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Theta Functions with Harmonic Coefficients over Number Fields

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We investigate theta functions attached to quadratic forms over a number field K. We establish a functional equation by regarding the theta functions as specializations of symplectic theta functions. By applying a differential operator to the functional equation, we show how theta functions with harmonic coefficients over K behave under modular transformations.

1. INTRODUCTION

Let Q be a positive-definite $n \times n$ matrix with integral entries and even diagonal entries defining the quadratic form $Q[x] = {}^t xQx$. Suppose ϕ is a spherical function of weight v with respect to Q. It is well known that the theta function

$$\theta_{Q,\phi}(z) = \sum_{g \in \mathbb{Z}^n} \phi(g) \exp\{\pi i\, Q[g]\, z\}, \qquad \operatorname{Im} z > 0 \tag{1}$$

is a modular form of weight $n/2 + v$ on $\Gamma_0(N)$, where $\Gamma = SL_2(\mathbb{Z})$ and N is the level of Q, i.e., NQ^{-1} is integral and NQ^{-1} has even diagonal entries. This was proved by Schoeneberg [13] for even n and by Pfetzer [9] for odd n. Shimura [14] generalizes their results for arbitrary n and also computes the theta multiplier explicitly.

Andrianov and Maloletkin [1, 2] generalize (1) and define theta series of higher degree. In [1], they construct Siegel modular forms by regarding theta series corresponding to positive-definite quadratic forms as specializations of symplectic theta functions. In addition, they apply a differential operator to the functional equation of the theta functions to show that theta series of higher degree with harmonic coefficients are also Siegel modular forms. In [2], they obtain analogous results for theta functions corresponding to indefinite quadratic forms. Stark [15] computes the theta multiplier for the symplectic theta function. As an application, he explicitly determines the theta multiplier of Andrianov's and Maloletkin's theta functions.

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One can also generalize (1) by considering theta functions of quadratic forms over number fields. Eichler [4] and Stopple [17] construct modular forms over real number fields using theta functions corresponding to positive-definite quadratic forms and indefinite quadratic forms. In [10, 11], we take the approach described in Andrianov and Maloletkin [1, 2] to construct modular forms over number fields. We define theta functions of quadratic forms over real number fields and over complex quadratic number fields, and we regard these theta functions as symplectic theta functions to determine the behavior under modular transformations. This elegant method has been used frequently in the literature, see also Friedberg [5], Imamoḡlu [7,8], and Stark [15,16].

In [10, 11], we do not consider theta functions with spherical functions. In this paper we fill that gap. We define theta functions corresponding to quadratic forms over an arbitrary number field K. We prove a functional equation for those theta functions by regarding them as symplectic theta functions and we use the main result of Stark [15] to determine the eighth root of unity which arises under modular transformations. In particular, we generalize the main results of [10,11]. Furthermore, we apply a differential operator to that functional equation. This allows us to show how theta functions with harmonic coefficients over K behave under modular transformations.

2. STATEMENT OF THE RESULTS

2.1. Notation. Let K be an algebraic number field with r_1 real conjugates and r_2 pairs of complex conjugates. The real conjugates of an element α in K are given by $\alpha^{(1)}, \dots, \alpha^{(r_1)}$ and the complex conjugates are given by $\alpha^{(r_1+1)}, \dots, \alpha^{(r_1+2r_2)}$, where $\alpha^{(j+r_2)} = \overline{\alpha^{(j)}}$ for $r_1 + 1 \le j \le r_1 + r_2$. Let δ_K be the different of K and $\mathfrak{O}_K$ be the ring of integers of K. Denote the field of complex numbers by $\mathbb{C}$ and let $\mathbb{H} = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$ be the usual upper half-plane. Let $\mathbb{H}_{\mathcal{Q}} = \{x + yk \mid x \in \mathbb{C},\ y \in \mathbb{R}^+\}$ be the quaternionic upper half-plane consisting of quaternions with no j-component and positive k-component. Set $\mathfrak{H} = \mathbb{H}^{r_1}\mathbb{H}_{\mathcal{Q}}^{r_2}$ and write a typical element as $z = (z_1, \dots, z_{r_1+r_2}) \in \mathfrak{H}$ where $z_j = x_j + iy_j \in \mathbb{H}$ for $j = 1, \dots, r_1$ and $z_j = x_j + y_jk \in \mathbb{H}_{\mathcal{Q}}$ for $j = r_1 + 1, \dots, r_1 + r_2$. We have $\bar{z} = (\overline{z_1}, \dots, \overline{z_{r_1+r_2}})$ where, as usual, $\overline{z_j} = x_j - iy_j$ for $j = 1, \dots, r_1$ and $\overline{z_j} = \overline{x_j} - y_jk$ for $j = r_1 + 1, \dots, r_1 + r_2$. The action of a matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma = \mathrm{Sl}_2(\mathfrak{O}_K)$$

on $z = (z_1, \dots, z_{r_1+r_2}) \in \mathfrak{H}$ is given by

$$M \circ z = (M^{(1)} \circ z_1, \dots, M^{(r_1+r_2)} \circ z_{r_1+r_2}), \tag{2}$$

where

$$M^{(j)} \circ z_j = (\alpha^{(j)} z_j + \beta^{(j)})(\gamma^{(j)} z_j + \delta^{(j)})^{-1}.$$

Note that $M \circ z \in \mathfrak{H}$. Furthermore, for γ and δ in K and z in $\mathfrak{H}$, we define

$$\mathcal{N}(\gamma z + \delta) = \prod_{j=1}^{r_1} (\gamma^{(j)} z_j + \delta^{(j)}) \prod_{j=r_1+1}^{r_1+r_2} \|\gamma^{(j)} z_j + \delta^{(j)}\|^2,$$

where $\|\gamma^{(j)} z_j + \delta^{(j)}\|^2 = |\gamma^{(j)} x_j + \delta^{(j)}|^2 + |\gamma^{(j)}|^2 y_j^2$ is the usual norm of the quaternion, and in particular,

$$\mathcal{N}(\gamma z + \delta)^{1/2} = \prod_{j=1}^{r_1} (\gamma^{(j)} z_j + \delta^{(j)})^{1/2} \prod_{j=r_1+1}^{r_1+r_2} \|\gamma^{(j)} z_j + \delta^{(j)}\|,$$

where each of the r_1 square roots on the right is given by the principal value.

