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Differential operators on Hilbert modular forms

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Abstract

We investigate differential operators and their compatibility with subgroups of $\mathrm{SL}_2(\mathbb{R})^n$. In particular, we construct Rankin–Cohen brackets for Hilbert modular forms, and more generally, multilinear differential operators on the space of Hilbert modular forms. As an application, we explicitly determine the Rankin–Cohen bracket of a Hilbert–Eisenstein series and an arbitrary Hilbert modular form. We use this result to compute the Petersson inner product of such a bracket and a Hilbert modular cusp form.

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1. Introduction

It is well known how to obtain modular forms from the derivative of N modular forms. In 1956, Rankin [14] discussed this in detail for $N = 1$ and in 1975, Cohen [7] explored the case $N = 2$ and he constructed a bilinear operator to obtain modular forms. Later, Zagier [19] investigated algebraic properties of this operator and called it the Rankin–Cohen bracket. Rankin–Cohen brackets have been studied for elliptic modular forms, Jacobi forms, and Siegel modular forms. For more details, see [1–5,10,19].

In this paper, we use differential operators to construct various maps between spaces of holomorphic functions which are equivariant under the action of subgroups of $\mathrm{SL}_2(\mathbb{R})^n$. Our results

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include a construction of Hilbert modular forms, which then allows us to define Rankin–Cohen brackets for Hilbert modular forms, and furthermore, multilinear differential operators on the space of Hilbert modular forms. In the last section, we give an explicit application: We compute the Petersson inner product of a Rankin–Cohen bracket against a Hilbert modular cusp form. We proceed as in Zagier [18] and Choie and Kohnen [6], where such an application is presented for elliptic modular forms and Jacobi forms, respectively.

2. Hilbert modular forms and Jacobi-like forms of several variables

In this section, we briefly discuss modular forms and Jacobi-like forms of several variables. Let $\mathbb{H} \subset \mathbb{C}$ be the usual complex upper half plane. For variables $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)\right) \in \mathrm{SL}_2(\mathbb{R})^n$, define the actions

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \tau = \left(\frac{a_1\tau_1 + b_1}{c_1\tau_1 + d_1}, \ldots, \frac{a_n\tau_n + b_n}{c_n\tau_n + d_n}\right) \tag{1}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (\tau, z) = \left(\frac{a_1\tau_1 + b_1}{c_1\tau_1 + d_1}, \ldots, \frac{a_n\tau_n + b_n}{c_n\tau_n + d_n}, \frac{z_1}{c_1\tau_1 + d_1}, \ldots, \frac{z_n}{c_n\tau_n + d_n}\right). \tag{2}$$

The trace and the norm of an element $\alpha \in \mathbb{C}^n$ are given by the sum and by the product of its components, respectively. More generally, if $c = (c_1, \ldots, c_n)$, $d = (d_1, \ldots, d_n)$, $\mathfrak{k} = (k_1, \ldots, k_n)$, and $\mathfrak{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$, then the norm and trace are given by

$$\mathcal{N}(c\tau + d)^{\mathfrak{k}} = \prod_{j=1}^{n} (c_j\tau_j + d_j)^{k_j} \tag{3}$$

and

$$\operatorname{tr}\left(\mathfrak{m}\frac{cz^2}{c\tau + c}\right) = \sum_{j=1}^{n} m_j \frac{c_j z_j^2}{c_j\tau_j + d_j}. \tag{4}$$

We define the following two slash operators. For functions $f : \mathbb{H}^n \to \mathbb{C}$, for fixed $\mathfrak{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, and for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})^n$, set

$$\left(f\Big|_{\mathfrak{k}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau) = \mathcal{N}(c\tau + d)^{-\mathfrak{k}} f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \tau\right). \tag{5}$$

Furthermore, for functions $\phi : \mathbb{H}^n \times \mathbb{C}^n \to \mathbb{C}$, for fixed $\mathfrak{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ and $\mathfrak{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$, and for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})^n$, set

$$\begin{aligned}
&\left(\phi\Big|_{\mathfrak{k},\mathfrak{m}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau, z) \\
&\quad = \mathcal{N}(c\tau + d)^{-\mathfrak{k}} \exp\left\{-2\pi i \operatorname{tr}\left(\mathfrak{m}\frac{\gamma z^2}{c\tau + d}\right)\right\}\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (\tau, z)\right).
\end{aligned} \tag{6}$$

The slash operators (5) and (6) define (for each $\mathfrak{k}, \mathfrak{m} \in \mathbb{N}_0^n$) a group action of $\mathrm{SL}_2(\mathbb{R})^n$ on the set of functions $f: \mathbb{H}^n \to \mathbb{C}$ and on the set of functions $\phi: \mathbb{H}^n \times \mathbb{C}^n \to \mathbb{C}$, respectively. We write $\mathcal{M}_{\mathfrak{k}} = \{f: \mathbb{H}^n \to \mathbb{C} \text{ holomorphic}\}$ and $\mathcal{J}_{\mathfrak{k},\mathfrak{m}} = \{\phi: \mathbb{H}^n \times \mathbb{C}^n \to \mathbb{C} \text{ holomorphic}\}$ to indicate the specific group actions given in (5) and (6), which depend on fixed $\mathfrak{k}, \mathfrak{m} \in \mathbb{N}_0^n$.

Special consideration belongs to the invariant elements of $\mathcal{M}_{\mathfrak{k}}$ and $\mathcal{J}_{\mathfrak{k},\mathfrak{m}}$. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{R})^n$. A modular form $f: \mathbb{H}^n \to \mathbb{C}$ of weight $\mathfrak{k} \in \mathbb{N}_0^n$ with respect to Γ is an element of $\mathcal{M}_{\mathfrak{k}}$ invariant under the action of Γ, i.e., for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$\left(f\bigg|_{\mathfrak{k}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau) = f(\tau). \tag{7}$$

A Jacobi-like form $\phi: \mathbb{H}^n \times \mathbb{C}^n \to \mathbb{C}$ of weight $\mathfrak{k} \in \mathbb{N}_0^n$ and index $\mathfrak{m} \in \mathbb{N}_0^n$ on Γ is an element of $\mathcal{J}_{\mathfrak{k},\mathfrak{m}}$ invariant under the action of Γ, i.e., for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

$$\left(\phi\bigg|_{\mathfrak{k},\mathfrak{m}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau, z) = \phi(\tau, z). \tag{8}$$

Jacobi-like forms where $n = 1$ were introduced by Zagier [19] and Cohen, Manin, and Zagier [9]. Note that Jacobi forms satisfy (8) and also an elliptic transformation law (see [10,17]). For our purposes, it will suffice to consider only Jacobi-like forms.

