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# ON CONGRUENCES OF JACOBI FORMS

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ABSTRACT. We consider congruences and filtrations of Jacobi forms. More specifically, we extend Tate's theory of theta cycles to Jacobi forms, which allows us to prove a criterion for an analog of Atkin's $U$-operator applied to a Jacobi form to be nonzero modulo a prime.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

The theory of modular forms modulo a prime $p$ has a long history and has been the source for many fruitful applications. Ono [10] gives a good overview, and he discusses several applications of congruences that involve Atkin's $U$-operator (see also Ahlgren and Ono [1], Elkies, Ono, and Yang [5], and Guerzhoy [6]). Chida and Kaneko [2] use Serre and Swinnerton-Dyer's theory of filtrations of modular forms modulo $p$ (see [13] and [15]) in combination with Tate's theory of theta cycles (see §7 of [7]) to establish a criterion for when the $p$-th coefficient of a modular form is not divisible by $p$. The purpose of this paper is to prove an analogous result for Jacobi forms of index 1.

In [12], we study the action of the heat operator $L := \frac{1}{(2\pi i)^2}\left(8\pi i \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}\right)$ on Jacobi forms of index 1. If $\phi$ is such a Jacobi form, then $L^n(\phi)$ (for all $n \in \mathbb{N}$) is a quasi-Jacobi form in the sense of Kawai and Yoshioka [9]. More precisely, there exist unique quasimodular forms $f$ and $g$ of weights $2n+k-4$ and $2n+k-6$, respectively, such that $L^n(\phi) = fE_{4,1} + gE_{6,1}$, where $E_{4,1}$ and $E_{6,1}$ are the Jacobi Eisenstein series of index 1 and weights 4 and 6, respectively. For details on quasimodular forms, see Kaneko and Zagier [8]. We adopt the notation in [2]. Let $E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\left(\sum_{d|n} d^{k-1}\right) q^n$ ($q := e^{2\pi i\tau}$, $k \geq 2$) denote the usual Eisenstein series. In particular, $E_2$ is the "quasimodular" Eisenstein series of weight 2. If $f \in \mathbb{C}[E_2, E_4, E_6]$ is quasimodular, then let $F(f; X, Y, Z)$ be the polynomial in $X$, $Y$, and $Z$ such that $f(\tau) = F\big(f; E_2(\tau), E_4(\tau), E_6(\tau)\big)$ and let $F^{(0)} := F^{(0)}(f; Y, Z) := F(f; 0, Y, Z)$ be the "modular part". Note that if $p \geq 5$ is a prime, then the Hasse invariant $H_p(Y, Z) := F^{(0)}(E_{p-1}; Y, Z)$ has $p$-integral rational coefficients.

Now we can state the main result of our paper.

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**Theorem 1.** *Let*

$$\phi(\tau, z) = \sum_{\substack{n,r \\ 4n-r^2 \geq 0}} c(n,r) q^n \zeta^r \in \mathbb{Z}[[q,\zeta]]$$

*be a Jacobi form of weight $k$ and index 1, where $q := e^{2\pi i \tau}$ and $\zeta := e^{2\pi i z}$ ($\tau \in \mathbb{H}$, $z \in \mathbb{C}$). Let $p > k$ be a prime such that $\phi \not\equiv 0 \pmod{p}$, and let*

$$\phi(\tau, z) \,\big|\, U_p := \sum_{\substack{n,r \\ 4n-r^2 \geq 0 \\ p \,|\, (4n-r^2)}} c(n,r) q^n \zeta^r$$

*be the analog of Atkin's $U$-operator for Jacobi forms (see also* [12]*). If $p > 2k-5$, then $\phi \,\big|\, U_p \not\equiv 0 \pmod{p}$. Suppose that $p < 2k-5$, and let us write $L^{\frac{3p+3}{2}-k}(\phi) = fE_{4,1} + gE_{6,1}$. Then the quasimodular forms $f$ and $g$ (and hence also $F^{(0)}$ and $G^{(0)}$) have $p$-integral rational coefficients and the following conditions are equivalent:*

1. $\phi \,\big|\, U_p \not\equiv 0 \pmod{p}$,
2. $H_p(Y,Z) \nmid F^{(0)} \pmod{p}$ *or* $H_p(Y,Z) \nmid G^{(0)} \pmod{p}$.

In Section 2, we give a closed formula for the quasi-Jacobi form $L^n(\phi)$. In Section 3, we extend Tate's theory of theta cycles to Jacobi forms. More specifically, we establish a result on heat cycles of Jacobi forms which is a key ingredient in our proof of Theorem 1. Finally, in Section 4, we apply Theorem 1 to the explicit example $\phi_{10,1} := \frac{1}{144}(E_6 E_{4,1} - E_4 E_{6,1})$. We find that $\phi_{10,1} \,\big|\, U_p \equiv 0 \pmod{p}$ for $p = 5, 11, 13$, while $\phi_{10,1} \,\big|\, U_p \not\equiv 0 \pmod{p}$ for $p = 7$ and all primes $p > 13$.