We will construct functions $f : \mathfrak{H} \to \mathbb{C}$ which transform in the following way: For

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma,$$

$$f(M \circ z) = \chi(M)\, \mathcal{N}(\gamma z + \delta)^{k_1} \mathcal{N}(\overline{\gamma z + \delta})^{k_2} f(z), \tag{3}$$

where $\chi(M)$ is a root of unity.

Moreover, if K is totally complex (i.e., if $r_1 = 0$), we will also investigate vector-valued functions with a more complicated multiplier system. For a vector $\binom{s}{t}$, set

$$\begin{pmatrix} s \\ t \end{pmatrix}^{(\kappa)} = \begin{pmatrix} s^{\kappa} \\ s^{\kappa-1} t \\ s^{\kappa-2} t^2 \\ \vdots \\ t^{\kappa} \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a complex matrix and let s and t be variables such that $\binom{u}{v} = A\binom{s}{t}$. We define the κ-fold symmetric product $A^{(\kappa)}$ to be the $(\kappa + 1) \times (\kappa + 1)$ matrix given by

$$\begin{pmatrix} u \\ v \end{pmatrix}^{(\kappa)} = A^{(\kappa)} \begin{pmatrix} s \\ t \end{pmatrix}^{(\kappa)}.$$

For a quaternion $z_j = x_j + y_j k$ with $x_j, y_j \in \mathbb{C}$, define the representation

$$\tilde{\rho}(z_j) = \begin{pmatrix} x_j & i y_j \\ i\overline{y_j} & \overline{x_j} \end{pmatrix}$$

and let $\tilde{\rho}^{(\kappa)}(z_j) = \tilde{\rho}(z_j)^{(\kappa)}$ be the κ-fold symmetric product representation of the quaternion. Let $U \otimes V = (u_{mn} V)$ denote the Kronecker product of two vectors or matrices U and V. Finally, for $z = (z_1, \ldots, z_{r_2})$ where $z_j = x_j + y_j k$ with $x_j, y_j \in \mathbb{C}$, we define the representation

$$\rho^{(\kappa)}(z) = \bigotimes_{1 \le j \le r_2} \tilde{\rho}^{(\kappa)}(z_j).$$

For K totally complex, we construct functions $f : \mathbb{H}_{\mathcal{Q}}^{r_2} \to \mathbb{C}^{r_2(\kappa+1)}$ which transform in the following way: For

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma,$$

$$f(M \circ z) = \chi(M)\, \mathcal{N}(\gamma z + \delta)^{k_1} \rho^{(\kappa)}(\gamma z + \delta) f(z), \tag{4}$$

where $\chi(M)$ is a root of unity. Note that if K is totally complex, then $\mathcal{N}(\gamma z + \delta) = \mathcal{N}(\overline{\gamma z + \delta})$.

2.2. Theta Functions of Quadratic Forms. It will be useful to define $U[V] = {}^t V U V$ and $U\{V\} = {}^t V U \overline{V}$ for any vector or matrix V and any matrix U. Suppose Q is a symmetric $n \times n$ matrix with entries in $\mathfrak{O}_K$ defining the quadratic form $Q[x]$, where $x \in \mathbb{C}^n$. If, in addition, Q has diagonal entries which are divisible by 2, we say that Q is of level $\mathfrak{N}$ ($\mathfrak{N}$ an ideal in $\mathfrak{O}_K$) whenever the following two conditions are satisfied:

(a) The matrix ηQ^{-1} has entries in $\mathfrak{O}_K$ and the diagonal entries of ηQ^{-1} are divisible by 2 for all $\eta \in \mathfrak{N}$.

(b) If $\mathfrak{M}$ is any integral ideal satisfying (a), i.e., μQ^{-1} has entries in $\mathfrak{O}_K$ and 2 divides the diagonal entries of μQ^{-1} for all $\mu \in \mathfrak{M}$, then $\mathfrak{N}$ divides $\mathfrak{M}$.

Suppose that all of the real conjugates of Q are of the same type (k, l). Then there exist matrices S_j in $GL_n(\mathbb{R})$ such that

$$Q^{(j)} = {}^t S_j E_{k,l} S_j \qquad \text{for } j = 1, \ldots, r_1 \tag{5}$$

and there exist matrices S_j in $GL_n(\mathbb{C})$ such that

$$Q^{(j)} = {}^t S_j S_j \qquad \text{for } j = r_1 + 1, \ldots, r_1 + r_2, \tag{6}$$

where

$$E_{k,l} = \begin{pmatrix} I_k & \\ & -I_l \end{pmatrix},$$

and I_k and I_l are the $k \times k$ and $l \times l$ identity matrices, respectively. We set

$$R_j = {}^t S_j \overline{S_j}. \tag{7}$$

For all j, R_j is a majorant of $Q^{(j)}$, i.e.,

$$\overline{R_j} Q^{(j)^{-1}} R_j = \overline{Q^{(j)}} \qquad \text{and} \qquad {}^t R_j = \overline{R_j} > 0.$$

We define a theta function corresponding to a quadratic form by

Definition 1. Let Q be a symmetric $n \times n$ matrix with entries in $\mathfrak{O}_K$ such that 2 divides the diagonal entries of Q and such that Q is of level $\mathfrak{N}$. Assume that all of the real conjugates $Q^{(j)}$ of Q have the same signature (k, l) and set R_j as in (7). Let $u_1, \ldots, u_{\deg K}$ and $v_1, \ldots, v_{\deg K}$ be vectors in $\mathbb{C}^n$ and set $u = {}^t({}^t u_1, \ldots, {}^t u_{\deg K})$ and $v = {}^t({}^t v_1, \ldots, {}^t v_{\deg K})$. We abuse notation and write $u_{j+r_2} = \bar{u}_j$ and $v_{j+r_2} = \bar{v}_j$ for $j = r_1 + 1, \ldots, r_1 + r_2$. For an ideal $\mathfrak{I} \subset \mathfrak{O}_K$ and for $z = (z_1, \ldots, z_{r_1+r_2}) \in \mathfrak{H}$, we define

$$\begin{aligned}
&\Theta_{Q,R}\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right) \\
&= \sum_{\iota \in \mathfrak{I}^n} \exp\Bigg\{\pi i \Bigg(\sum_{j=1}^{r_1} Q^{(j)}[\iota^{(j)} + v_j] x_j + i R_j [\iota^{(j)} + v_j] y_j \\
&+ \sum_{j=r_1+1}^{r_1+r_2} Q^{(j)}[\iota^{(j)} + v_j] x_j + \overline{Q^{(j)}}[\overline{\iota^{(j)}} + \bar{v}_j] \overline{x_j} + 2 i R_j \{\iota^{(j)} + v_j\} y_j \\
&- \sum_{j=1}^{\deg K} 2\, {}^t \iota^{(j)} Q^{(j)} u_j + {}^t v_j Q^{(j)} u_j \Bigg) \Bigg\},
\end{aligned} \tag{8}$$

where $\iota = {}^t(\iota_1, \ldots, \iota_n)$ and $\iota^{(j)} = {}^t(\iota_1^{(j)}, \ldots, \iota_n^{(j)})$. We abuse notation again by writing $R_j\{\iota^{(j)} + v_j\} = {}^t(\iota^{(j)} + v_j) R_j (\overline{\iota^{(j)}} + \bar{v}_j)$.

