THE ACTION OF THE HEAT OPERATOR ON JACOBI FORMS

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ABSTRACT. We investigate the action of the heat operator on Jacobi forms.
In particular, we present two explicit characterizations of this action on Jacobi
forms of index 1. Furthermore, we study congruences and filtrations of Jacobi
forms. As an application, we determine when an analog of Atkin's U-operator
applied to a Jacobi form is nonzero modulo a prime.

1. INTRODUCTION

The Ramanujan theta operator $\Theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$ plays an important role in
the theory of modular forms. If f is a modular form of weight k, then it is easy to
see that

$$(1) \qquad \Theta(f) = \frac{k}{12} fE_2 + \hat{f},$$

where E_2 is the "quasimodular" Eisenstein series of weight 2 and $\hat{f}$ is a modular
form of weight $k + 2$. In 1916, Ramanujan [14] noticed the identities

$$(2) \qquad \Theta(E_4) = \frac{1}{3}(E_4 E_2 - E_6) \quad \text{and} \quad \Theta(E_6) = \frac{1}{2}(E_6 E_2 - E_8),$$

where E_4, E_6, and E_8 are the usual (modular) Eisenstein series of weights 4, 6, and
8, respectively. However, for an arbitrary modular form f, an explicit description
of $\hat{f}$ in Equation (1) was not known until 2004 when Bruiner, Ono, and Kohnen [3]
established an exact formula.

In this paper, we consider an analogous situation for Jacobi forms. The heat
operator $L_m := \frac{1}{(2\pi i)^2}\left(8\pi im\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)$ is a natural tool in the study of Jacobi
forms (see, for example, §3 of Eichler and Zagier [7] and Choie [4], [5], among
others). If ϕ is a Jacobi form of weight k and index m, then a direct computation
reveals that

$$(3) \qquad L_m(\phi) = \frac{(2k-1)\,m}{6}\,\phi E_2 + \hat{\phi},$$

where E_2 is again the "quasimodular" Eisenstein series of weight 2 and $\hat{\phi}$ is a
Jacobi form of weight $k + 2$ and index m. This observation justifies the following
two remarks:

a) If $p \geq 5$ is a prime, then E_2 is a p-adic modular form of weight 2 in the sense
of Serre [15]. Hence Equation (3) implies that if ϕ is a Jacobi form of weight k and

index m, then $L_m(\phi)$ is a p-adic Jacobi form of weight $k+2$ and index m in the sense of Guerzhoy [9].

b) The Ramanujan theta operator preserves the ring of quasimodular forms (see Kaneko and Zagier [11]). Equation (3) shows that the heat operator preserves the ring of quasi-Jacobi forms (in the sense of Kawai and Yoshioka [12]).

In this paper, we take a closer look at the action of $L := L_1$ on Jacobi forms of weight k and index 1. Section 2 gives two different explicit characterizations of $\hat{\phi}$ in Equation (3) when $m = 1$. In particular, we have the following generalizations of the Ramanujan identities (2):

$$(4) \qquad L(E_{4,1}) = \frac{7}{6}\left(E_{4,1}E_2 - E_{6,1}\right) \quad \text{and} \quad L(E_{6,1}) = \frac{11}{6}\left(E_{6,1}E_2 - E_{8,1}\right),$$

where $E_{4,1}$, $E_{6,1}$, and $E_{8,1}$ denote the Jacobi Eisenstein series of index 1 and weights 4, 6, and 8, respectively.

In Section 3, we investigate congruences of Jacobi forms. We prove two results on filtrations of Jacobi forms, which allow us to generalize work of Ahlgren and Ono [1]. More specifically, we define an analog of Atkin's U-operator for Jacobi forms and we determine when that operator applied to a Jacobi form is nonzero modulo a prime.