We denote the vector space of holomorphic modular forms of weight $\mathfrak{k}$ on Γ by $\mathcal{M}_{\mathfrak{k}}(\Gamma)$ and the vector space of holomorphic Jacobi-like forms of weight $\mathfrak{k}$ and index $\mathfrak{m}$ on Γ by $\mathcal{J}_{\mathfrak{k},\mathfrak{m}}(\Gamma)$. Of particular interest is the case when Γ is a subgroup of finite index of $\mathrm{SL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a totally real number field K of degree n, and where $\mathrm{SL}_2(\mathcal{O}_K)$ is embedded into $\mathrm{SL}_2(\mathbb{R})^n$ using the n different embeddings of K into $\mathbb{R}$. In that case, the elements of $\mathcal{M}_{\mathfrak{k}}(\Gamma)$ are called *Hilbert modular forms* (see [11,12]).

3. Taylor expansions of Jacobi-like forms

In this section we investigate Taylor coefficients of functions $\phi(\tau, z)$ around $z = 0$ using ideas from §3 of Eichler and Zagier [10]. We use linear combinations of these Taylor coefficients to define an equivariant map from $\mathcal{J}_{\mathfrak{k},\mathfrak{m}}$ to $\mathcal{M}_{\mathfrak{k}+\mathfrak{v}}$, which provides a construction of Hilbert modular forms. In addition, we construct equivariant maps which yield (see Section 4) Rankin–Cohen brackets and multilinear operators on the space of Hilbert modular forms.

As before, let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{R})^n$. We use standard notation with multi-indices: If $\mathfrak{v} = (v_1, \ldots, v_n) \in \mathbb{N}_0^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then

$$|\mathfrak{v}| = \sum_{j=1}^n v_j, \qquad \mathfrak{v}! = \prod_{j=1}^n v_j! \quad \text{and} \quad z^{\mathfrak{v}} = \prod_{j=1}^n z_j^{v_j}.$$

Furthermore, for an integer $x \in \mathbb{N}_0$, we set $\vec{x} = (x, \ldots, x) \in \mathbb{N}_0^n$ and with a slight abuse of notation, we write $\chi^{(\mathfrak{l})}(\tau) = \frac{\partial^{|\mathfrak{l}|}}{\partial\tau_1^{l_1}\cdots\partial\tau_n^{l_n}}\chi(\tau)$, where $\chi: \mathbb{H}^n \to \mathbb{C}$ is holomorphic. We have the following theorem:

Theorem 1. *Let* $\phi(\tau, z) = \sum_{|\mathfrak{v}| \geqslant 0} \chi_{\mathfrak{v}}(\tau) z^{\mathfrak{v}}$ *be holomorphic and set*

$$\xi_{\mathfrak{v}}[\phi](\tau) = \sum_{\substack{\mathfrak{l} \\ 0\leqslant l_j\leqslant \frac{v_j}{2}}} \frac{(-2\pi i \mathfrak{m})^{\mathfrak{l}}(\mathfrak{k}+\mathfrak{v}-\vec{2}-\mathfrak{l})!}{\mathfrak{l}!(\mathfrak{k}+\mathfrak{v}-\vec{2})!}\chi^{(\mathfrak{l})}_{\mathfrak{v}-2\mathfrak{l}}(\tau), \tag{9}$$

where the sum on the right side is over all vectors $\mathfrak{l}=(l_1,\dots,l_n)\in\mathbb{N}_0^n$ *such that* $0\leqslant l_j\leqslant v_j/2$ *for* $j=1,\dots,n$. *Then for all* $M\in\mathrm{SL}_2(\mathbb{R})^n$,

$$\xi_{\mathfrak{v}}[\phi]\big|_{\mathfrak{k}+\mathfrak{v}}M = \xi_{\mathfrak{v}}\big[\phi\big|_{\mathfrak{k},\mathfrak{m}}M\big],$$

i.e., the map $\xi_{\mathfrak{v}}:\mathcal{J}_{\mathfrak{k},\mathfrak{m}}\to\mathcal{M}_{\mathfrak{k}+\mathfrak{v}}$ *is equivariant. In particular, if* $\phi\in\mathcal{J}_{\mathfrak{k},\mathfrak{m}}(\Gamma)$, *then* $\xi_{\mathfrak{v}}[\phi]\in\mathcal{M}_{\mathfrak{k}+\mathfrak{v}}(\Gamma)$.

Proof. We follow the proof of Theorem 3.2 in Eichler and Zagier [10]. Let $\mathcal{J}_{\mathfrak{k},\mathfrak{m}} = \mathcal{J}^+_{\mathfrak{k},\mathfrak{m}}\oplus\mathcal{J}^-_{\mathfrak{k},\mathfrak{m}}$, where $\mathcal{J}^+_{\mathfrak{k},\mathfrak{m}}$ is the subset of $\mathcal{J}_{\mathfrak{k},\mathfrak{m}}$ consisting of all functions $\phi(\tau,z)$ whose Taylor series expansion around $z=0$ is only over multi-indices of the form $2\mathfrak{v}$. If $\phi^-(\tau,z)\in\mathcal{J}^-_{\mathfrak{k},\mathfrak{m}}$, then $\phi^-(\tau,z)=z^{\mathfrak{h}}\phi_1(\tau,z)$, where $\mathfrak{h}\neq 0$ is a vector of zeros and ones and $\phi_1(\tau,z)\in\mathcal{J}^+_{\mathfrak{k}+\mathfrak{h},\mathfrak{m}}$. Note that the corresponding functions $\chi_{\mathfrak{v}}(\tau)$ and $\xi_{\mathfrak{v}}[\phi](\tau)$ for $\phi^-(\tau,z)$ and $\phi_1(\tau,z)$ are the same except for the shift $\mathfrak{k}\to\mathfrak{k}+\mathfrak{h}$ and $\mathfrak{v}\to\mathfrak{v}-\mathfrak{h}$. Thus, it will suffice to consider $\mathcal{J}^+_{\mathfrak{k},\mathfrak{m}}$ and deduce that $\xi_{2\mathfrak{v}}[\phi]\big|_{\mathfrak{k}+2\mathfrak{v}}M = \xi_{2\mathfrak{v}}\big[\phi\big|_{\mathfrak{k},\mathfrak{m}}M\big]$.

