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HERMITIAN JACOBI FORMS AND $U(p)$ CONGRUENCES

OLAV K. RICHTER AND JAYANTHA SENADHEERA

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ABSTRACT. We introduce a new space of Hermitian Jacobi forms, and we determine its structure. Moreover, we characterize $U(p)$ congruences of Hermitian Jacobi forms, and we discuss an explicit example.

1. Introduction and statement of main results

Jacobi forms appear naturally in various areas of mathematics and physics, and they connect different types of automorphic forms. In particular, they occur as Fourier-Jacobi coefficients of Siegel modular forms of degree 2, a fact that figured prominently in the solution of the Saito-Kurokawa conjecture (see [2, 8, 15–17, 29]).

Hermitian modular forms are generalizations of Siegel modular forms, and Hermitian Jacobi forms occur as Fourier-Jacobi coefficients of Hermitian modular forms of degree 2. Haverkamp [10, 11] systematically studied Hermitian Jacobi forms, and [5–7, 23] contributed further to the theory of such forms.

The usual heat operator is an important device in the study of classical Jacobi forms (see, for example, [3, 4, 8]), and its action on Jacobi forms can be "corrected" so that Jacobi forms of weight k are mapped to Jacobi forms of weight $k+2$ (see [22]). The heat operator

$$L_m := \frac{1}{(2\pi i)^2}\Big(8\pi i m\frac{\partial}{\partial\tau} - 4\frac{\partial^2}{\partial w\partial z}\Big) \tag{1.1}$$

is a natural tool in the theory of Hermitian Jacobi forms, and it plays a vital role in Section 3 of this paper. Equation (3.7) of [14] implicitly gives an action of L_m on Hermitian Jacobi forms, but unfortunately, (3.7) of [14] is not quite correct. In fact, the action of L_m on the Hermitian Jacobi forms in [5, 6, 10, 23] cannot be "corrected" as in the case of classical Jacobi forms, and one needs a different notion of Hermitian Jacobi form.

In this paper, we introduce the more general space $J^{\delta}_{k,m}$ of Hermitian Jacobi forms over $\mathbb{Q}(i)$ of weight k, index m, and parity $\delta=\pm$, and we set $J_{k,m} := J^{+}_{k,m}\oplus J^{-}_{k,m}$ (see Definition 2.1). A direct computation shows that if $\phi\in J^{\delta}_{k,m}$, then

$$L_m(\phi) = \frac{(k-1)m}{3}E_2\phi + \widehat{\phi}, \tag{1.2}$$

where E_2 is the usual quasimodular Eisenstein series, and where $\widehat{\phi}\in J^{-\delta}_{k+2,m}$. The fact that ϕ and $\widehat{\phi}$ in (1.2) are Hermitian Jacobi forms of different parities explains the need for our Definition 2.1.

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We now introduce notation necessary to state our main results. Throughout, k is an even integer, and M_k and S_k are the weight k spaces of elliptic modular forms and cusp forms, respectively. We denote the Taylor coefficients of a Hermitian Jacobi form of parity δ by $\chi^{\delta}_{\mu,\nu}$ (see Section 2.3), and $\zeta^{\delta}_{\mu,\nu}$ are the combinations of Taylor coefficients in Proposition 2.7. Our first result gives the structure of Hermitian Jacobi forms over $\mathbb{Q}(i)$ of index 1. More specifically, it asserts that $J_{k,1}$ is a free module of rank 4 over the ring of modular forms, where a set of generators is given by the Hermitian Jacobi forms $\phi^{+}_{4,1}$, $\phi^{-}_{6,1}$, $\phi^{+}_{8,1}$, and $\phi^{+,cusp}_{10,1}$ (defined in (2.5) and Lemma 2.4).

Theorem 1.1. *If $k \equiv 0 \pmod 4$, then both linear maps*

$$\zeta : J_{k,1} \to M_k \oplus S_{k+2} \oplus S_{k+2} \oplus S_{k+4},$$
$$\phi \mapsto (\chi^{+}_{0,0},\ \zeta^{+}_{1,1},\ \chi^{-}_{2,0},\ \zeta^{+}_{2,2} - 12\chi^{+}_{4,0})$$

and

$$\eta : M_{k-4} \oplus M_{k-6} \oplus M_{k-8} \oplus M_{k-10} \to J_{k,1},$$
$$(e,f,g,h) \mapsto (e\phi^{+}_{4,1} + f\phi^{-}_{6,1} + g\phi^{+}_{8,1},\ h\phi^{+,cusp}_{10,1})$$

are isomorphisms.

If $k \equiv 2 \pmod 4$, then both linear maps

$$\zeta : J_{k,1} \to M_k \oplus S_{k+2} \oplus S_{k+2} \oplus S_{k+4},$$
$$\phi \mapsto (\chi^{-}_{0,0},\ \zeta^{-}_{1,1},\ \chi^{+}_{2,0},\ \zeta^{-}_{2,2} - 12\chi^{-}_{4,0})$$

and

$$\eta : M_{k-4} \oplus M_{k-6} \oplus M_{k-8} \oplus M_{k-10} \to J_{k,1},$$
$$(e,f,g,h) \mapsto (h\phi^{+,cusp}_{10,1},\ e\phi^{+}_{4,1} + f\phi^{-}_{6,1} + g\phi^{+}_{8,1})$$

are isomorphisms.

Theorem 1.1 allows us to investigate congruences and filtrations of Hermitian Jacobi forms of index 1. We introduce more necessary notation. Throughout, $p \geq 5$ is a prime. Let $\mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$ be the local ring of p-integral rational numbers, $J^{\delta}_{k,1}(\mathbb{Z}_{(p)})$ the space of forms in $J^{\delta}_{k,1}$ that have p-integral rational coefficients, Ω the filtration of a Hermitian Jacobi form in $J^{\delta}_{k,1}(\mathbb{Z}_{(p)})$ (see Definition 3.4), and $U(p)$ the following analog of Atkin's U-operator:

$$\Bigg(\sum_{\substack{n\in\mathbb{Z},\, r\in\mathcal{O}^{\#} \\ n-|r|^2\geq 0}} c(n,r) q^n \zeta^{\overline{r}} (\zeta')^r \Bigg) \Bigg| U(p) := \sum_{\substack{n\in\mathbb{Z},\, r\in\mathcal{O}^{\#} \\ n-|r|^2\geq 0 \\ p|4(n-|r|^2)}} c(n,r) q^n \zeta^{\overline{r}} (\zeta')^r,$$

where here (and in the following) $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi i z}$, $\zeta' := e^{2\pi i w}$, and $\mathcal{O}^{\#} := \frac{1}{2}\mathbb{Z}[i]$ is the inverse different of $\mathcal{O} := \mathbb{Z}[i]$. Finally, for convenience, we write $L := L_1$. Our next theorem provides a criterion for the existence of $U(p)$ congruences of Hermitian Jacobi forms of index 1.