2. Explicit description of the action of L

We introduce some standard notation. Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and nonnegative integers, respectively, and let $\mathbb{H} \subset \mathbb{C}$ be the complex upper half plane. Set $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_0(N) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c \equiv 0 \pmod{N}\right\}$, and let $M_k\left(\Gamma_0(N)\right)$ be the vector space of modular forms of weight k on $\Gamma_0(N)$. If $N = 1$, then we write $M_k := M_k\left(\Gamma_0(1)\right)$. Now let us recall the definition of a Jacobi form (for more details, see [7]).

Definition 1. A Jacobi form of weight k and index m ($k, m \in \mathbb{N}$) on Γ is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the transformation laws

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k\, e^{2\pi i m \frac{cz^2}{c\tau + d}}\phi(\tau, z), \text{ for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

and

$$\phi\left(\tau, z + \lambda\tau + \mu\right) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\phi(\tau, z), \text{ for all } (\lambda, \mu) \in \mathbb{Z}^2.$$

Furthermore, one requires that a Jacobi form has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} c(n, r)q^n\zeta^r,$$

where here (and in the following) $q := e^{2\pi i \tau}$ and $\zeta := e^{2\pi i z}$ ($\tau \in \mathbb{H}$, $z \in \mathbb{C}$). A Jacobi form that satisfies the stronger condition $c(n, r) \neq 0 \Rightarrow 4nm > r^2$ is called a cusp form. We denote the vector space of Jacobi forms of weight k and index m by $J_{k,m}$ and the space of Jacobi cusp forms of weight k and index m by $J_{k,m}^{cusp}$.

Basic examples of Jacobi forms are given by the Jacobi-Eisenstein series $E_{k,m}$ (see §2 of [7]). In particular, if $m = 1$, then one has the following Fourier expansions:

$$\begin{aligned} E_{4,1}(\tau, z) &= 1 + \left(\zeta^2 + 56\zeta + 126 + 56\zeta^{-1} + \zeta^{-2}\right)q + \dots \\ E_{6,1}(\tau, z) &= 1 + \left(\zeta^2 - 88\zeta - 330 - 88\zeta^{-1} + \zeta^{-2}\right)q + \dots \\ E_{8,1}(\tau, z) &= 1 + \left(\zeta^2 + 56\zeta + 366 + 56\zeta^{-1} + \zeta^{-2}\right)q + \dots. \end{aligned}$$

Apply L to the Fourier expansions of $E_{4,1}$ and $E_{6,1}$, observe that

$$E_2(\tau) = 1 - 24q + \dots,$$

and note that $J_{6,1}$ and $J_{8,1}$ are one-dimensional to verify that the generalizations of the Ramanujan identities in (4) hold.

Let ϕ be an arbitrary Jacobi form of weight k and index 1. We seek an explicit formula for the Jacobi form $\hat{\phi}$ in Equation (3). Note that there exist unique $f \in M_{k-4}$ and $g \in M_{k-6}$ (see Theorem 3.5 in [7]) such that $\phi = fE_{4,1} + gE_{6,1}$. As an immediate consequence of the identities in (4), we obtain the following characterization of $\hat{\phi}$ in Equation (3).

Theorem 1. *If $\phi = fE_{4,1} + gE_{6,1} \in J_{k,1}$ ($f \in M_{k-4}$, $g \in M_{k-6}$), then*

$$(5) \qquad L(\phi) = \frac{2k-1}{6}\phi E_2 + \left(4\hat{f} - \frac{11}{6}gE_4\right)E_{4,1} + \left(4\hat{g} - \frac{7}{6}f\right)E_{6,1},$$

where $\hat{f}$ and $\hat{g}$ are defined by Equation (1) and explicitly determined in [3].

