IMRN International Mathematics Research Notices
2001, No. 7

A Remark on the Behavior of Theta Series of Degree n under Modular Transformations

Olav K. Richter

1 Introduction

A. Andrianov and G. Maloletkin [3], [4] and Andrianov [1], [2] investigate transformation properties of theta series corresponding to quadratic forms. Let F be a symmetric, integral matrix of rank m with even diagonal entries, and let q be the level of F; that is, qF^{-1} is integral and qF^{-1} has even diagonal entries. Suppose that F is of type (k, l), and let H be a majorant of F; that is, $HF^{-1}H = F$ and ${}^tH = H > 0$.

For Z in the Siegel upper half-plane, $\mathfrak{H}^{(n)} = \{Z \in M_{n,n}(\mathbb{C}) \mid Z = {}^tZ \text{ and } \operatorname{Im}(Z) > 0\}$, and for $\zeta_+, \zeta_- \in M_{m,n}(\mathbb{C})$ (with $m > n$), Andrianov and Maloletkin [4] define the theta series

$$\theta^{(\kappa,\lambda)}_{F,H,\zeta_+,\zeta_-}(Z) = \sum_{N \in M_{m,n}(\mathbb{Z})} \det\left({}^tNF\zeta_+\right)^{\kappa} \det\left({}^tNF\zeta_-\right)^{\lambda} \mathbf{e}\left\{\sigma\left(F[N]\operatorname{Re}(Z) + iH[N]\operatorname{Im}(Z)\right)\right\}, \quad (1)$$

where κ, λ are nonnegative integers, $\mathbf{e}\{x\} = \exp(\pi i x)$, $U[V] = {}^tVUV$ for any matrices U and V, and $\sigma(W)$ is the trace of the matrix W. Note that $\theta^{(\kappa,\lambda)}_{F,H,\zeta_+,\zeta_-}(Z)$ is identically zero if both n and $(\kappa + \lambda)$ are odd.

If F is positive definite and if $F[\zeta_+] = 0$, Andrianov and Maloletkin [3] show that $\theta^{(\kappa)}_{F,\zeta_+}(Z) = \theta^{(\kappa,0)}_{F,H,\zeta_+,\zeta_-}(Z)$ is a Siegel modular form on $\Gamma_0^{(n)}(q)$, where $\Gamma^{(n)} = \mathrm{Sp}_n(\mathbb{Z})$ and $\Gamma_0^{(n)}(q) = \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(n)}, C \equiv 0 \mod q\right\}$. Andrianov and Maloletkin [4] assume that $F\zeta_+ = H\zeta_+$, $F\zeta_- = -H\zeta_-$, and $F[\zeta_+] = H[\zeta_-] = 0$. They then determine the behavior of $\theta^{(\kappa,\lambda)}_{F,H,\zeta_+,\zeta_-}(Z)$ under modular transformations. We determine the behavior of $\theta^{(\kappa,\lambda)}_{F,H,\zeta_+,\zeta_-}(Z)$

Received 17 August 2000.
Communicated by Dennis Hejhal.

in the more general situation when $F[\zeta_+] \neq 0 \neq H[\zeta_-]$. In this case, $\theta^{(\kappa,\lambda)}_{F,H,\zeta_+,\zeta_-}(Z)$ is not "modular" but can be used to construct a function $\Theta_{F,H,\zeta_+,\zeta_-}(Z,X)$, which satisfies a transformation property that generalizes the transformation law of Jacobi-like forms introduced by D. Zagier [10] and P. Cohen, Y. Manin, and Zagier [6]. Furthermore, we see that functions that share the transformation property of $\Theta_{F,H,\zeta_+,\zeta_-}(Z,X)$ provide a method to construct Siegel modular forms.