Theorem 1.2. *Let $\phi \in J^{\delta}_{k,1}(\mathbb{Z}_{(p)})$ such that $\phi \not\equiv 0 \pmod p$. If $p > k$, then*

$$\Omega\Big(L^{p+2-k}(\phi)\Big) = \begin{cases} 2p+4-k, & \text{if } \ \phi\,\big|\,U(p) \not\equiv 0 \pmod p, \\ \ \ p+5-k, & \text{if } \ \phi\,\big|\,U(p) \equiv 0 \pmod p. \end{cases}$$

Recall that $U(p)$ congruences of elliptic modular forms have applications in the context of traces of singular moduli and class equations (see [1, 9, 18]). It would be interesting to see if $U(p)$ congruences for Hermitian Jacobi forms also find further applications.

The paper is organized as follows. In Section 2.1, we give a new definition of Hermitian Jacobi forms over $\mathbb{Q}(i)$. In Section 2.2, we discuss the theta decomposition of Hermitian Jacobi forms. In Section 2.3, we prove Theorem 1.1. In Section 3.1, we explore congruences of Hermitian Jacobi forms of index 1, and we provide the ingredients to prove Theorem 1.2. Finally, in Section 3.2, we present an explicit example to illustrate Theorem 1.2.

Many of the results here have also been reported in the second author's University of North Texas doctoral dissertation [24].

2. Hermitian Jacobi forms over $\mathbb{Q}(i)$

Hermitian Jacobi forms appear as Fourier-Jacobi coefficients of Hermitian modular forms of degree 2 over a complex quadratic field (see [7, 10]). In this paper, we restrict ourselves to the case where the complex quadratic field is the Gaussian number field $\mathbb{Q}(i)$. Throughout, k and m are nonnegative integers, and if $s \in \mathbb{C}$, then $\overline{s}$ denotes its complex conjugate.

2.1. A new definition.

Recall from the introduction that $\mathcal{O} := \mathbb{Z}[i]$ is the ring of Gaussian integers with inverse different $\mathcal{O}^{\#} := \frac{1}{2}\mathbb{Z}[i]$. Let $\mathcal{O}^{\times} := \{1, -1, i, -i\}$ be the group of units of $\mathcal{O}$ and $\Gamma(\mathcal{O}) := \big\{\epsilon M \mid \epsilon \in \mathcal{O}^{\times},\ M \in SL_2(\mathbb{Z})\big\}$ be the Hermitian modular group.

Our following extension of Haverkamp's [10] notion of a Hermitian Jacobi form depends on a parity $\delta = \pm$.

Definition 2.1. A holomorphic function $\phi : \mathbb{H} \times \mathbb{C}^2 \to \mathbb{C}$ is a Hermitian Jacobi form of weight k, index m, and parity δ if it satisfies the transformation laws

$$\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{\epsilon z}{c\tau+d}, \frac{\epsilon^{-1}w}{c\tau+d}\Big) = \sigma(\epsilon)\epsilon^k(c\tau+d)^k e^{\frac{2\pi i mczw}{c\tau+d}}\phi(\tau,z,w),$$

$$\text{for all} \quad \epsilon\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma(\mathcal{O}),$$

where $\sigma(\epsilon) := \begin{cases} 1 & \text{if} \quad \delta = + \\ \epsilon^2 & \text{if} \quad \delta = - \end{cases}$, and

$$\phi(\tau, z+\lambda\tau+\mu, w+\overline{\lambda}\tau+\overline{\mu}) = e^{2\pi i m(\lambda\overline{\lambda}\tau + z\overline{\lambda} + \lambda w)}\phi(\tau,z,w), \text{ for all } [\lambda,\mu] \in \mathcal{O}^2.$$

Furthermore, one requires that ϕ has a Fourier series expansion of the form

$$\phi(\tau,z,w) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathcal{O}^{\#} \\ nm-|r|^2\geq 0}} c(n,r)q^n\zeta^{\overline{r}}(\zeta')^r.$$

A Hermitian Jacobi form is called a cusp form if $c(n,r) = 0$ unless $mn - |r|^2 > 0$. We denote the space of Hermitian Jacobi forms of weight k, index m, and parity δ by $J^{\delta}_{k,m}$, and the space of cusp forms in $J^{\delta}_{k,m}$ by $J^{\delta,cusp}_{k,m}$. Finally, the space of Hermitian Jacobi forms of weight k and index m is defined by

$$J_{k,m} := J^+_{k,m} \oplus J^-_{k,m} = \Big\{(\phi^+, \phi^-) \mid \phi^+ \in J^+_{k,m},\ \phi^- \in J^-_{k,m}\Big\}.$$

Remark 2.2. The space of Hermitian Jacobi forms in [5, 6, 10, 23] coincides with the space of Hermitian Jacobi forms of positive parity, i.e., with $J^{+}_{k,m}$.

The next proposition follows exactly as Propositions 1.3 and 1.4 of [10] (see also Lemma 1 of [23]). We omit the proof, which is contained in [24].

