PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 701–708
S 0002-9939(99)05619-1
Article electronically published on September 27, 1999

THETA FUNCTIONS OF INDEFINITE QUADRATIC FORMS OVER REAL NUMBER FIELDS

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(Communicated by Dennis A. Hejhal)

ABSTRACT. We define theta functions attached to indefinite quadratic forms over real number fields and prove that these theta functions are Hilbert modular forms by regarding them as specializations of symplectic theta functions. The eighth root of unity which arises under modular transformations is determined explicitly.

1. INTRODUCTION

We construct a theta function attached to a quadratic form over a totally real number field and show that this theta function is a modular form. If the quadratic form is not totally positive, the usual sum over some ideal in the ring of integers will not converge. Stopple [8] solves this problem by introducing the analogue of a spherical harmonic for the theta function to ensure convergence. Furthermore, he follows Eichler [2] to show that his theta function is a modular form on some Γ_0^0 subgroups. Following Siegel [5] and [6], Friedberg [3] defines a theta function of indefinite quadratic forms over $\mathbb{C}$ by using the majorants of the quadratic forms to guarantee that the theta function will converge. By converting this theta function into a symplectic theta function, Friedberg proves that his theta function is indeed a modular form on some special subgroups of $SL_2(\mathbb{R})$. The advantage of his method is that one can compute the theta multiplier explicitly without too much effort using the main result of Stark [7]. We will follow his method and obtain results similar to his over totally real number fields, i.e. we will prove that the theta function we will construct is a Hilbert modular form on some Γ_0 subgroups and also compute the theta multiplier explicitly using the main result of [7].

2. SYMPLECTIC THETA FUNCTIONS

The symplectic group, $\mathrm{Sp}_n(\mathbb{R})$, consists of those $2n \times 2n$ real matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(each entry is $n \times n$) such that

$$^tMJM = J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Received by the editors April 29, 1998.
1991 *Mathematics Subject Classification.* Primary 11F41; Secondary 11F27.

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The corresponding symmetric space is the Siegel upper half plane $\mathfrak{H}^{(n)}$ which consists of $n\times n$ symmetric complex matrices Z with $\mathrm{Im}(Z)>0$ (positive definite). The action of M on Z is given by

$$M\circ Z=(AZ+B)(CZ+D)^{-1}.$$

Let $\Gamma^{(n)}=\mathrm{Sp}_n(\mathbb{Z})$. The theta subgroup $\Gamma_\vartheta^{(n)}$ of $\Gamma^{(n)}$ is the set of all $\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ in $\Gamma^{(n)}$ such that both $A\,{}^tB$ and $C\,{}^tD$ have even diagonal entries. The subgroup acts on the symplectic theta function,

$$\vartheta\left(Z,\begin{pmatrix}u\\v\end{pmatrix}\right)=\sum_{m\in\mathbb{Z}^n}\exp\left\{\pi i\left[{}^t(m+v)Z(m+v)-2\,{}^tmu-{}^tvu\right]\right\},$$

where u and v are column vectors in $\mathbb{C}^n$. It is well known (see Eichler [1], for example) that for

$$M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\ \text{in}\ \Gamma_\vartheta^{(n)},$$

$$\vartheta\left(M\circ Z,M\begin{pmatrix}u\\v\end{pmatrix}\right)=\chi(M)\left[\det(CZ+D)\right]^{1/2}\vartheta\left(Z,\begin{pmatrix}u\\v\end{pmatrix}\right)\tag{1}$$

where $\chi(M)$ is an eighth root of unity which depends upon the chosen square root of $\det(CZ+D)$, but which is otherwise independent of Z, u, and v. It is also known that $\chi(M)$ can be expressed in terms of Gaussian sums. Stark [7] determined $\chi(M)$ in the important special case that pD^{-1} is integral for some odd prime p. The main result in [7] is