For $1\leqslant j\leqslant n$, set $e_j=(0,\dots,0,1,0,\dots,0)\in\mathbb{N}_0^n$ and define the differential operator

$$L^{(j)}_{k_j,\mathfrak{m}^{(j)}} = 8\pi i\mathfrak{m}^{(j)}\frac{\partial}{\partial\tau_j} - \frac{\partial^2}{\partial z_j^2} - \frac{2k_j-1}{z_j}\frac{\partial}{\partial z_j},$$

where $\mathfrak{k}=(k_1,\dots,k_n)$. Then (as in [10])

$$L^{(j)}_{k_j,\mathfrak{m}^{(j)}}\big(\phi\big|_{\mathfrak{k},\mathfrak{m}}M\big) = L^{(j)}_{k_j,\mathfrak{m}^{(j)}}(\phi)\big|_{\mathfrak{k}+2e_j,\mathfrak{m}}M \tag{10}$$

for all $M\in\mathrm{SL}_2(\mathbb{R})^n$, which shows that $L^{(j)}_{k_j,\mathfrak{m}^{(j)}}:\mathcal{J}^+_{\mathfrak{k},\mathfrak{m}}\to\mathcal{J}^+_{\mathfrak{k}+2e_j,\mathfrak{m}}$.

Note that $\mathfrak{v}=(v_1,\dots,v_n)=\sum_{j=1}^n v_je_j$. As in [10], one can construct a map $L_{\mathfrak{k},\mathfrak{m},\mathfrak{v}}:\mathcal{J}^+_{\mathfrak{k},\mathfrak{m}}\to\mathcal{J}^+_{\mathfrak{k}+2\mathfrak{v},\mathfrak{m}}$ that sends

$$\sum_{|\mathfrak{w}|\geqslant 0}\chi_{\mathfrak{w}}(\tau)z^{2\mathfrak{w}}$$

to

$$\sum_{|\mathfrak{w}|\geqslant 0}\Bigg(\sum_{\substack{\mathfrak{l} \\ 0\leqslant l_j\leqslant v_j}}(8\pi i\mathfrak{m})^{\mathfrak{l}}(-4)^{|\mathfrak{v}-\mathfrak{l}|}\binom{\mathfrak{v}}{\mathfrak{l}}\frac{(\mathfrak{w}+\mathfrak{v}-\mathfrak{l})!}{\mathfrak{w}!}\frac{(\mathfrak{w}+\mathfrak{k}+2\mathfrak{v}-\vec{2}-\mathfrak{l})!}{(\mathfrak{w}+\mathfrak{k}+\mathfrak{v}-\vec{2})!}\chi^{(\mathfrak{l})}_{\mathfrak{w}+\mathfrak{v}-\mathfrak{l}}(\tau)\Bigg)z^{2\mathfrak{w}}.$$

Composing with the map $\phi(\tau,z)\to\phi(\tau,0)$ yields

$$\xi_{2\mathfrak{v}}[\phi]\big|_{\mathfrak{k}+2\mathfrak{v}}M = \xi_{2\mathfrak{v}}\big[\phi\big|_{\mathfrak{k},\mathfrak{m}}M\big],$$

which completes the proof. □

Remark. In [10, §7], Eichler and Zagier show that Jacobi theta functions are Jacobi forms. The "2υth development coefficient" is (up to a constant factor) the analog of the $\xi_{2\mathfrak{v}}[\phi](\tau)$. In the case of a Jacobi theta function, one can show that this coefficient is given by a theta function with harmonic coefficients. In [16], the authors use Jacobi theta functions over a number field K to construct Jacobi forms over K. When K is totally real, then one finds that the corresponding $\xi_{2\mathfrak{v}}[\phi](\tau)$ are (up to a constant factor) given by theta functions with harmonic coefficients over K, which are indeed Hilbert modular forms (see [15] for more details).

Note that (9) can be inverted. We find that

$$\chi_{\mathfrak{v}}(\tau) = \sum_{\substack{\mathfrak{l} \\ 0 \leqslant l_j \leqslant \frac{v_j}{2}}} \frac{(2\pi i \mathfrak{m})^{\mathfrak{l}}(\mathfrak{k}+\mathfrak{v}-\vec{1}-2\mathfrak{l})!}{\mathfrak{l}!(\mathfrak{k}+\mathfrak{v}-\vec{1}-\mathfrak{l})!}\xi_{\mathfrak{v}-2\mathfrak{l}}[\phi]^{(\mathfrak{l})}(\tau). \tag{11}$$

We choose $\xi_0[\phi](\tau) = f(\tau)$ and $\xi_{\mathfrak{v}}[\phi](\tau) = 0$ for $\mathfrak{v} \neq 0$, and we extend a result by Cohen and Kuznetsov (see, for example, [10, Theorem 3.3]):

Theorem 2. *For a holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ and for $\mathfrak{k}, \mathfrak{m} \in \mathbb{N}_0^n$, set*

$$\tilde{f}(\tau, z) = \tilde{f}(\mathfrak{k}, \mathfrak{m}, \tau, z) = \sum_{|\mathfrak{v}| \geqslant 0} \frac{(2\pi i \mathfrak{m})^{\mathfrak{v}}(\mathfrak{k}-\vec{1})!}{\mathfrak{v}!(\mathfrak{k}+\mathfrak{v}-\vec{1})!} f^{(\mathfrak{v})}(\tau) z^{2\mathfrak{v}}. \tag{12}$$

Then for all $M \in \mathrm{SL}_2(\mathbb{R})^n$,

$$\widetilde{\left(f\big|_{\mathfrak{k}} M\right)} = \tilde{f}\big|_{\mathfrak{k},\mathfrak{m}} M, \tag{13}$$

i.e., the map $\sim : \mathcal{M}_{\mathfrak{k}} \to \mathcal{J}_{\mathfrak{k},\mathfrak{m}}$ is equivariant. In particular, if $f \in \mathcal{M}_{\mathfrak{k}}(\Gamma)$, then $\tilde{f} \in \mathcal{J}_{\mathfrak{k},\mathfrak{m}}(\Gamma)$.

Remarks.

(a) Note that Theorem 2 also follows directly from the following identity:

$$f^{(\mathfrak{v})}\big|_{\mathfrak{k}+2\mathfrak{v}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{\substack{\mathfrak{l} \\ 0 \leqslant l_j \leqslant v_j}} \binom{\mathfrak{v}}{\mathfrak{l}} \left(\frac{c}{c\tau+d}\right)^{\mathfrak{v}-\mathfrak{l}} \frac{(\mathfrak{k}+\mathfrak{v}-\vec{1})!}{(\mathfrak{k}+\mathfrak{l}-\vec{1})!} \left(f\bigg|_{\mathfrak{k}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{(\mathfrak{l})}, \tag{14}$$

which is easily proved by an induction on $|\mathfrak{v}|$ (as before, $\mathfrak{v} = (v_1, \ldots, v_n)$).