Proposition 2.3. *Let $\phi(\tau, z, w) = \sum c(n,r) q^n \zeta^{\overline{r}} (\zeta')^r \in J^{\delta}_{k,m}$. Then we have the following:*

(i) The coefficient $c(n,r)$ depends only on $nm - |r|^2$ and on $r \pmod{m\mathcal{O}}$.
(ii) If $\epsilon \in \mathcal{O}^{\times}$, then $\sigma(\epsilon)\epsilon^k c(n,r) = c(n, \overline{\epsilon} r)$.
(iii) If $m = 1$, $k \equiv 0 \pmod 4$, and $\delta = +$, then $c(n,r)$ depends only on $n - |r|^2$.
(iv) If $m = 1$, $k \equiv 2 \pmod 4$, and $\delta = -$, then $c(n,r)$ depends only on $n - |r|^2$.
(v) If $m = 1$ and k is odd, then $\phi = 0$.

2.2. **The theta decomposition.** Haverkamp [10] establishes the so-called *theta decomposition* for Hermitian Jacobi forms of positive parity. Set

$$\theta^{H}_{m,s}(\tau, z, w) := \sum_{\substack{r \in \mathcal{O}^{\#} \\ r \equiv s \ (\mathrm{mod}\, m\mathcal{O})}} q^{|r|^2/m} \zeta^{\overline{r}} (\zeta')^r.$$

Exactly as in [10], one finds that $\phi = \sum_{n,r} c(n,r) q^n \zeta^{\overline{r}} (\zeta')^r \in J^{\delta}_{k,m}$ has the theta decomposition

$$\phi(\tau, z, w) = \sum_{s \in \mathcal{O}^{\#}/m\mathcal{O}} h_s(\tau) \theta^{H}_{m,s}(\tau, z, w), \tag{2.1}$$

where h_s are certain vector-valued modular forms of weight $k-1$ (for more properties of h_s, see [24]). Note that we suppress the dependence on δ and we write h_s instead of h^{δ}_s.

For the remainder, we are only interested in Hermitian Jacobi forms of index 1. Observe that if $m = 1$, then $\{0, \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2}\}$ is a set of representatives for the set of cosets $\mathcal{O}^{\#}/m\mathcal{O}$. We now recall the theta decompositions for important examples of Hermitian Jacobi forms of positive parity. Consider the Jacobi theta function

$$\theta_{a,b}(\tau, z) := \sum_{n \in \mathbb{Z}} e^{\pi i (a+n)^2 \tau + 2\pi i (n+a)(z+b)} \qquad (a, b \in \mathbb{R}), \tag{2.2}$$

and its following specializations (theta constants):

$$\begin{aligned} x &:= \theta_{0,0}(\tau, 0) = 1 + 2\sum_{n=1}^{\infty} q^{\frac{n^2}{2}}, \\ y &:= \theta_{0,\frac{1}{2}}(\tau, 0) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{\frac{n^2}{2}}, \\ z &:= \theta_{\frac{1}{2},0}(\tau, 0) = 2q^{\frac{1}{8}} \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}. \end{aligned} \tag{2.3}$$

It is well known (see, for example, [12]) that

$$\begin{aligned} x^4 &= y^4 + z^4, \\ E_4 &= \tfrac{1}{2}(x^8 + y^8 + z^8), \\ E_6 &= \tfrac{1}{2}(x^4 + y^4)(x^4 + z^4)(y^4 - z^4), \end{aligned} \tag{2.4}$$

where $E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\left(\sum_{d|n} d^{k-1}\right) q^n$ denotes the usual Eisenstein series.

Sasaki [23] provides the theta decomposition of several Hermitian Jacobi forms of index 1. In particular, he considers (up to normalization) the following Hermitian Jacobi forms $\phi^{+}_{k,1} \in J^{+}_{k,1}$ for $k = 4, 8, 12$ and $\phi^{+,cusp}_{10,1} \in J^{+,cusp}_{10,1}$:

$$\begin{aligned}
\phi^{+}_{4,1} &:= \tfrac{1}{2}(x^6+y^6)\theta^H_{1,0} + \tfrac{1}{2}z^6\big(\theta^H_{1,\frac{1}{2}} + \theta^H_{1,\frac{i}{2}}\big) + \tfrac{1}{2}(x^6-y^6)\theta^H_{1,\frac{1+i}{2}},\\
\phi^{+}_{8,1} &:= \tfrac{1}{2}(x^{14}+y^{14})\theta^H_{1,0} + \tfrac{1}{2}z^{14}\big(\theta^H_{1,\frac{1}{2}} + \theta^H_{1,\frac{i}{2}}\big) + \tfrac{1}{2}(x^{14}-y^{14})\theta^H_{1,\frac{1+i}{2}},\\
\phi^{+}_{12,1} &:= \tfrac{1}{2}(x^{22}+y^{22})\theta^H_{1,0} + \tfrac{1}{2}z^{22}\big(\theta^H_{1,\frac{1}{2}} + \theta^H_{1,\frac{i}{2}}\big) + \tfrac{1}{2}(x^{22}-y^{22})\theta^H_{1,\frac{i+1}{2}},\\
\phi^{+,\,cusp}_{10,1} &:= \tfrac{1}{64}x^6y^6z^6\big(\theta^H_{1,\frac{1}{2}} - \theta^H_{1,\frac{i}{2}}\big).
\end{aligned} \tag{2.5}$$

Hermitian Jacobi forms of negative parity have not been studied rigorously in the literature, but they do arise via Fourier-Jacobi coefficients of Hermitian modular forms of degree 2 with certain characters (see [7]). Hermitian Eisenstein series are examples of such Hermitian modular forms of degree 2. In particular, there exists such a Hermitian Eisenstein series of weight 6, whose first Fourier-Jacobi coefficient $\phi^{-}_{6,1}$ is a Hermitian Jacobi form of negative parity, weight 6, and index 1. It is somewhat demanding to explicitly compute the Fourier series coefficients of the Hermitian Eisenstein series. We determined $\phi^{-}_{6,1}$ via a different approach. We used SAGE [27] and the SAGE code written by Martin Raum to calculate some Fourier series coefficients of $\phi^{-}_{6,1}$, which allowed us to guess (and then prove) the correct theta decomposition of $\phi^{-}_{6,1}$.

