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ON RANKIN-COHEN BRACKETS FOR SIEGEL MODULAR FORMS

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ABSTRACT. We determine an explicit formula for a Rankin-Cohen bracket for Siegel modular forms of degree n on a certain subgroup of the symplectic group. Moreover, we lift that bracket via a Poincaré series to a Siegel cusp form on the full symplectic group.

1. INTRODUCTION

In 1956, Rankin [10] showed that certain polynomials in the derivatives of modular forms are again modular forms. In 1977, Cohen [6] defined for each $\nu \geq 0$ an operator which assigns to two modular forms f and g of weight k and l a modular form $[f, g]_\nu$ of weight $k + l + 2\nu$. This operator is known as the Rankin-Cohen bracket, and operators similar in nature are called Rankin-Cohen-type brackets.

Explicit formulas for Rankin-Cohen brackets have been found for Jacobi forms ([1] and [2]), Siegel modular forms of degree 2 ([3] and [4]), and for Jacobi forms (of higher degree) on $\mathbb{H} \times \mathbb{C}^n$ ([5]). Eholzer and Ibukiyama [7] prove that there exists a unique Rankin-Cohen bracket for Siegel modular forms of arbitrary degree n. A closed formula for the Rankin-Cohen bracket is known only for $n = 1$ and 2, even though a system of recursion relations is given for any degree n in [7].

In this paper, we consider Fourier-Jacobi expansions of Siegel modular forms. We demonstrate how a Rankin-Cohen bracket for Jacobi forms on $\mathbb{H} \times \mathbb{C}^{n-1}$ can be lifted to a Rankin-Cohen bracket for Siegel modular forms of degree n. In particular, for a certain subgroup of the symplectic group, we determine a closed formula for the Rankin-Cohen bracket. When $n = 2$, that formula holds for the full symplectic group and our result coincides with Theorem 1.4 in [4], but if $n > 2$, then our formula is valid only for the subgroup. However, in Theorem 3, we use our result to define a Poincaré series on the full symplectic group, which yields (for any $\nu \geq 0$) an operator that sends two Siegel modular forms (degree n) of weight k_1 and k_2 and a modular cusp form (degree 1) of weight k_3 to a Siegel cusp form (degree n) of weight $k_1 + k_2 + k_3 + 2\nu$.

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2. Jacobi forms of higher degree

Let $\mathbb{A}$ be a commutative ring with unity and let $M_{m,n}(\mathbb{A})$ be the set of $m \times n$ matrices with entries in $\mathbb{A}$. If $U \in M_{n,n}(\mathbb{A})$, let $tr(U)$ be the trace of U and let $|U|$ be the determinant of U. We denote the symplectic group over the integers of degree m by $\Gamma_m = \mathrm{Sp}_m(\mathbb{Z})$. Let $\Gamma_{m,j}$ be the subgroup of Γ_m that consists of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where $A = \left(\begin{smallmatrix} A_1 & 0 \\ A_3 & A_4 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} B_1 & B_2 \\ B_3 & B_4 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} C_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $D = \left(\begin{smallmatrix} D_1 & D_2 \\ 0 & D_4 \end{smallmatrix}\right)$, where $A_1, B_1, C_1, D_1 \in M_{j,j}(\mathbb{Z})$, $B_2, D_2 \in M_{j,m-j}(\mathbb{Z})$, $A_3, B_3 \in M_{m-j,j}(\mathbb{Z})$, and $A_4, B_4, D_4 \in M_{m-j,m-j}(\mathbb{Z})$. The subgroup $\Gamma_{m,j}$ plays an important role in the theory of Siegel modular forms. For more details, see Freitag [9], chapter I, §5 and chapter II, §2. Let $\mathbb{H}_m$ be the Siegel upper half plane of degree m, $G : \mathbb{H}_m \to \mathbb{C}$, and let Γ be a subgroup of Γ_m. As usual, for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ and for a positive integer k, we define the slash operator