(b) We should also point out that both Theorems 1 and 2 can be deduced from the classical $n = 1$ case (see [10, Theorems 3.2 and 3.3]): Write an arbitrary $M = (M_1, \ldots, M_n) \in \mathrm{SL}_2(\mathbb{R})^n$ as a product of the (commutative) elements $(1, \ldots, 1, M_j, 1, \ldots, 1)$ and apply the $n = 1$ case to each pair of variables (τ_j, z_j) separately.

4. Generalized Rankin–Cohen brackets

Theorems 1 and 2 can be combined to give various types of brackets of Hilbert modular forms. The first one is a direct generalization of the classical Rankin–Cohen bracket to the Hilbert modular case.

Corollary 1. *Let $f_r:\mathbb{H}^n \to \mathbb{C}$ be holomorphic and $\mathfrak{k}_r \in \mathbb{N}_0^n$ for $r = 1, 2$. For all $\mathfrak{v} = (v_1, \ldots, v_n) \in \mathbb{N}_0^n$, define the Rankin–Cohen bracket*

$$[f_1, f_2]_{\mathfrak{v}} = \sum_{\substack{\mathfrak{l} \\ 0\leqslant l_j\leqslant v_j}} (-1)^{|\mathfrak{l}|} \binom{\mathfrak{k}_1+\mathfrak{v}-\vec{1}}{\mathfrak{v}-\mathfrak{l}} \binom{\mathfrak{k}_2+\mathfrak{v}-\vec{1}}{\mathfrak{l}} f_1^{(\mathfrak{l})}(\tau) f_2^{(\mathfrak{v}-\mathfrak{l})}(\tau). \tag{15}$$

Then for all $M \in \mathrm{SL}_2(\mathbb{R})^n$,

$$\bigl[\bigl(f_1\big|_{\mathfrak{k}_1}M\bigr), \bigl(f_2\big|_{\mathfrak{k}_2}M\bigr)\bigr]_{\mathfrak{v}} = [f_1, f_2]_{\mathfrak{v}}\big|_{\mathfrak{k}_1+\mathfrak{k}_2+2\mathfrak{v}}M, \tag{16}$$

i.e., the map $[\cdot,\cdot]_{\mathfrak{v}}:\mathcal{M}_{\mathfrak{k}_1} \otimes \mathcal{M}_{\mathfrak{k}_2} \to \mathcal{M}_{\mathfrak{k}_1+\mathfrak{k}_2+2\mathfrak{v}}$ is equivariant. In particular, if Γ is a subgroup of $\mathrm{SL}_2(\mathbb{R})^n$ and $f_r \in \mathcal{M}_{\mathfrak{k}_r}(\Gamma)$ for $r = 1, 2$, then $[f_1, f_2]_{\mathfrak{v}} \in \mathcal{M}_{\mathfrak{k}_1+\mathfrak{k}_2+2\mathfrak{v}}(\Gamma)$.

Proof. Compute the coefficient of $z^{2\mathfrak{v}}$ in $\tilde{f}_1(\mathfrak{k}_1, \mathfrak{m}, \tau, z)\tilde{f}_2(\mathfrak{k}_2, \mathfrak{m}, \tau, iz)$. $\square$

Remarks.

(a) As examples, if $n = 2$, $\mathfrak{k}_1 = (k_1, \bar{k}_1)$, and $\mathfrak{k}_2 = (k_2, \bar{k}_2)$, then we have

$$[f_1, f_2]_{(0,0)} = f_1 f_2, \qquad [f_1, f_2]_{(1,0)} = k_1 f_1 \frac{\partial f_2}{\partial \tau_1} - k_2 \frac{\partial f_1}{\partial \tau_1} f_2,$$

$$[f_1, f_2]_{(0,1)} = \bar{k}_1 f_1 \frac{\partial f_2}{\partial \tau_2} - \bar{k}_2 \frac{\partial f_1}{\partial \tau_2} f_2,$$

$$[f_1, f_2]_{(1,1)} = k_1\bar{k}_1 f_1 \frac{\partial^2 f_2}{\partial \tau_1 \partial \tau_2} - k_1\bar{k}_2 \frac{\partial f_1}{\partial \tau_2}\frac{\partial f_2}{\partial \tau_1} - \bar{k}_1 k_2 \frac{\partial f_1}{\partial \tau_1}\frac{\partial f_2}{\partial \tau_2} + k_2\bar{k}_2 \frac{\partial^2 f_1}{\partial \tau_1 \partial \tau_2} f_2, \quad \text{etc.}$$

(b) It is well known that the vector space of Hilbert modular forms has the structure of a commutative graded ring, which corresponds to the 0th bracket. Note that $[\cdot,\cdot]_{e_j}$, where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^n$ satisfies the Jacobi identity, which gives the vector space of Hilbert modular forms the structure of a graded Lie algebra in n different ways (but with the grading shifted by a different amount for each structure).

(c) Note that Lee [13] discovered Corollary 1 independently in the case where Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})^n$ and $f_r \in \mathcal{M}_{\mathfrak{k}_r}(\Gamma)$ for $r = 1, 2$.

The next corollary, which is more general than Corollary 1, is an application of Theorems 1 and 2 and shows how to construct multilinear operators on the space of Hilbert modular forms, depending on a number of auxiliary parameters. It seems to be new even if $n = 1$.

Corollary 2. *Let* $\mathfrak{a} = (a_1, \ldots, a_s) \in \mathbb{C}^s$, $\mathfrak{k}_r \in \mathbb{N}_0^n$, $f_r \in \mathcal{M}_{\mathfrak{k}_r}$ *for* $r = 1, \ldots, s$ *and set* $\mathfrak{k} = \sum_{r=1}^{s} \mathfrak{k}_r$. *For all* $\mathfrak{v} \in \mathbb{N}_0^n$ *define*

$$[f_1, \ldots, f_s]_{\mathfrak{v}}^{[\mathfrak{a}]} = \sum_{\mathfrak{l}_1+\cdots+\mathfrak{l}_s+\mathfrak{l}=\mathfrak{v}} \frac{(-|\mathfrak{a}|)^{|\mathfrak{l}|}(\mathfrak{k}+2\mathfrak{v}-\vec{2}-\mathfrak{l})!}{\mathfrak{l}!(\mathfrak{k}+2\mathfrak{v}-\vec{2})!} \left(\prod_{r=1}^{s} \frac{(\mathfrak{k}_r-\vec{1})!\, a_r^{|\mathfrak{l}_r|}}{\mathfrak{l}_r!(\mathfrak{k}_r-\vec{1}+\mathfrak{l}_r)!} \right) \times \left(f_1^{(\mathfrak{l}_1)} \cdots f_s^{(\mathfrak{l}_s)}\right)^{(\mathfrak{l})}, \tag{17}$$

