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CLASSIFICATION OF THE SPACE SPANNED BY THETA SERIES AND APPLICATIONS

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Abstract. We determine a class of functions spanned by theta series of higher degree. We give two applications: A simple proof of the inversion formula of such theta series and a classification of skew-holomorphic Jacobi forms.

1. Introduction

Theta functions play an important role in the study of holomorphic and skew-holomorphic Jacobi forms. In particular, Jacobi forms decompose into theta functions and certain half-integral weight modular forms (see [8] and [14]; for Jacobi forms of higher degree, see [16], [12], and Theorem 3 below). The theta functions in this decomposition have the form

$$\theta_{2\mathcal{M},\,\mu}(\tau,z) = \sum_{\kappa\in\mathbb{Z}^{(n,l)}} \exp\left\{2\pi i\, tr\left(\mathcal{M}\left({}^{t}(\kappa+\mu)\tau(\kappa+\mu)+2\,{}^{t}(\kappa+\mu)z\right)\right)\right\}, \tag{1}$$

where tr denotes the trace, $2\mathcal{M}\in\mathbb{Z}^{(l,l)}$ is symmetric and positive definite, $\mu\in\mathcal{N}$, where $\mathcal{N}$ is a complete system of representatives for the set of cosets $\mathbb{Z}^{(n,l)}(2\mathcal{M})^{-1}/\mathbb{Z}^{(n,l)}$, and finally, τ and z are variables in $\mathbb{H}_n$ (Siegel upper half plane of degree n) and $\mathbb{C}^{(n,l)}$, respectively.

The Poisson summation formula is a standard tool in determining the inversion formula for $\theta_{2\mathcal{M},\,\mu}(\tau,z)$. Bellman and Lehman [4] (for $l=1$, $\mathcal{M}=\frac{1}{2}$, and arbitrary n), Couwenberg [6] (for $n=l=1$ and $\mathcal{M}=\frac{1}{2}$), and Choie and Taguchi [5] (for $n=l=1$ and arbitrary $2\mathcal{M}\in\mathbb{Z}$) discovered a proof of the inversion formula which does not require Poisson summation and instead uses the heat operator. In this paper, we identify functions spanned by $\theta_{2\mathcal{M},\,\mu}(\tau,z)$ (Theorem 1), which allows us to establish the inversion formula for $\theta_{2\mathcal{M},\,\mu}(\tau,z)$ for arbitrary n, l, and $2\mathcal{M}\in\mathbb{Z}^{(l,l)}$ (Theorem 2). More specifically, we extend the approach in [5], and we use a (matrix-valued) generalization of the classical heat operator instead of the Poisson summation formula.

Various types of theta functions can be regarded as specializations of the symplectic theta function $\vartheta(\tau)=\theta_{1,\,0}(\tau,0)$ (i.e., $l=1$) as first observed by Eichler [7], and then also by many others (see for example [1], [2], [15], [10], and [13]). This

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yields an elegant way to prove transformation laws for such theta functions. Note that such proofs contain no "hidden" Poisson summation formula, since our proof (as well as the proof in [4]) of the inversion formula for $\vartheta(\tau)$ does not require it.

As another application of Theorem 1, we classify the space of skew-holomorphic Jacobi forms of higher degree (Theorem 3). This allows us to show that some combinations of theta functions in (1) and also theta functions attached to certain indefinite quadratic forms are examples of skew-holomorphic Jacobi forms.

2. Class of functions spanned by theta series

As in the previous section, let $2\mathcal{M} \in \mathbb{Z}^{(l,l)}$ be symmetric and positive definite. For variables $\tau = (\tau_{ij}) \in \mathbb{H}_n$ and $z = (z_{ij}) \in \mathbb{C}^{(n,l)}$, we write $\partial_\tau = (1+\delta_{ij})\frac{\partial}{\partial \tau_{ij}}$ and $\partial_z = \frac{\partial}{\partial z_{ij}}$. The differential operator

$$L_{\mathcal{M}} = 8\pi i\, \partial_\tau - \partial_z \mathcal{M}^{-1}\, {}^t\partial_z \tag{2}$$

(see [11]) is a generalization of the classical heat operator. We have the following theorem.