The following theorem gives the transformation law of $\Theta_{Q,R}(z, \binom{u}{v})$ under modular transformations and, furthermore, generalizes the main results of [10,11].

THEOREM 1. *For*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(\mathfrak{I}^2 \delta_K \mathfrak{N}),$$

we have

$$\begin{aligned} &\Theta_{Q,R}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right) \\ &\quad = \chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right) \mathcal{N}(\gamma z + \delta)^{k/2} \mathcal{N}(\overline{\gamma z + \delta})^{l/2} \Theta_{Q,R}\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right), \end{aligned} \tag{9}$$

where

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right)$$

is an eighth root of unity. In particular, if $\delta \gg 0$ *is a first degree prime of norm* p, *then*

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right) = \varepsilon_p^{-n}\left(\frac{(p\delta^{-1}2\gamma)^n \det(Q)}{\delta}\right), \tag{10}$$

where $\mathcal{N}(\delta) = p$, *and* $\varepsilon_p = 1$ *for* $p \equiv 1 \bmod 4$ *and* $\varepsilon_p = i$ *for* $p \equiv 3 \bmod 4$ *and* $(\frac{\cdot}{\delta})$ *is the quadratic symbol.*

Actually, in Section 3.3 we will see that

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right)$$

can be determined even if δ is not a totally positive first degree prime.

Now we will introduce spherical functions over number fields. We would like to emphasize that the situation is different for real and complex number fields and, therefore, we treat these cases separately.

Let K be totally real (i.e., $r_2 = 0$). Suppose $w_1^+, \ldots, w_{r_1}^+$ and $w_1^-, \ldots, w_{r_1}^-$ are vectors in $\mathbb{C}^n$, such that $R_j[w_j^+] = R_j[w_j^-] = 0$, $Q^{(j)} w_j^+ = R_j w_j^+$, and $Q^{(j)} w_j^- = -R_j w_j^-$ for $j = 1, \ldots, r_1$. We call the function

$$\phi(X) = \phi(X_1, \ldots, X_{r_1}) = \prod_{j=1}^{r_1} ({}^t X_j Q^{(j)} w_j^+)^\kappa ({}^t X_j Q^{(j)} w_j^-)^\lambda \tag{11}$$

a spherical function of weight (κ, λ) relative to the pair (Q, R) over K, where $X_j \in \mathbb{C}^n$ are vectors of variables. We generalize the theta function in (1) and we define a theta function with harmonic coefficients over a totally real number field K by

DEFINITION 2. Let Q be a symmetric $n \times n$ matrix with entries in $\mathfrak{O}_K$ such that 2 divides the diagonal entries of Q and such that Q is of level $\mathfrak{N}$. Assume that all conjugates $Q^{(j)}$ of Q have the same signature (k, l) and set R_j as in (7). Let $\phi(X)$ be a harmonic function of weight (κ, λ) as in (11). For an ideal $\mathfrak{I} \subset \mathfrak{O}_K$ and for $z = (z_1, \ldots, z_{r_1}) \in \mathbb{H}^{r_1}$, we define

$$\Theta^{(r_1)}_{Q,R,\phi}(z) = \sum_{\iota \in \mathfrak{I}^n} \phi(\iota) \exp\left\{ \pi i \sum_{j=1}^{r_1} Q^{(j)}[\iota^{(j)}] x_j + i R_j[\iota^{(j)}] y_j \right\}, \tag{12}$$

where $\iota = {}^t(\iota_1, \ldots, \iota_n)$ and $\iota^{(j)} = {}^t(\iota_1^{(j)}, \ldots, \iota_n^{(j)})$.

We have the following theorem:

THEOREM 2. *For*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(\mathfrak{I}^2 \delta_K \mathfrak{N}),$$

we have

$$\Theta^{(r_1)}_{Q,R,\phi}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z \right) = \chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q \right) \mathcal{N}(\gamma z + \delta)^{k/2+\kappa} \mathcal{N}(\gamma \bar{z} + \delta)^{l/2+\lambda}\, \Theta^{(r_1)}_{Q,R,\phi}(z), \tag{13}$$

where

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q \right)$$

is the same root of unity as in Theorem 1 *and is given explicitly in* (10).

Now let K be a totally complex number field (i.e., $r_1 = 0$). Suppose $w_1, \ldots, w_{r_2}$ are vectors in $\mathbb{C}^n$ such that $R_j\{w_j\} = 0$ and $Q^{(j)} w_j = R_j \overline{w_j}$, for $j = 1, \ldots, r_2$. We call the vector-valued function

$$\phi(X) = \phi(X_1, \ldots, X_{r_2}) = \bigotimes_{1 \le j \le r_2} \begin{pmatrix} {}^t X_j Q^{(j)} w_j \\ {}^t \overline{X_j} \overline{Q^{(j)}} \overline{w_j} \end{pmatrix}^{(\kappa)}, \tag{14}$$

a spherical function of weight κ over K, where $X_j \in \mathbb{C}^n$ are vectors of variables. Again, we generalize the theta function in (1) and we define a theta function with harmonic coefficients over a totally complex field K by

DEFINITION 3. Let Q be a symmetric $n \times n$ matrix with entries in $\mathfrak{O}_K$ such that 2 divides the diagonal entries of Q and such that Q is of level $\mathfrak{N}$. Set R_j as in (7). Let $\phi(X)$ be a spherical function of weight κ as in (14). For an ideal $\mathfrak{I} \subset \mathfrak{O}_K$ and for $z = (z_1, \ldots, z_{r_2}) \in \mathbb{H}_{\mathcal{Q}}^{r_2}$, we define

$$\Theta_{Q,R,\phi}^{(r_2)}(z) = \sum_{\iota \in \mathfrak{I}^n} \phi(\iota) \exp\left\{ \pi i \sum_{j=1}^{r_2} Q^{(j)}[\iota^{(j)}]x_j + \overline{Q^{(j)}}[\overline{\iota^{(j)}}]\overline{x_j} + 2iR_j\{\iota^{(j)}\} y_j \right\}, \quad (15)$$

where $\iota = {}^t(\iota_1, \ldots, \iota_n)$ and $\iota^{(j)} = {}^t(\iota_1^{(j)}, \ldots, \iota_n^{(j)})$.