$$G\Big|_k M = G\left((AZ+B)(CZ+D)^{-1}\right)|CZ+D|^{-k}.$$

Let F be a Siegel modular form of weight k and degree $j+n-1$ on Γ_{j+n-1}, i.e., F is holomorphic and $F\big|_k M = F$ for all $M \in \Gamma_{j+n-1}$. We write $Z \in \mathbb{H}_{j+n-1}$ as $Z = \left(\begin{smallmatrix} \tau & z \\ {}^t z & W \end{smallmatrix}\right)$, where $\tau \in \mathbb{H}_j$, $z \in M_{j,\,n-1}(\mathbb{C})$, and $W \in \mathbb{H}_{n-1}$. The Fourier-Jacobi expansion of F is given by

$$F(Z) = F(\tau, z, W) = \sum_{\substack{\mathcal{M} = {}^t\mathcal{M} \geq 0 \\ \mathcal{M}\,\text{even}}} \Phi_{\mathcal{M}}(\tau, z) e^{\pi i\, tr(\mathcal{M}W)}, \tag{1}$$

where the sum is over symmetric, semi-positive definite, integral, and even $(n-1)\times(n-1)$ matrices $\mathcal{M}$. Note that $\Phi_{\mathcal{M}}$ is a Jacobi form of weight k and index $\mathcal{M}$ in the sense of Ziegler [11], and that if $j=1$ and $n=2$, then $\Phi_{\mathcal{M}}$ is a Jacobi form in the sense of Eichler and Zagier [8]. Of particular interest is the case where $j=1$ and $n \geq 2$ is arbitrary. We denote the vector space of such Jacobi forms by $\mathcal{J}_{k,\,\mathcal{M}}(\Gamma_1)$.

Choie and Kim [5] provide an explicit formula for the Rankin-Cohen bracket for Jacobi forms on $\mathcal{J}_{k,\,\mathcal{M}}(\Gamma_1)$. We need the following definition to state their main result.

Definition 1. Let $\tau \in \mathbb{H}_1$ and $z = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$. Suppose $\mathcal{M} = (m_{s\,t})$ is a symmetric, positive definite, integral, and even $(n-1)\times(n-1)$ matrix, where $\mathcal{M}_{s\,t}$ is the cofactor of the entry $m_{s\,t}$. The heat operator $L_{\mathcal{M}}$ is defined by

$$L_{\mathcal{M}} = 4\pi i\,|\mathcal{M}|\partial_\tau - \sum_{1\leq s,\,t\leq n-1} \mathcal{M}_{s\,t}\partial_{z_s}\partial_{z_t}, \tag{2}$$

where $\partial_x = \frac{\partial}{\partial x}$.

Theorem 1 (Choie, Kim)**.** *Let $\phi_1 \in \mathcal{J}_{k_1,\,\mathcal{M}_1}(\Gamma_1)$ and $\phi_2 \in \mathcal{J}_{k_2,\,\mathcal{M}_2}(\Gamma_1)$. For each nonnegative integer ν and $Y \in \mathbb{C}$, set*

$$\begin{aligned}&[[\phi_1, \phi_2]]_{Y,\,\nu} \\ &= \sum_{r+s+p=\nu} C_{r,\,s,\,p}(k_1,k_2) D_{r,\,s}(\mathcal{M}_1,\,\mathcal{M}_2,\,Y) L^p_{\mathcal{M}_1+\mathcal{M}_2}\left(L^r_{\mathcal{M}_1}(\phi_1) L^s_{\mathcal{M}_2}(\phi_2)\right),\end{aligned} \tag{3}$$

where

$$C_{r,\,s,\,p}(k_1,k_2) = (-1)^p \frac{(\gamma+2\nu-p-2)!}{r!\,s!\,p!\,(\alpha+r-1)!\,(\beta+s-1)!\,(\gamma+2\nu-2)!},$$

$$D_{r,\,s}(\mathcal{M}_1,\,\mathcal{M}_2,\,Y) = \left(\frac{|\mathcal{M}_1+\mathcal{M}_2|}{|\mathcal{M}_1|+|\mathcal{M}_2|}+|\mathcal{M}_2|\,Y\right)^r\left(\frac{|\mathcal{M}_1+\mathcal{M}_2|}{|\mathcal{M}_1|+|\mathcal{M}_2|}-|\mathcal{M}_1|\,Y\right)^s,$$

and where $\alpha = k_1 - \frac{n-1}{2}$, $\beta = k_2 - \frac{n-1}{2}$, *and* $\gamma = k_1+k_2-\frac{n-1}{2}$. *Then,* $[[\phi_1,\phi_2]]_{Y,\,\nu} \in \mathcal{J}_{k_1+k_2+2\nu,\,\mathcal{M}_1+\mathcal{M}_2}(\Gamma_1)$.

