A(\Gamma')$ is a covering of proj. $A(\Gamma(4))$, we see that S is integral over the analytic local ring of proj. $A(\Gamma(4))$ at the image point of ∞. This contradicts Theorem 4.

We note that the above proof can be made more formal using the "purity of branch loci" due to Zariski and Nagata [cf. 8, pp. 158-68]. At any rate, since we know in general that $\Gamma_g(n)$ has no element of finite order for $n \geqq 3$, Corollary 2 can be applied to the analytic local rings of proj. $A(\Gamma(n))$ at the image points of ∞ for $n \geqq 3$. In this way, Theorem 4 becomes a special case of the more general Corollary 2.

5. Concluding remark. As before, let Γ be commensurable with $\Gamma(1)$. We shall denote by $v(\Gamma)$ the volume of the quotient variety $\Gamma\backslash\mathfrak{S}$ with respect to the invariant measure of the homogeneous space $\mathfrak{S}=Sp(2,\boldsymbol{R})/U(2)$. We know that $v(\Gamma)$ is finite [10]. Moreover, if Γ contains another Γ' as a subgroup of finite index, the quotient $v(\Gamma')/v(\Gamma)$ is equal to the index of $\Gamma'(\pm 1_4)$ in $\Gamma(\pm 1_4)$ and also to the degree of the covering proj. $A(\Gamma')$ $\rightarrow$ proj. $A(\Gamma)$. We normalize the constant factor in $v(\Gamma)$ so that we get

$$v(\Gamma(1)) = 1/2^6 3^2 5.$$

Actually, this "intrinsic volume" can be defined in general and in the elliptic case we have $v(\Gamma(1)) = 1/2^2 3$.

THEOREM 5. *According as Γ may or does not contain elements of finite orders, we have*

$$\dim_{\boldsymbol{C}} A(\Gamma)_k = v(\Gamma)\cdot(k-1)(k-2)(\tfrac{1}{3}\cdot k-\tfrac{1}{2}) + 0(k^2), 0(k).$$

In the second case, it may be necessary that k tends to infinity as multiples of a certain integer.

We shall not give a proof of this theorem but leave it as an exercise to the reader. The proof we have in mind depends on Leray's spectral sequence, coherency of direct images of coherent sheaves and on the comparison of characteristic functions of coherent sheaves on a normal projective variety which coincide on some Zariski open set and finally on the following formula:

$$\dim_{\boldsymbol{C}} A(\Gamma(4,8))_k = \begin{cases} 2^{11}3^{-1}k^3 - 2^{10}3k^2 + 2^8 3^{-1}67k - 2^7 31 \\ 1, 55, 695, 3969 \end{cases}$$

in which the first line is used for $k \geqq 4$ and the second line is used for

$k = 0, 1, 2, 3$. This result itself can be derived easily from Theorem 1 using some part of the proof of Lemma 1.

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