Alternatively, we can also use the results in Atkinson [2] to describe $\hat{\phi}$ in Equation (3) in terms of its theta decomposition. We give this description in the next theorem, after introducing necessary notation. For $\mu = 0, 1$, set

$$\theta_{1,\mu}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \,(\mathrm{mod}\,2)}} q^{\frac{r^2}{4}} \zeta^r.$$

If $\phi \in J_{k,1}$, then

$$(6) \qquad \phi(\tau, z) = h_0(\tau)\theta_{1,0}(\tau, z) + h_1(\tau)\theta_{1,1}(\tau, z),$$

where

$$(7) \qquad h_0(\tau) = \sum_{N=N_0}^{\infty} c_0(N)q^N \quad \text{and} \quad h_1(\tau) = q^{-\frac{1}{4}} \sum_{N=N_1}^{\infty} c_1(N)q^N$$

are, loosely speaking, modular forms of weights $k-\frac{1}{2}$ (for details, see §5 of [7]). Note that if $\phi \not\equiv 0$, then there exist minimal $N_0, N_1 \in \mathbb{N}$ such that $C_0(N_0) \neq 0 \neq C_1(N_1)$. Set $M := \min(4N_0, 4N_1 - 1)$ and set

$$(8) \qquad \alpha_{h_0} := -\frac{5}{6}\left(k - \frac{1}{2}\right) + 2N_0 + 2M, \quad \beta_{h_0} := \frac{2}{3}\left(k - \frac{1}{2}\right) - \frac{8}{3}N_0 - \frac{4}{3}M,$$

and

$$(9) \qquad \alpha_{h_1} := -\frac{5}{6}\left(k - \frac{1}{2}\right) + 2N_1 + 2M, \quad \beta_{h_1} := \frac{2}{3}\left(k - \frac{1}{2}\right) - \frac{8}{3}N_1 - \frac{4}{3}M.$$

Finally, let $\mathcal{F}$ be a fundamental domain for $\Gamma_0(4)$, let $H_{\tau'}(\tau)$ be the (meromorphic) modular form of weight 2 discussed in detail in [2], and let

$$(10) \qquad (h_\mu)_\theta := \sum_{\tau' \in \mathcal{F}} \mathrm{ord}_{\tau'}(h_\mu)H_{\tau'}.$$

Now we are in a position to characterize $\hat{\phi}$ in Equation (3) in terms of its theta decomposition.

Theorem 2. *Let $\phi(\tau, z) = h_0(\tau)\theta_{1,0}(\tau, z) + h_1(\tau)\theta_{1,1}(\tau, z) \in J_{k,1}$. Then*

$$(11) \qquad L(\phi) = \frac{2k-1}{6}\phi E_2 - 4\sum_{\mu=0}^{1} h_\mu \left((h_\mu)_\theta + \alpha_{h_\mu} E_2^{(2)} + \beta_{h_\mu} E_2^{(4)} \right) \theta_{1,\mu},$$

where α_{h_μ} and β_{h_μ} are as in (8) and (9), $(h_\mu)_\theta$ is as in (10), and $E_2^{(2)}(\tau) := E_2(2\tau) - \frac{1}{2}E_2(\tau) \in M_2\big(\Gamma_0(2)\big)$ and $E_2^{(4)}(\tau) := E_2(4\tau) - \frac{1}{4}E_2(\tau) \in M_2\big(\Gamma_0(4)\big)$.

Proof. One can proceed exactly as in Atkinson [2]. More precisely, one can modify the proof of Theorem 3 in [2] to determine the action of the Ramanujan theta operator on h_μ in Equation (7). For brevity, we omit the detailed proof and only state the final result:

$$(12) \qquad \frac{\Theta(h_\mu)}{h_\mu}(\tau) = -(h_\mu)_\theta(\tau) - \alpha_{h_\mu} E_2(2\tau) - \beta_{h_\mu} E_2(4\tau) - \gamma_{h_\mu} E_2(\tau),$$

where $\gamma_{h_0} := \frac{1}{6}\left(k - \frac{1}{2}\right) - \frac{1}{3}N_0 - \frac{2}{3}M$ and $\gamma_{h_1} := \frac{1}{6}\left(k - \frac{1}{2}\right) - \frac{1}{3}N_1 - \frac{2}{3}M$. Now, $L\big(\theta_{1,\mu}(\tau, z)\big) = 0$, and hence

$$(13) \qquad L\big(\phi(\tau, z)\big) = 4\Theta\big(h_0(\tau)\big)\theta_{1,0}(\tau, z) + 4\Theta\big(h_1(\tau)\big)\theta_{1,1}(\tau, z).$$

Thus Equations (12) and (13) imply Equation (11). $\qquad\square$

Remark. Choie and Kohnen [6] (following [3]) give an explicit characterization of $\frac{z\frac{\partial}{\partial z}\phi(\tau,z)}{\phi(\tau,z)}$ ($\phi \in J_{k,m}$), which also involves the "quasimodular" Eisenstein series E_2. However, our discussion of the action of the heat operator on Jacobi forms seems quite unrelated to the results in [6].