Remark. Choie and Kim [5] also construct bilinear operators of the form $\mathcal{J}_{k_1,\,\mathcal{M}_1}(\Gamma_1)\times\mathcal{J}_{k_2,\,\mathcal{M}_2}(\Gamma_1)\to\mathcal{J}_{k_1+k_2+2\nu+1,\,\mathcal{M}_1+\mathcal{M}_2}(\Gamma_1)$, and they determine multilinear operators on $\mathcal{J}_{k,\,\mathcal{M}}(\Gamma_1)$.

3. Rankin-Cohen type brackets for Siegel modular forms

We will demonstrate how Theorem 1 can be used to construct an explicit formula for Rankin-Cohen brackets of holomorphic functions $F:\mathbb{H}_n\to\mathbb{C}$ that satisfy $F\big|_k M = F$ for all $M\in\Gamma_{n,1}$ for arbitrary $n\ge 2$.

As before, $Z = \left(\begin{smallmatrix}\tau & z\\ {}^tz & W\end{smallmatrix}\right)\in\mathbb{H}_n$, where $\tau\in\mathbb{H}_1$, $z=(z_1,\ldots,z_{n-1})\in\mathbb{C}^{n-1}$, and $W=(w_{ij})\in\mathbb{H}_{n-1}$. Set

$$\mathbb{D} = \left|\begin{smallmatrix}2\partial_\tau & \partial_z\\ {}^t\partial_z & \partial_W\end{smallmatrix}\right|,$$

where $\partial_z = (\partial z_1,\ldots,\partial z_{n-1})$ and $\partial_W = (1+\delta_{ij})\frac{\partial}{\partial\, w_{ij}}$. Furthermore, let $|\partial_W|^{-1}$ be the inverse to the operator $|\partial_W|$ on the space of analytic functions of the form

$$G(W) = \sum_{\substack{\mathcal{M}={}^t\mathcal{M}>0\\ \mathcal{M}\,\mathrm{even}}} a(\mathcal{M})\,e^{\pi i\,tr(\mathcal{M}W)}. \tag{4}$$

More precisely, if $l\in\mathbb{Z}$, then

$$|\partial_W|^l\,G(W) = (2\pi i)^{(n-1)l}\sum_{\substack{\mathcal{M}={}^t\mathcal{M}>0\\ \mathcal{M}\,even}}|\mathcal{M}|^l a(\mathcal{M})\,e^{\pi i\,tr(\mathcal{M}W)}. \tag{5}$$

Note that $|\partial_W|\,|\partial_W|^{-1}\,G(W) = |\partial_W|^{-1}\,|\partial_W|\,G(W) = G(W)$. We have the following theorem.

Theorem 2. *Let* $F_l:\mathbb{H}_n\to\mathbb{C}$ *be holomorphic such that* $F_l\big|_{k_l}M = F_l$ *for all* $M\in\Gamma_{n,1}$ *where* $l=1,\,2$. *For each nonnegative integer* ν, *set*

$$F = [F_1,F_2]_\nu = \sum_{r+s+p=\nu} C_{r,\,s,\,p}(k_1,k_2)\,\mathbb{D}^p\left(|\partial_W|^{-p}\sum_{l=0}^{p}\binom{p}{l}\mathbb{D}^r(|\partial_W|^l F_1)\,\mathbb{D}^s(|\partial_W|^{p-l}F_2)\right), \tag{6}$$

where $C_{r,\,s,\,p}(k_1,k_2)$ *is as in Theorem 1. Then* $F\big|_{k_1+k_2+2\nu}M = F$ *for all* $M\in\Gamma_{n,1}$. *Moreover, if* $\nu>0$, *then* F *is in the kernel of the Siegel* ϕ*-operator.*