Theorem 1. *Suppose $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ is in $\Gamma_\vartheta^{(n)}$ where C^{-1} and D^{-1} exist. Suppose further that for some odd prime p, pD^{-1} is integral. Then (mod p), the symmetric matrix $pD^{-1}C$ has rank h where $\det(D)=\pm p^h$. Let $\left(pD^{-1}C\right)^{(h)}$ be a nonsingular (mod p) $h\times h$ principal submatrix of $pD^{-1}C$ and let s be the signature (the number of positive eigenvalues minus the number of negative eigenvalues) of $C^{-1}D$. Then*

$$\begin{aligned}&\chi(M)\left[\det(CZ+D)\right]^{1/2}\\&=\varepsilon_p^{-h}\left(\frac{2^h\det\left[\left(pD^{-1}C\right)^{(h)}\right]}{p}\right)e^{\frac{\pi is}{4}}|\det(C)|^{1/2}\{\det[-iC^{-1}(CZ+D)]\}^{1/2},\end{aligned}$$

where $\varepsilon_p=1$ for $p\equiv1$ mod 4, $\varepsilon_p=i$ for $p\equiv3$ mod 4, $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, $|\det(C)|^{1/2}$ is positive and $\left\{\det\left[-iC^{-1}(CZ+D)\right]\right\}^{1/2}$ is given by analytic continuation from the principal value when $Z=-C^{-1}D+iY$. Alternatively, if just C^{-1} exists and pC^{-1} is integral, $\det(C)=\pm p^h$, then $pC^{-1}D$ (mod p) has rank h and

$$\begin{aligned}&\chi(M)\left[\det(CZ+D)\right]^{1/2}\\&=\varepsilon_p^{-h}\left(\frac{-2}{p}\right)^h\left(\frac{\det\left[\left(pC^{-1}D\right)^{(h)}\right]}{p}\right)|\det(C)|^{1/2}\{\det[-iC^{-1}(CZ+D)]\}^{1/2}.\end{aligned}$$

3. Theta functions attached to indefinite quadratic forms

Let K be a totally real number field of degree r_1. Let Δ_K be the discriminant of K, let δ_K be the different of K, and let $\mathfrak{O}_K$ be the ring of integers of K. The algebraic conjugates of an algebraic number α in K are given by $\alpha^{(1)}, \dots, \alpha^{(r_1)}$. Furthermore, let $\Gamma = SL_2(\mathfrak{O}_K)$ and as usual, for an integral ideal $\mathfrak{N}$, let

$$\Gamma_0(\mathfrak{N}) = \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\ M \in \Gamma \text{ and } \gamma \in \mathfrak{N} \right\}.$$

We define the upper half plane $\mathfrak{H} = \mathbb{H}^{r_1}$, where $\mathbb{H} = \{z \in \mathbb{C}\ ,\ \mathrm{Im} z > 0\}$ is the usual upper half plane. The matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$$

acts on $z = (z_1, \dots, z_{r_1}) \in \mathfrak{H}$ by

$$M \circ z = \left(M^{(1)} \circ z_1, \dots, M^{(r_1)} \circ z_{r_1} \right),$$

where

$$M^{(j)} = \begin{pmatrix} \alpha^{(j)} & \beta^{(j)} \\ \gamma^{(j)} & \delta^{(j)} \end{pmatrix}$$

and

$$M^{(j)} \circ z_j = \left(\alpha^{(j)} z_j + \beta^{(j)} \right) \left(\gamma^{(j)} z_j + \delta^{(j)} \right)^{-1}.$$

For γ and δ in K and z in $\mathfrak{H}$, we define

$$\mathcal{N}(\gamma z + \delta) = \prod_{j=1}^{r_1} \left(\gamma^{(j)} z_j + \delta^{(j)} \right)$$

and

$$\mathcal{N}\left[(\gamma z + \delta)^{1/2} \right] = \prod_{j=1}^{r_1} \left(\gamma^{(j)} z_j + \delta^{(j)} \right)^{1/2}, \tag{2}$$

where each of the r_1 square roots is given by the principal value.

Let Q be a symmetric $n \times n$ matrix defining the quadratic form $Q[x] = {}^t x Q x$, where $x \in \mathbb{R}^n$. If Q has entries in $\mathfrak{O}_K$ and diagonal entries which are divisible by 2, we say that Q is of level N ($N \in \mathfrak{O}_K$) whenever the following two conditions are satisfied:

a) The matrix NQ^{-1} has entries in $\mathfrak{O}_K$, and 2 divides the diagonal entries of NQ^{-1}.
b) For any $M \in \mathfrak{O}_K$, N divides M whenever MQ^{-1} has entries in $\mathfrak{O}_K$ and 2 divides the diagonal entries of MQ^{-1}.

