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Generators of Jacobi forms are Poincaré series

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Abstract The ring of Jacobi forms of even weights is generated by the weak Jacobi forms $\phi_{-2,1}$ and $\phi_{0,1}$. Bringmann and the first author expressed $\phi_{-2,1}$ as a specialization of a Maass–Jacobi–Poincaré series. In this paper, we extend the domain of absolute convergence of Maass–Jacobi–Poincaré series which allows us to show that $\phi_{0,1}$ is also a Poincaré series.

Keywords Weak Jacobi forms · Maass–Jacobi–Poincaré series · Theta decomposition

Mathematics Subject Classification Primary 11F50; Secondary 11F27 · 11F37

1 Introduction

Eichler and Zagier [7] proved that the ring of Jacobi forms of even weights is generated (over the ring of modular forms) by two weak Jacobi forms of index 1 and weights −2 and 0, which we denote by $\phi_{-2,1}$ and $\phi_{0,1}$, respectively. Bringmann and the first author [5] employed Maass–Jacobi–Poincaré series $P_{k,m,s}^{(n,r)}$ (defined in (5)) to give exact formulas for Fourier series coefficients of holomorphic parts of harmonic Maass–

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Jacobi forms of even weights $k < 0$. As an application, they realized $\phi_{-2,1}$ as the following Jacobi–Poincaré series: If $k = -2$, then

$$\phi_{k,1} = \frac{1}{2\Gamma\left(\frac{5}{2} - \frac{k}{2}\right)} P^{(0,1)}_{k,1,\frac{5}{4}-\frac{k}{2}}, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Note that $P^{(n,r)}_{k,m,s}$ in (5) converges absolutely and uniformly on $\mathrm{Re}(s) > \frac{5}{4}$, but it does not converge absolutely on $\mathrm{Re}(s) \leq \frac{5}{4}$. In particular, if $k = 0$, then $P^{(0,1)}_{k,1,s}$ does not converge absolutely at the harmonic point $s = \frac{5}{4} - \frac{k}{2}$, which is the reason that [5] could not establish (1) for $k = 0$. In this paper, we enlarge the domain of absolute convergence of $P^{(n,r)}_{k,m,s}$ as follows:

Theorem 1 *The theta decomposition of* $P^{(n,r)}_{k,m,s}$ *in* (13) *converges absolutely for* $\mathrm{Re}(s) > 1$.

Theorem 1 permits the extension of [5] to also include the case of $k = 0$. However, our focus in this paper is slightly different, and we do not provide exact formulas for Fourier series coefficients of holomorphic parts of harmonic Maass–Jacobi forms. Instead, we apply Theorem 1 and tools from the theory of harmonic Maass–Jacobi forms (see [4]) to prove that (1) holds also for $k = 0$, i.e., $\phi_{0,1}$ is also a Poincaré series.

2 Weak Jacobi forms

We begin by introducing some standard notation for this paper. Throughout, we write $\tau = x + iy \in \mathbb{H}$ (the usual complex upper half plane), $z \in \mathbb{C}$, $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi i z}$, $k \in \mathbb{Z}$, and $D := r^2 - 4mn$ ($r \in \mathbb{Z}, m, n \in \mathbb{N}$). Let $\Gamma^J := \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ be the Jacobi group. For fixed integers k and m, define the following slash operator on functions $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$:

$$\begin{aligned}&\left(\phi\big|_{k,m} A\right)(\tau, z)\\ &:= \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right)(c\tau + d)^{-k}\, e^{2\pi i m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z\right)}\end{aligned} \tag{2}$$

for all $A = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right] \in \Gamma^J$.

We follow [5] and call a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ a weak Jacobi form of weight k and index m if ϕ is invariant under (2) and if it has a Fourier series expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\ D\ll\infty}} c(n, r) q^n \zeta^r. \tag{3}$$

If in addition, the Fourier series in (3) is only over $D \leq 0$, then ϕ is a Jacobi form as in [7]. Furthermore,

$$P_\phi(\tau, z) := \sum_{\substack{n,r\in\mathbb{Z}\\ D>0}} c(n,r) q^n \zeta^r$$

is the principal part of ϕ.