Theorem 1. *Let $f : \mathbb{H}_n \times \mathbb{C}^{(n,l)} \to \mathbb{C}$ be real analytic in $\tau \in \mathbb{H}_n$ and holomorphic in $z \in \mathbb{C}^{(n,l)}$ and suppose that for all $\lambda \in \mathbb{Z}^{(n,l)}$,*
(i) *(**shift property**)* $f(\tau, z+\lambda) = f(\tau, z)$,
(ii) *(**elliptic property**)* $f(\tau, z+\tau\lambda) = e^{-2\pi i\, tr\left(\mathcal{M}({}^t\lambda\tau\lambda + 2\,{}^t\lambda z)\right)} f(\tau, z)$,
(iii) *(**heat kernel**)* $L_{\mathcal{M}}(f(\tau, z)) = 0$.
Then $f(\tau, z)$ belongs to the vector space spanned by

$$\{b_\mu(-\overline{\tau})\, \theta_{2\mathcal{M},\,\mu}(\tau, z) \,|\, \mu \in \mathcal{N}\}$$

over $\mathbb{C}$, and where $b_\mu(-\overline{\tau})$ is a holomorphic function in $\overline{\tau}$.

Proof. Since f is periodic in z (shift property), f has a Fourier series with respect to z. It is easy to verify that the elliptic property implies that

$$f(\tau, z) = \sum_{\mu \in \mathcal{N}} b_\mu(\tau, -\overline{\tau})\, \theta_{2\mathcal{M},\,\mu}(\tau, z),$$

where b_μ is a function of τ and $\overline{\tau}$. Property (iii) together with the fact that $L_{\mathcal{M}}\,(\theta_{2\mathcal{M},\,\mu}(\tau, z)) = 0$ imply that $\partial_\tau\,(b_\mu(\tau, -\overline{\tau})) = 0$, which completes the proof. □

Remarks. a) If f is also holomorphic in τ, then $f(\tau, z) = \sum_{\mu\in\mathcal{N}} c_\mu\, \theta_{2\mathcal{M},\,\mu}(\tau, z)$ for some constants $c_\mu \in \mathbb{C}$.

b) It is clear that the converse of Theorem 1 also holds, i.e., any element that belongs to the vector space spanned by

$$\{b_\mu(-\overline{\tau})\, \theta_{2\mathcal{M},\,\mu}(\tau, z) \,|\, \mu \in \mathcal{N}\}$$

over $\mathbb{C}$, where $b_\mu(-\overline{\tau})$ is real analytic in τ, and satisfies conditions (i), (ii), and (iii) of Theorem 1.

3. Applications

3.1. **A simple proof of the inversion formula for theta series.** We apply Theorem 1 to prove (without using the Poisson summation formula) the following well-known inversion formula for $\theta_{2\mathcal{M},\,\mu}(\tau, z)$.

Theorem 2. *We have*

$$\begin{aligned}\theta_{2\mathcal{M},\,\mu}(-\tau^{-1},\tau^{-1}z) = &\det(2\mathcal{M})^{-\frac{n}{2}}\det(-i\tau)^{\frac{l}{2}}\, e^{2\pi i\, tr\left(\mathcal{M}({}^tz\tau^{-1}z)\right)} \\ &\times \sum_{\nu\in\mathcal{N}} e^{-2\pi i\, tr(2\mathcal{M}\,{}^t\nu\mu)}\,\theta_{2\mathcal{M},\,\nu}(\tau,z),\end{aligned} \tag{3}$$

where $\det(-i\tau)^{\frac{1}{2}}$ *is positive if* $\tau = iy$ *for positive definite* $y\in\mathbb{R}^{(n,n)}$.