where the sum is over all vectors $\mathfrak{l}_1, \ldots, \mathfrak{l}_s$, $\mathfrak{l} \in \mathbb{N}_0^n$ *with* $\mathfrak{l}_1 + \cdots + \mathfrak{l}_s + \mathfrak{l} = \mathfrak{v}$. *Then for all* $M \in \mathrm{SL}_2(\mathbb{R})^n$,

$$\left[\left(f_1\big|_{\mathfrak{k}_1} M\right), \ldots, \left(f_s\big|_{\mathfrak{k}_s} M\right)\right]_{\mathfrak{v}}^{[\mathfrak{a}]} = [f_1, \ldots, f_s]_{\mathfrak{v}}^{[\mathfrak{a}]}\big|_{\mathfrak{k}+2\mathfrak{v}} M, \tag{18}$$

i.e., the map $[\cdot, \ldots, \cdot]_{\mathfrak{v}}^{[\mathfrak{a}]} : \mathcal{M}_{\mathfrak{k}_1} \otimes \cdots \otimes \mathcal{M}_{\mathfrak{k}_s} \to \mathcal{M}_{\mathfrak{k}+2\mathfrak{v}}$ *is equivariant. In particular, if* Γ *is a subgroup of* $\mathrm{SL}_2(\mathbb{R})^n$ *and* $f_r \in \mathcal{M}_{\mathfrak{k}_r}(\Gamma)$ *for* $r = 1, \ldots, s$, *then* $[f_1, \ldots, f_s]_{\mathfrak{v}}^{[\mathfrak{a}]} \in \mathcal{M}_{\mathfrak{k}+2\mathfrak{v}}(\Gamma)$.

Proof. Note that Theorems 1 and 2 hold actually for all $\mathfrak{m} \in \mathbb{C}^n$ (using (6) for $\mathfrak{m} \in \mathbb{C}^n$) and not just for $\mathfrak{m} \in \mathbb{N}_0^n$. Set

$$\phi[f_1, \ldots, f_s] = \prod_{r=1}^{s} \tilde{f}_r(\mathfrak{k}_r, a_r\vec{1}, \tau, z) = \sum_{|\mathfrak{v}| \geqslant 0} \chi_{2\mathfrak{v}}(\tau) z^{2\mathfrak{v}}, \tag{19}$$

where $\tilde{f}_r(\mathfrak{k}_r, a_r\vec{1}, \tau, z)$ (for $r = 1, \ldots, s$) is as in Theorem 2. One finds that

$$\chi_{2\mathfrak{v}}(\tau) = (2\pi i)^{|\mathfrak{v}|} \sum_{\mathfrak{l}_1+\cdots+\mathfrak{l}_s=\mathfrak{v}} \prod_{r=1}^{s} \frac{(\mathfrak{k}_r-\vec{1})!}{\mathfrak{l}_r!(\mathfrak{k}_r-\vec{1}+\mathfrak{l}_r)!} a_r^{|\mathfrak{l}_r|} f_r^{(\mathfrak{l}_r)}(\tau).$$

Theorem 2 implies

$$\phi\left[\left(f_1\big|_{\mathfrak{k}_1} M\right), \ldots, \left(f_s\big|_{\mathfrak{k}_s} M\right)\right] = \phi[f_1, \ldots, f_s]\big|_{\mathfrak{k},(|\mathfrak{a}|,\ldots,|\mathfrak{a}|)} M$$

and our claim then follows from Theorem 1. □

Remark. If $s = 2$ and $|\mathfrak{a}| = 0$, then the multilinear bracket (17) reduces by definition to the Rankin–Cohen bracket (15).

The fact that the classical Rankin–Cohen brackets are the unique differential operators compatible with modularity implies that the generalized brackets (17) can be expressed (in many ways!) in terms of the standard 2-variable brackets $[\cdot,\cdot]_{\mathfrak{v}}$. In particular, there is a formula

$$[f_1, \ldots, f_s]_{\mathfrak{v}}^{[\mathfrak{a}]} = \sum_{\mathfrak{v}_1+\cdots+\mathfrak{v}_{s-1}=\mathfrak{v}} c(\mathfrak{a}; \mathfrak{v}_1, \ldots, \mathfrak{v}_{s-1}; \mathfrak{k}_1, \ldots, \mathfrak{k}_s) \times \left[\left[\ldots, \left[[f_1, f_2]_{\mathfrak{v}_1}, f_3\right]_{\mathfrak{v}_2}, \ldots\right]_{\mathfrak{v}_{s-2}}, f_s\right]_{\mathfrak{v}_{s-1}} \tag{20}$$

expressing $[f_1, \dots, f_s]_{\mathfrak{v}}^{[\mathfrak{a}]}$ for any s and any $\mathfrak{a} = (a_1, \dots, a_s)$ as a linear combination of iterated 2-argument Rankin–Cohen brackets. It might be of interest to compute the coefficients $c(\mathfrak{a}; \mathfrak{v}_1, \dots, \mathfrak{v}_{s-1}; \mathfrak{k}_1, \dots, \mathfrak{k}_s)$ and study their combinatorial structure explicitly.

In the special case where $s = 2$, (20) simply says that the bracket $[f, g]_{\mathfrak{v}}^{[(a,b)]}$ (where $(f, g) \in \mathcal{M}_{\mathfrak{k}} \times \mathcal{M}_{\mathfrak{l}}$ and $(a, b) \in \mathbb{C}^2$) is a scalar-multiple of the bracket $[f, g]_{\mathfrak{v}}$. More specifically, we find that

$$[f, g]_{\mathfrak{v}}^{[(a,b)]} = \frac{H_{\mathfrak{v}}(\mathfrak{k}, \mathfrak{l}; a, b)}{(\mathfrak{k})_{\mathfrak{v}}(\mathfrak{l})_{\mathfrak{v}}(\mathfrak{k} + \mathfrak{l} + \mathfrak{v} - \vec{1})_{\mathfrak{v}}}[f, g]_{\mathfrak{v}}, \tag{21}$$

where

$$H_{\mathfrak{v}}(\mathfrak{k}, \mathfrak{l}; a, b) = \sum_{\substack{\mathfrak{l} \\ 0 \leqslant l_j \leqslant v_j}} (-1)^{|\mathfrak{l}|} \binom{\mathfrak{k} + \mathfrak{v} - \vec{1}}{\mathfrak{v} - \mathfrak{l}} \binom{\mathfrak{l} + \mathfrak{v} - \vec{1}}{\mathfrak{l}} a^{|\mathfrak{l}|} b^{|\mathfrak{v} - \mathfrak{l}|}$$

is the Rankin–Cohen polynomial and

$$(\mathfrak{k})_{\mathfrak{v}} = \frac{(\mathfrak{k} + \mathfrak{v} - \vec{1})!}{(\mathfrak{k} - \vec{1})!}$$

is the ascending Pochhammer symbol. Note that if $n = 1$, then (21) is given implicitly in [19, §3].