For example, the weak Jacobi forms

$$\phi_{-2,1}(\tau, z) = \zeta - \ 2 + \zeta^{-1} + \cdots,$$
$$\phi_{0,1}(\tau, z) = \zeta + 10 + \zeta^{-1} + \cdots$$

have the same principal part, which is given by

$$\sum_{\substack{n,r\in\mathbb{Z}\\ D=1}} q^n \zeta^r = \sum_{r\equiv 1 \pmod 2} q^{\frac{r^2-1}{4}} \zeta^r = q^{-\frac{1}{4}} \theta_{1,1}(\tau, z), \tag{4}$$

where the theta function $\theta_{1,1}$ is defined in (10).

3 Maass–Jacobi–Poincaré series

We briefly recall the Maass–Jacobi–Poincaré series of [4]. Let $M_{\nu,\mu}$ be the usual M-Whittaker function. For $s \in \mathbb{C}$, $\kappa \in \frac{1}{2}\mathbb{Z}$, and $t \in \mathbb{R} \setminus \{0\}$, define

$$\mathcal{M}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}} M_{\mathrm{sgn}(t)\frac{\kappa}{2}, s-\frac{1}{2}}(|t|)$$

and

$$\phi^{(n,r)}_{k,m,s}(\tau, z) := \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi D y}{m}\right) e^{2\pi i r z - \frac{\pi r^2 y}{2m} + 2\pi i n x},$$

where $D = r^2 - 4mn \neq 0$. Set $\Gamma^J_\infty := \left\{\left[\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}, (0, \mu)\right] \mid b, \mu \in \mathbb{Z}\right\}$, and define the Poincaré series

$$P^{(n,r)}_{k,m,s}(\tau, z) := \sum_{A\in\Gamma^J_\infty\backslash\Gamma^J} \left(\phi^{(n,r)}_{k,m,s}\big|_{k,m} A\right)(\tau, z). \tag{5}$$

Recall that a set of representatives of $\Gamma^J_\infty\backslash\Gamma^J$ is given by $\left\{\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}, (a\lambda, b\lambda)\right]\right\}$, where $c, d \in \mathbb{Z}$ with $\gcd(c, d) = 1$, $\lambda \in \mathbb{Z}$, and where for each pair (c, d), the integers a, b are chosen such that $ad - bc = 1$. With this set of representatives of $\Gamma^J_\infty\backslash\Gamma^J$, one easily verifies the following explicit expression.

$$P_{k,m,s}^{(n,r)}(\tau,z) = \sum_{\substack{\gcd(c,d)=1\\ \lambda\in\mathbb{Z}}} (c\tau+d)^{-k} e^{2\pi i m\left(\lambda^2\frac{a\tau+b}{c\tau+d}+\frac{2\lambda z}{c\tau+d}-\frac{cz^2}{c\tau+d}\right)} \phi_{k,m,s}^{(n,r)}\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda(a\tau+b)}{c\tau+d}\right). \tag{6}$$

If $1-2s \notin \mathbb{N}$, then [8, 13.14.14] implies the asymptotic

$$\mathcal{M}_{s,k-\frac{1}{2}}(y) \sim y^{s-\frac{2k-1}{4}} \qquad \text{as} \quad y \to 0, \tag{7}$$

and one finds that $P_{k,m,s}^{(n,r)}$ converges absolutely and uniformly on $\mathrm{Re}(s) > \frac{5}{4}$ (see also [2, Lemma 2.27]). In particular, if $s = \frac{k}{2} - \frac{1}{4}, k > 3$ or if $s = \frac{5}{4} - \frac{k}{2}, k < 0$, then $P_{k,m,s}^{(n,r)}$ is a harmonic Maass–Jacobi form of weight k and index m. These are functions that are invariant under the action in (2), which are in the kernel of the cubic Jacobi Casimir operator, and which satisfy certain exponential growth conditions (for details, see [4] and also [3] for an improved definition).

The domain of absolute convergence of $P_{k,m,s}^{(n,r)}$ cannot be extended by its definition (5) (see also [2, Lemma 2.27]): Consider the point $(\tau, z) = (i, 0)$. The sum of absolute values of $P_{k,m,s}^{(n,r)}(i, 0)$ in (6) is given by

$$\sum_{\gcd(c,d)=1} (c^2+d^2)^{-\frac{k}{2}} \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi D}{m(c^2+d^2)}\right) \sum_{\lambda\in\mathbb{Z}} e^{-\frac{2\pi m}{c^2+d^2}\left(\lambda+\frac{r}{2m}\right)^2}. \tag{8}$$

The asymptotic (7) together with the integral test applied to the sum over λ in (8) shows that (8) can be estimated from below by

$$\sum_{\gcd(c,d)=1} (c^2+d^2)^{\frac{1}{4}-\mathrm{Re}(s)},$$

which diverges for $\mathrm{Re}(s) \leq \frac{5}{4}$.