Proof. Let $\lambda\in\mathbb{Z}^{(n,l)}$. As an immediate consequence of (1) we record

$$\theta_{2\mathcal{M},\,\mu}\left(\tau, z+\lambda(2\mathcal{M})^{-1}\right) = e^{2\pi i\, tr\left({}^t\lambda\mu\right)}\,\theta_{2\mathcal{M},\,\mu}(\tau,z) \tag{4}$$

and

$$\begin{aligned}&\theta_{2\mathcal{M},\,\mu}\left(\tau, z+\tau\lambda(2\mathcal{M})^{-1}\right)\\ &= e^{-2\pi i\, tr({}^t\lambda\tau\lambda(4\mathcal{M})^{-1})}\, e^{-2\pi i\, tr\left({}^t\lambda z\right)}\,\theta_{2\mathcal{M},\,\mu+\lambda(2\mathcal{M})^{-1}}(\tau,z).\end{aligned} \tag{5}$$

Set

$$F(\tau,z) = \det(2\mathcal{M})^{\frac{n}{2}}\det(-i\tau)^{-\frac{l}{2}}\, e^{-2\pi i\, tr\left(\mathcal{M}({}^tz\tau^{-1}z)\right)}\theta_{2\mathcal{M},\,\mu}\left(-\tau^{-1},\tau^{-1}z\right).$$

Straightforward computations show that F satisfies the conditions (i), (ii), and (iii) in Theorem 1. Hence there exist complex numbers $c_{\nu,\,\mu}$ such that

$$\begin{aligned}&\theta_{2\mathcal{M},\,\mu}\left(-\tau^{-1},\tau^{-1}z\right)\\ &= \det(2\mathcal{M})^{-\frac{n}{2}}\det(-i\tau)^{\frac{l}{2}}\, e^{2\pi i\, tr\left(\mathcal{M}({}^tz\tau^{-1}z)\right)}\sum_{\nu\in\mathcal{N}} c_{\nu,\,\mu}\,\theta_{2\mathcal{M},\,\nu}(\tau,z).\end{aligned} \tag{6}$$

Replacing z by $z+\tau\lambda(2\mathcal{M})^{-1}$ in (6) and applying (5) leads to

$$\begin{aligned}&\theta_{2\mathcal{M},\,\mu}\left(-\tau^{-1},\tau^{-1}z+\lambda(2\mathcal{M})^{-1}\right)\\ &= \det(2\mathcal{M})^{-\frac{n}{2}}\det(-i\tau)^{\frac{l}{2}}\, e^{2\pi i\, tr\left(\mathcal{M}({}^tz\tau^{-1}z)\right)}\sum_{\nu\in\mathcal{N}} c_{\nu-\lambda(2\mathcal{M})^{-1},\,\mu}\,\theta_{2\mathcal{M},\,\nu}(\tau,z).\end{aligned} \tag{7}$$

Using (4), we obtain

$$\begin{aligned}\theta_{2\mathcal{M},\,\mu}\left(-\tau^{-1},\tau^{-1}z\right) = &\det(2\mathcal{M})^{-\frac{n}{2}}\det(-i\tau)^{\frac{l}{2}}\, e^{2\pi i\, tr\left(\mathcal{M}({}^tz\tau^{-1}z)\right)}\\ &\times e^{-2\pi i\, tr\left({}^t\lambda\mu\right)}\sum_{\nu\in\mathcal{N}} c_{\nu-\lambda(2\mathcal{M})^{-1},\,\mu}\,\theta_{2\mathcal{M},\,\nu}(\tau,z).\end{aligned} \tag{8}$$

Combing (6) and (8) gives us

$$\begin{aligned}\theta_{2\mathcal{M},\,\mu}\left(-\tau^{-1},\tau^{-1}z\right) = &\det(2\mathcal{M})^{-\frac{n}{2}}\det(-i\tau)^{\frac{l}{2}}\, e^{2\pi i\, tr\left(\mathcal{M}({}^tz\tau^{-1}z)\right)}\\ &\times c_{0,\,\mu}\sum_{\nu\in\mathcal{N}} e^{-2\pi i\, tr\left(2\mathcal{M}\,{}^t\nu\mu\right)}\,\theta_{2\mathcal{M},\,\nu}(\tau,z).\end{aligned} \tag{9}$$

