

## A COMBINATORIAL CHARACTERIZATION OF JACOBI FORMS

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ABSTRACT. We present a combinatorial characterization of Jacobi forms à la Henri Cohen.

**1. Introduction.** Cohen [3] establishes a lift from elliptic modular forms to Hilbert modular forms over real quadratic fields. A key tool in Cohen's proof is a combinatorial characterization of modular forms in two variables. In this paper, we extend Cohen's combinatorial characterization of modular forms to Jacobi forms on  $\mathbf{H}^2 \times \mathbf{C}^2$ . It would be interesting to see if our result could be used to construct a lift from Jacobi forms in the sense of Eichler and Zagier [4] to Jacobi forms over real quadratic fields.

**2. Notation and statement of results.** Let  $\mathbf{H} \subset \mathbf{C}$  be the usual complex upper half plane. We write variables  $\vec{\tau} = (\tau_1, \tau_2) \in \mathbf{H}^2$  and  $\vec{z} = (z_1, z_2) \in \mathbf{C}^2$ . We define the following slash operator on functions  $f : \mathbf{H}^2 \rightarrow \mathbf{C}$ . For fixed  $\mathfrak{k} = (k_1, k_2) \in \mathbf{N}_0^2$ , where  $\mathbf{N}_0$  denotes the set of nonnegative integers, set

$$(1) \quad (f \mid_{\mathfrak{k}} M)(\vec{\tau}) = \prod_{j=1}^2 (c_j \tau_j + d_j)^{-k_j} f\left(\frac{a_1 \tau_1 + b_1}{c_1 \tau_1 + d_1}, \frac{a_2 \tau_2 + b_2}{c_2 \tau_2 + d_2}\right)$$

for  $M = \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in \mathrm{SL}_2(\mathbf{R})^2$ . Furthermore, we define the following two slash operators on functions  $f : \mathbf{H}^2 \times \mathbf{C}^2 \rightarrow \mathbf{C}$ . For a

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fixed  $\mathfrak{k} = (k_1, k_2) \in \mathbf{N}_0^2$  and  $\mathfrak{m} = (m_1, m_2) \in \mathbf{N}_0^2$ , set

$$(2) \quad (f |_{\mathfrak{k}, \mathfrak{m}} M) (\vec{\tau}, \vec{z}) = \prod_{j=1}^2 (c_j \tau_j + d_j)^{-k_j} \exp \left\{ -2\pi i \sum_{j=1}^2 m_j \frac{c_j z_j^2}{c_j \tau_j + d_j} \right\} \\ \cdot f \left( \frac{a_1 \tau_1 + b_1}{c_1 \tau_1 + d_1}, \frac{a_2 \tau_2 + b_2}{c_2 \tau_2 + d_2}, \frac{z_1}{c_1 \tau_1 + d_1}, \frac{z_2}{c_2 \tau_2 + d_2} \right)$$

for  $M = \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in \mathrm{SL}_2(\mathbf{R})^2$ , and also

$$(3) \quad (f |_{\mathfrak{m}} [\lambda, \mu]) (\vec{\tau}, \vec{z}) = \exp \left\{ 2\pi i \sum_{j=1}^2 m_j (\lambda_j^2 \tau_j + 2\lambda_j z_j) \right\} \\ \cdot f (\tau_1, \tau_2, z_1 + \lambda_1 \tau_1 + \mu_1, z_2 + \lambda_2 \tau_2 + \mu_2)$$

for  $[\lambda, \mu] = (\lambda_1, \lambda_2, \mu_1, \mu_2) \in \mathbf{R}^4$ .

Of particular interest are the elements invariant under the actions in (1), (2) and (3). The most important case occurs when one restricts to  $M \in \Gamma$  and  $[\lambda, \mu] \in \mathcal{O}_K^2$ , where  $\Gamma$  is a subgroup of finite index of  $\mathrm{SL}_2(\mathcal{O}_K)$  and where  $\mathcal{O}_K$  is the ring of integers of a real quadratic field  $K$ . In that case, a Hilbert modular form  $f : \mathbf{H}^2 \rightarrow \mathbf{C}$  is invariant under the action (1) and a Hilbert Jacobi form  $f : \mathbf{H}^2 \times \mathbf{C}^2 \rightarrow \mathbf{C}$  is invariant under (2) and (3). For more details, see Freitag [5], Garrett [6] and Skogman [7].