Lemma 2.4. *Set*

$$\phi^{-}_{6,1} := h_0\theta^H_{1,0} + h_{\frac{1}{2}}\theta^H_{1,\frac{1}{2}} + h_{\frac{i}{2}}\theta^H_{1,\frac{i}{2}} + h_{\frac{1+i}{2}}\theta^H_{1,\frac{1+i}{2}},$$

where

$$\begin{aligned}
h_0 &:= -\tfrac{1}{2}(x^2+y^2)(x^8-x^6y^2-x^4y^4-x^2y^6+y^8),\\
h_{\frac{1}{2}} &:= \tfrac{1}{2}z^6(z^4-2x^4),\\
h_{\frac{i}{2}} &:= \tfrac{1}{2}z^6(z^4-2x^4),\\
h_{\frac{1+i}{2}} &:= -\tfrac{1}{2}(x^2-y^2)(x^8+x^6y^2-x^4y^4+x^2y^6+y^8),
\end{aligned}$$

and where x, y, and z are as in (2.3). *Then $\phi^{-}_{6,1} \in J^{-}_{6,1}$.*

Proof. Consider $\psi^{+}_{12,1} := -2E_4^2\phi^{+}_{4,1} + \frac{15}{2}E_4\phi^{+}_{8,1} - \frac{9}{2}\phi^{+}_{12,1} \in J^{+}_{12,1}$, and let

$$\psi^{+}_{12,1} = \hat{h}_0\theta^H_{1,0} + \hat{h}_{\frac{1}{2}}\theta^H_{1,\frac{1}{2}} + \hat{h}_{\frac{i}{2}}\theta^H_{1,\frac{i}{2}} + \hat{h}_{\frac{1+i}{2}}\theta^H_{1,\frac{1+i}{2}}$$

be its theta decomposition. Then using (2.4) and (2.5), one finds that

$$\begin{aligned}
\hat{h}_0 &= E_6h_0,\\
\hat{h}_{\frac{1}{2}} &= E_6h_{\frac{1}{2}},\\
\hat{h}_{\frac{i}{2}} &= E_6h_{\frac{i}{2}},\\
\hat{h}_{\frac{1+i}{2}} &= E_6h_{\frac{1+i}{2}}.
\end{aligned}$$

Hence $\psi^{+}_{12,1} = E_6\phi^{-}_{6,1}$. Observe that the modular Eisenstein series E_6 can also be viewed as a weight 6 and index 0 Hermitian Jacobi form of negative parity. We conclude that $\phi^{-}_{6,1} \in J^{-}_{6,1}$. □

We end this subsection with the initial Fourier series expansions of the Hermitian Jacobi forms $\phi_{4,1}^{+}$, $\phi_{6,1}^{-}$, $\phi_{8,1}^{+}$, and $\phi_{10,1}^{+,cusp}$ (for more coefficients of these forms, see [24]).

Remark 2.5. We have the following initial Fourier series expansions:

$$\begin{aligned}
\phi_{4,1}^{+} = 1 + q\Big(60 &+ 32\big(\zeta^{\frac{1}{2}}(\zeta')^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}(\zeta')^{-\frac{1}{2}} + \zeta^{-\frac{i}{2}}(\zeta')^{\frac{i}{2}} + \zeta^{\frac{i}{2}}(\zeta')^{-\frac{i}{2}}\big)\\
&+ \big(\zeta\zeta' + \zeta^{-1}(\zeta')^{-1} + \zeta^{-i}(\zeta')^{i} + \zeta^{i}(\zeta')^{-i}\big)\\
&+ 12\big(\zeta^{\frac{1+i}{2}}(\zeta')^{\frac{1-i}{2}} + \zeta^{\frac{-1+i}{2}}(\zeta')^{\frac{-1-i}{2}} + \zeta^{\frac{1-i}{2}}(\zeta')^{\frac{1+i}{2}} + \zeta^{\frac{-1-i}{2}}(\zeta')^{\frac{-1+i}{2}}\big)\Big)\\
&+ \cdots,\\
\phi_{6,1}^{-} = 1 + q\Big(-204 &- 64\big(\zeta^{\frac{1}{2}}(\zeta')^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}(\zeta')^{-\frac{1}{2}} + \zeta^{-\frac{i}{2}}(\zeta')^{\frac{i}{2}} + \zeta^{\frac{i}{2}}(\zeta')^{-\frac{i}{2}}\big)\\
&+ \big(\zeta\zeta' + \zeta^{-1}(\zeta')^{-1} + \zeta^{-i}(\zeta')^{i} + \zeta^{i}(\zeta')^{-i}\big)\\
&- 12\big(\zeta^{\frac{1+i}{2}}(\zeta')^{\frac{1-i}{2}} + \zeta^{\frac{-1+i}{2}}(\zeta')^{\frac{-1-i}{2}} + \zeta^{\frac{1-i}{2}}(\zeta')^{\frac{1+i}{2}} + \zeta^{\frac{-1-i}{2}}(\zeta')^{\frac{-1+i}{2}}\big)\Big)\\
&+ \cdots,\\
\phi_{8,1}^{+} = 1 + q\Big(364 &+ \big(\zeta\zeta' + \zeta^{-1}(\zeta')^{-1} + \zeta^{-i}(\zeta')^{i} + \zeta^{i}(\zeta')^{-i}\big)\\
&+ 28\big(\zeta^{\frac{1+i}{2}}(\zeta')^{\frac{1-i}{2}} + \zeta^{\frac{-1+i}{2}}(\zeta')^{\frac{-1-i}{2}} + \zeta^{\frac{1-i}{2}}(\zeta')^{\frac{1+i}{2}} + \zeta^{\frac{-1-i}{2}}(\zeta')^{\frac{-1+i}{2}}\big)\Big)\\
&+ \cdots,\\
\phi_{10,1}^{+,cusp} = \ & q\Big(\zeta^{\frac{1}{2}}(\zeta')^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}(\zeta')^{-\frac{1}{2}} - \zeta^{-\frac{i}{2}}(\zeta')^{\frac{i}{2}} - \zeta^{\frac{i}{2}}(\zeta')^{-\frac{i}{2}}\Big)\\
&+ q^2\Big(-18\big(\zeta^{\frac{1}{2}}(\zeta')^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}(\zeta')^{-\frac{1}{2}} - \zeta^{-\frac{i}{2}}(\zeta')^{\frac{i}{2}} - \zeta^{\frac{i}{2}}(\zeta')^{-\frac{i}{2}}\big)\\
&+ \zeta^{\frac{1+2i}{2}}(\zeta')^{\frac{1-2i}{2}} + \zeta^{\frac{-1+2i}{2}}(\zeta')^{\frac{-1-2i}{2}} + \zeta^{\frac{1-2i}{2}}(\zeta')^{\frac{1+2i}{2}} + \zeta^{\frac{-1-2i}{2}}(\zeta')^{\frac{-1+2i}{2}}\\
&- \zeta^{\frac{2+i}{2}}(\zeta')^{\frac{2-i}{2}} - \zeta^{\frac{-2+i}{2}}(\zeta')^{\frac{-2-i}{2}} - \zeta^{\frac{2-i}{2}}(\zeta')^{\frac{2+i}{2}} - \zeta^{\frac{-2-i}{2}}(\zeta')^{\frac{-2+i}{2}}\Big) + \cdots. \qquad \blacksquare
\end{aligned}$$