It remains to show that $c_{0,\mu}=1$. We replace z by τz and then τ by $-\tau^{-1}$ in (9) (note that $\theta_{2\mathcal{M},\nu}(\tau,-z)=\theta_{2\mathcal{M},-\nu}(\tau,z)$) and find that

$$\begin{aligned}
&\theta_{2\mathcal{M},\mu}(\tau,z)\\
&=\det(2\mathcal{M})^{-\frac{n}{2}}\det\left(i\tau^{-1}\right)^{\frac{l}{2}}e^{-2\pi i\,tr\left(\mathcal{M}(\,{}^tz\tau^{-1}z)\right)}\\
&\quad\times c_{0,\mu}\sum_{\nu\in\mathcal{N}}e^{2\pi i\,tr\left(2\mathcal{M}\,{}^t\nu\mu\right)}\theta_{2\mathcal{M},\nu}\left(-\tau^{-1},\tau^{-1}z\right)\\
&\underset{(9)}{=}\det(2\mathcal{M})^{-n}c_{0,\mu}\sum_{\eta\in\mathcal{N}}\sum_{\nu\in\mathcal{N}}c_{0,\nu}\,e^{2\pi i\,tr\left(2\mathcal{M}\,{}^t\nu(\mu-\eta)\right)}\theta_{2\mathcal{M},\eta}(\tau,z)\\
&=\det(2\mathcal{M})^{-n}c_{0,\mu}\sum_{\nu\in\mathcal{N}}c_{0,\nu}\,\theta_{2\mathcal{M},\mu}(\tau,z),
\end{aligned}\tag{10}$$

where the last equality follows from the fact that the functions $\theta_{2\mathcal{M},\mu}(\tau,z)$ are linearly independent over $\mathbb{C}$. Hence

$$1=\det(2\mathcal{M})^{-n}c_{0,\mu}\sum_{\nu\in\mathcal{N}}c_{0,\nu}.\tag{11}$$

In particular, $c_{0,\mu}$ does not depend on μ and (11) simplifies to

$$1=\det(2\mathcal{M})^{-n}\left(c_{0,\mu}\right)^2\sum_{\nu\in\mathcal{N}}1=\left(c_{0,\mu}\right)^2.\tag{12}$$

Setting $\tau=iI_n$ and $z=0$ in (9) shows that $c_{0,\mu}>0$ and hence $c_{0,\mu}=1$. □

3.2. **Skew-holomorphic Jacobi forms.** We recall the definition of skew-holomorphic Jacobi forms. For more details, see also [14], [3], [11] and [12]. Denote the symplectic group over the integers of degree n by $\mathrm{Sp}_n(\mathbb{Z})$. Let $k\in\mathbb{N}_0$ and $2\mathcal{M}\in\mathbb{Z}^{(l,l)}$ be symmetric and semi-positive definite.

Definition 1. A skew-holomorphic Jacobi form of weight k and index $\mathcal{M}$ is a function $F:\mathbb{H}_n\times\mathbb{C}^{(n,l)}\to\mathbb{C}$ that is real analytic in $\tau\in\mathbb{H}_n$, holomorphic in $z\in\mathbb{C}^{(n,l)}$, and which satisfies the following conditions:

(J1) For all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{Sp}_n(\mathbb{Z})$,

$$\begin{aligned}
&F\left((a\tau+b)(c\tau+d)^{-1},(c\tau+d)^{-1}z\right)\\
&=\det(c\overline{\tau}+d)^{k-l}\left|\det(c\tau+d)\right|^l e^{2\pi i\,tr\left(\mathcal{M}(\,{}^tz(c\tau+d)^{-1}cz)\right)}F\left(\tau,z\right).
\end{aligned}$$

(J2) For all $\lambda_1,\lambda_2\in\mathbb{Z}^{(n,l)}$,

$$F(\tau,z+\tau\lambda_1+\lambda_2)=e^{-2\pi i\,tr\left(\mathcal{M}(\,{}^t\lambda_1\tau\lambda_1+2\,{}^t\lambda_1z)\right)}F(\tau,z).$$

(J3) $L_{\mathcal{M}}(F(\tau,z))=0$.

In addition, if $n=1$, then the function $\frac{e^{-2\pi y\mathcal{M}\,{}^ty/v}}{1+v^{-k}}F(\tau,z)$ is bounded on $\mathbb{H}_1\times\mathbb{C}^{(1,l)}$, where $v=\mathrm{Im}(\tau)$ and $y=\mathrm{Im}(z)$.