If $Q^{(j)}$ has signature (p, q) for $j = 1, \dots, r_1$, then there exist matrices L_j in $GL_n(\mathbb{R})$ such that $Q^{(j)} = {}^t L_j E_{p,q} L_j$, where

$$E_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix},$$

and I_p and I_q are the $p \times p$ and $q \times q$ identity matrices, respectively.

Set $R_j = {}^t L_j L_j$. Then R_j is a majorant of $Q^{(j)}$, i.e.

$$R_j {Q^{(j)}}^{-1} R_j = Q^{(j)} \text{ and } {}^t R_j = R_j > 0.$$

For the vector $\lambda = {}^t(\lambda_1, \dots, \lambda_n)$, we set $\lambda^{(j)} = {}^t(\lambda_1^{(j)}, \dots, \lambda_n^{(j)})$ where $\lambda_1, \dots, \lambda_n$ are in K. We define the theta function Θ_Q of an indefinite quadratic form by

Definition 1. Let Q be a symmetric $n \times n$ matrix with entries in $\mathfrak{O}_K$ such that 2 divides the diagonal entries of Q and such that Q is of level N. Furthermore, assume that each $Q^{(j)}$ has the same signature (p, q) for $j = 1, \dots, r_1$. Let $u_1, \dots, u_{r_1}$ and $v_1, \dots, v_{r_1}$ be vectors in $\mathbb{C}^n$. For an ideal $\mathfrak{I} \subset \mathfrak{O}_K$ and $z = (z_1, \dots, z_{r_1}) \in \mathfrak{H}$ set

$$\Theta_Q\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \left(\prod_{j=1}^{r_1} y_j\right)^{q/2} \sum_{\lambda \in \mathfrak{I}^n} \exp\Bigg\{\pi i \Bigg[\sum_{j=1}^{r_1} Q^{(j)}[\lambda^{(j)} + v_j] x_j + i R_j[\lambda^{(j)} + v_j] y_j - 2\, {}^t\lambda^{(j)} Q^{(j)} u_j - {}^t v_j Q^{(j)} u_j\Bigg]\Bigg\}, \tag{3}$$

where $\lambda = {}^t(\lambda_1, \dots, \lambda_n)$, $u = {}^t({}^tu_1, \dots, {}^tu_{r_1})$ and $v = {}^t({}^tv_1, \dots, {}^tv_{r_1})$.

Note that for any algebraic integer $t \in K$, $\sum_{j=1}^{r_1} Q^{(j)}[\lambda^{(j)}]t^{(j)} = \mathrm{tr}\,(Q[\lambda]t)$ is an even rational integer, and thus $\Theta_Q(z)$ is invariant under linear transformations, i.e.

$$\Theta_Q\left(z + t, \begin{pmatrix} u + tv \\ v \end{pmatrix}\right) = \Theta_Q\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right). \tag{4}$$

The first task is to convert Θ_Q into a symplectic theta function. Let $\omega_1, \dots, \omega_{r_1}$ be an integral basis of the ideal $\mathfrak{I} \subset \mathfrak{O}_K$ and define the vector $\omega^{(j)} = (\omega_1^{(j)}, \dots, \omega_{r_1}^{(j)})$. We define the $n \times nr_1$ matrix

$$W_j = \begin{pmatrix} \omega^{(j)} & & \\ & \ddots & \\ & & \omega^{(j)} \end{pmatrix}$$

and the $nr_1 \times nr_1$ matrix $W = {}^t({}^tW_1, \dots, {}^tW_{r_1})$. Note that W^{-1} has entries in $\mathfrak{I}^{-1}\delta_K^{-1}$.

For $z = (z_1, \dots, z_{r_1}) \in \mathfrak{H}$, set

$$Z_j^* = \begin{pmatrix} z_j I_p & \\ & -\overline{z_j} I_q \end{pmatrix}$$

and

$$Z^* = \begin{pmatrix} Z_1^* & & \\ & \ddots & \\ & & Z_{r_1}^* \end{pmatrix},$$

where $\overline{z_j}$ is the complex conjugate of z_j. Define

$$L = \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_{r_1} \end{pmatrix}$$

and set

$$T = LW \text{ and } Z = {}^tTZ^*T = \begin{pmatrix} {}^tT & 0 \\ 0 & T^{-1} \end{pmatrix} \circ Z^*.$$