2.3. **Proof of Theorem 1.1.** We proceed as in [23] (see also §3 of [8]) to prove Theorem 1.1. Consider the Taylor series of a Hermitian Jacobi form $\phi \in J_{k,1}^{\delta}$ around $(z, w) = (0, 0)$:

$$\phi(\tau, z, w) = \sum_{\mu,\nu=0}^{\infty} \chi_{\mu,\nu}^{\delta}(\tau) z^{\mu} w^{\nu}.$$

Then $\chi_{\mu,\nu}^{\delta} = 0$ unless $\mu - \nu$ is even, and if $\epsilon \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathcal{O})$, then one finds that

$$\begin{aligned}
&\chi_{\mu,\nu}^{\delta}\Big(\frac{a\tau + b}{c\tau + d}\Big)\\
&= \sigma(\varepsilon)\varepsilon^{k-\mu+\nu}(c\tau + d)^{k+\mu+\nu}\Big(\chi_{\mu,\nu}^{\delta}(\tau) + \frac{2\pi ic}{c\tau + d}\chi_{\mu-1,\nu-1}^{\delta}(\tau)\\
&\qquad + \frac{1}{2!}\Big(\frac{2\pi ic}{c\tau + d}\Big)^2 \chi_{\mu-2,\nu-2}^{\delta}(\tau) + \dots\Big).
\end{aligned}$$

The two following propositions on the Taylor coefficients $\chi_{\mu,\nu}^{\delta}$ are easy to verify, and we omit their proofs (see also [24]).

Proposition 2.6. *Let* $\phi(\tau, z, w) = \sum_{\mu,\nu=0}^{\infty} \chi_{\mu,\nu}^{\delta}(\tau) z^{\mu} w^{\nu} \in J_{k,1}^{\delta}$.

If $k \equiv 0 \pmod 4$ *and* $\delta = +$, *then* $\chi_{0,2}^{+}(\tau) = \chi_{2,0}^{+}(\tau) = 0$.

If $k \equiv 0 \pmod 4$ *and* $\delta = -$*, then* $\chi^{-}_{0,0}(\tau) = \chi^{-}_{1,1}(\tau) = \chi^{-}_{4,0}(\tau) = \chi^{-}_{0,4}(\tau) = \chi^{-}_{2,2}(\tau) = 0$.

If $k \equiv 2 \pmod 4$ *and* $\delta = -$*, then* $\chi^{-}_{0,2}(\tau) = \chi^{-}_{2,0}(\tau) = 0$.

If $k \equiv 2 \pmod 4$ *and* $\delta = +$*, then* $\chi^{+}_{0,0}(\tau) = \chi^{+}_{1,1}(\tau) = \chi^{+}_{4,0}(\tau) = \chi^{+}_{0,4}(\tau) = \chi^{+}_{2,2}(\tau) = 0$.

Proposition 2.7. *Let* $\phi(\tau,z,w) = \sum_{\mu,\nu=0}^{\infty} \chi^{\delta}_{\mu,\nu}(\tau) z^{\mu} w^{\nu} \in J^{\delta}_{k,1}$*. Then*

$$\chi^{\delta}_{0,0}(\tau) \in M_k,$$
$$\chi^{\delta}_{2,0}(\tau), \chi^{\delta}_{0,2}(\tau) \in S_{k+2},$$
$$\chi^{\delta}_{4,0}(\tau), \chi^{\delta}_{0,4}(\tau) \in S_{k+4},$$
$$\zeta^{\delta}_{1,1}(\tau) := \chi^{\delta}_{1,1}(\tau) - \frac{2\pi i}{k}(\chi^{\delta}_{0,0})'(\tau) \in S_{k+2},$$
$$\zeta^{\delta}_{2,2}(\tau) := \chi^{\delta}_{2,2}(\tau) - \frac{2\pi i}{k+2}(\chi^{\delta}_{1,1})'(\tau) + \frac{(2\pi i)^2}{2(k+1)(k+2)}(\chi^{\delta}_{0,0})''(\tau) \in S_{k+4}.$$

Remember the definition of the Jacobi theta function $\theta_{a,b}$ in (2.2) to verify the factorization

$$\theta^{H}_{1,\frac{a+bi}{2}}(\tau,z,w) = \theta_{\frac{a}{2},0}(2\tau, z+w)\,\theta_{\frac{b}{2},0}(2\tau, i(w-z)). \tag{2.6}$$

Let $\phi \in J^{\delta}_{k,1}$. Expand the theta decomposition

$$\phi(\tau,z,w) = \sum_{a,b\in\{0,1\}} h_{\frac{a+bi}{2}}(\tau)\,\theta^{H}_{1,\frac{a+bi}{2}}(\tau,z,w) \underset{(2.6)}{=} \sum_{a,b\in\{0,1\}} h_{\frac{a+bi}{2}}(\tau)\,\theta_{\frac{a}{2},0}(2\tau,z+w)\,\theta_{\frac{b}{2},0}(2\tau,i(w-z))$$

into a Taylor series, and compare its coefficients with the coefficients of the Taylor series

$$\phi(\tau,z,w) = \chi^{\delta}_{0,0}(\tau) + \chi^{\delta}_{1,1}(\tau)zw + \big(\chi^{\delta}_{0,2}(\tau) + \chi^{\delta}_{2,0}(\tau)\big)(z^2+w^2) + \chi^{\delta}_{2,2}(\tau)z^2w^2 + \big(\chi^{\delta}_{0,4}(\tau) + \chi^{\delta}_{4,0}(\tau)\big)(z^4+w^4) + \cdots .$$