Remark. Note that condition (J3) may be replaced by a condition requiring that F have a certain Fourier series expansion, which yields an equivalent definition of skew-holomorphic Jacobi forms.

The following theorem classifies skew-holomorphic Jacobi forms.

Theorem 3. *A function G is a skew-holomorphic Jacobi form of weight k and index $\mathcal{M}$ if and only if*

$$G(\tau,z)=\sum_{\mu\in\mathcal{N}} b_\mu(-\overline{\tau})\theta_{2\mathcal{M},\mu}(\tau,z), \tag{13}$$

where $b_\mu(-\overline{\tau})$ is holomorphic in $-\overline{\tau}\in\mathbb{H}_n$ such that

(i) $b_\mu(\overline{\tau}^{-1})=(-i)^{\frac{nl}{2}}\det(2\mathcal{M})^{-\frac{n}{2}}\det(-\overline{\tau})^{k-\frac{l}{2}}\sum_{\nu\in\mathcal{N}} e^{2\pi i\, tr\left(2\mathcal{M}\,{}^t\nu\mu\right)}\, b_\nu(-\overline{\tau})$,

(ii) *for all symmetric $S\in\mathbb{Z}^{(n,n)}$, $b_\mu(-\overline{\tau}-S)=e^{-2\pi i tr\left(\mathcal{M}({}^t\mu S\mu)\right)}b_\mu(-\overline{\tau})$,*

and in addition, if $n=1$, then the function $\frac{e^{-2\pi y\mathcal{M}\,{}^ty/v}}{1+v^{-k}}G(\tau,z)$ is bounded on $\mathbb{H}_1\times\mathbb{C}^{(1,l)}$, where again $v=Im(\tau)$ and $y=Im(z)$.

Proof. If G is a skew-holomorphic Jacobi form of weight k and index $\mathcal{M}$, then Theorem 1 implies that $G(\tau,z)=\sum_{\mu\in\mathcal{N}} b_\mu(-\overline{\tau})\theta_{2\mathcal{M},\mu}(\tau,z)$ and conditions (i) and (ii) are immediate consequences of (J1), Theorem 2, and the fact that

$$\theta_{2\mathcal{M},\mu}(\tau+S,z)=e^{2\pi i tr\left(\mathcal{M}({}^t\mu S\mu)\right)}\theta_{2\mathcal{M},\mu}(\tau,z). \tag{14}$$

On the other hand, one can reverse the previous argument. If

$$G(\tau,z)=\sum_{\mu\in\mathcal{N}} b_\mu(-\overline{\tau})\theta_{2\mathcal{M},\mu}(\tau,z),$$

then conditions (i) and (ii), Theorem 2, and (14) guarantee that G satisfies condition (J1) for $\left(\begin{smallmatrix}0 & I_n\\ -I_n & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}I_n & S\\ 0 & I_n\end{smallmatrix}\right)$, where $S\in\mathbb{Z}^{(n,n)}$ is symmetric. Note that $Sp_n(\mathbb{Z})$ is generated by these matrices (see for example Anhang V of Freitag [9]). We conclude that condition (J1) holds for all $M\in Sp_n(\mathbb{Z})$. Moreover, $\theta_{2\mathcal{M},\mu}(\tau,z)$ satisfies conditions (J2) and (J3) in Definition 1 and hence so does G. □

Remarks. a) Equation (13) provides an isomorphism between the space of skew-holomorphic Jacobi forms of weight k and index $\mathcal{M}$ and the space of vector-valued Siegel modular forms of half-integral weight satisfying (i), (ii), and the additional growth condition for $n=1$ (see also Hayashida [12] for the case $l=\mathcal{M}=1$).

b) Recall that a holomorphic Jacobi form of weight k and index $\mathcal{M}$ satisfies (J1) and (J2) in Definition 1, except with the factor $\det(c\overline{\tau}+d)^{k-l}\,|\det(c\tau+d)|^l$ in (J1) replaced by $\det(c\tau+d)^k$. Ziegler [16] proves an analogous version of Theorem 3 for holomorphic Jacobi forms, which yields an isomorphism between such Jacobi forms and vector-valued Siegel modular forms of half-integral weight satisfying conditions similar to (i) and (ii).