## 2. Jacobi forms and the heat operator

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and nonnegative integers, respectively, and let $\mathbb{H} \subset \mathbb{C}$ be the complex upper half plane. We recall the definition of a Jacobi form (for more details, see Eichler and Zagier [4]).

**Definition 1.** A Jacobi form of weight $k$ and index $m$ ($k, m \in \mathbb{N}$) is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the transformation laws

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \, e^{2\pi i m \frac{cz^2}{c\tau+d}} \phi(\tau,z), \text{ for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}),$$

and

$$\phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2 \tau + 2\lambda z)} \phi(\tau, z), \text{ for all } (\lambda, \mu) \in \mathbb{Z}^2.$$

Furthermore, one requires that a Jacobi form has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm-r^2 \geq 0}} c(n,r) q^n \zeta^r.$$

We denote the vector space of Jacobi forms of weight $k$ and index $m$ by $J_{k,m}$.

In this paper we are solely interested in Jacobi forms of index 1. If $\phi \in J_{k,1}$, then

$$\mathcal{D}\phi := L(\phi) - \frac{2k-1}{6} \phi E_2 \in J_{k+2,1}.$$

It is not important for our purposes, but $\mathcal{D}\phi$ can be explicitly characterized (for details, see [12]). For $\phi \in J_{k,1}$, define the sequence $\phi_r \in J_{k+2r,1}$ recursively by

$$\phi_{r+1} := \mathcal{D}\phi_r - \frac{r\left(r+k-\frac{3}{2}\right)}{9} E_4 \phi_{r-1} \ \ (r \geq 0)$$

with initial condition $\phi_0 = \phi$. Then the formula for $f_r$ on page 1272 of Choie and Eholzer [3] (with $\Phi = -\frac{1}{9}E_4$ and $\phi = \frac{1}{3}E_2$) yields the following closed formula.

**Lemma 1.** *Let $\phi \in J_{k,1}$. Then for all $n \in \mathbb{N}$, we have*

$$L^n(\phi) = n! \sum_{j=0}^{n} \binom{k-\frac{3}{2}+n}{j} \frac{\phi_{n-j}}{(n-j)!} \left(\frac{E_2}{3}\right)^j. \tag{1}$$

## 3. Heat cycles and the proof of Theorem 1

We study heat cycles of Jacobi forms (generalizing Tate's theory of theta cycles), which allows us to prove Theorem 1. Let us briefly review congruences and filtrations of Jacobi forms. Throughout, let $p \geq 5$ be a prime. Set

$$\widetilde{J}_{k,1} := \left\{\phi \pmod{p} \,:\, \phi(\tau,z) \in J_{k,1} \cap \mathbb{Z}[[q,\zeta]]\right\}.$$

If $\phi \in J_{k,1}$ has $p$-integral rational coefficients, then we denote its filtration modulo $p$ by

$$\Omega\big(\phi\big) := \inf\left\{k \,:\, \phi \pmod{p} \in \widetilde{J}_{k,1}\right\}.$$

Recall the following facts:

**Proposition 1** (Sofer [14])**.** *Let $\phi(\tau,z) \in J_{k,1} \cap \mathbb{Z}[[q,\zeta]]$ and $\psi(\tau,z) \in J_{k',1} \cap \mathbb{Z}[[q,\zeta]]$ such that $0 \not\equiv \phi \equiv \psi \pmod{p}$. Then $k \equiv k' \pmod{p-1}$.*

**Proposition 2** ([12])**.** *If $\phi(\tau,z) \in J_{k,1} \cap \mathbb{Z}[[q,\zeta]]$, then $L(\phi) \pmod{p}$ is the reduction of a Jacobi form modulo $p$. Moreover, we have*

$$\Omega\big(L(\phi)\big) \leq \Omega\big(\phi\big) + p + 1, \tag{2}$$

*with equality if and only if $p \nmid \big(2\Omega\big(\phi\big) - 1\big)$.*

Propositions 1 and 2 play an important role in our following investigation of heat cycles of Jacobi forms.