Proof. We begin by noting that

$$\sum_{l=0}^{p}\binom{p}{l}\mathbb{D}^r(|\partial_W|^l F_1(Z))\,\mathbb{D}^s(|\partial_W|^{p-l}F_2(Z))$$

has a Fourier series as in (4). Hence applying $|\partial_W|^{-p}$ in (6) is well defined. If $\nu=0$, then $[F_1,F_2]_0 = F_1F_2$. From now on we assume that $\nu>0$. For $l=1,\,2$ let

$$F_l(Z) = \sum_{\substack{T={}^tT\ge 0\\ T\,\mathrm{even}}} a_l(T)e^{\pi i\,tr(TZ)} = \sum_{\substack{\mathcal{M}={}^t\mathcal{M}\ge 0\\ \mathcal{M}\,\mathrm{even}}}\Phi^{(l)}_{\mathcal{M}}(\tau,z)e^{\pi i\,tr(\mathcal{M}W)}.$$

It is crucial to realize that if $\phi : \mathbb{H} \times \mathbb{C}^{n-1} \to \mathbb{C}$ is analytic, then

$$\mathbb{D}\left(\phi\, e^{\pi i\, tr(\mathcal{M}W)}\right) = (2\pi i)^{n-2} L_{\mathcal{M}}(\phi)\, e^{\pi i\, tr(\mathcal{M}W)}, \tag{7}$$

where $L_{\mathcal{M}}$ is the heat operator in (2). The key step in our proof is to use (7) to express $F = [F_1, F_2]_\nu$ as

$$F(Z) = \sum_{\substack{\mathcal{M}_1 = {}^t\mathcal{M}_1 \geq 0,\\ \mathcal{M}_2 = {}^t\mathcal{M}_2 \geq 0,\\ \mathcal{M}_1, \mathcal{M}_2 \text{ even}\\ \mathcal{M}_1 + \mathcal{M}_2 > 0}} \phi_{\mathcal{M}_1\,,\,\mathcal{M}_2}(\tau, z)\, e^{\pi i\, tr((\mathcal{M}_1 + \mathcal{M}_2)W)}, \tag{8}$$

where

$$\phi_{\mathcal{M}_1\,,\,\mathcal{M}_2}(\tau, z) = (2\pi i)^{(n-2)\nu} \left(\frac{|\mathcal{M}_1| + |\mathcal{M}_2|}{|\mathcal{M}_1 + \mathcal{M}_2|}\right)^\nu [[\phi^{(1)}_{\mathcal{M}_1}(\tau, z), \phi^{(2)}_{\mathcal{M}_2}(\tau, z)]]_{0,\,\nu}.$$

To verify that $F(Z)$ is indeed a modular form of weight $k_1 + k_2 + 2\nu$ on $\Gamma_{n,1}$, it suffices to check the behavior under modular transformations for a set of generators of $\Gamma_{n,1}$: For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, set

$$\overline{\gamma} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & I_{n-1} & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix} \in \Gamma_{n,1} \tag{9}$$

and for $\lambda = (\lambda_1, \ldots, \lambda_{n-1}), \mu = (\mu_1, \ldots, \mu_{n-1}) \in \mathbb{Z}^{n-1}$ set

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \mu \\ {}^t\lambda & I_{n-1} & {}^t\mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix} \in \Gamma_{n,1}. \tag{10}$$

By Theorem 1, $[[\phi^{(1)}_{\mathcal{M}_1}(\tau, z), \phi^{(2)}_{\mathcal{M}_2}(\tau, z)]]_{0,\,\nu} \in \mathcal{J}_{k_1+k_2+2\nu,\,\mathcal{M}_1+\mathcal{M}_2}(\Gamma_1)$. This implies that $F\big|_{k_1+k_2+2\nu}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = F$ and also that $F\big|_{k_1+k_2+2\nu}\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) = F$.

Observe that Γ_n is generated by

$$\left\{ \begin{pmatrix} {}^tX & {}^tXS \\ 0 & X^{-1} \end{pmatrix} \,\middle|\, X \in \mathrm{GL}_n(\mathbb{Z}),\, S = {}^tS \in \mathrm{M}_{n,n}(\mathbb{Z}) \right\} \cup \{\overline{\gamma} \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$$

and that $\mathrm{GL}_n(\mathbb{Z})$ is generated by matrices of the following type:

1. $\{(1 + r\delta_{ij})I_n \mid 1 \leq i < j \leq n,\, r \in \mathbb{Z}\}$.
2. $\{P_\sigma = \left(\delta_{j\sigma(j)}\right) \mid \sigma \in \mathbb{S}_n\}$.
3. $\left\{\left(\begin{smallmatrix} \epsilon & \\ & I_{n-1} \end{smallmatrix}\right) \,\middle|\, \epsilon = \pm 1\right\}$.