Note that $\Theta_{Q,R,\phi} : \mathbb{H}_{\mathcal{Q}}^{r_2} \to \mathbb{C}^{r_2(\kappa+1)}$.

We have the following theorem:

THEOREM 3. *For*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(\mathfrak{I}^2 \delta_K \mathfrak{N}),$$

we have

$$\Theta_{Q,R,\phi}^{(r_2)}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z \right) = \chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q \right) \mathcal{N}(\gamma z + \delta)^{n/2} \rho(\gamma z + \delta)^{(\kappa)} \Theta_{Q,R,\phi}^{(r_2)}(z), \quad (16)$$

where

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q \right)$$

is the same root of unity as in Theorem 1 *and is given explicitly in* (10).

3. PROOF OF THEOREM 1

3.1. Symplectic Theta Functions. We will use the transformation property of the symplectic theta function to prove Theorem 1. Let us recall some basic facts. The symplectic group

$$\mathrm{Sp}_n(\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n,2n}(\mathbb{R}) \,|\, J[M] = J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \right\},$$

where I_n is the $n \times n$ identity matrix, acts on the Siegel upper half-plane,

$$\mathfrak{H}^{(n)} = \{ Z \in M_{n,n}(\mathbb{C}) \,|\, Z = {}^tZ \text{ and } \mathrm{Im}(Z) > 0 \}.$$

The action of M on Z is given by

$$M \circ Z = (AZ + B)(CZ + D)^{-1}.$$

Let $\Gamma^{(n)} = \mathrm{Sp}_n(\mathbb{Z})$. The theta subgroup,

$$\Gamma_\vartheta^{(n)} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \,|\, A\,{}^tB,\ C\,{}^tD \text{ have even diagonal entries} \right\},$$

acts on the symplectic theta function,

$$\vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \sum_{m \in \mathbb{Z}^n} \exp\{ \pi i (Z[m + v] - 2\,{}^tmu - {}^tvu) \}, \tag{17}$$

where u and v are column vectors in $\mathbb{C}^n$. It is well known (see, for example in Eichler [3]) that for

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\vartheta^{(n)},$$

$$\vartheta\left(M \circ Z, M \begin{pmatrix} u \\ v \end{pmatrix} \right) = \chi(M) [\det(CZ + D)]^{1/2} \vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix} \right), \tag{18}$$

where $\chi(M)$ is an eighth root of unity which depends upon the chosen square root of $\det(CZ + D)$ but which is otherwise independent of Z, u, and v. Stark [15] determines $\chi(M)$ in the important special case that both C and D are nonsingular and pD^{-1} is integral for some odd prime p. We will use the following result of [15] to compute explicitly the theta multipliers of $\Theta_{Q,R}(z, \binom{u}{v})$, $\Theta_{Q,R,\phi}^{(r_1)}(z)$, and $\Theta_{Q,R,\phi}^{(r_2)}(z)$.

THEOREM 4 (Stark [15]). *Suppose*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is in $\Gamma_\vartheta^{(n)}$ *where* C^{-1} *and* D^{-1} *exist. Suppose further that for some odd prime* p, pD^{-1} *is integral. Then* (mod p), *the symmetric matrix* $pD^{-1}C$ *has* rank h *where* $\det(D) = \pm p^h$. *Let* $(pD^{-1}C)^{(h)}$ *be a nonsingular* (mod p) $h \times h$ *principal submatrix of* $pD^{-1}C$ *and* s *be the signature* (*the number of positive eigenvalues minus the number of negative eigenvalues*) *of* $C^{-1}D$. *Then*

$$\chi(M)[\det(CZ+D)]^{1/2}$$

$$= \varepsilon_p^{-h}\left(\frac{2^h \det[(pD^{-1}C)^{(h)}]}{p}\right) e^{\pi i s/4} |\det(C)|^{1/2} \{\det[-iC^{-1}(CZ+D)]\}^{1/2},$$

where $\varepsilon_p = 1$ for $p \equiv 1 \bmod 4$, $\varepsilon_p = i$ *for* $p \equiv 3 \bmod 4$, $(\frac{\cdot}{p})$ *is the Legendre symbol*, $|\det(C)|^{1/2}$ *is positive and* $\{\det[-iC^{-1}(CZ+D)]\}^{1/2}$ *is given by analytic continuation from the principal value when* $Z = -C^{-1}D + iY$.

3.2. The Modular Transformation. We will proceed as in [10,11] and we will convert $\Theta_{Q,R}(z, \binom{u}{v})$ into a symplectic theta function. Applying (18) then will prove (9).

Let $\omega_1, \ldots, \omega_{\deg K}$ be an integral basis of the ideal $\mathfrak{J} \subset \mathfrak{O}_K$ and define the vector $\omega^{(j)} = (\omega_1^{(j)}, \ldots, \omega_{\deg K}^{(j)})$. We use the $n \times n \deg K$ matrix

$$W_j = \begin{pmatrix} \omega^{(j)} & & \\ & \ddots & \\ & & \omega^{(j)} \end{pmatrix}$$

to define the $n \deg K \times n \deg K$ matrix $W = {}^t({}^tW_1, \ldots, {}^tW_{\deg K})$. Note that W^{-1} has entries in $\mathfrak{J}^{-1}\delta_K^{-1}$.