A direct calculation reveals that (see [24] for more details)

$$\big(\chi^{\delta}_{0,0}, \chi^{\delta}_{1,1}, \chi^{\delta}_{2,0}, \tfrac{1}{2}(\chi^{\delta}_{2,2} - 12\chi^{\delta}_{4,0})\big) = \big(h_0, h_{\frac{1}{2}}, h_{\frac{i}{2}}, h_{\frac{1+i}{2}}\big)A, \tag{2.7}$$

where

$$A := \begin{pmatrix} {T_0}^2 & 2T_0{T_0}' & 0 & {T_0'}^2 \\ T_0T_1 & T_0{T_1}' + {T_0}'T_1 & \frac{1}{4}({T_1}'T_0 - T_1{T_0}') & 2{T_0}'{T_1}' \\ T_0T_1 & T_0{T_1}' + {T_0}'T_1 & -\frac{1}{4}({T_1}'T_0 - T_1{T_0}') & 2{T_0}'{T_1}' \\ {T_1}^2 & 2T_1{T_1}' & 0 & {T_1'}^2 \end{pmatrix} \tag{2.8}$$

with

$$T_{2a} := \theta_{a,0}(2\tau), \quad {T_{2a}}' := 2\pi i \frac{d}{d\tau}\theta_{a,0}(2\tau), \quad {T_{2a}}'' := (2\pi i)^2 \frac{d^2}{d\tau^2}\theta_{a,0}(2\tau),$$

and where T_{2b}, ${T_{2b}}'$, and ${T_{2b}}''$ are defined analogously. Moreover, one finds that

$$\det A = -\tfrac{1}{2}(T_1{T_0}' - T_0{T_1}')^2\big((T_1{T_0}')^2 - 4T_0{T_0}'T_1{T_1}'\big) \neq 0.$$

Now we are in a position to prove Theorem 1.1. We only consider the case $k \equiv 0 \pmod 4$, since the proof of the case $k \equiv 2 \pmod 4$ is completely analogous.

Note that Proposition 2.7 shows that the map ζ is well-defined. First, we demonstrate that ζ is injective. Let $\phi = (\phi^+, \phi^-) \in J_{k,1}$ such that $\zeta(\phi) = 0$. Then $\chi^+_{0,0} = \zeta^+_{1,1} = \chi^-_{2,0} = \zeta^+_{2,2} - 12\chi^+_{4,0} = 0$. Proposition 2.7 implies that $\chi^+_{1,1} = 0$, $\zeta^+_{2,2} = \chi^+_{2,2}$, and hence $\frac{1}{2}(\chi^+_{2,2} - 12\chi^+_{4,0}) = 0$. Furthermore, Proposition 2.6 gives that $\chi^+_{2,0} = 0$ and $\chi^-_{0,0} = \chi^-_{1,1} = \chi^-_{2,2} = \chi^-_{4,0} = 0$. Thus, for $\delta = \pm$ we have

$$(0,0,0,0) = (\chi^\delta_{0,0}, \chi^\delta_{1,1}, \chi^\delta_{2,0}, \tfrac{1}{2}(\chi^\delta_{2,2} - 12\chi^\delta_{4,0})) \underset{(2.7)}{=} (h_0, h_{\frac{1}{2}}, h_{\frac{i}{2}}, h_{\frac{1+i}{2}})A,$$

where A is as in (2.8). Since $\det A \neq 0$, we obtain $\phi = 0$.

Next we show the injectivity of η. Let $(e, f, g, h) \in M_{k-4} \oplus M_{k-6} \oplus M_{k-8} \oplus M_{k-10}$ such that $e\phi^+_{4,1} + f\phi^-_{6,1} + g\phi^+_{8,1} + h\phi^{+,cusp}_{10,1} = 0$. Observe the theta decompositions of $\phi^+_{4,1}$, $\phi^+_{8,1}$, and $\phi^{+,cusp}_{10,1}$ in (2.5) and of $\phi^-_{6,1}$ in Lemma 2.4 to discover that

$$(e, f, g, h)H = (0,0,0,0),$$

where

$$H := \begin{pmatrix} \frac{1}{2}(x^6 + y^6) & \frac{1}{2}z^6 & \frac{1}{2}z^6 & \frac{1}{2}(x^6 - y^6) \\ h_0 & h_{\frac{1}{2}} & h_{\frac{i}{2}} & h_{\frac{1+i}{2}} \\ \frac{1}{2}(x^{14} + y^{14}) & \frac{1}{2}z^{14} & \frac{1}{2}z^{14} & \frac{1}{2}(x^{14} - y^{14}) \\ 0 & \frac{1}{64}x^6y^6z^6 & -\frac{1}{64}x^6y^6z^6 & 0 \end{pmatrix}$$

with h_0, $h_{\frac{1}{2}}$, $h_{\frac{i}{2}}$, and $h_{\frac{1+i}{2}}$ as in Lemma 2.4. One finds that (note that $x^4 = y^4 + z^4$)

$$\det H = -\tfrac{9}{128}x^{16}y^{16}z^{16} \neq 0,$$

which shows that $e = f = g = h = 0$.

Finally,

$$\begin{aligned} &\dim M_k + \dim S_{k+2} + \dim S_{k+2} + \dim S_{k+4} \\ &\quad = \dim M_{k-4} + \dim M_{k-6} + \dim M_{k-8} + \dim M_{k-10}, \end{aligned}$$

and we conclude that ζ and η are isomorphisms.

Remark 2.8. The proof of Lemma 2.4 reveals that $E_6\phi^-_{6,1} = -2E_4^2\phi^+_{4,1} + \frac{15}{2}E_4\phi^+_{8,1} - \frac{9}{2}\phi^+_{12,1}$. Thus, the restriction of the maps η and ζ in Theorem 1.1 to the case of positive parity yields the structure of $J^+_{k,1}$ in [23].