Define (for fixed F, H, ζ_+, and ζ_-) a function of $Z \in \mathfrak{H}^{(n)}$ and $X \in M_{n,n}(\mathbb{C})$ by

$$\Theta_{F,H,\zeta_+,\zeta_-}(Z,X) = \sum_{\kappa \geq 0}\sum_{\lambda \geq 0}\left(\frac{2}{n!}\right)^{\kappa+\lambda}\frac{\theta^{(2\kappa,2\lambda)}_{F,H,\zeta_+\zeta_-}(Z)}{(2\kappa)!(2\lambda)!}\det(2\pi i X)^{\kappa}\det(2\pi i\overline{X})^{\lambda}. \tag{2}$$

Note that $\Theta_{F,H,\zeta_+,\zeta_-}(Z,X)$ does not vanish identically.

Our main result is the following theorem.

Theorem 1. Suppose $F\zeta_+ = H\zeta_+$ and $F\zeta_- = -H\zeta_-$. Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(q)$. Choose T integral and symmetric such that, for $D^* = CT + D$, $\det D^* = \pm p$ for an odd prime p. Then

$$\begin{aligned}\Theta_{F,H,\zeta_+,\zeta_-}&\left((AZ+B)(CZ+D)^{-1},(CZ+D)^{-2}X\right)\\ &= \phi(M,Z)\exp\left\{\frac{\det\left(F[\zeta_+]CX\right)}{\det(CZ+D)} + \frac{\det\left(H[\zeta_-]C\overline{X}\right)}{\det(C\overline{Z}+D)}\right\}\Theta_{F,H,\zeta_+,\zeta_-}(Z,X),\end{aligned} \tag{3}$$

where

$$\phi(M,Z) = \chi_F(M)\det(CZ+D)^{k/2}\det(C\overline{Z}+D)^{l/2}$$

and where $\chi_F(M)$ is an eighth root of unity. More precisely,

$$\begin{aligned}\phi(M,Z) = \varepsilon_p^{-m}&\left(\frac{2^m c^m \det F}{p}\right)\mathbf{e}\left\{\frac{(k-l)s}{4}\right\}\\ &\times\left|\det(C)\right|^{m/2}\left\{\det\left[-iC^{-1}(CZ+D)\right]\right\}^{k/2}\left\{\det\left[iC^{-1}\left(C\overline{Z}+D\right)\right]\right\}^{l/2},\end{aligned}$$

where $\varepsilon_p = 1$ for $p \equiv 1 \mod 4$, $\varepsilon_p = i$ for $p \equiv 3 \mod 4$, $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, c is any diagonal element of $(pD^{*-1}C)$ with $(c,p) = 1$, and s is the signature of $D^{*-1}C$. If C is singular, then C^{-1} is interpreted as ${}^tD(C\,{}^tD)^{-1}$, where $(C\,{}^tD)^{-1}$ is the Moore-Penrose generalized inverse, and the determinants are interpreted as the product of the nonzero eigenvalues. Furthermore, $|\det(C)|^{1/2}$ is positive, and $\{\det[-iC^{-1}(CZ+D)]\}^{1/2}$ and $\{\det[iC^{-1}(C\overline{Z}+D)]\}^{1/2}$ are given by analytic continuation from the principal value when $Z = -C^{-1}D + iY$. □

2 The symplectic theta function

The symplectic group

$$\mathrm{Sp}_n(\mathbb{R}) = \left\{ M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mid M \in M_{2n,2n}(\mathbb{R}) \text{ such that } J[M] = J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right) \right\},$$

where I_n is the $(n \times n)$-identity matrix, acts on the Siegel upper half-plane

$$\mathfrak{H}^{(n)} = \left\{ Z \in M_{n,n}(\mathbb{C}) \mid Z = {}^tZ \text{ and } \operatorname{Im}(Z) > 0 \right\}.$$

The action of M on Z is given by

$$M \circ Z = (AZ + B)(CZ + D)^{-1}.$$

Let $\Gamma^{(n)} = \mathrm{Sp}_n(\mathbb{Z})$. The theta subgroup

$$\Gamma_\vartheta^{(n)} = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(n)} \mid A\,{}^tB, C\,{}^tD \text{ have even diagonal entries} \right\}$$

acts on the symplectic theta function

$$\vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \sum_{m \in \mathbb{Z}^n} \mathbf{e}\left\{ Z[m + v] - 2\,{}^tmu - {}^tvu \right\}, \tag{4}$$

where u and v are column vectors in $\mathbb{C}^n$. It is well known (see, e.g., [7]) that, for