Let $z = (z_1, \ldots, z_{r_1+r_2}) \in \mathfrak{H}$. For $j = 1, \ldots, r_1$, set

$$Z_j^* = \begin{pmatrix} z_j I_p & \\ & -\overline{z_j} I_q \end{pmatrix},$$

and for $j = r_1+1, \ldots, r_1+r_2$, define the $n \times n$ matrices $X_j = x_j I_n$ and $\overline{X_j} = \overline{x_j} I_n$. For all j, set $Y_j = y_j I_n$. Furthermore, define the

$n \deg K \times n \deg K$ matrix

$$Z^* = \begin{pmatrix} Z_1^* & & & & & & & & \\ & \ddots & & & & & & & \\ & & Z_{r_1}^* & & & & & & \\ & & & X_{r_1+1} & & & iY_{r_1+1} & & \\ & & & & \ddots & & & \ddots & \\ & & & & & X_{r_1+r_2} & & & iY_{r_1+r_2} \\ & & & iY_{r_1+1} & & & \overline{X_{r_1+1}} & & \\ & & & & \ddots & & & \ddots & \\ & & & & & iY_{r_1+r_2} & & & \overline{X_{r_1+r_2}} \end{pmatrix}.$$

Let

$$S = \begin{pmatrix} S_1 & & \\ & \ddots & \\ & & S_{\deg K} \end{pmatrix}, \tag{19}$$

where $S_{j+r_2} = \overline{S_j}$ for $j = r_1 + 1, \ldots, r_1 + r_2$, and let $T = SW$ and

$$Z = {}^tTZ^*T = \begin{pmatrix} {}^tT & 0 \\ 0 & T^{-1} \end{pmatrix} \circ Z^*. \tag{20}$$

It is not difficult to see that $Z \in \mathfrak{H}^{(n \deg K)}$ (see also [10, 11]).

We have

$$\Theta_{Q,R}\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \vartheta\left(Z, \begin{pmatrix} {}^tW\tilde{Q}u \\ W^{-1}v \end{pmatrix}\right), \tag{21}$$

where

$$\tilde{Q} = \begin{pmatrix} Q^{(1)} & & \\ & \ddots & \\ & & Q^{(\deg K)} \end{pmatrix}. \tag{22}$$

We need a symplectic matrix which expresses the action of

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$$

on our new variables u, v and Z. For

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma,$$

set

$$A^* = \begin{pmatrix} \alpha^{(1)} I_n & & \\ & \ddots & \\ & & \alpha^{(\deg K)} I_n \end{pmatrix}, \qquad D^* = \begin{pmatrix} \delta^{(1)} I_n & & \\ & \ddots & \\ & & \delta^{(\deg K)} I_n \end{pmatrix},$$

$$B^* = \begin{pmatrix} \beta^{(1)} E_{k,l} & & & & & \\ & \ddots & & & & \\ & & \beta^{(r_1)} E_{k,l} & & & \\ & & & \beta^{(r_1+1)} I_n & & \\ & & & & \ddots & \\ & & & & & \beta^{(\deg K)} I_n \end{pmatrix}$$

and

$$C^* = \begin{pmatrix} \gamma^{(1)} E_{k,l} & & & & & \\ & \ddots & & & & \\ & & \gamma^{(r_1)} E_{k,l} & & & \\ & & & \gamma^{(r_1+1)} I_n & & \\ & & & & \ddots & \\ & & & & & \gamma^{(\deg K)} I_n \end{pmatrix}.$$

We make the variable change and set

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}^tT & 0 \\ 0 & T^{-1} \end{pmatrix} \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} \begin{pmatrix} {}^tT & 0 \\ 0 & T^{-1} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} {}^tTA^*\,{}^tT^{-1} & {}^tTB^*T \\ T^{-1}C^*\,{}^tT^{-1} & T^{-1}D^*T \end{pmatrix}. \tag{23}$$

It is easy to check that the diagrams

$$\begin{array}{ccc} z & \to & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z \\ \downarrow & & \downarrow \\ Z^* & \to & \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} \circ Z^* \end{array}$$

and

$$\begin{array}{ccc} \begin{pmatrix} u \\ v \end{pmatrix} & \to & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} {}^tW\tilde{Q}u \\ W^{-1}v \end{pmatrix} & \to & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} {}^tW\tilde{Q}u \\ W^{-1}v \end{pmatrix} \end{array}$$

commute, where the horizontal arrows are linear fractional transformations in the first diagram and matrix multiplication in the second, and the vertical arrows are given by the specified variable changes. Hence,

$$z \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ z$$

in $\mathfrak{H}$ corresponds to

$$Z \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z$$

in $\mathfrak{H}^{(n \deg K)}$.

The entries of A, B, C and D are rational integers and $A\,{}^tB$ and $C\,{}^tD$ have even diagonal entries if γ is in the ideal $\mathfrak{I}^2\delta_K N$ (see also [10, 11]). Hence, if

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(\mathfrak{I}^2\delta_K N),$$

then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\vartheta^{(n \deg K)}.$$

Furthermore, we find that

$$\det(CZ+D)=\prod_{j=1}^{r_1}(\gamma^{(j)}z_j+\delta^{(j)})^k(\gamma^{(j)}\overline{z_j}+\delta^{(j)})^l\prod_{j=r_1+1}^{r_1+r_2}\|\gamma^{(j)}z_j+\delta^{(j)}\|^{2n}.$$

Thus, Eqs. (18) and (21) imply that

$$\begin{aligned}&\Theta_{Q,R}\left(\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\circ z,\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}\right)\\&=\chi\left(\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix},Q\right)\mathcal{N}(\gamma z+\delta)^{k/2}\mathcal{N}(\overline{\gamma z+\delta})^{l/2}\Theta_{Q,R}\left(z,\begin{pmatrix}u\\v\end{pmatrix}\right),\end{aligned}\tag{24}$$

where

$$\chi\left(\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix},Q\right)$$

is an eighth root of unity depending on

$$\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$$

and Q.

3.3. The Eighth Root of Unity. To complete the proof of Theorem 1, we have to determine

$$\chi\left(\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix},Q\right).$$

Suppose that

$$\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in\Gamma_0(\mathfrak{I}^2\delta_K N),$$

where $\delta\gg 0$ is a first degree prime in $\mathfrak{O}_K$ of norm p. Then C^{-1} and D^{-1} exist and pD^{-1} is integral. We can therefore use Theorem 4 to determine the eighth root of unity in (24). We find that

$$\begin{aligned}&|\det(C)|^{1/2}\{\det[-iC^{-1}(CZ+D)]\}^{1/2}e^{\pi is/4}\\&=\mathcal{N}(\gamma z+\delta)^{k/2}\mathcal{N}(\overline{\gamma z+\delta})^{l/2},\end{aligned}$$

where s is the signature of $C^{-1}D$, and also that

$$\det(pD^{-1}C)^{(n)}\equiv(p\delta^{-1}\gamma)^n(v)^{2n}\det(Q)^{-1}\pmod{\delta},$$

where $v \in \mathfrak{I}^{-1}\delta_K^{-1}$. By Theorem 4 (and since $\mathfrak{O}_K/\delta \cong \mathbb{Z}/p\mathbb{Z}$),

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right) = \varepsilon_p^{-n}\left(\frac{(p\delta^{-1}2\gamma)^n \det(Q)}{\delta}\right), \tag{25}$$

where $\varepsilon_p = 1$ for $p \equiv 1 \bmod 4$ and $\varepsilon_p = i$ for $p \equiv 3 \bmod 4$.