Let $\phi(\tau,z) \in J_{k,1} \cap \mathbb{Z}[[q,\zeta]]$ such that $\phi \not\equiv 0 \pmod{p}$ and suppose that $\phi \,\big|\, U_p \equiv 0 \pmod{p}$. Then $L^{p-1}(\phi) \equiv \phi \pmod{p}$. We use the terminology of §7 of [7] and we call $\phi_1$ a *low point* of its heat cycle if it occurs immediately after a fall, i.e., if $\phi_1 = L^A(\phi)$ and $2\Omega\big(L^{A-1}\phi\big) \equiv 1 \pmod{p}$. Let $\phi_1$ be a low point of its heat cycle and let $c_j - 1 \in \mathbb{N}$ be minimal such that

$$2\Omega\big(L^{c_j-1}(\phi_1)\big) = 2\left(\Omega\big(\phi_1\big) + (c_j-1)(p+1)\right) \equiv 1 \pmod{p},$$

and let $b_j \in \mathbb{N}$ be defined by

$$\Omega\big(L^{c_j}(\phi_1)\big) = \Omega\big(\phi_1\big) + c_j(p+1) - b_j(p-1).$$

An argument as in §7 of [7] shows that there is either one fall with $c_1 = p-1$ and $b_1 = p+1$, or there are two falls with $b_1 = p - c_2$ and $b_2 = p - c_1$. The first case occurs if and only if $2\Omega\big(\phi_1\big) \equiv 5 \pmod{p}$. In the second case, $\Omega\big(\phi_1\big) = ap + B$ with $1 \leq B \leq p$ and $p \neq 2B - 5$. If $p > 2B - 5$, then we calculate that $c_1 = \frac{p+3}{2} - B$, $c_2 = \frac{p-5}{2} + B$, and $\Omega\left(L^{\frac{p+3}{2}-B}(\phi_1)\right) = ap - B + 4$. If $p < 2B - 5$, then we obtain that $c_1 = \frac{3p+3}{2} - B$, $c_2 = -\frac{p+5}{2} + B$, and $\Omega\left(L^{\frac{3p+3}{2}-B}(\phi_1)\right) = (a+2)p - B + 4$. In particular, if $a = 0$, then $B \geq 4$ and we necessarily have that $p < 2B - 5$.

On the other hand, if $\phi(\tau, z) = \sum c(n,r) q^n \zeta^r \in J_{k,1} \cap \mathbb{Z}[[q,\zeta]]$ such that $\Omega(\phi) = k < p < 2k - 5$ and $\phi \,\big|\, U_p \not\equiv 0 \pmod{p}$, then $L(\phi)$ must be a low point of its heat cycle. If it were not, then $\psi(\tau, z) := \sum_{p \nmid (4n - r^2)} c(n,r) q^n \zeta^r$ would satisfy $\psi \,\big|\, U_p \equiv 0 \pmod{p}$, $L(\psi) \equiv L(\phi) \pmod{p}$, and $\Omega(\psi) = \Omega(\phi)$. Then $\Omega(\psi - \phi) = \Omega(\phi)$ and the fact that $2\Omega(\phi) \not\equiv 1 \pmod{p}$ would imply that

$$\Omega\big(L(\psi - \phi)\big) \underset{(2)}{=} \Omega\big(\psi - \phi\big) + p + 1 = \Omega\big(\phi\big) + p + 1,$$

which is a contradiction, since $L(\psi - \phi) \equiv 0 \pmod{p}$. We conclude that $L(\phi)$ is a low point of its heat cycle and we find that $\Omega\left(L^{\frac{3p+3}{2}-k}(\phi)\right) = 3p - k + 3$.

We summarize our investigation of heat cycles:

**Proposition 3.** *Let $\phi(\tau, z) \in J_{k,1} \cap \mathbb{Z}[[q,\zeta]]$ such that $\phi \not\equiv 0 \pmod{p}$. If $p > 2k - 5$, then $\phi \,\big|\, U_p \not\equiv 0 \pmod{p}$. If $k < p < 2k - 5$, then*

$$\Omega\left(L^{\frac{3p+3}{2}-k}(\phi)\right) = \begin{cases} 3p - k + 3 & \text{if } \phi \,\big|\, U_p \not\equiv 0 \pmod{p}, \\ 2p - k + 4 & \text{if } \phi \,\big|\, U_p \equiv 0 \pmod{p}. \end{cases}$$

Note that $E_{4,1}$ and $E_{6,1}$ form a basis (over the ring of modular forms) of $J_{k,1}$. It is easy to see that $E_{4,1}$ and $E_{6,1}$ are relatively prime modulo $p$, and hence Lemma 5 of [15] immediately implies the following result.