One can check that $\Gamma_{n,1}$ is generated by

$$\left\{ \begin{pmatrix} {}^tX & {}^tXS \\ 0 & X^{-1} \end{pmatrix} \,\middle|\, X \in \mathrm{GL}'_n(\mathbb{Z}),\, S = {}^tS \in \mathrm{M}_{n,n}(\mathbb{Z}) \right\} \cup \{\overline{\gamma} \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\},$$

where $\mathrm{GL}'_n(\mathbb{Z})$ is generated by matrices of the following type:

1. $\{(1 + r\delta_{ij})I_n \mid 1 \leq i < j \leq n,\, r \in \mathbb{Z}\}$.

2′. $\{P_\sigma = \left(\delta_{j\sigma(j)}\right) \mid \sigma \in \mathbb{S}_n,\, \sigma(1) = 1\}$.

3. $\left\{\left(\begin{smallmatrix} \epsilon & \\ & I_{n-1} \end{smallmatrix}\right) \,\middle|\, \epsilon = \pm 1\right\}$.

A straightforward computation reveals that

$$F(Z) = (2\pi i)^{n\nu} \sum_{r+s+p=\nu} C_{r,\,s,\,p}(k_1, k_2) \sum_{\substack{T={}^tT>0 \\ T\text{even}}} C_{r,\,s,\,p}(T) e^{\pi i\, tr(TZ)}, \tag{11}$$

where

$$C_{r,\,s,\,p}(T) = \sum_{\substack{T_1+T_2=T, \\ T_1={}^tT_1\geq 0, \\ T_2={}^tT_2\geq 0, \\ T_1,\,T_2 \text{ even}}} a_1(T_1)a_2(T_2)|T_1|^r|T_2|^s|T_1+T_2|^p \left(\frac{|T_1^*|+|T_2^*|}{|T_1^*+T_2^*|}\right)^p,$$

and where $T_l = \left(\begin{smallmatrix} * & * \\ * & T_l^* \end{smallmatrix}\right)$ with $T_l^* = {}^tT_l^* \in M_{n-1,n-1}(\mathbb{Z})$. Consequently, $F(Z+S) = F(Z)$ for all symmetric $S \in \mathrm{M}_{n,n}(\mathbb{Z})$.

Note that if X is a generator of $\mathrm{GL}'_n(\mathbb{Z})$, then $|(X^{-1}T\,{}^tX^{-1})^*| = |T^*|$ for all symmetric $T \in M_{n,n}(\mathbb{Z})$. Since

$$a_1(X^{-1}T_1\,{}^tX^{-1})a_2(X^{-1}T_2\,{}^tX^{-1}) = |X|^{k_1+k_2}a_1(T_1)a_2(T_2),$$

we find that $C_{r,\,s,\,p}(X^{-1}T\,{}^tX^{-1}) = |X|^{k_1+k_2}C_{r,\,s,\,p}(T)$. Hence by equation (11),

$$F({}^tXZX) = |X|^{k_1+k_2+2\nu}F(Z). \tag{12}$$

We conclude that $F\big|_{k_1+k_2+2\nu}M = F$ for all $M \in \Gamma_{n,1}$. Moreover, equation (11) shows that $F(Z)$ is in the kernel of the ϕ-operator. □

Remarks. 1) If $n = 2$, then $\frac{|T_1^*|+|T_2^*|}{|T_1^*+T_2^*|} = 1$ in equation (11), which yields that equation (12) is valid for all $X \in \mathrm{GL}_2(\mathbb{Z})$. Hence $F\big|_{k_1+k_2+2\nu}M = F$ for all $M \in \Gamma_2$, F differs only by a constant from the Rankin-Cohen bracket in [4], and Theorem 2 reduces to Theorem 1.4 of [4].

2) If $n > 2$, then $[F_1, F_2]_\nu$ is not necessarily a Siegel modular form on the full symplectic group Γ_n. For example, if $F_1 = F_2$ is the theta function associated to an even unimodular lattice, then $[F_1, F_2]_\nu$ is a theta function with polynomial coefficients. However, equation (11) implies that the polynomial is not harmonic, and hence $[F_1, F_2]_\nu$ is not a Siegel modular form on Γ_n.