Observe that $\left(\begin{smallmatrix} {}^tT & 0 \\ 0 & T^{-1} \end{smallmatrix}\right)$ is in $\mathrm{Sp}_{nr_1}(\mathbb{R})$. Furthermore, Z^* is in the Siegel upper half plane $\mathfrak{H}^{(nr_1)}$ and therefore Z is in $\mathfrak{H}^{(nr_1)}$ as well. We have

$$\left(\prod_{j=1}^{r_1} y_j\right)^{-q/2} \Theta_Q\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \vartheta\left(Z, \begin{pmatrix} {}^tW\widetilde{Q}u \\ W^{-1}v \end{pmatrix}\right), \tag{5}$$

where

$$\widetilde{Q} = \begin{pmatrix} Q^{(1)} & & \\ & \ddots & \\ & & Q^{(r_1)} \end{pmatrix}.$$

In order to apply Theorem 1, we need a symplectic matrix which expresses the action of $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma = SL_2(\mathfrak{O}_k)$ on our new variables u, v and Z. For $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$, set

$$\begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} = \begin{pmatrix} \alpha^{(1)}I_n & & & \beta^{(1)}E_{p,q} & & \\ & \ddots & & & \ddots & \\ & & \alpha^{(r_1)}I_n & & & \beta^{(r_1)}E_{p,q} \\ \gamma^{(1)}E_{p,q} & & & \delta^{(1)}I_n & & \\ & \ddots & & & \ddots & \\ & & \gamma^{(r_1)}E_{p,q} & & & \delta^{(r_1)}I_n \end{pmatrix}. \tag{6}$$

It is easy to check that the diagrams

$$\begin{array}{ccc} z & \longrightarrow & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z \\ \big\downarrow & & \big\downarrow \\ Z^* & \longrightarrow & \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} \circ Z^* \end{array}$$

and

$$\begin{array}{ccc} \begin{pmatrix} u \\ v \end{pmatrix} & \longrightarrow & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\ \big\downarrow & & \big\downarrow \\ \begin{pmatrix} {}^tW\widetilde{Q}u \\ W^{-1}v \end{pmatrix} & \longrightarrow & \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} {}^tW\widetilde{Q}u \\ W^{-1}v \end{pmatrix} \end{array}$$

commute, where

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} {}^tT & 0 \\ 0 & T^{-1} \end{pmatrix}\begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix}\begin{pmatrix} {}^tT & 0 \\ 0 & T^{-1} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} {}^tTA^*\,{}^tT^{-1} & {}^tTB^*T \\ T^{-1}C^*\,{}^tT^{-1} & T^{-1}D^*T \end{pmatrix}. \end{aligned} \tag{7}$$

Hence

$$z \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z$$

in $\mathfrak{H}$ corresponds to

$$Z \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z$$

in $\mathfrak{H}^{(nr_1)}$.

Let us introduce some more notation to show conditions under which the matrix in (7) is in the theta subgroup. Assume that $S = (s_{il})_{i,l=1,\dots,r_1}$ and $R = (r_{km})_{k,m=1,\dots,n}$ are matrices with entries in K. We define the matrix $R \odot S = ((\mathrm{tr}(r_{km}s_{il})_{i,l=1,\dots,r_1}))_{k,m=1,\dots,n}$. Note that the entries of $R \odot S$ are rational numbers. Computation shows that $A = I_n \odot A'$, $B = Q \odot B'$, $C = Q^{-1} \odot C'$ and $D = I_n \odot D'$, where A', B', C' and D' are given by

$$A' = \begin{pmatrix} \omega_1\nu_1\alpha & \cdots & \omega_1\nu_{r_1}\alpha \\ \vdots & & \vdots \\ \omega_{r_1}\nu_1\alpha & \cdots & \omega_{r_1}\nu_{r_1}\alpha \end{pmatrix},$$

$$B' = \begin{pmatrix} \omega_1\omega_1\beta & \cdots & \omega_1\omega_{r_1}\beta \\ \vdots & & \vdots \\ \omega_{r_1}\omega_1\beta & \cdots & \omega_{r_1}\omega_{r_1}\beta \end{pmatrix},$$

$$C' = \begin{pmatrix} \nu_1\nu_1\gamma & \cdots & \nu_1\nu_{r_1}\gamma \\ \vdots & & \vdots \\ \nu_{r_1}\nu_1\gamma & \cdots & \nu_{r_1}\nu_{r_1}\gamma \end{pmatrix},$$

$$D' = \begin{pmatrix} \omega_1\nu_1\delta & \cdots & \omega_{r_1}\nu_1\delta \\ \vdots & & \vdots \\ \omega_1\nu_{r_1}\delta & \cdots & \omega_{r_1}\nu_{r_1}\delta \end{pmatrix}.$$