3. Congruences and reduction mod p

For the remainder of our work, let $p \geq 5$ be a prime and (for simplicity) assume throughout that k is even. For Jacobi forms $\phi(\tau, z) = \sum c(n, r)q^n\zeta^r$ and $\psi(\tau, z) = \sum c'(n, r)q^n\zeta^r$ with p-integral rational coefficients, we write $\phi \equiv \psi \pmod{p}$ when $c(n, r) \equiv c'(n, r) \pmod{p}$ for all n, r. Recall the following fundamental fact regarding congruences of Jacobi forms.

Proposition 1 (Sofer [16]). *Let $\phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}[[q, \zeta]]$ and $\psi(\tau, z) \in J_{k',m'} \cap \mathbb{Z}[[q, \zeta]]$ such that $0 \not\equiv \phi \equiv \psi \pmod{p}$. Then $k \equiv k' \pmod{p-1}$ and $m = m'$.*

Since there are congruences among Jacobi forms of different weights, it is desirable to find the smallest weight in which the (coefficient-wise) reduction of a Jacobi form modulo p exists.

Definition 2. Let $\widetilde{M_k} := \left\{ f \pmod{p} : f(\tau) \in M_k \cap \mathbb{Z}[[q]] \right\}$. Similarly, let $\widetilde{J}_{k,m} := \left\{ \phi \pmod{p} : \phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}[[q, \zeta]] \right\}$. For modular forms and Jacobi forms with p-integral rational coefficients, we define the filtrations modulo p by

$$\omega(f) := \inf\left\{ k : f \pmod{p} \in \widetilde{M_k} \right\}$$

and

$$\Omega(\phi) := \inf\left\{ k : \phi \pmod{p} \in \widetilde{J}_{k,m} \right\}.$$

Serre [15] and Swinnerton-Dyer [17] thoroughly investigate filtrations of modular forms. Ono [13] gives a good overview of this theory and, in addition, discusses many interesting applications. Several of these applications rely on the crucial fact that if $f(\tau) \in M_k \cap \mathbb{Z}[[q]]$, then $\omega\big(\Theta(f)\big) \leq \omega(f) + p + 1$, with equality if and only if $p \nmid \omega(f)$ (see Lemma 5 of [17]). Our next proposition gives an analogous result for Jacobi forms.

Proposition 2. *If $\phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}[[q, \zeta]]$, then $L_m(\phi)$ (mod p) is the reduction of a Jacobi form modulo p. Moreover, we have*

$$(14) \qquad \Omega\big(L_m(\phi)\big) \leq \Omega(\phi) + p + 1,$$

with equality if and only if $p \nmid (2\Omega(\phi) - 1)\, m$.

Proof. Note that $E_{p-1} \equiv 1$ (mod p) and $E_2 \equiv E_{p+1}$ (mod p). If $\Omega(\phi) = k' < k$, then $k - k' = l(p - 1)$ by Proposition 1 and $\phi \equiv (E_{p-1})^l \psi$ (mod p), where $\psi \in J_{k',m}$ with $\Omega(\psi) = k'$. We find that $L_m(\phi) \equiv L_m(\psi)$ (mod p), i.e., $\Omega\big(L_m(\phi)\big) = \Omega\big(L_m(\psi)\big)$, and hence we may (and do) assume that $\Omega(\phi) = k$. Equation (3) shows that $L_m(\phi)$ (mod p) $\in \widetilde{J}_{k+p+1,m}$, i.e., $\Omega\big(L_m(\phi)\big) \leq k + p + 1$.