3) Choie and Kim [5] also determine multilinear operators on $\mathcal{J}_{k,\,\mathcal{M}}(\Gamma_1)$. One can use their result to construct multilinear operators on Siegel modular forms of degree n. This allows one to generalize the main result in Choie [3] to the case where $n > 2$.

Next we proceed as in Freitag [9] to lift $[F_1, F_2]_\nu$ to a Siegel modular form on Γ_n. If $Z = X + iY \in \mathbb{H}_n$, then let z_1 and y_1 be the $(1,1)$-entries of Z and Y, respectively.

Theorem 3. *Let F_l be Siegel modular forms of weight k_l and degree n on Γ_n for $l = 1, 2$, and let f_3 be a modular cusp form of weight k_3 and degree 1 on Γ_1. For each nonnegative integer ν, set $F(Z) = [F_1, F_2]_\nu$ and $F_3(Z) = f_3(z_1)$. If $k_1 + k_2 + k_3 + 2\nu > 2n^2 - n + 6$, then*

$$P_{F_1,\,F_2,\,f_3}(Z) = \sum_{M\in\Gamma_{n,1}\backslash\Gamma_n} (FF_3)\Big|_{k_1+k_2+k_3+2\nu} M \tag{13}$$

is a Siegel modular form of weight $k_1 + k_2 + k_3 + 2\nu$ on Γ_n. Moreover, if $\nu > 0$, then $P_{F_1,\,F_2,\,f_3}(Z)$ is a Siegel cusp form on Γ_n.

Remarks. 1) If $n = 2$, $F\big|_{k_1+k_2+2\nu} M = F$ for all $M \in \Gamma_2$ and F differs only by a constant from the Rankin-Cohen bracket in [4]. Furthermore, $P_{F_1, F_2, f_3}(Z) = F(Z)E_{f_3}(Z)$, where $E_{f_3}(Z) = \sum_{M \in \Gamma_{2,1}\backslash\Gamma_2} F_3\Big|_{k_3} M$ is the Klingen-Eisenstein series attached to the cusp form f_3. For $n > 2$, as in the classical case of Poincaré series, it seems difficult to determine when the series in equation (13) vanishes identically.

2) If $n > 2$, then Theorem 3 does not coincide with the Rankin-Cohen bracket in [7]. However, it gives a construction of a Siegel cusp form of weight $k_1+k_2+k_3+2\nu$ attached to two Siegel cusp forms of weight k_1 and k_2 and an elliptic cusp form of weight k_3.

Proof. We have $F\big|_{k_1+k_2+2\nu} M = F$ and $F_3\big|_{k_3} M = F_3$ for all $M \in \Gamma_{n,1}$. Hence $P_{F_1, F_2, f_3}(Z)$ transforms like a Siegel modular form of weight $k = k_1 + k_2 + k_3 + 2\nu$ on Γ_n. It remains to show that $P_{F_1, F_2, f_3}(Z)$ converges.

If $\nu = 0$, then $P_{F_1, F_2, f_3}(Z) = F_1(Z)F_2(Z)E_{f_3}(Z)$ converges whenever $k_3 > n + 2$ (see I. 5.4 in [9]). For $\nu > 0$, set $G(Z) = y_1^{\left(\frac{k}{2}-N\right)}|Y|^N \|F(Z)\|$, where $(n-1)^2 + 1 < N < k/2$, and where $\|\cdot\|$ denotes the absolute value. We will show that $G(Z)$ is bounded on $\mathbb{H}_n$, which then yields the convergence of $P_{F_1, F_2, f_3}(Z)$. Note that $G\Big|_k M = G$ for all $M \in \Gamma_{n,1}$. There exists $u > 1$ such that $\mathcal{F}_{n,1}[u]$ is a fundamental set for $\Gamma_{n,1}$ (for details see I. §5 in [9]). Let $Z = X + iY \in \overline{\mathcal{F}_{n,1}[u]}$. Hence $Y = \left(\begin{smallmatrix} 1 & 0 \\ {}^t b & I_{n-1}\end{smallmatrix}\right)\left(\begin{smallmatrix} y_1 & 0 \\ 0 & Y_2\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & I_{n-1}\end{smallmatrix}\right)$, where $y_1 \geq \frac{1}{u}$, $b\,{}^t b \leq u$, and $Y_2 = {}^t Y_2 = \begin{pmatrix} y_{22} & \cdots & y_{2n} \\ \vdots & & \vdots \\ y_{2n} & \cdots & y_{nn}\end{pmatrix} > 0$ with $y_{jj} \leq u y_{j+1\,j+1}$ $(2 \leq j \leq n-1)$, $\|y_{ij}\| \leq u y_{jj}$ $(2 \leq i, j \leq n)$, and $y_{22} \cdots y_{nn} \leq u|Y_2|$.