Clearly, ${}^tAC = {}^tCA$, ${}^tBD = {}^tDB$, and ${}^tDA - {}^tBC = I_{nr_1}$. Hence $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{nr_1}(\mathbb{R})$. Furthermore, $A\,{}^tB = {}^tTA^*B^*T = Q\odot(\alpha B')$ and $C\,{}^tD = T^{-1}C^*D^*\,{}^tT^{-1} = Q^{-1}\odot(\delta C')$. It follows that the entries of A, B, C and D are rational integers and that $A\,{}^tB$ and $C\,{}^tD$ have even diagonal entries if γ is in the ideal $\mathfrak{I}^2\delta_K N$. Hence for

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0\left(\mathfrak{I}^2\delta_K N\right),$$

we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\vartheta^{(nr_1)}.$$

It is easy to verify that

$$\det(CZ+D) = \det(C^*Z^*+D^*) = \mathcal{N}\left(\gamma z+\delta\right)^p \mathcal{N}\left(\gamma\overline{z}+\delta\right)^q,$$

and therefore by equations (1) and (5),

$$\begin{aligned} &\Theta_Q\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right) \\ (8)\qquad &= \chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right) \mathcal{N}\left(\gamma z+\delta\right)^{(p-q)/2} \Theta_Q\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right), \end{aligned}$$

where $\chi((\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}), Q)$ is an eighth root of unity depending on $(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix})$ and Q. Thus, $\Theta_Q(z, (\begin{smallmatrix}u \\ v\end{smallmatrix}))$ is a (nonanalytic) Hilbert modular form on $\Gamma_0\left(\mathfrak{I}^2\delta_K N\right)$ of weight $(p-q)/2$.

4. The eighth root of unity

Now we determine $\chi((\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}), Q)$. Let us assume that $\delta \gg 0$ is a first degree prime in $\mathfrak{O}_k$ of norm d, where d is a positive odd prime in $\mathbb{Z}$. In this case, dD^{-1} is integral. Note that $\det(D) = \det(D^*) = d^n$, and thus by Theorem 1, $dD^{-1}C$ has rank n (mod d). Hence for $Q^{-1} = (r_{il})_{i,l=1,\dots,n}$, we see that

$$\left(dD^{-1}C\right)^{(n)} = \begin{pmatrix} \operatorname{tr}\left(r_{11}\nu_1\nu_1 d\delta^{-1}\gamma\right) & \cdots & \operatorname{tr}\left(r_{1n}\nu_1\nu_1 d\delta^{-1}\gamma\right) \\ \vdots & & \vdots \\ \operatorname{tr}\left(r_{1n}\nu_1\nu_1 d\delta^{-1}\gamma\right) & \cdots & \operatorname{tr}\left(r_{nn}\nu_1\nu_1 d\delta^{-1}\gamma\right) \end{pmatrix}$$

and

$$\det\left(dD^{-1}C\right)^{(n)} \equiv (d\delta^{-1}\gamma)^n(\nu_1\nu_1)^n \det(Q)^{-1} \pmod{\delta}.$$

Some computation shows that

$$|\det(C)|^{1/2}\{\det[-iC^{-1}(CZ+D)]\}^{1/2} e^{\frac{\pi i s}{4}} = \mathcal{N}(\gamma z+\delta)^{(p-q)/2}\,|\mathcal{N}(\gamma z+\delta)|^q.$$

Hence

$$\chi\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, Q\right) = \varepsilon_d^{-n}\left(\frac{(d\delta^{-1}2\gamma)^n \det(Q)}{\delta}\right). \tag{9}$$