Finally, we apply Theorem 3 to construct some concrete examples of skew-holomorphic Jacobi forms.

Examples. 1) We find that $b_\mu(-\overline{\tau})=\theta_{2\mathcal{M},\,\mu}(-\overline{\tau},0)$ satisfies conditions (i) and (ii) in Theorem 3 with $k=l$ (observe that $\theta_{2\mathcal{M},\,\mu}(-\overline{\tau},0)=\theta_{2\mathcal{M},\,-\mu}(-\overline{\tau},0)$). Hence

$$G(\tau,z)=\sum_{\mu\in\mathcal{N}}\theta_{2\mathcal{M},\,\mu}(-\overline{\tau},0)\,\theta_{2\mathcal{M},\mu}(\tau,z) \tag{15}$$

is a skew-holomorphic Jacobi form of weight l and index $\mathcal{M}$. Note that this construction does in general not produce holomorphic Jacobi forms. If $F(\tau,z)=\sum_{\mu\in\mathcal{N}}\theta_{2\mathcal{M},\,\mu}(\tau,0)\,\theta_{2\mathcal{M},\mu}(\tau,z)$, then only if 4 divides nl do we have $F\left(-\tau^{-1},\tau^{-1}z\right)$ $=\det(-\tau)^l\,e^{2\pi i\,tr\left(\mathcal{M}({}^tz\tau^{-1}z)\right)}\,F(\tau,z)$ and only if $|\mathcal{N}|=1$ do we have $F(\tau+S,z)=F(\tau,z)$ for all $S\in\mathbb{Z}^{(n,n)}$.

2) Suppose that $2\mathcal{M} \in \mathbb{Z}^{(l,l)}$ is symmetric, positive definite, unimodular (i.e, $|\mathcal{N}| = 1$), and even. If $b(\tau)$ is a Siegel modular form of weight $k - l/2$, then $b(-\overline{\tau})$ satisfies conditions (i) and (ii) in Theorem 3 and $G(\tau, z) = b(-\overline{\tau})\,\theta_{2\mathcal{M},0}(\tau, z)$ is a skew holomorphic Jacobi form of weight k and index $\mathcal{M}$. In particular, if $q \in \mathbb{Z}^{(2k-l,2k-l)}$ is symmetric, positive definite, unimodular, and even, then it is well known that the theta function

$$\vartheta_q(\tau) = \sum_{N \in \mathbb{Z}^{(2k-l,n)}} e^{\pi i\, tr\left({}^tNqN\tau\right)}$$

is a Siegel modular form of weight $k - l/2$. Set $Q = \left(\begin{smallmatrix} 2\mathcal{M} & 0 \\ 0 & -q \end{smallmatrix}\right) \in \mathbb{Z}^{(2k,2k)}$ and $R = \left(\begin{smallmatrix} 2\mathcal{M} & 0 \\ 0 & q \end{smallmatrix}\right) \in \mathbb{Z}^{(2k,2k)}$. Hence Q is symmetric, unimodular, even, of type $(l, 2k-l)$, and R is a corresponding majorant, i.e., $RQ^{-1}R = Q$ and R is symmetric and positive definite. Set $W = \left(\begin{smallmatrix} I_l \\ 0 \end{smallmatrix}\right) \in \mathbb{Z}^{(2k,l)}$. Then

$$\begin{aligned} G(\tau, z) &= \vartheta_q(-\overline{\tau})\,\theta_{2\mathcal{M},0}(\tau, z) \\ &= \sum_{N \in \mathbb{Z}^{(2k,n)}} e^{\pi i\, tr\left({}^tNQNu + i\,{}^tNRNv + 2\,{}^tNQW\,{}^tz\right)} \quad (\tau = u + iv) \end{aligned} \tag{16}$$

is the Jacobi theta function attached to the indefinite quadratic form Q (studied also in [3] and [13]), and by the above reasoning, G is a skew-holomorphic Jacobi form of weight k and index $\mathcal{M}$.

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