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{in } \Gamma_\vartheta^{(n)},$$

$$\vartheta\left(M \circ Z, M\begin{pmatrix} u \\ v \end{pmatrix}\right) = \chi(M)\left[\det(CZ + D)\right]^{1/2} \vartheta\left(Z, \begin{pmatrix} u \\ v \end{pmatrix}\right), \tag{5}$$

where $\chi(M)$ is an eighth root of unity which depends upon the chosen square root of $\det(CZ + D)$ but which is otherwise independent of Z, u, and v. It is also known that $\chi(M)$ can be expressed in terms of Gaussian sums. H. Stark [8] determines $\chi(M)$ in the important special case when both C and D are nonsingular and when pD^{-1} is integral for some odd prime p. R. Styer [9] extends Stark's results and includes the case where C is singular. We use the following theorem of [9] to compute the explicit theta multiplier of $\Theta_{F,H,\zeta_+,\zeta_-}(Z, X)$.

Theorem 2 (Stark, Styer). Suppose $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is in $\Gamma_\vartheta^{(n)}$, where D^{-1} exists. Suppose further that pD^{-1} is integral and that $\det D = \pm p^h$ for some odd prime p. Then

$$\chi(M)\left[\det(CZ+D)\right]^{1/2} = \varepsilon_p^{-h}\left(\frac{2^h\det\left[(pD^{-1}C)^{(h)}\right]}{p}\right)\mathbf{e}\left\{\frac{s}{4}\right\}\left|\det(C)\right|^{1/2}\left\{\det\left[-iC^{-1}(CZ+D)\right]\right\}^{1/2}, \tag{6}$$

where $\varepsilon_p = 1$ for $p \equiv 1 \mod 4$, $\varepsilon_p = i$ for $p \equiv 3 \mod 4$, $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, $(pD^{-1}C)^{(h)}$ is any $(h \times h)$-principal submatrix of $pD^{-1}C$ which is nonsingular mod p, and s is the signature (the number of positive eigenvalues minus the number of negative eigenvalues) of $D^{-1}C$. If C is singular, then C^{-1} is interpreted as ${}^tD(C\,{}^tD)^{-1}$, where $(C\,{}^tD)^{-1}$ is the Moore-Penrose generalized inverse (see [5]), and the determinants are interpreted as the product of the nonzero eigenvalues. Furthermore, $|\det(C)|^{1/2}$ is positive and $\{\det[-iC^{-1}(CZ+D)]\}^{1/2}$ is given by analytic continuation from the principal value when $Z = -C^{-1}D + iY$. □

3 Proof of Theorem 1

Now we turn to the proof of Theorem 1. Let F be a symmetric, integral matrix of rank m with even diagonal entries, and let q be the level of F; that is, qF^{-1} is integral and qF^{-1} has even diagonal entries. Suppose that F is of type (k,l), and let H be a majorant of F; that is, $HF^{-1}H = F$ and ${}^tH = H > 0$.

Andrianov and Maloletkin [4] regard $\theta_{F,H}(Z) = \theta^{(0,0)}_{F,H,\zeta_+,\zeta_-}(Z)$ as a symplectic theta function and then apply (5). Let $U \otimes V = (u_{ij}V)$ denote the Kronecker product of two matrices U and V. For $Z = X + iY \in \mathfrak{H}^{(n)}$, set $\widetilde{Z} = X \otimes F + iY \otimes H \in \mathfrak{H}^{(nm)}$. One verifies that $\theta_{F,H}(Z) = \vartheta\big(\widetilde{Z}, \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\big)$. Furthermore, if $M = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \Gamma_0^{(n)}(q)$, then $\widetilde{M} = \left(\begin{smallmatrix}\widetilde{A} & \widetilde{B}\\ \widetilde{C} & \widetilde{D}\end{smallmatrix}\right) = \left(\begin{smallmatrix}A\otimes I_m & B\otimes F\\ C\otimes F^{-1} & D\otimes I_m\end{smallmatrix}\right) \in \Gamma_\vartheta^{(nm)}$, and Andrianov and Maloletkin [4] show that