Note that $\Theta_{Q,R}(z, \binom{u}{v})$ is invariant under linear transformations, i.e., for any algebraic integer $\mu \in K$,

$$\Theta_{Q,R}\left(z + \mu, \begin{pmatrix} u + \mu v \\ v \end{pmatrix}\right) = \Theta_{Q,R}\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right). \tag{26}$$

From (24), it follows that for

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(\mathfrak{I}^2\delta_K N)$$

and for all algebraic integers μ,

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, Q\right) = \chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right). \tag{27}$$

Note that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(\mathfrak{I}^2\delta_K N)$$

implies that $(\gamma, \delta) = 1$ and by Dirichlet's primes in progression theorem for number fields (see, for example in Hecke [6]), the arithmetic progression $\{\gamma\mu + \delta\}_{\mu \in \mathfrak{O}_k}$ contains infinitely many totally positive first degree primes. Hence, the eighth root of unity is determined explicitly by (27) after locating a totally positive first degree prime with positive odd norm in the arithmetic progression $\{\gamma\mu + \delta\}_{\mu \in \mathfrak{O}_k}$.

4. PROOF OF THEOREMS 2 AND 3

4.1. Proof of Theorem 2. Now we turn to the proof of Theorem 2. Suppose K is totally real. We will define a differential operator and apply the operator to (9).

LEMMA 1. *Let* $E_j, H_j \in \mathbb{C}$, $F_j \in \mathbb{C}^n$, *and* $\eta_j \in \mathbb{C}^n$ *such that* ${}^t\eta_j\eta_j = 0$ *for* $j = 1, \ldots, r$. *Suppose* $\xi_j = {}^t(\xi_1^{(j)}, \ldots, \xi_n^{(j)}) \in \mathbb{C}^n$ *for* $j = 1, \ldots, r$ *are* *vectors of variables (accounting for a total of nr independent variables) and suppose*

$$\partial_j = {}^t\left(\frac{\partial}{\partial \xi_1^{(j)}}, \ldots, \frac{\partial}{\partial \xi_n^{(j)}}\right)$$

are the corresponding vectors of differentiation operators. Define the differential operator

$$\mathscr{L} = \mathscr{L}_\eta = \prod_{j=1}^{r} {}^t\eta_j \partial_j.$$

Then for $v \geq 1$,

$$\mathscr{L}^v\left(\exp\left\{\pi i \sum_{j=1}^{r} E_j\, {}^t\xi_j\xi_j + 2\,{}^tF_j\xi_j + H_j\right\}\right)$$
$$= \prod_{j=1}^{r} (2\pi i(E_j\, {}^t\xi_j + {}^tF_j)\eta_j)^v \exp\left\{\pi i \sum_{j=1}^{r} E_j\, {}^t\xi_j\xi_j + 2\,{}^tF_j\xi_j + H_j\right\}. \qquad (28)$$

Proof. It is easy to check that

$$\mathscr{L}\left(\exp\left\{\pi i \sum_{j=1}^{r} E_j\, {}^t\xi_j\xi_j + 2\,{}^tF_j\xi_j + H_j\right\}\right)$$
$$= \left(\prod_{j=1}^{r} 2\pi i(E_j\, {}^t\xi_j + {}^tF_j)\eta_j\right)\exp\left\{\pi i \sum_{j=1}^{r} E_j\, {}^t\xi_j\xi_j + 2\,{}^tF_j\xi_j + H_j\right\}$$

and that

$$\mathscr{L}\left(\prod_{j=1}^{r} 2\pi i(E_j\, {}^t\xi_j + {}^tF_j)\eta_j\right) = \prod_{j=1}^{r} 2\pi i\, E_j\, {}^t\eta_j\eta_j = 0,$$

which implies (28). ∎

Since K is totally real, we let $1 \leq j \leq r_1$. Let

$$\xi_j = \begin{pmatrix} \xi_j^+ \\ \xi_j^- \end{pmatrix} \in \mathbb{C}^n,$$

where $\xi_j^+ \in \mathbb{C}^k$ and $\xi_j^- \in \mathbb{C}^l$, and let ∂_j^+ and ∂_j^- be the corresponding vectors of differentiation operators. Set $\tilde{\eta}_j^+ = S_j w_j^+$ and $\tilde{\eta}_j^- = S_j w_j^-$, where S_j is defined in (5). Note that

$$\tilde{\eta}_j^+ = \begin{pmatrix} \eta_j^+ \\ 0 \end{pmatrix}$$

and

$$\tilde{\eta}_j^- = \begin{pmatrix} 0 \\ \eta_j^- \end{pmatrix}$$

for $\eta_j^+ \in \mathbb{C}^k$ and $\eta_j^- \in \mathbb{C}^l$, since $Q^{(j)} w_j^+ = R_j w_j^+$, and $Q^{(j)} w_j^- = -R_j w_j^-$.

Set $u = 0$ and $v = S^{-1}\xi$ in Eq. (9) where S is defined in (19) and $\xi = {}^t({}^t\xi_1, \ldots, {}^t\xi_{r_1})$:

$$\begin{aligned}
&\sum_{\iota \in \mathfrak{I}^n} \left(\exp\left\{ \pi i \sum_{j=1}^{r_1} E_j^{\oplus}\, {}^t\xi_j^+ \xi_j^+ + 2\,{}^tF_j^{\oplus} \xi_j^+ + H_j^{\oplus} \right\} \right. \\
&\quad \left. \times \exp\left\{ \pi i \sum_{j=1}^{r_1} E_j^{\ominus}\, {}^t\xi_j^- \xi_j^- + 2\,{}^tF_j^{\ominus} \xi_j^- + H_j^{\ominus} \right\} \right) \\
&= \mathscr{J}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, z \right) \sum_{\iota \in \mathfrak{I}^n} \left(\exp\left\{ \pi i \sum_{j=1}^{r_1} E_j^+\, {}^t\xi_j^+ \xi_j^+ + 2\,{}^tF_j^+ \xi_j^+ + H_j^+ \right\} \right. \\
&\quad \left. \times \exp\left\{ \pi i \sum_{j=1}^{r_1} E_j^-\, {}^t\xi_j^- \xi_j^- + 2\,{}^tF_j^- \xi_j^- + H_j^- \right\} \right),
\end{aligned} \tag{29}$$