If p divides $(2k - 1)\, m$, then $\Omega\big(L_m(\phi)\big) \leq k + 2 < k + p + 1$ by Equation (3). On the other hand, if $\Omega\big(L_m(\phi)\big) < k + p + 1$, then $\Omega\left(\frac{(2k-1)\, m}{6} \phi E_2\right) < k + p + 1$ by Equation (3). It remains to show that $\Omega\left(\phi E_2\right) = k + p + 1$, which then implies that p divides $(2k - 1)\, m$. Recall that every Jacobi form (of even weight) can be written as

$$(15) \qquad \phi = \sum_{j=0}^{m} f_j \left(\phi_{-2,1}\right)^j \left(\phi_{0,1}\right)^{m-j},$$

where $\phi_{-2,1}(\tau, z) \in \mathbb{Z}[[q, \zeta]]$ and $\phi_{0,1}(\tau, z) \in \mathbb{Z}[[q, \zeta]]$ are weak Jacobi forms of index 1 and weights -2 and 0, respectively, and where each $f_j \in M_{k+2j}$ has p-integral rational coefficients and is uniquely determined (see §8 and §9 of [7] for details and also for the corresponding result for Jacobi forms of odd weight). Moreover, there exists an f_j such that $\omega\big(f_j\big) = k + 2j$, since otherwise $\Omega(\phi) < k$. Hence (using Theorem 2 and Lemma 5 of [17]), $\omega\big(f_j E_2\big) = k + 2j + p + 1$, and we find that (by Lemma 2.2 of [16]) $\Omega\left(\phi E_2\right) = k + p + 1$, which completes the proof. $\qquad \square$

Let us introduce an analog of Atkin's U-operator for Jacobi forms:

Definition 3. For $\phi(\tau, z) = \displaystyle\sum_{\substack{n,r \\ 4nm - r^2 \geq 0}} c(n,r) q^n \zeta^r \in J_{k,m}$, we define

$$(16) \qquad \phi(\tau, z)\,\big|\, U_p := \sum_{\substack{n,r \\ 4nm - r^2 \geq 0 \\ p\,|\,(4nm - r^2)}} c(n,r) q^n \zeta^r.$$

Remark. Note that the operator in Definition 3 is different from the operator U_l in §4 of [7].

Our following result generalizes Proposition 6.3 of [1] and provides a useful tool in studying congruences defined by the action of the operator U_p.

Proposition 3. *Let* $\phi(\tau,z) = \displaystyle\sum_{\substack{n,r \\ 4nm-r^2\geq 0 \\ n\geq n_0}} c(n,r)q^n\zeta^r \in J_{k,m}^{cusp} \cap \mathbb{Z}[[q,\zeta]]$ *such that*
$p \nmid mn_0c(n_0,0).$ *Then for every* $s \in \mathbb{N}$, *we have* $\Omega\big(L_m^s(\phi)\big) \geq 12n_0 - 2m.$

Proof. To see that $k \geq 12n_0 - 2m$, apply the valence formula to the cusp form $\phi(\tau,0) \in M_k$ or to some $f_j \not\equiv 0$ in the decomposition (15) of ϕ if $\phi(\tau,0) \equiv 0$ (in which case $f_0(\tau) \equiv 0$, but $f_j(\tau) \not\equiv 0$ for some $j \neq 0$). Let $\Delta(\tau) := q - 24q^2 + \ldots$ be the usual cusp form of weight 12. The Ramanujan identities (2) imply that $\Theta(\Delta) = \Delta E_2$, and a straightforward computation shows that

$$\frac{L_m^s(\phi)(\tau,z)}{\Delta^{n_0}(\tau)} = (4mn_0)^s c(n_0,0) + \sum_{(n,r)\neq(0,0)} C(n,r)q^n\zeta^r,$$

where $C(n,r) \in \mathbb{Z}$. In particular, $\Delta^{-n_0}L_m^s(\phi)$ is congruent modulo p to a weak Jacobi form $\psi \not\equiv 0 \pmod{p}$. Note that the weight of a nonzero weak Jacobi form is at least $-2m$ and hence $\Omega\big(L_m^s(\phi)\big) \geq 12n_0 - 2m.$ $\square$