$$\theta_{F,H}(M \circ Z) = \chi\big(\widetilde{M}\big)\det\big(\widetilde{C}\widetilde{Z} + \widetilde{D}\big)^{1/2}\vartheta\big(\widetilde{Z}, \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\big) = \phi(M,Z)\theta_{F,H}(Z), \tag{7}$$

where $\phi(M,Z) = \chi_F(M)\det(CZ+D)^{k/2}\det(C\overline{Z}+D)^{l/2}$ and $\chi_F(M)$ is an eighth root of unity. Unfortunately, Andrianov and Maloletkin [4] can determine $\chi_F(M)$ only when m is even. In the special case where F is positive definite ($F = H$), Styer [9] uses Theorem 2 to determine $\chi_F(M)$ for all m. It is easy to see that Styer's method can also be applied to determine $\chi_F(M)$ for all m, even if F is indefinite.

Styer [9] shows that if $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \Gamma^{(n)}$, then there exists a symmetric, integral matrix T such that $\det(CT + D) = \pm p$ for some arbitrarily large prime p. As in Styer [9], we set $Z^* = Z - T$, $M^* = M\left(\begin{smallmatrix}I_n & T\\ 0 & I_n\end{smallmatrix}\right)$, and we observe that

$$\theta_{F,H}(M \circ Z) = \chi\big(\widetilde{M^*}\big)\det\big(\widetilde{C}\widetilde{Z^*} + \widetilde{D^*}\big)^{1/2}\theta_{F,H}(Z). \tag{8}$$

We apply Theorem 2 and find that

$$\phi(M,Z) = \varepsilon_p^{-m}\left(\frac{2^m c^m \det F}{p}\right) \mathbf{e}\left\{\frac{(k-l)s}{4}\right\} \times \left|\det(C)\right|^{m/2}\left\{\det\left[-iC^{-1}(CZ+D)\right]\right\}^{k/2}\left\{\det\left[iC^{-1}(C\overline{Z}+D)\right]\right\}^{l/2}, \tag{9}$$

where $\varepsilon_p = 1$ for $p \equiv 1 \mod 4$, $\varepsilon_p = i$ for $p \equiv 3 \mod 4$, $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, c is any diagonal element of $(pD^{*-1}C)$ with $(c,p) = 1$, and s is the signature of $D^{*-1}C$. If C is singular, then C^{-1} is interpreted as ${}^tD(C\,{}^tD)^{-1}$, where $(C\,{}^tD)^{-1}$ is the Moore-Penrose generalized inverse, and the determinants are interpreted as the product of the nonzero eigenvalues. Furthermore, $|\det(C)|^{1/2}$ is positive, and $\{\det[-iC^{-1}(CZ+D)]\}^{1/2}$ and $\{\det[iC^{-1}(C\overline{Z}+D)]\}^{1/2}$ are given by analytic continuation from the principal value when $Z = -C^{-1}D + iY$.

Note that if m is even and if $\det D = \pm p$, formula (9) matches the result from Andrianov and Maloletkin [4], and we have

$$\phi(M,Z) = (\operatorname{sgn}(\det D))^{(k-l)/2}\left(\frac{(-1)^{m/2}\det F}{|\det D|}\right)\det(CZ+D)^{(k-l)/2}\left|\det(C\overline{Z}+D)\right|^{l}. \tag{10}$$

To prove Theorem 1, we proceed as in Andrianov and Maloletkin [4], and we differentiate (7). For this purpose, we state [3, Lemma 3] in a slightly more general form.