We have proved

Theorem 2. *Suppose that* $(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}) \in \Gamma_0\left(\mathfrak{I}^2\delta_K N\right)$*, where* δ *is a first degree prime in* $\mathfrak{O}_K$ *of norm* d *(*d *is a positive odd prime in* $\mathbb{Z}$*). For* $z \in \mathfrak{H}$*, we have*

$$\begin{aligned} &\Theta_Q\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \circ z, \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}u \\ v\end{pmatrix}\right) \\ &= \varepsilon_d^{-n}\left(\tfrac{(d\delta^{-1}2\gamma)^n \det(Q)}{\delta}\right)\mathcal{N}(\gamma z+\delta)^{(p-q)/2}\,\Theta_Q\left(z, \begin{pmatrix}u \\ v\end{pmatrix}\right), \end{aligned} \tag{10}$$

where $\varepsilon_d = 1$ *for* $d \equiv 1$ *mod* 4 *and* $\varepsilon_d = i$ *for* $d \equiv 3$ *mod* 4.

Actually, we have determined the eighth root of unity more explicitly than it seems. In (4), we showed that for all algebraic integers t,

$$\Theta_Q\left(\begin{pmatrix}1 & t \\ 0 & 1\end{pmatrix} \circ z, \begin{pmatrix}1 & t \\ 0 & 1\end{pmatrix}\begin{pmatrix}u \\ v\end{pmatrix}\right) = \Theta_Q\left(z, \begin{pmatrix}u \\ v\end{pmatrix}\right).$$

Together with (8) this implies that for $(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}) \in \Gamma_0\left(\mathfrak{I}^2\delta_K N\right)$ and for all algebraic integers t,

$$\chi\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & t \\ 0 & 1\end{pmatrix}, Q\right) = \chi\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, Q\right). \tag{11}$$

Furthermore, Hecke [4] gives a proof of Dirichlet's primes in progression theorem for number fields. Hence for algebraic integers γ and δ with $(\gamma, \delta) = 1$, the arithmetic progression $\{\gamma t+\delta\}_{t\in\mathfrak{O}_k}$ contains infinitely many primes $\pi \gg 0$ such that $\mathcal{N}(\pi)$ is a positive odd prime in $\mathbb{Z}$. Hence the theta multiplier is determined explicitly after locating a totally positive first degree prime in the arithmetic progression $\{\gamma t+\delta\}_{t\in\mathfrak{O}_k}$.

There is a special case which also should be mentioned. Let $\delta \gg 0$ be a prime in $\mathfrak{O}_K$ with $\mathcal{N}(\delta) = d^{r_1}$, where d is a positive odd prime in $\mathbb{Z}$. As before, we observe that dD^{-1} has rational integers as entries and hence D is of level d. We see that $\det(D) = \det(D^*) = d^{nr_1}$ and by Theorem 1, $dD^{-1}C$ has rank $nr_1 \pmod{d}$. Thus,

$$\det\left(dD^{-1}C\right) = (\mathcal{N}(\det(Q)))^{-1}\left(\Delta_K\left(\mathcal{N}(\mathcal{I})^2\right)\right)^{-n}(\mathcal{N}(\gamma^n)).$$

Hence

$$\chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, Q\right) = \varepsilon_d^{-nr_1}\left(\frac{(\Delta_K)^n\mathcal{N}\left((2\gamma)^n\det(Q)\right)}{d}\right). \tag{12}$$

In the case that n is even, we should see that (9) and (12) yield the same result. For an odd rational prime d, an element a is a square in $\mathbb{F}_{d^{r_1}}$ (the field of d^{r_1} elements) iff $\mathcal{N}_{\mathbb{F}_{d^{r_1}}/\mathbb{F}_d}(a)$ is a square in $\mathbb{F}_d$ (the field of d elements). This can be seen by observing that the mapping $N : \mathbb{F}^*_{d^{r_1}} \to \mathbb{F}^*_d$ given by $a \to N(a) := \mathcal{N}_{\mathbb{F}_{d^{r_1}}/\mathbb{F}_d}(a)$ is an epimorphism. Hence in the special case that n is even, (9) becomes

$$\chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, Q\right) = \left(\frac{(-1)^{n/2}\det(Q)}{\delta}\right) \tag{13}$$

and (12) becomes

$$\chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, Q\right) = \left(\frac{\mathcal{N}\left((-1)^{n/2}\det(Q)\right)}{d}\right), \tag{14}$$

and the result from (9) coincides with the result from (12).

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