where

$$E_j^{\oplus} = \frac{\delta^{(j)} z_j}{\gamma^{(j)} z_j + \delta^{(j)}}, \qquad E_j^{\ominus} = \frac{-\delta^{(j)} \overline{z_j}}{\gamma^{(j)} \overline{z_j} + \delta^{(j)}}, \qquad E_j^+ = z_j, \qquad E_j^- = -\overline{z_j},$$

$$F_j^{\oplus} = \frac{z_j}{\gamma^{(j)} z_j + \delta^{(j)}}\, {}^t\!\begin{pmatrix} I_k \\ 0 \end{pmatrix} S_j \iota^{(j)}, \qquad F_j^{\ominus} = \frac{-\overline{z_j}}{\gamma^{(j)} \overline{z_j} + \delta^{(j)}}\, {}^t\!\begin{pmatrix} 0 \\ I_l \end{pmatrix} S_j \iota^{(j)},$$

$$F_j^+ = z_j\, {}^t\!\begin{pmatrix} I_k \\ 0 \end{pmatrix} S_j \iota^{(j)}, \qquad F_j^- = -\overline{z_j}\, {}^t\!\begin{pmatrix} 0 \\ I_l \end{pmatrix} S_j \iota^{(j)},$$

$$H_j^{\oplus} = \operatorname{Re}\left(\begin{pmatrix} \alpha^{(j)} & \beta^{(j)} \\ \gamma^{(j)} & \delta^{(j)} \end{pmatrix} \circ z_j \right) Q^{(j)}[\iota^{(j)}], \quad H_j^{\ominus} = i \operatorname{Im}\left(\begin{pmatrix} \alpha^{(j)} & \beta^{(j)} \\ \gamma^{(j)} & \delta^{(j)} \end{pmatrix} \circ z_j \right) R_j[\iota^{(j)}],$$

$$H_j^+ = x_j Q^{(j)}[\iota^{(j)}], \qquad H_j^- = i y_j R_j[\iota^{(j)}],$$

and where

$$\mathscr{J}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, z \right) = \chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q \right) \mathscr{N}(\gamma z + \delta)^{k/2} \mathscr{N}(\gamma \bar{z} + \delta)^{l/2}$$

and

$$\chi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, Q\right)$$

as in Theorem 1. Applying Lemma 1 with

$$\mathscr{L}_+^\kappa \mathscr{L}_-^\lambda = \prod_{j=1}^{r_1} ({}^t\eta_j^+ \partial_j^+)^\kappa \prod_{j=1}^{r_1} ({}^t\eta_j^- \partial_j^-)^\lambda$$

and setting $\xi = 0$ then leads to (13).

4.2. Proof of Theorem 3. Now, we turn to the proof of Theorem 3. Suppose K is totally complex. We define another differential operator and apply the differential operator to (9).

LEMMA 2. *Let $E_j, \tilde{E}_j, G_j, H_j \in \mathbb{C}$, $F_j, \tilde{F}_j \in \mathbb{C}^n$, and $\eta_j \in \mathbb{C}^n$ such that ${}^t\eta_j\eta_j = 0$ and ${}^t\eta_j\overline{\eta_j} = 0$ for $j = 1, \ldots, r$. Suppose $\xi_j \in \mathbb{C}^n$ for $j = 1, \ldots, 2r$ are vectors of independent variables and suppose ∂_j are the corresponding vectors of differentiation operators. We abuse notation and write $\xi_{j+r} = \bar{\xi}_j$ and $\partial_{j+r} = \bar{\partial}_j$ for $j = 1, \ldots, r$. For $v \geq 1$, we define the vector-valued differential operator*

$$\mathscr{L}^{(v)} = \mathscr{L}_\eta^{(v)} = \bigotimes_{1 \le j \le r} \begin{pmatrix} {}^t\eta_j \partial_j \\ {}^t\overline{\eta_j}\bar{\partial}_j \end{pmatrix}^{(v)}.$$

Then

$$\mathscr{L}^{(v)}(f(\xi, \bar{\xi})) = \bigotimes_{1 \le j \le r} \begin{pmatrix} 2\pi i\ (E_j\, {}^t\xi_j + {}^tF_j + G_j\, {}^t\bar{\xi}_j)\eta_j \\ 2\pi i\ (\tilde{E}_j\, {}^t\bar{\xi}_j + {}^t\tilde{F}_j + G_j\, {}^t\xi_j)\overline{\eta_j} \end{pmatrix}^{(v)} f(\xi, \bar{\xi}), \qquad (30)$$

where

$$f(\xi, \bar{\xi}) = \exp\left\{\pi i \sum_{j=1}^{r} E_j\, {}^t\xi_j\xi_j + \tilde{E}_j\, {}^t\bar{\xi}_j\bar{\xi}_j + 2\,{}^tF_j\xi_j + 2\,{}^t\tilde{F}_j\bar{\xi}_j + 2G_j\, {}^t\xi_j\bar{\xi}_j + H_j\right\}.$$

Proof. Let $1 \le j_0 \le r$. For all $1 \le j \le r$, we have

$${}^t\eta_j\partial_j(2\pi i\ (E_{j_0}\, {}^t\xi_{j_0} + {}^tF_{j_0} + G_{j_0}\, {}^t\bar{\xi}_{j_0})\eta_{j_0}) = 0,$$

$${}^t\eta_j\partial_j(2\pi i\ (\tilde{E}_{j_0}\, {}^t\bar{\xi}_{j_0} + {}^t\tilde{F}_{j_0} + G_{j_0}\, {}^t\xi_{j_0})\overline{\eta_{j_0}}) = 0,$$

$${}^t\overline{\eta_j}\bar{\partial}_j(2\pi i\ (E_{j_0}\, {}^t\xi_{j_0} + {}^tF_{j_0} + G_{j_0}\, {}^t\bar{\xi}_{j_0})\eta_{j_0}) = 0,$$

$${}^t\overline{\eta_j}\bar{\partial}_j(2\pi i\ (\tilde{E}_{j_0}\, {}^t\bar{\xi}_{j_0} + {}^t\tilde{F}_{j_0} + G_{j_0}\, {}^t\xi_{j_0})\overline{\eta_{j_0}}) = 0.$$