We now introduce the function on  $\mathbf{H} \times \mathbf{C}$  which will play the main role in our characterization of Jacobi forms. Let  $f : \mathbf{H}^2 \times \mathbf{C}^2 \rightarrow \mathbf{C}$  be holomorphic and  $k, m \in \mathbf{N}_0$ . Set  $\vec{k} = (k, k)$ ,  $\vec{m} = (m, m)$ ,

$$\alpha = k - \frac{1}{2} \quad \text{and} \quad L_j = 8\pi i m \partial_{\tau_j} - \partial_{z_j}^2 \quad (\text{heat operator}),$$

where

$$\partial_{\tau_j} = \frac{\partial}{\partial \tau_j}, \quad \partial_{z_j} = \frac{\partial}{\partial z_j},$$

and where  $j = 1, 2$ . If  $x \in \mathbf{C}$ , then we write  $x! = \Gamma(x + 1)$ , where  $\Gamma(\cdot)$  denotes the Gamma function. For every  $v \in \mathbf{N}_0$ , define  $\Phi_v[f] : \mathbf{H} \times \mathbf{C} \rightarrow \mathbf{C}$  by

$$(4) \quad \Phi_v[f](\tau, z) = \sum_{v_1 + v_2 = v} (-1)^{v_1} \frac{(L_1^{v_1} L_2^{v_2} f)(\tau, \tau, z, z)}{(v_1)! (v_2)! (\alpha + v_1 - 1)! (\alpha + v_2 - 1)!}.$$

The following theorem gives a combinatorial characterization of Jacobi forms on  $\mathbf{H}^2 \times \mathbf{C}^2$  in terms of functions on  $\mathbf{H}$  and on  $\mathbf{H} \times \mathbf{C}$ . In particular, we extend Theorem 2.2 in Cohen [3] to Jacobi forms.

**Theorem 1.** *Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbf{R}),$$

and set

$$M = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

Then statements (G1) and (G2) are equivalent:

(G1):

$$\left\{ \begin{array}{l} \text{(i) For all } v \in \mathbf{N}_0, \\ \Phi_v[f] \left( \frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d} \right) = (c\tau+d)^{2k+2v} \exp \left\{ 4\pi i m \frac{cz^2}{c\tau+d} \right\} \Phi_v[f](\tau, z); \\ \text{(ii) For all } u_1, u_2, v_1, v_2 \in \mathbf{N}_0 \text{ and } r, s \in \{0, 1\}, \\ \left( \partial_{\tau_1}^{u_1} \partial_{\tau_2}^{u_2} \partial_{z_1}^r \partial_{z_2}^s L_1^{v_1} L_2^{v_2} [f |_{\vec{k}, \vec{m}} M - f] \right) (\tau, \tau, 0, 0) \\ = \left( \partial_{\tau_1}^{u_2} \partial_{\tau_2}^{u_1} \partial_{z_1}^s \partial_{z_2}^r L_1^{v_1} L_2^{v_2} [f |_{\vec{k}, \vec{m}} M - f] \right) (\tau, \tau, 0, 0); \\ \text{(iii) If } v_1, v_2 \in \mathbf{N}_0 \text{ such that} \\ \left( L_1^{v_1} L_2^{v_2} [f |_{\vec{k}, \vec{m}} M - f] \right) (\tau, \tau, z, z) = 0, \\ \text{then } \left( \partial_{z_1} \partial_{z_2} L_1^{v_1} L_2^{v_2} [f |_{\vec{k}, \vec{m}} M - f] \right) (\tau, \tau, z, z) = 0. \end{array} \right.$$

(G2):

$$\left( f |_{\vec{k}, \vec{m}} M \right) (\tau_1, \tau_2, z_1, z_2) = f(\tau_1, \tau_2, z_1, z_2).$$

Let  $(\lambda, \mu) \in \mathbf{Z}^2$  and  $[\lambda, \mu] = (\lambda, \lambda, \mu, \mu)$ . Then statements (H1) and (H2) are equivalent:

(H1):

$$\left\{ \begin{array}{l} \text{(i) For all } v \in \mathbf{N}_0, \\ \quad \Phi_v[f](\tau, z + \lambda\tau + \mu) = \exp\{-4\pi im(\lambda^2\tau + 2\lambda z)\} \Phi_v[f](\tau, z); \\ \text{(ii) For all } u_1, u_2, v_1, v_2 \in \mathbf{N}_0 \text{ and } r, s \in \{0, 1\}, \\ \quad (\partial_{\tau_1}^{u_1} \partial_{\tau_2}^{u_2} \partial_{z_1}^r \partial_{z_2}^s L_1^{v_1} L_2^{v_2} [f |_{\bar{m}} [\lambda, \mu] - f]) (\tau, \tau, 0, 0) \\ \quad = (\partial_{\tau_1}^{u_2} \partial_{\tau_2}^{u_1} \partial_{z_1}^r \partial_{z_2}^s L_1^{v_1} L_2^{v_2} [f |_{\bar{m}} [\lambda, \mu] - f]) (\tau, \tau, 0, 0); \\ \text{(iii) If } v_1, v_2 \in \mathbf{N}_0 \text{ such that} \\ \quad (L_1^{v_1} L_2^{v_2} [f |_{\bar{m}} [\lambda, \mu] - f]) (\tau, \tau, z, z) = 0, \\ \quad \text{then } (\partial_{z_1} \partial_{z_2} L_1^{v_1} L_2^{v_2} [f |_{\bar{m}} [\lambda, \mu] - f]) (\tau, \tau, z, z) = 0. \end{array} \right.$$

(H2):

$$(f |_{\bar{m}} [\lambda, \mu]) (\tau_1, \tau_2, z_1, z_2) = f(\tau_1, \tau_2, z_1, z_2).$$

In particular,  $f$  is a Jacobi form of weight  $\vec{k}$  and index  $\bar{m}$  with respect to the subgroup  $\text{SL}_2(\mathbf{Z}) \times \mathbf{Z}^2$  (acting diagonally) of the Hilbert Jacobi group of a real quadratic field if and only if  $f$  satisfies (G1) and (H1) for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbf{Z})$  and  $(\lambda, \mu) \in \mathbf{Z}^2$ .

*Remark 1.* Note that all of the conditions in (G1) and (H1) are necessary. For example,

$$\begin{aligned} f_1(\tau_1, \tau_2, z_1, z_2) &= (z_1 - z_2)^2, \\ f_2(\tau_1, \tau_2, z_1, z_2) &= z_1^2 + z_2^2, \end{aligned}$$

and

$$f_3(\tau_1, \tau_2, z_1, z_2) = \frac{1}{4\pi im} (\tau_1 - \tau_2) + z_1^2 - z_2^2$$

do not satisfy (G2) and (H2). On the other hand:

(1)  $f_1$  satisfies (G1) (i), (G1) (ii), (H1) (i) and (H1) (ii), but  $f_1$  does not satisfy (G1) (iii) and (H1) (iii) for  $v_1 = v_2 = 0$ .

(2)  $f_2$  satisfies (G1) (ii), (G1) (iii), (H1) (ii) and (H1) (iii), but  $f_2$  does not satisfy (G1) (i) and (H1) (i) for  $v = 0$ .

(3)  $f_3$  satisfies (G1) (i), (G1) (iii), (H1) (i) and (H1) (iii), but  $f_3$  does not satisfy (G1) (ii) and (H1) (ii) for  $u_2 = r = s = v_1 = v_2 = 0$  and  $u_1 = 1$ .

As a conclusion, none of the conditions in (G1) and (H1) can be omitted. However, if one additionally assumes that  $f(\tau_1, \tau_2, z_1, z_2) = f(\tau_2, \tau_1, z_2, z_1)$ , then conditions (G1) (ii) and (H1) (ii) are automatically satisfied. Hence conditions (G1) (i), (G1) (iii), (H1) (i) and (H1) (iii) alone already characterize such symmetric Jacobi forms on  $\mathbf{H}^2 \times \mathbf{C}^2$ .

**3. Proof of Theorem 1.** First we need the following proposition:

**Proposition 1.** *Let  $M = \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in \text{SL}_2(\mathbf{R})^2$ , and suppose that  $f : \mathbf{H}^2 \times \mathbf{C}^2 \rightarrow \mathbf{C}$  satisfies*

$$(5) \quad (f |_{\mathfrak{k}, \mathfrak{m}} M) (\vec{\tau}, \vec{z}) = f(\vec{\tau}, \vec{z})$$

for some  $\mathfrak{k} = (k_1, k_2)$ ,  $\mathfrak{m} = (m_1, m_2) \in \mathbf{N}_0^2$ . Define