**Lemma 2.** *If $\phi = fE_{4,1} + gE_{6,1} \in J_{k,1}$ has integral coefficients, then the following conditions hold:*
*(1) If $\Omega(\phi) = k$, then $H_p(Y,Z) \nmid F^{(0)} \pmod{p}$ or $H_p(Y,Z) \nmid G^{(0)} \pmod{p}$.*
*(2) If $\Omega(\phi) < k$, then $H_p(Y,Z) \mid F^{(0)} \pmod{p}$ and $H_p(Y,Z) \mid G^{(0)} \pmod{p}$.*

Now we are in a position to prove Theorem 1. Let $\phi(\tau, z) \in J_{k,1} \cap \mathbb{Z}[[q,\zeta]]$ such that $k < p < 2k - 5$. Then Lemma 1 with $n = \frac{3p+3}{2} - k$ shows that

$$L^{\frac{3p+3}{2}-k}(\phi) = fE_{4,1} + gE_{6,1} \equiv F^{(0)} E_{4,1} + G^{(0)} E_{6,1} \pmod{p},$$

i.e., $L^{\frac{3p+3}{2}-k}(\phi) \pmod{p} \in \widetilde{J}_{3p-k+3,1}$, and Theorem 1 follows from Proposition 3 and Lemma 2.

## 4. An example

We illuminate Theorem 1 and Proposition 3 with a concrete example. Let $\phi_{10,1} = \frac{1}{144}(E_6 E_{4,1} - E_4 E_{6,1})$ be the unique (up to a scalar) Jacobi cusp form of weight 10 and index 1. Recall Ramanujan's [11] identities and their Jacobi form generalizations in [12]:

$$\begin{aligned} L(E_2) &= 4\Theta(E_2) = \tfrac{1}{3}\big(E_2^2 - E_4\big), \\ L(E_4) &= 4\Theta(E_4) = \tfrac{4}{3}\big(E_4 E_2 - E_6\big), \\ L(E_6) &= 4\Theta(E_6) = 2\big(E_6 E_2 - E_4^2\big), \\ L(E_{4,1}) &= \tfrac{7}{6}\big(E_{4,1} E_2 - E_{6,1}\big), \\ L(E_{6,1}) &= \tfrac{11}{6}\big(E_{6,1} E_2 - E_4 E_{4,1}\big), \end{aligned} \tag{3}$$

where $\Theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$ is the Ramanujan theta operator. If $p = 11$, then $E_4E_6 = E_{10} \equiv 1 \pmod{11}$ and a direct computation (with Mathematica) using the formulas in (3) reveals that

$$\begin{aligned} L^8(\phi_{10,1}) &\equiv E_4E_6\big(3E_4^3 + 2E_6^2\big)E_{4,1} + 6\big(E_4E_6\big)^2 E_{6,1} \\ &\equiv \big(3E_4^3 + 2E_6^2\big)E_{4,1} + \big(6E_{10}\big)E_{6,1} \pmod{11}. \end{aligned}$$

Hence $\Omega\big(L^8(\phi_{10,1})\big) = 16$ and $\phi_{10,1}\,\big|\,U_{11} \equiv 0 \pmod{11}$. Similarly if $p = 13$, then $6E_4^3 + 8E_6^2 \equiv E_{12} \equiv 1 \pmod{13}$, and we find that

$$\begin{aligned} L^{11}(\phi_{10,1}) &\equiv 9E_4\big(6E_4^3 + 8E_6^2\big)^2 E_{4,1} + \big(6E_4^3 + 8E_6^2\big)\big(4E_4^2E_6\big)E_{6,1} \\ &\equiv \big(9E_4E_{12}\big)E_{4,1} + \big(4E_4^2E_6\big)E_{6,1} \pmod{13}. \end{aligned}$$

Hence $\Omega\big(L^{11}(\phi_{10,1})\big) = 20$ and $\phi_{10,1}\,\big|\,U_{13} \equiv 0 \pmod{13}$. Furthermore, Proposition 3 implies that $\phi_{10,1}\,\big|\,U_p \not\equiv 0 \pmod{p}$ for all primes $p > 13$.

Note that Theorem 1 and Proposition 3 provide no information on primes $p < 11$. Nevertheless, the table of coefficients of Jacobi forms of index 1 on p. 141 of [4] shows that $\phi_{10,1}\,\big|\,U_7 \not\equiv 0 \pmod{7}$. Finally, if $p = 5$, then we calculate that $\Omega\big(L(\phi_{10,1})\big) = 16$, $\Omega\big(L^2(\phi_{10,1})\big) = 22$, $\Omega\big(L^3(\phi_{10,1})\big) = 28$, and $\Omega\big(L^4(\phi_{10,1})\big) = \Omega\big(\phi_{10,1}\big) = 10$. If $\phi_{10,1}\,\big|\,U_5 \not\equiv 0 \pmod{5}$, then (with the same argument as before) $\Omega\big(L(\phi_{10,1})\big)$ would be a low point of its heat cycle, which is obviously not the case. Hence $\phi_{10,1}\,\big|\,U_5 \equiv 0 \pmod{5}$.

We conclude that $\phi_{10,1}\,\big|\,U_p \equiv 0 \pmod{p}$ for $p = 5, 11, 13$, while $\phi_{10,1}\,\big|\,U_p \not\equiv 0 \pmod{p}$ for $p = 7$ and all primes $p > 13$.

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