We apply a standard estimate (Lemma 2.6 of [11]) to the Jacobi cusp form $\phi_{\mathcal{M}_1, \mathcal{M}_2}(\tau, z)$ in equation (8) and find

$$G(Z) \leq C y_1^{\frac{k_3}{2}} \|f_3(\tau)\|\, |Y_2|^N \sum_{\substack{\mathcal{M}={}^t\mathcal{M}>0 \\ \mathcal{M}\text{ even}}} e^{-\pi\, tr(\mathcal{M}Y_2)}, \tag{14}$$

for some $C > 0$. The right-hand side of (14) clearly vanishes when $y_1 \to \infty$ (f_3 is a cusp form) or when $y_{nn} \to \infty$. It remains to show that $G(Z)$ vanishes when $y_{22} \to 0$. Let $\tilde{Y}_2 = \begin{pmatrix} y_{22} & & \\ & \ddots & \\ & & y_{nn}\end{pmatrix}$. It is not difficult to see that there exists $\delta_n > 0$ such that $\delta_n \tilde{Y}_2 \leq Y_2 \leq \delta_n{}^{-1}\tilde{Y}_2$. Let

$$\sigma(n, l) = \#\{\mathcal{M} \,|\, \mathcal{M} \in M_{n-1,n-1}(\mathbb{Z}),\ {}^t\mathcal{M} = \mathcal{M} > 0, \mathcal{M}\text{ even }, tr(\mathcal{M}) = l\}.$$

Then

$$\begin{aligned} &|Y_2|^N \sum_{\substack{\mathcal{M}={}^t\mathcal{M}>0 \\ \mathcal{M}\text{ even}}} e^{-\pi\, tr(\mathcal{M}Y_2)} \\ &\quad \leq (\delta_n^{-n}\, y_{22}\cdots y_{nn})^N \sum_{\substack{\mathcal{M}={}^t\mathcal{M}>0 \\ \mathcal{M}\ even}} e^{-\pi\,\delta_n\, y_{22}\, tr(\mathcal{M})}, \\ &\quad = (\delta_n^{-n}\, y_{22}\cdots y_{nn})^N \sum_{l>0} \sigma(n, l) e^{-\pi\,\delta_n\, y_{22}\, l}, \\ &\quad \leq (\delta_n^{-n}\, y_{22}\cdots y_{nn})^N \sum_{l>0} l^{(n-1)^2} \left(e^{-\pi\,\delta_n\, y_{22}}\right)^l. \end{aligned} \tag{15}$$

If $N > (n-1)^2+1$, then the right side vanishes when $y_{22} \to 0$. Hence $G(Z)$ is bounded on $\overline{\mathcal{F}_{n,1}[u]}$ and consequently also on $\mathbb{H}_n$ (see also I. 3.11 in [9]).

Let $h(Z) = y_1^{-\left(\frac{k}{2}-N\right)}$ and $H(Z) = |Y|^{-N}$. Then

$$E_{hH}(Z) = \sum_{M \in \Gamma_{n,1}\backslash\Gamma_n} \left\| (hH)\Big|_k M \right\| = H(Z) \sum_{M \in \Gamma_{n,1}\backslash\Gamma_n} \left\| h\Big|_{k-2N} M \right\|$$

converges absolutely (see I. 5.4_1 in [9]) if $k-2N > n+2$. Hence $P_{F_1,F_2,f_3}(Z)$ converges if $k > 2n^2-n+6$. Finally, if $\nu > 0$, then it is easy to check that $P_{F_1,F_2,f_3}(Z)$ is in the kernel of the ϕ-operator (compare with p. 72 in [9]), i.e., $P_{F_1,F_2,f_3}(Z)$ is a Siegel cusp form. □

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