Equation (20) yields $s!/2$ ways of writing $[f_1, \dots, f_s]_{\mathfrak{v}}^{[\mathfrak{a}]}$ as a linear combination of iterated 2-argument Rankin–Cohen brackets (permute the functions $f_1, \dots, f_s$ in any way, except that the order of the first two functions does not matter). However, there are other expressions in terms of 2-argument brackets. If $1 \leqslant r < s$, then $\phi[f_1, \dots, f_s] = \phi[f_1, \dots, f_r]\phi[f_{r+1}, \dots, f_s]$, where ϕ is as in (19), and applying Theorems 2 and 1 implies that

$$\begin{aligned} &[f_1, \dots, f_s]_{\mathfrak{v}}^{[\mathfrak{a}]} \\ &\quad = \sum_{\mathfrak{v}_1 + \mathfrak{v}_2 + \mathfrak{v}_3 = \mathfrak{v}} C(\mathfrak{a}; \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3; \mathfrak{k}_1, \dots, \mathfrak{k}_s) \\ &\qquad \times \big[[f_1, \dots, f_r]_{\mathfrak{v}_1}^{[(a_1, \dots, a_r)]}, [f_{r+1}, \dots, f_s]_{\mathfrak{v}_2}^{[(a_{r+1}, \dots, a_s)]}\big]_{\mathfrak{v}_3}^{[(a_1 + \dots + a_r, a_{r+1} + \dots + a_s)]}, \end{aligned} \tag{22}$$

where the coefficients $C(\mathfrak{a}; \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3; \mathfrak{k}_1, \dots, \mathfrak{k}_s)$ can be computed explicitly. Iterating (22) yields a large variety of expressions in terms of 2-argument brackets. In particular, iterating the special case $r = s - 1$ in (22) gives (20).

We conclude this section with a slight generalization of Corollaries 1 and 2. This gives in particular maps from Hilbert modular forms over L to Hilbert modular forms over K, where L is a totally real extension field of a totally real number field K.

Corollary 3. *Let $\mathfrak{a} = (a_1, \dots, a_s) \in \mathbb{C}^s$ and $\mathfrak{k} = \sum_{r=1}^s \mathfrak{k}_r$ where $\mathfrak{k}_r \in \mathbb{N}_0^n$. For a holomorphic function $f : \mathbb{H}^{sn} \to \mathbb{C}$ and for $\mathfrak{v} \in \mathbb{N}_0^n$, define*

$$\Phi_{\mathfrak{v}}^{[\mathfrak{a}]}[f](\tau) = \sum_{\mathfrak{l}_1 + \dots + \mathfrak{l}_s + \mathfrak{l} = \mathfrak{v}} c(\mathfrak{l}_1, \dots, \mathfrak{l}_s, \mathfrak{l}) \left(\sum_{r=1}^s \frac{\partial}{\partial \mathfrak{t}_r} \right)^{\mathfrak{l}} \frac{\partial^{|\mathfrak{v} - \mathfrak{l}|}}{\partial \mathfrak{t}_1^{\mathfrak{l}_1} \cdots \partial \mathfrak{t}_s^{\mathfrak{l}_s}} f(\mathfrak{t}_1, \dots, \mathfrak{t}_s) \Big|_{\tau = \mathfrak{t}_1 = \dots = \mathfrak{t}_s}, \tag{23}$$

where $\tau, \mathfrak{t}_1, \ldots, \mathfrak{t}_s$ *are variables in* $\mathbb{H}^n$ *and where*

$$c(\mathfrak{l}_1, \ldots, \mathfrak{l}_s, \mathfrak{l}) = \frac{(-|\mathfrak{a}|)^{|\mathfrak{l}|}(\mathfrak{k} + 2\mathfrak{v} - \vec{2} - \mathfrak{l})!}{\mathfrak{l}!(\mathfrak{k} + 2\mathfrak{v} - \vec{2})!} \prod_{r=1}^{s} \frac{(\mathfrak{k}_r - \vec{1})!\, a_r^{|\mathfrak{l}_r|}}{\mathfrak{l}_r!(\mathfrak{k}_r - \vec{1} + \mathfrak{l}_r)!}.$$

Then for all $M \in \mathrm{SL}_2(\mathbb{R})^n$,

$$\Phi_{\mathfrak{v}}^{[\mathfrak{a}]}\big[f\big|_{(\mathfrak{k}_1,\ldots,\mathfrak{k}_s)}(M, \ldots, M)\big] = \Phi_{\mathfrak{v}}^{[\mathfrak{a}]}[f]\big|_{\mathfrak{k}+2\mathfrak{v}} M,$$

i.e., the map $\Phi_{\mathfrak{v}}^{[\mathfrak{a}]} : \mathcal{M}_{(\mathfrak{k}_1,\ldots,\mathfrak{k}_s)} \to \mathcal{M}_{\mathfrak{k}+2\mathfrak{v}}$ *is equivariant. In particular, if* Γ *is a subgroup of* $\mathrm{SL}_2(\mathbb{R})^n$ *and* $f \in \mathcal{M}_{(\mathfrak{k}_1,\ldots,\mathfrak{k}_s)}(\Delta(\Gamma))$, *where* $\Delta : \mathrm{SL}_2(\mathbb{R})^n \to (\mathrm{SL}_2(\mathbb{R})^n)^s$ *is the diagonal map, then* $\Phi_{\mathfrak{v}}^{[\mathfrak{a}]}[f] \in \mathcal{M}_{\mathfrak{k}+2\mathfrak{v}}(\Gamma)$.