Observe that the domain of absolute convergence of $P_{k,m,s}^{(n,r)}$ can be extended via its Fourier series expansion in [4], which requires the estimates of certain Kloosterman sums and Bessel functions. In the next section, we present a different and also quite direct argument to enlarge the domain of absolute convergence of $P_{k,m,s}^{(n,r)}$.

4 The theta decomposition and the proof of Theorem 1

Every classical Jacobi form ϕ has a theta decomposition of the form

$$\phi(\tau,z) = \sum_{\mu \pmod{2m}} h_\mu(\tau)\,\theta_{m,\mu}(\tau,z), \tag{9}$$

where $(h_\mu)_\mu$ are certain vector-valued modular forms (for details, see [7]) with the theta functions

$$\theta_{m,\mu}(\tau,z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2m}}} q^{\frac{r^2}{4m}} \zeta^r. \tag{10}$$

Recall that the vector $\Theta(\tau,z) := (\theta_{m,\mu}(\tau,z))_\mu$ is a vector-valued Jacobi form of weight $\frac{1}{2}$, index m, and of type $\check{\rho}_m$, where $\check{\rho}_m$ is the dual of the Weil representation ρ_m of the metaplectic cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ associated to the Jacobi index m. In particular, if $M \in \Gamma$, then there exists a unitary matrix $U(M)$ such that

$$\begin{aligned}\left(\Theta\big|_{\frac{1}{2},m}[M,(0,0)]\right)(\tau,z) &:= \left(\left(\theta_{m,\mu}\big|_{\frac{1}{2},m}[M,(0,0)]\right)(\tau,z)\right)_\mu \\ &= U(M)\Theta(\tau,z),\end{aligned} \tag{11}$$

where the slash operator $|_{\frac{1}{2},m}$ is the usual half-integral weight extension of (2).

Harmonic Maass–Jacobi forms that are holomorphic in the Jacobi variable $z \in \mathbb{C}$ also have a theta decomposition as in (9) (see [3,4]). We now follow [6] to determine the theta decomposition of $P^{(n,r)}_{k,m,s}$: Write $w^{\frac{1}{2}} := \sqrt{t}\cdot e^{\frac{i\varphi}{2}}$ if $w = te^{i\varphi}$ with $-\varphi < \pi \le \varphi$. Let $\Gamma_\infty := \left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \mid b \in \mathbb{Z}\right\}$ and let $\chi^i_j(M)$ denote the (i,j)-th entry of the matrix $U(M)$ in (11) to define the Poincaré series

$$\begin{aligned}&\mathbb{P}^{(n,r)}_{k,m,s,\mu}(\tau)\\ &:= \sum_{M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} \chi^r_\mu(M)\,(c\tau+d)^{-\left(k-\frac{1}{2}\right)} e^{2\pi i\left(-\frac{D}{4m}\,\mathrm{Re}\left(\frac{a\tau+b}{c\tau+d}\right)\right)} \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi D}{m}\,\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)\right).\end{aligned} \tag{12}$$

A direct computation (as in [6]) shows that

$$P^{(n,r)}_{k,m,s}(\tau,z) = \sum_{\mu \pmod{2m}} \mathbb{P}^{(n,r)}_{k,m,s,\mu}(\tau)\,\theta_{m,\mu}(\tau,z). \tag{13}$$

Note that the asymptotic (7) implies that $\mathbb{P}^{(n,r)}_{k,m,s,\mu}$ in (12) and hence also $P^{(n,r)}_{k,m,s}$ in (13) converge for $\mathrm{Re}(s) > 1$, which proves Theorem 1.