As an application of Propositions 2 and 3, we now determine when U_p applied to a Jacobi form is nonzero modulo p. Our result is of the same flavor as Theorem 1.7 of Ahlgren and Ono [1] (see Elkies, Ono, and Yang [8] and Guerzhoy [10] for general discussions of congruences that involve Atkin's U-operator).

Theorem 3. *Let* $\phi(\tau,z) = \displaystyle\sum_{\substack{n,r \\ 4nm-r^2\geq 0 \\ n\geq n_0}} c(n,r)q^n\zeta^r \in J_{k,m}^{cusp} \cap \mathbb{Z}[[q,\zeta]].$ *Suppose that*
$p \geq \max(5, k - 12n_0 + 2m)$ *is a prime such that* $p \nmid mn_0c(n_0,0).$ *If* $2\Omega\big(L_m(\phi)\big) \equiv 3,5 \pmod{p}$, *then* $\phi\,\big|\,U_p \not\equiv 0 \pmod{p}.$

Proof. We proceed as in [1]. Suppose that $\phi\,\big|\,U_p \equiv 0 \pmod{p}$. Note that $\phi\,\big|\,U_p \equiv \phi - L_m^{p-1}(\phi) \pmod{p}$ for any ϕ and hence

(17) $0 \not\equiv L_m^{p-1}(\phi) \equiv \phi \pmod{p}.$

If $p \nmid \big(2\Omega(L_m^{p-2}(\phi)) - 1\big)$, then $\Omega\big(L_m^{p-1}(\phi)\big) = \Omega\big(L_m^{p-2}(\phi)\big) + p + 1$ by Proposition 2. However, Equation (17) shows that $\Omega\big(L_m^{p-1}(\phi)\big) \leq k$ while Proposition 3 gives $\Omega\big(L_m^{p-2}(\phi)\big) \geq 12n_0 - 2m$, which contradicts the assumption that $p \geq k - 12n_0 + 2m.$

Consequently, we have that p divides $\big(2\Omega\big(L_m^{p-2}(\phi)\big) - 1\big)$. Let $j \leq p - 3$ be the smallest positive integer such that p divides $\big(2\Omega(L_m^{j+1}(\phi)) - 1\big)$. Proposition 2 implies that

$$2\Omega\big(L_m^{j+1}(\phi)\big) = 2\big(\Omega\big(L_m(\phi)\big) + j(p+1)\big) \equiv 2\Omega\big(L_m(\phi)\big) + 2j \equiv 1 \pmod{p},$$

which contradicts the assumption that $2\Omega\big(L_m(\phi)\big) \equiv 3,5 \pmod{p}$. We conclude that $\phi\,\big|\,U_p \not\equiv 0 \pmod{p}.$ $\square$

For example, if

$$\phi_{10,1}(\tau,z) = \big(\zeta - 2 + \zeta^{-1}\big)q + \ldots \in J_{10,1}^{cusp} \cap \mathbb{Z}[[q,\zeta]]$$

and $p = 17$, then $2\Omega\big(L(\phi_{10,1})\big) = 56 \equiv 5 \pmod{17}$ and Theorem 3 implies that $\phi_{10,1}\,\big|\,U_{17} \not\equiv 0 \pmod{17}$. Similarly, if

$$\phi_{12,1}(\tau,z) = \big(\zeta + 10 + \zeta^{-1}\big)q + \ldots \in J_{12,1}^{cusp} \cap \mathbb{Z}[[q,\zeta]]$$

and $p = 7$, then $2\Omega\bigl(L(\phi_{12,1})\bigr) = 40 \equiv 5 \pmod 7$ and Theorem 3 implies that $\phi_{12,1} \bigm| U_7 \not\equiv 0 \pmod 7$.

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