$$(6) \quad \tilde{f}(\vec{\tau}, \vec{z}, \vec{x}) = \sum_{v_1+v_2 \geq 0} \frac{L_{m_1}^{v_1} L_{m_2}^{v_2} (f) (\vec{\tau}, \vec{z})}{(v_1)!(v_2)!(\alpha_1 + v_1 - 1)!(\alpha_2 + v_2 - 1)!} x_1^{v_1} x_2^{v_2},$$

where  $\vec{x} = (x_1, x_2) \in \mathbf{C}^2$ ,

$$a_j = k_j - \frac{1}{2}$$

and  $L_{m_j} = 8\pi i m_j \partial_{\tau_j} - \partial_{z_j}^2$  for  $j = 1, 2$ . Then

$$(7) \quad \tilde{f} \left( \frac{a_1 \tau_1 + b_1}{c_1 \tau_1 + d_1}, \frac{a_2 \tau_2 + b_2}{c_2 \tau_2 + d_2}, \frac{z_1}{c_1 \tau_1 + d_1}, \frac{z_2}{c_2 \tau_2 + d_2}, \frac{x_1}{(c_1 \tau_1 + d_1)^2}, \frac{x_2}{(c_2 \tau_2 + d_2)^2} \right) \\ = \prod_{j=1}^2 (c_j \tau_j + d_j)^{k_j} \exp \left\{ 2\pi i \sum_{j=1}^2 m_j \left( \frac{c_j (z_j^2 + 4x_j)}{c_j \tau_j + d_j} \right) \right\} \tilde{f}(\vec{\tau}, \vec{z}, \vec{x}).$$

*Proof.* Let  $\mathbf{v} = (v_1, v_2) \in \mathbf{N}_0^2$ . An induction on  $v_1 + v_2$  shows that

$$\begin{aligned}
 (8) \quad & (L_{m_1}^{v_1} L_{m_2}^{v_2}(f) |_{\mathfrak{t}+2\mathbf{v}, \mathfrak{m}} M)(\vec{\tau}, \vec{z}) \\
 &= \sum_{\substack{(l_1, l_2) \in \mathbf{N}_0^2 \\ 0 \leq l_j \leq v_j}} \left( \prod_{j=1}^2 \frac{(v_j)!}{(l_j)!} \binom{\alpha_j + v_j - 1}{v_j - l_j} \left( \frac{8\pi i m_j c_j}{c_j \tau_j + d_j} \right)^{v_j - l_j} \right) \\
 &\quad \cdot L_{m_1}^{l_1} L_{m_2}^{l_2}(f)(\vec{\tau}, \vec{z}),
 \end{aligned}$$

where the sum on the right side is over all vectors  $(l_1, l_2) \in \mathbf{N}_0^2$  such that  $0 \leq l_j \leq v_j$  for  $j = 1, 2$ . It is easy to check that (8) implies (7).  $\square$

*Remark 2.* (1) Proposition 1 can obviously be extended to functions  $f : \mathbf{H}^n \times \mathbf{C}^n \rightarrow \mathbf{C}$ , for any  $n \geq 2$ .

(2) Let  $f_1$  and  $f_2$  be Hilbert Jacobi forms over a real quadratic field  $K$ . Then  $f_1$  and  $f_2$  satisfy (5) for all  $M \in \text{SL}_2(\mathcal{O}_K)$  and computing the coefficient of  $x_1^{v_1} x_2^{v_2}$  in  $f_1(\vec{\tau}, \vec{z}, m_2 \vec{x}) f_2(\vec{\tau}, \vec{z}, -m_1 \vec{x})$  yields an explicit formula for the Rankin-Cohen bracket for Hilbert Jacobi forms. For details on Rankin-Cohen brackets for Jacobi forms on  $\mathbf{H} \times \mathbf{C}$ , see [1, 2].

Now we turn to the proof of Theorem 1. Assume that (G2) is true. Then (G1) (i) follows from Proposition 1 after setting  $\tau = \tau_1 = \tau_2$ ,  $z = z_1 = z_2$ ,  $k = k_1 = k_2$ ,  $m = m_1 = m_2$ ,  $x_1 = -mx$  and  $x_2 = mx$ . Clearly, (G1) (ii) and (G1) (iii) hold.