5 Weak Jacobi forms as Poincaré series

The Fourier series expansion of a harmonic Maass–Jacobi form features a holomorphic part and a nonholomorphic part, where the holomorphic part has an expansion of a weak Jacobi form in (3) (see [4]). The Fourier series expansion of the Poincaré series $\mathbb{P}^{(n,r)}_{k,m,s,\mu}$ can be computed in a standard way (see also [6]). We are here only interested in the case that $s = \frac{5}{4} - \frac{k}{2}$ ($k = -2$ or $k = 0$), $m = 1$, $n = 0$, and $r = 1$ (in which case $D = 1$). Examine $c = 0$ in (12): Note that $U(I_2) = I_2$, $U(-I_2) = -iI_2$ (where I_2 stands for the 2×2 identity matrix), and observe the identity

$$\mathcal{M}_{\frac{5}{4}-\frac{k}{2},k-\frac{1}{2}}(-y) = e^{\frac{y}{2}}\Gamma\left(\frac{5}{2}-k\right) + \left(k-\frac{3}{2}\right)e^{\frac{y}{2}}\Gamma\left(\frac{3}{2}-k, y\right),$$

where $\Gamma(s, x) := \int_x^\infty e^{-t}t^{s-1}\,dt$ is the incomplete Gamma function. We discover that the principal part of the holomorphic part of $P^{(0,1)}_{k,1,\frac{5}{4}-\frac{k}{2}}$ in (13) is given by

$$(1 + (-1)^k)\Gamma\left(\frac{5}{2}-k\right)q^{-\frac{1}{4}}\theta_{1,1}(\tau, z).$$

In particular, if $k = -2$ or $k = 0$, then (4) shows that $\phi_{k,1}$ and $\frac{1}{2\Gamma\left(\frac{5}{2}-\frac{k}{2}\right)}P^{(0,1)}_{k,1,\frac{5}{4}-\frac{k}{2}}$ have the same principal parts. Consider the weight k and index 1 harmonic Maass–Jacobi form

$$\phi := \phi_{k,1} - \frac{1}{2\Gamma\left(\frac{5}{4}-\frac{k}{2}\right)}P^{(0,1)}_{k,1,\frac{5}{4}-\frac{k}{2}}.$$

The differential operator

$$\xi_{k,m} := y^{k-3/2}\left(-2iy\partial_{\overline{\tau}} - 2iv\partial_{\overline{z}} + \frac{y}{4\pi m}\partial_{\overline{zz}}\right)$$

maps harmonic Maass–Jacobi forms of weight k and index m to (weak) skew-holomorphic Jacobi forms of weight $3-k$ and index m (see [4]). One finds that $\xi_{k,1}(\phi)$ is a skew-holomorphic Jacobi cusp form of weight $3-k$ and index 1 (see also [4]). The space of such skew-holomorphic Jacobi cusp forms is isomorphic to the space of elliptic modular cusp forms (on $\mathrm{SL}_2(\mathbb{Z})$) of weight $2(3-k)-2 < 12$ (see [1,9]). Hence $\xi_{k,1}(\phi) = 0$. Furthermore, $\xi_{k,m}$ (trivially) annihilates the holomorphic parts of harmonic Maass–Jacobi forms. Thus, ϕ is a harmonic Maass–Jacobi form with no nonholomorphic part, i.e., ϕ is a weak Jacobi form. By construction, ϕ has also no principal part and hence ϕ is a holomorphic Jacobi form (as in [7]) of weight $k \le 0$. We conclude that $\phi = 0$, and (1) holds for $k = -2$ and $k = 0$. In particular, $\phi_{0,1}$ is a Poincaré series.

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References

1. Boylan, H., Skoruppa, N.-P., Zhou, H.: Arithmetic theory of skew-holomorphic Jacobi forms. Preprint (2016)
2. Bringmann, K.: Applications of Poincaré series on Jacobi groups. PhD thesis, University of Cologne, Germany (2004)

3. Bringmann, K., Raum, M., Richter, O.: Harmonic Maass–Jacobi forms with singularities and a theta-like decomposition. Trans. Am. Math. Soc. **367**(9), 6647–6670 (2015)
4. Bringmann, K., Richter, O.: Zagier-type dualites and lifting maps for harmonic Maass–Jacobi forms. Adv. Math. **225**(4), 2298–2315 (2010)
5. Bringmann, K., Richter, O.: Exact formulas for Fourier coefficients of Jacobi forms. Int. J. Number Theory **7**(3), 825–833 (2011)
6. Bringmann, K., Yang, T.: On Jacobi Poincaré series of small weight. Int. Math. Res. Notices **6**, 2891–2912 (2007)
7. Eichler, M., Zagier, D.: The Theory of Jacobi Forms. Birkhäuser, Boston (1985)
8. NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.10 of 2015-08-07
9. Skoruppa, N.-P.: Developments in the Theory of Jacobi Forms. Acad. Sci. USSR, Inst. Appl. Math., Khabarovsk, pp. 167–185 (1990)