Proof. We apply Theorems 1 and 2. Let $\mathfrak{t} = (\mathfrak{t}_1, \ldots, \mathfrak{t}_s) \in \mathbb{H}^{sn}$ and $\mathcal{Z} = (\mathcal{Z}_1, \ldots, \mathcal{Z}_s) \in \mathbb{C}^{sn}$ be variables. Set

$$\phi[f] = \tilde{f}(\mathcal{K}, \mathcal{A}, \mathfrak{t}, \mathcal{Z})\Big|_{\substack{\tau=\mathfrak{t}_1=\cdots=\mathfrak{t}_s \\ z=\mathcal{Z}_1=\cdots=\mathcal{Z}_s}} = \sum_{|\mathfrak{v}|\geqslant 0} \chi_{2\mathfrak{v}}(\tau) z^{2\mathfrak{v}}, \tag{24}$$

where $\mathcal{K} = (\mathfrak{k}_1, \ldots, \mathfrak{k}_s)$, $\mathcal{A} = (a_1\vec{1}, \ldots, a_s\vec{1})$, and where $\tilde{f}$ is as in Theorem 2. Then (compare the proof of Corollary 2)

$$\chi_{2\mathfrak{v}}(\tau) = (2\pi i)^{|\mathfrak{v}|} \sum_{\mathfrak{l}_1+\cdots+\mathfrak{l}_s=\mathfrak{v}} \prod_{r=1}^{s} \frac{(\mathfrak{k}_r - \vec{1})!}{\mathfrak{l}_r!(\mathfrak{k}_r - \vec{1} + \mathfrak{l}_r)!} a_r^{|\mathfrak{l}_r|} \frac{\partial^{|\mathfrak{l}_r|}}{\partial \mathfrak{t}_r^{\mathfrak{l}_r}} f(\mathfrak{t})\bigg|_{\tau=\mathfrak{t}_1=\cdots=\mathfrak{t}_s}.$$

Theorem 2 implies

$$\phi\big[f\big|_{(\mathfrak{k}_1,\ldots,\mathfrak{k}_s)}(M, \ldots, M)\big] = \phi[f]\big|_{\mathfrak{k},(|\mathfrak{a}|,\ldots,|\mathfrak{a}|)} M$$

and Theorem 1 then yields (23). □

Corollary 3 can be specialized in many ways. If $n = 1$, $s = 2$, and $|\mathfrak{a}| = 0$, then Corollary 3 reduces to Theorem 2.2(a) in [8]. If $f = \prod_{j=1}^{s} f_r$ where $f_r \in \mathcal{M}_{\mathfrak{k}_r}$, then (23) coincides with the multilinear bracket (17). At the other extreme, we can take f to be a Hilbert modular form of multiweight $(\mathfrak{k}_1, \ldots, \mathfrak{k}_s)$ on a totally real field extension L of K of degree s, where $[K : \mathbb{Q}] = n$; then Corollary 3 gives a collection of maps

$$\mathcal{M}_{(\mathfrak{k}_1,\ldots,\mathfrak{k}_s)}\big(\mathrm{SL}_2(\mathcal{O}_L)\big) \to \mathcal{M}_{\mathfrak{k}_1+\cdots+\mathfrak{k}_s+2\mathfrak{v}}\big(\mathrm{SL}_2(\mathcal{O}_K)\big).$$

More generally, we also get multilinear maps from tensor products of Hilbert modular forms on $L_1, \ldots, L_r$ to Hilbert modular forms on K, where $L_1, \ldots, L_r$ are totally real extensions of K with $\sum_{i=1}^{r} [L_i : K] = s$.

5. Rankin–Cohen brackets and the Petersson inner product

In this section we present an application of Corollary 1. We explicitly determine the Rankin–Cohen bracket of a Hilbert–Eisenstein series and an arbitrary Hilbert modular form. Furthermore, we compute the Petersson inner product of such a bracket and a Hilbert modular cusp form. We follow Zagier [18] and Choie and Kohnen [6], where a similar computation is shown for elliptic modular forms and Jacobi forms, respectively.

Throughout this section, let Γ be a subgroup of finite index of $\mathrm{SL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a totally real number field K of degree n. Let $\mathfrak{d}_K$ be the different of K. If $\alpha \in \mathcal{O}_K$, then $\alpha \succcurlyeq 0$ denotes that $\alpha = 0$ or α is totally positive, and $\alpha \gg 0$ means that α is totally positive. The trace and the norm of an element $\alpha \in K$ are given by the sum and by the product of its conjugates $\alpha^{(1)}, \dots, \alpha^{(n)}$, respectively. We also use the more general definitions of the trace and norm in (3) and (4) by identifying $\alpha \in K$ with $(\alpha^{(1)}, \dots, \alpha^{(n)})$. Finally, $\mathcal{M}_{\mathfrak{k}}^{\mathrm{cusp}}(\Gamma)$ is the vector space of Hilbert modular cusp forms of weight $\mathfrak{k}$ on Γ (for more details see [11,12]).

Set $\Gamma_\infty = \left\{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma \mid t \in \mathcal{O}_K\right\}$. Let $k \in \mathbb{N}_0$, then

$$E_k(\tau) = \sum_{M \in \Gamma_\infty \backslash \Gamma} (1|_{\vec{k}} M)(\tau) = \sum_{M = \left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} \mathcal{N}(\gamma\tau + \delta)^{-\vec{k}} \tag{25}$$

is the Hilbert–Eisenstein series of weight $\vec{k}$ on Γ. It is well known that $E_k(\tau) \in \mathcal{M}_{\vec{k}}(\Gamma)$ if $k > 2$ (see, for example, Freitag [11]). For $\nu \in \mathcal{O}_K$, $\nu \succcurlyeq 0$, set $p_\nu(\tau) = \exp\{2\pi i \,\mathrm{tr}(\nu\tau)\}$. Let $\mathfrak{l} = (l_1, \dots, l_n) \in \mathbb{N}_0^n$. Then

$$\begin{aligned} \mathcal{P}_{\mathfrak{l},\nu}(\tau) &= \sum_{M \in \Gamma_\infty \backslash \Gamma} (p_\nu|_{\mathfrak{l}} M)(\tau) \\ &= \sum_{M = \left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} \mathcal{N}(\gamma\tau + \delta)^{-\mathfrak{l}} \exp\bigl\{2\pi i \,\mathrm{tr}\bigl(\nu(M \circ \tau)\bigr)\bigr\} \end{aligned} \tag{26}$$

is the νth Hilbert–Poincaré series of weight $\mathfrak{l}$ with respect to Γ. It is well known that $\mathcal{P}_{\mathfrak{l},\nu}(\tau) \in \mathcal{M}_{\mathfrak{l}}^{\mathrm{cusp}}(\Gamma)$ if $\nu \gg 0$ and $l_j > 2$ for $j = 1, \dots, n$ (see, for example, Garrett [12]). Note that if $l \in \mathbb{N}_0$, then $\mathcal{P}_{\vec{l},0}(\tau) = E_l(\tau)$.