3. Hermitian Jacobi forms modulo p

3.1. **Congruences and filtrations.** In this section, we explore congruences and filtrations of Hermitian Jacobi forms. In particular, we establish the necessary tools to prove Theorem 1.2.

For Hermitian Jacobi forms $\phi(\tau, z, w) = \sum c(n,r)q^n\zeta^{\overline{r}}(\zeta')^r$ and $\psi(\tau, z, w) = \sum c'(n,r)q^n\zeta^{\overline{r}}(\zeta')^r$ with p-integral rational coefficients, we write $\phi \equiv \psi \pmod{p}$ whenever $c(n,r) \equiv c'(n,r) \pmod{p}$ for all n, r.

Proposition 3.1. *If $\phi \in J^\delta_{k,1}(\mathbb{Z}_{(p)})$ such that $\phi = e\phi^+_{4,1} + f\phi^-_{6,1} + g\phi^+_{8,1}$ (resp. $\phi = h\phi^{+,cusp}_{10,1}$), then the elliptic modular forms e, f, and g (resp. h) have p-integral rational coefficients.*

Moreover, if $\phi \equiv 0 \pmod{p}$, then $e \equiv f \equiv g \equiv 0 \pmod{p}$ (resp. $h \equiv 0 \pmod{p}$).

Proof. The initial Fourier series expansions in Remark 2.5 imply that the generators $\phi_{4,1}^{+}$, $\phi_{6,1}^{-}$, $\phi_{8,1}^{+}$, and $\phi_{10,1}^{+,cusp}$ are linearly independent over the field $\mathbb{Z}/p\mathbb{Z}$ (see [24] for more details).

Suppose that $\phi = e\phi_{4,1}^{+} + f\phi_{6,1}^{-} + g\phi_{8,1}^{+}$ (the case $\phi = h\phi_{10,1}^{+,cusp}$ is analogous). Note that the elliptic modular forms e, f, and g have bounded denominators. If e, f, or g do not have p-integral rational coefficients, then there exists some integer $t \geq 1$ such that $0 \equiv p^t\phi \equiv p^te\phi_{4,1}^{+} + p^tf\phi_{6,1}^{-} + p^tg\phi_{8,1}^{+} \pmod{p}$. This yields a nontrivial linear dependence relation for $\phi_{4,1}^{+}$, $\phi_{6,1}^{-}$, and $\phi_{8,1}^{+}$, which contradicts the above.

Similarly, if $\phi \equiv 0 \pmod{p}$ such that e, f, or g do not vanish modulo p, then one also obtains a nontrivial linear dependence relation for $\phi_{4,1}^{+}$, $\phi_{6,1}^{-}$, and $\phi_{8,1}^{+}$, which is again a contradiction. ☐

An argument as in the proof of Lemma 2.1 of Sofer [26] shows that if two Hermitian Jacobi forms of indices m and m' are congruent modulo p, then $m = m'$. We now give an analog of Sofer's Lemma 2.1 in the case $m = 1$.

Corollary 3.2. *Let $\phi \in J_{k,1}^{\delta}(\mathbb{Z}_{(p)})$ and $\psi \in J_{k',1}^{\delta'}(\mathbb{Z}_{(p)})$ such that $0 \not\equiv \phi \equiv \psi \pmod{p}$. Then $k \equiv k' \pmod{(p-1)}$.*

Proof. Recall that if two elliptic modular forms $f_i \in M_{k_i}$ $(i = 1, 2)$ have p-integral rational coefficients such that $0 \not\equiv f_1 \equiv f_2 \pmod{p}$, then $k_1 \equiv k_2 \pmod{(p-1)}$ (see [25, 28]). This fact in combination with Proposition 3.1 implies the claim. ☐

We also record the following consequence of Theorem 1.1 and Proposition 3.1.

Corollary 3.3. *Let $\phi \in J_{k,1}^{\delta}(\mathbb{Z}_{(p)})$ and $\psi \in J_{k',1}^{\delta'}(\mathbb{Z}_{(p)})$ such that $\phi \equiv \psi \pmod{p}$. If $\delta \neq \delta'$ and $k \equiv k' \pmod{4}$, then $\phi \equiv \psi \equiv 0 \pmod{p}$.*

Corollary 3.2 shows that there are congruences among Hermitian Jacobi forms of different weights, and one wishes to find the smallest weight in which the (coefficient-wise) reduction of a Hermitian Jacobi form modulo p exists.

Definition 3.4. Set $\widetilde{J_{k,1}^{\delta}} := \left\{ \phi \pmod{p} : \phi \in J_{k,1}^{\delta}(\mathbb{Z}_{(p)}) \right\}$. For Hermitian Jacobi forms with p-integral rational coefficients, we define the filtration modulo p by

$$\Omega(\phi) := \inf \left\{ k : \ \phi \pmod{p} \in \widetilde{J_{k,1}^{\delta}} \right\}.$$

Next we generalize Proposition 2 of [22] (see also Proposition 2.15 of [20]) to the case of Hermitian Jacobi forms of index 1.

Proposition 3.5. *If $\phi \in J_{k,1}^{\delta}(\mathbb{Z}_{(p)})$, then $L(\phi) \pmod{p}$ is the reduction of a Hermitian Jacobi form modulo p. Moreover, we have*

$$\Omega\big(L(\phi)\big) \leq \Omega(\phi) + p + 1,$$

with equality if and only if $p \nmid \ \Omega(\phi) - 1$.

Proof. It is easy to adapt the proofs in [20, 22] to the case of Hermitian Jacobi forms. Specifically, one employs (1.2), Proposition 3.1, Corollary 3.2, and Theorem 2 and Lemma 5 of [28]. We omit further details, which are contained in [24]. ☐

Tate's theory of theta cycles (see §7 of [13]) was extended to Jacobi forms (see [21]) and Jacobi forms of higher degree (see [20]). The arguments in [20, 21] apply also to Hermitian Jacobi forms, and Corollary 3.2 and Proposition 3.5 are the key ingredients in proving Theorem 1.2. We omit the detailed proof of Theorem 1.2, which is contained in [24].