Conversely, suppose that (G1) is true. Define  $g : \mathbf{H}^2 \times \mathbf{C}^2 \rightarrow \mathbf{C}$  by

$$(9) \quad g(\tau_1, \tau_2, z_1, z_2) = \left( f |_{\vec{k}, \vec{m}} M \right) (\tau_1, \tau_2, z_1, z_2) - f(\tau_1, \tau_2, z_1, z_2).$$

Consider the Taylor expansion of  $g$  around  $(\tau_1, \tau_2, 0, 0)$  to see that

$$\begin{aligned}
 g \equiv 0 \iff & \text{For all } \tau \in \mathbf{H}, u_1, u_2, v_1, v_2 \in \mathbf{N}_0, \text{ and } r, s \in \{0, 1\}, \\
 & (\partial_{\tau_1}^{u_1} \partial_{\tau_2}^{u_2} \partial_{z_1}^r \partial_{z_2}^s L_1^{v_1} L_2^{v_2} g)(\tau, \tau, 0, 0) = 0.
 \end{aligned}$$

Now we come to the key step. We will verify that  $(L_1^{v_1} L_2^{v-v_1} g)(\tau, \tau, z, z) = 0$  for all  $\tau \in \mathbf{H}$ ,  $z \in \mathbf{C}$  and  $v_1 \leq v$  by inducting on  $v$ . Set

$v = 0$  in (G1) (i) to see that  $g(\tau, \tau, z, z) = 0$ . Suppose that  $(L_1^{v_1} L_2^{v_1 - v_1} g)(\tau, \tau, z, z) = 0$ . With the help of (G1) (i) and (G1) (iii), one may check that

$$(10) \quad (L_1^{v_1} L_2^{v - v_1} g)(\tau, \tau, z, z) = (-1)^{v_1} (L_2^v g)(\tau, \tau, z, z).$$

An induction on  $v_1 + v_2$  shows that

$$\begin{aligned} & L_1^{v_1} L_2^{v_2} g(\tau_1, \tau_2, z_1, z_2) + L_1^{v_1} L_2^{v_2} (f)(\vec{\tau}, \vec{z}) \\ &= \sum_{\substack{(l_1, l_2) \in \mathbf{N}_0^2 \\ 0 \leq l_j \leq v_j}} \left( \prod_{j=1}^2 \binom{v_j}{l_j} \frac{(\alpha + v_j - 1)!}{(\alpha + l_j - 1)!} \frac{(-8\pi i mc)^{v_j - l_j}}{(c\tau_j + d)^{v_j + l_j + k}} \exp \left\{ -2\pi i m \frac{cz_j^2}{c\tau_j + d} \right\} \right) \\ & \cdot L_1^{l_1} L_2^{l_2} (f) \left( \frac{a_1 \tau_1 + b_1}{c_1 \tau_1 + d_1}, \frac{a_2 \tau_2 + b_2}{c_2 \tau_2 + d_2}, \frac{z_1}{c_1 \tau_1 + d_1}, \frac{z_2}{c_2 \tau_2 + d_2} \right), \end{aligned}$$

which, in combination with (G1) (i), yields (with a computation like in [3]) that

$$\begin{aligned} 0 &= \sum_{v_1 + v_2 = v} (-1)^{v_1} \frac{(L_1^{v_1} L_2^{v_2} g)(\tau, \tau, z, z)}{(v_1)! (v_2)! (\alpha + v_1 - 1)! (\alpha + v_2 - 1)!} \\ & \stackrel{(10)}{=} \left( \sum_{v_1 + v_2 = v} \frac{1}{(v_1)! (v_2)! (\alpha + v_1 - 1)! (\alpha + v_2 - 1)!} \right) (L_2^v g)(\tau, \tau, z, z). \end{aligned}$$

Hence,  $0 = (L_2^v g)(\tau, \tau, z, z) \stackrel{(10)}{=} (L_1^{v_1} L_2^{v - v_1} g)(\tau, \tau, z, z)$ . Then (G1) (ii) and (G1) (iii) imply that  $(\partial_{\tau_1}^{u_1} \partial_{\tau_2}^{u_2} \partial_{z_1}^r \partial_{z_2}^s L_1^{v_1} L_2^{v_2} g)(\tau, \tau, 0, 0) = 0$  for all  $\tau \in \mathbf{H}$ ,  $u_1, u_2, v_1, v_2 \in \mathbf{N}_0$  and  $r, s \in \{0, 1\}$ . We conclude that (G2) holds.

The proof of the equivalence of (H1) and (H2) is similar and we omit the detailed proof.  $\square$

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