Let $f, g \in \mathcal{M}_{\mathfrak{k}}(\Gamma)$ such that fg is cuspidal. Then the Petersson inner product is given by

$$\langle f, g\rangle = \int_{\Gamma \backslash \mathbb{H}^n} f(\tau)\overline{g(\tau)} y^{\mathfrak{k} - \vec{2}} \, dx \, dy, \tag{27}$$

where $\tau = x + iy$, $dx = dx_1 \cdots dx_n$, and $dy = dy_1 \cdots dy_n$. Note that the space $(\mathcal{M}_{\mathfrak{k}}^{\mathrm{cusp}}(\Gamma), \langle\, ,\, \rangle)$ is a finite-dimensional Hilbert space. It is also well known (see, for example, Garrett [12]) that if

$$f(\tau) = \sum_{\substack{\mathfrak{n} \in \mathfrak{d}_K^{-1} \\ \mathfrak{n} \gg 0}} a_{\mathfrak{n}} \exp\bigl\{2\pi i \,\mathrm{tr}(\mathfrak{n}\tau)\bigr\} \in \mathcal{M}_{\mathfrak{l}}^{\mathrm{cusp}}(\Gamma),$$

then

$$\langle f, \mathcal{P}_{\mathfrak{l},\nu}\rangle = a_\nu \operatorname{vol}\bigl(\Lambda \setminus \mathbb{R}^n\bigr)(4\pi\nu)^{\vec{1}-\mathfrak{l}}(\mathfrak{l}-\vec{2})!, \tag{28}$$

where $\Lambda = \bigl\{\mathfrak{t} \in \mathbb{R}^n \mid \bigl(\begin{smallmatrix}1 & \mathfrak{t}\\ 0 & 1\end{smallmatrix}\bigr) \in \Gamma\bigr\}$.

We will use the following lemma to compute the Rankin–Cohen bracket of a Hilbert–Eisenstein series and an arbitrary Hilbert modular form.

Lemma 1. *For $\vec{k}$ and $\mathfrak{r} \in \mathbb{N}_0^n$, we have*

$$E_k^{(\mathfrak{r})}(\tau) = \sum_{M=\left(\begin{smallmatrix}* & *\\ \gamma & \delta\end{smallmatrix}\right)\in\Gamma_\infty\setminus\Gamma} \left((-\gamma)^{\mathfrak{r}}\frac{(\vec{k}-\vec{1}+\mathfrak{r})!}{(\vec{k}-\vec{1})!}\right)(1|_{\vec{k}+\mathfrak{r}}M)(\tau). \tag{29}$$

Proof. Direct computation. □

The following proposition is an application of Corollary 1 and gives an explicit formula for the Rankin–Cohen bracket of a Hilbert–Eisenstein series and an arbitrary Hilbert modular form.

Proposition 1. *Let $E_k(\tau)$ be the Hilbert–Eisenstein series in* (25) *and let*

$$g_{\mathfrak{l}}(\tau) = \sum_{\substack{\mathfrak{n}\in\mathfrak{d}_K^{-1}\\ \mathfrak{n}\succcurlyeq 0}} b_{\mathfrak{n}} \exp\bigl\{2\pi i \operatorname{tr}(\mathfrak{n}\tau)\bigr\} \in \mathcal{M}_{\mathfrak{l}}(\Gamma).$$

Then, for all $\mathfrak{v} \in \mathbb{N}_0^n$, we have

$$[E_k, g_{\mathfrak{l}}]_{\mathfrak{v}} = \binom{\vec{k}+\mathfrak{v}-\vec{1}}{\mathfrak{v}} \sum_{\nu\succcurlyeq 0} (2\pi i\nu)^{\mathfrak{v}} b_\nu \mathcal{P}_{\vec{k}+\mathfrak{l}+2\mathfrak{v},\nu}(\tau). \tag{30}$$

Proof. One can check that

$$\begin{aligned}
&\sum_{\nu\succcurlyeq 0}(2\pi i\nu)^{\mathfrak{v}} b_\nu \mathcal{P}_{\vec{k}+\mathfrak{l}+2\mathfrak{v},\nu}(\tau)\\
&\quad= \sum_{M\in\Gamma_\infty\setminus\Gamma}\sum_{\nu\succcurlyeq 0}(2\pi i\nu)^{\mathfrak{v}}(1|_{\vec{k}+\mathfrak{l}+2\mathfrak{v}}M)(\tau) b_\nu \exp\bigl\{2\pi i \operatorname{tr}\bigl(\nu(M\circ\tau)\bigr)\bigr\}\\
&\quad= \sum_{M\in\Gamma_\infty\setminus\Gamma}(1|_{\vec{k}}M)(\tau)\bigl(g_{\mathfrak{l}}^{(\mathfrak{v})}\big|_{\mathfrak{l}+2\mathfrak{v}}M\bigr)(\tau)\\
&\quad= \frac{(\vec{k}-\vec{1})!\mathfrak{v}!}{(\vec{k}+\mathfrak{v}-\vec{1})!}[E_k, g_{\mathfrak{l}}]_{\mathfrak{v}}.
\end{aligned}$$

Note that Lemma 1 and Eq. (14) justify the last equation. □

As an immediate consequence of Theorem 1 and Eq. (28), we record:

Theorem 3. *Let* $\mathfrak{w} = \vec{k} + \mathfrak{l} + 2\mathfrak{v}$, *where* $k > 2$ *and* $\mathfrak{l}, \mathfrak{v} \in \mathbb{N}_0^n$. *Suppose that*

$$f_{\mathfrak{w}}(\tau) = \sum_{\substack{\mathfrak{n} \in \mathfrak{d}_K^{-1} \\ \mathfrak{n} \gg 0}} a_{\mathfrak{n}} \exp\left\{2\pi i \operatorname{tr}(\mathfrak{n}\tau)\right\} \in \mathcal{M}_{\mathfrak{w}}^{\mathrm{cusp}}(\Gamma)$$

and

$$g_{\mathfrak{l}}(\tau) = \sum_{\substack{\mathfrak{n} \in \mathfrak{d}_K^{-1} \\ \mathfrak{n} \succcurlyeq 0}} b_{\mathfrak{n}} \exp\left\{2\pi i \operatorname{tr}(\mathfrak{n}\tau)\right\} \in \mathcal{M}_{\mathfrak{l}}(\Gamma).$$

Then

$$\left\langle f_{\mathfrak{w}}, [E_k, g_{\mathfrak{l}}]_{\mathfrak{v}} \right\rangle = \operatorname{vol}\left(\Lambda \setminus \mathbb{R}^n\right)(2\pi i)^{|\mathfrak{v}|}(4\pi)^{|\vec{1}-\mathfrak{w}|} \frac{(\mathfrak{w} - \vec{2})!(\vec{k} + \mathfrak{v} - \vec{1})!}{(\vec{k} - \vec{1})!\mathfrak{v}!} \sum_{\nu \gg 0} \frac{a_\nu \overline{b_\nu}}{\nu^{\vec{k}+\mathfrak{l}+\mathfrak{v}-\vec{1}}}. \tag{31}$$

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