Furthermore, for all $1 \le j \le r$, one checks that

$$\begin{pmatrix} {}^t\eta_j\partial_j \\ {}^t\overline{\eta_j}\bar{\partial}_j \end{pmatrix}^{(v)} (f(\xi,\bar{\xi})) = \begin{pmatrix} 2\pi i\ (E_j\,{}^t\xi_j + {}^tF_j + G_j\,{}^t\bar{\xi}_j)\eta_j \\ 2\pi i\ (\tilde{E}_j\,{}^t\bar{\xi}_j + {}^t\tilde{F}_j + G_j\,{}^t\xi_j)\overline{\eta_j} \end{pmatrix}^{(v)} f(\xi,\bar{\xi}),$$

and (30) follows. ∎

Since K is totally complex, we let $1 \le j \le r_2$. Let $\xi_j, \xi_{j+r_2} \in \mathbb{C}^n$ be vectors of variables, and let ∂_j and ∂_{j+r_2} be the corresponding vectors of differentiation operators. As in Lemma 2, with an abuse of notation we write $\xi_{j+r_2} = \bar{\xi}_j$ and $\partial_{j+r_2} = \bar{\partial}_j$. Set $\eta_j = S_j w_j$, where S_j is defined in (5). Note that ${}^t\eta_j\eta_j = 0$ and ${}^t\eta_j\overline{\eta_j} = 0$.

Set $u = 0$ and $v = S^{-1}\xi$ in Eq. (9), where S is defined in (19) and $\xi = {}^t({}^t\xi_1, \ldots, {}^t\xi_{r_2}, {}^t\bar{\xi}_1, \ldots {}^t\bar{\xi}_{r_2})$. For $z_j \in \mathbb{H}_{\mathcal{Q}}$, we set

$$\begin{pmatrix} \alpha^{(j)} & \beta^{(j)} \\ \gamma^{(j)} & \delta^{(j)} \end{pmatrix} \circ z_j = x_j^* + y_j^* k,$$

and for a quaternion $z_j = x_j + y_j k$ with $x_j, y_j \in \mathbb{C}$, we write for convenience $\{z_j\}_{\mathbb{C}} = x_j$ and $\{z_j\}_{\mathcal{Q}} = y_j$. We find that

$$\begin{aligned} &\sum_{\iota\in\mathfrak{I}^n} \left(\exp\left\{ \pi i \sum_{j=1}^{r_2} E_j\,{}^t\xi_j\xi_j + \overline{E_j}\,{}^t\bar{\xi}_j\bar{\xi}_j \right\} \right. \\ &\quad \left. \times \exp\left\{ \pi i \sum_{j=1}^{r_2} 2\,{}^tF_j\xi_j + 2\,{}^t\tilde{F}_j\bar{\xi}_j + 2G_j\,{}^t\xi_j\bar{\xi}_j + H_j \right\} \right) \\ &\quad = \mathscr{J}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, z \right) \sum_{\iota\in\mathfrak{I}^n} \left(\exp\left\{ \pi i \sum_{j=1}^{r_2} E'_j\,{}^t\xi_j\xi_j + \overline{E'_j}\,{}^t\bar{\xi}_j\,\bar{\xi}_j \right\} \right. \\ &\quad \left. \times \exp\left\{ \pi i \sum_{j=1}^{r_2} 2\,{}^tF'_j\xi_j + 2\,{}^t\tilde{F}'_j\bar{\xi}_j + 2G'_j\,{}^t\xi_j\bar{\xi}_j + H'_j \right\} \right), \end{aligned} \tag{31}$$

where

$$E_j = \delta^{(j)}\{z_j(\gamma^{(j)}z_j + \delta^{(j)})^{-1}\}_{\mathbb{C}}, \qquad E'_j = x_j,$$

$$F_j = \{z_j(\gamma^{(j)}z_j + \delta^{(j)})^{-1}\}_{\mathbb{C}} S_j\iota^{(j)} + i\{z_j(\gamma^{(j)}z_j + \delta^{(j)})^{-1}\}_{\mathcal{Q}}\overline{S_j}\overline{\iota^{(j)}},$$

$$\tilde{F}_j = \overline{\{z_j(\gamma^{(j)}z_j + \delta^{(j)})^{-1}\}_{\mathbb{C}}}\,\overline{S_j}\overline{\iota^{(j)}} + i\overline{\{z_j(\gamma^{(j)}z_j + \delta^{(j)})^{-1}\}_{\mathcal{Q}}} S_j\iota^{(j)}$$

$$F'_j = x_jS_j\iota^{(j)} + iy_j\overline{S_j}\overline{\iota^{(j)}}, \qquad \tilde{F}'_j = \overline{x_j}\,\overline{S_j}\overline{\iota^{(j)}} + iy_jS_j\iota^{(j)},$$

$$G_j = i|\delta^{(j)}|^2 y_j^*, \qquad G_j' = i y_j,$$

$$H_j = Q^{(j)}[\iota^{(j)}]x_j^* + \overline{Q^{(j)}}[\overline{\iota^{(j)}}]\overline{x_j^*} + 2iR_j\{\iota^{(j)}\} y_j^*,$$

$$H_j' = Q^{(j)}[\iota^{(j)}]x_j + \overline{Q^{(j)}}[\overline{\iota^{(j)}}]\overline{x_j} + 2iR_j\{\iota^{(j)}\} y_j,$$

and where

$$\mathscr{J}\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, z\right) = \chi\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, Q\right)\mathscr{N}(\gamma z + \delta)^{n/2}$$

and

$$\chi\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}, Q\right)$$

is as in Theorem 1. Applying Lemma 2 with

$$\mathscr{L}^{(\kappa)} = \bigotimes_{1 \leq j \leq r_2} \begin{pmatrix} {}^t\eta_j \partial_j \\ {}^t\overline{\eta_j}\bar{\partial}_j \end{pmatrix}^{(\kappa)}$$

and setting $\xi = 0$ leads to (16).

Remark. In [12], we investigate a generalization of Andrianov's and Maloletkin's theta functions. We construct functions which satisfy a transformation property that generalizes the transformation law of Jacobi-like forms. Furthermore, we show how such functions can be used to construct Siegel modular forms. Theorems 2 and 3 can be extended in the same way. If one does not require that $R_j[w_j^+] = R_j[w_j^-] = 0$ in (11) and $R_j\{w_j\} = 0$ in (14), i.e., if the functions are not harmonic function over K, one can generalize (13) and (16). One can create "Jacobi-like forms over K" which can be used to construct modular forms over K.

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