3.2. **An example.** As in the case of elliptic modular forms, one does not know if a given Hermitian Jacobi form has $U(p)$ congruences for only finitely many primes. In the following example, we consider primes $5 \leq p < 100$, and we show that $E_4^2\phi_{10,1}^{+,cusp} \in J_{18,1}^+$ has $U(p)$ congruences for $p = 5, 7, 13, 23, 79$, and for no other primes $5 \leq p < 100$ (see [24] for more examples). If $p \neq 5, 7, 13, 23, 79$, then the table of Fourier series coefficients of $E_4^2\phi_{10,1}^{+,cusp}$ in the Appendix of [24] guarantees that $E_4^2\phi_{10,1}^{+,cusp} \,|\, U(p) \not\equiv 0 \pmod{p}$.

Recall the Ramanujan theta operator $\Theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$, and Ramanujan's [19] identities (up to a factor of 4)

$$\begin{aligned} L(E_2) &= 4\Theta(E_2) = \tfrac{1}{3}(E_2^2 - E_4), \\ L(E_4) &= 4\Theta(E_4) = \tfrac{4}{3}(E_2E_4 - E_6), \\ L(E_6) &= 4\Theta(E_6) = 2(E_2E_6 - E_4^2). \end{aligned} \tag{3.1}$$

Moreover, (1.2) together with Theorem 1.1 and Remark 2.5 yields the following identities

$$\begin{aligned} L(\phi_{4,1}^+) &= E_2\phi_{4,1}^+ - \phi_{6,1}^-, \\ L(\phi_{6,1}^-) &= \tfrac{5}{3}E_2\phi_{6,1}^- - \tfrac{8}{3}E_4\phi_{4,1}^+ + \phi_{8,1}^+, \\ L(\phi_{8,1}^+) &= \tfrac{7}{3}E_2\phi_{8,1}^+ - \tfrac{14}{9}E_6\phi_{4,1}^+ - \tfrac{7}{9}E_4\phi_{6,1}^-, \\ L(\phi_{10,1}^{+,cusp}) &= 3E_2\phi_{10,1}^{+,cusp}. \end{aligned} \tag{3.2}$$

We employ (3.1) and (3.2) in combination with the well-known congruences $E_{p-1} \equiv 1 \pmod{p}$ and $E_{p+1} \equiv E_2 \pmod{p}$ to establish (with the help of Mathematica) the $U(p)$ congruences for $E_4^2\phi_{10,1}^{+,cusp}$ for $p = 5, 7, 13, 23, 79$.

Let $p = 5$, 7, or 13. We cannot apply Theorem 1.2, since $p < 18$. However, straightforward calculations show that $L^{p-1}(E_4^2\phi_{10,1}^{+,cusp}) \equiv E_4^2\phi_{10,1}^{+,cusp} \pmod{p}$, which implies the desired $U(p)$ congruences.

If $p = 23$ or 79, then we apply Theorem 1.2.

Let $p = 23$. One finds that $E_{22} \equiv 10E_4^4E_6 + 14E_4E_6^3 \equiv 1 \pmod{23}$. Moreover, $p + 2 - 18 = 7$, and a direct calculation reveals that

$$L^7\big(E_4^2\phi_{10,1}^{+,cusp}\big) \equiv \big(20E_4^4E_6 + 5E_4E_6^3\big)\phi_{10,1}^{+,cusp} \equiv 2E_{22}\phi_{10,1}^{+,cusp} \equiv 2\phi_{10,1}^{+,cusp} \pmod{23}.$$

Hence $\Omega\big(L^7(E_4^2\phi_{10,1}^{+,cusp})\big) = 10 = p + 5 - 18$, and $E_4^2\phi_{10,1}^{+,cusp} \,|\, U(23) \equiv 0 \pmod{23}$.

Let $p = 79$. One finds that

$$\begin{aligned} E_{78} &\equiv 26E_4^{18}E_6 + 10E_4^{15}E_6^3 + 73E_4^{12}E_6^5 + 33E_4^9E_6^7 + 41E_4^6E_6^9 + 72E_4^3E_6^{11} + 62E_6^{13} \\ &\equiv 1 \pmod{79}. \end{aligned}$$

Moreover, $p + 2 - 18 = 63$, and a direct calculation shows that

$$\begin{aligned} &L^{63}\big(E_4^2\phi_{10,1}^{+,cusp}\big) \\ &\quad\equiv \Big(73E_4^{32}E_6 + 46E_4^{29}E_6^3 + 70E_4^{26}E_6^5 + 12E_4^{23}E_6^7 + 57E_4^{20}E_6^9 + 75E_4^{17}E_6^{11} \\ &\qquad\qquad + 61E_4^{14}E_6^{13} + 9E_4^{11}E_6^{15} + 16E_4^8E_6^{17} + 39E_4^5E_6^{19} + 31E_4^2E_6^{21}\Big)\phi_{10,1}^{+,cusp} \\ &\quad\equiv E_{78}\big(18E_4^{14} + 7E_4^{11}E_6^2 + 71E_4^8E_6^4 + 37E_4^5E_6^6 + 40E_4^2E_6^8\big)\phi_{10,1}^{+,cusp} \\ &\quad\equiv \big(18E_4^{14} + 7E_4^{11}E_6^2 + 71E_4^8E_6^4 + 37E_4^5E_6^6 + 40E_4^2E_6^8\big)\phi_{10,1}^{+,cusp} \pmod{79}. \end{aligned}$$

Hence $\Omega\big(L^{63}(E_4^2\phi_{10,1}^{+,cusp})\big) = 66 = 79 + 5 - 18$, and $E_4^2\phi_{10,1}^{+,cusp} \,|\, U(79) \equiv 0 \pmod{79}$.

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Department of Mathematics, University of North Texas, Denton, Texas 76203
E-mail address: `richter@unt.edu`

Department of Mathematics, University of North Texas, Denton, Texas 76203
E-mail address: `JayanthaSenadheera@my.unt.edu`
Current address: Department of Mathematics and Computer Science, Faculty of Natural Sciences, The Open University of Sri Lanka, Nawala 10250, Sri Lanka