Lemma 1. Let $1 \le n < m$. Let $P, R \in M_{n,n}(\mathbb{C})$, ${}^tP = P$, and $Q, \eta \in M_{m,n}(\mathbb{C})$. Denote by $\xi = (\xi_{\alpha\beta})$ an $(m \times n)$-variable matrix and by $\partial = (\partial/\partial\xi_{\alpha\beta})$ the corresponding matrix of differentiation operators. Set $L = L_\eta = \det({}^t\eta\partial)$. Then, for $\nu \ge 1$, we have

$$L^\nu\left(\mathbf{e}\left\{\sigma\left(P\,{}^t\xi\xi + 2\,{}^tQ\xi + R\right)\right\}\right) = f_{P,Q,\eta,\nu}(\xi)\,\mathbf{e}\left\{\sigma\left(P\,{}^t\xi\xi + 2\,{}^tQ\xi + R\right)\right\}, \tag{11}$$

where

$$f_{P,Q,\eta,\nu}(\xi) = \sum_{j=0}^{[\nu/2]} \nu!\left(\frac{n!}{2}\right)^j \frac{\det\left(2\pi i P\,{}^t\eta\eta\right)^j}{j!}\frac{\det\left(2\pi i\left(P\,{}^t\xi + {}^tQ\right)\eta\right)^{\nu-2j}}{(\nu-2j)!}. \qquad \square$$

Remark. If in addition ${}^t\eta\eta = 0$, then $f_{P,Q,\eta,\nu}(\xi) = \det(2\pi i(P\,{}^t\xi + {}^tQ)\eta)^\nu$ and our lemma simplifies to [3, Lemma 3].

Proof. Andrianov and Maloletkin [3] point out that

$$L\big(\mathbf{e}\{\sigma(P\,{}^t\xi\xi+2\,{}^tQ\xi+R)\}\big)=\det\big(2\pi i(P\,{}^t\xi+{}^tQ)\eta\big)\mathbf{e}\{\sigma(P\,{}^t\xi\xi+2\,{}^tQ\xi+R)\}$$

and that, for indices β, γ, μ, and ι,

$$\left(\sum_{\alpha=1}^{m}\eta_{\alpha\beta}\frac{\partial}{\partial\xi_{\alpha\gamma}}\right)\left(\sum_{x=1}^{m}(\xi P+Q)_{x\mu}\eta_{x\iota}\right)=P_{\gamma\mu}({}^t\eta\eta)_{\beta\iota}.$$

Hence

$$L\big(\det(2\pi i(P\,{}^t\xi+{}^tQ)\eta)\big)=\det(2\pi iP)L\big(\det({}^t\xi\eta)\big).$$

The operator L can be written as follows:

$$L=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\sum_{\gamma}\prod_{j=1}^{n}\eta_{\gamma(j)j}\frac{\partial}{\partial\xi_{\gamma(j)\sigma(j)}},$$

where the second summation is over all maps γ from $\{1,\dots,n\}$ to $\{1,\dots,m\}$. We find that

$$\begin{aligned}L\big(\det({}^t\xi\eta)\big)&=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\sum_{\gamma}\sum_{\tau\in S_n}\operatorname{sgn}(\sigma\circ\tau)\prod_{j=1}^{n}\eta_{\gamma(j)j}\,\eta_{\gamma\circ\tau(j)j}\\&=n!\sum_{\tau\in S_n}\operatorname{sgn}(\tau)\sum_{\gamma}\prod_{j=1}^{n}\eta_{\gamma(j)j}\,\eta_{\gamma(j)\tau(j)}\\&=n!\det({}^t\eta\eta),\end{aligned}$$

and therefore

$$L\big(\det(2\pi i(P\,{}^t\xi+{}^tQ)\eta)\big)=n!\det(2\pi iP\,{}^t\eta\eta).$$

Induction on ν then gives the desired result. ■

Now we are ready to determine the behavior of $\theta^{(\kappa,\lambda)}_{F,H,\zeta_+,\zeta_-}(Z)$ under modular transformations.

Theorem 3. Suppose $F\zeta_+=H\zeta_+$ and $F\zeta_-=-H\zeta_-$. Let $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma_0^{(n)}(q)$. Then

$$\begin{aligned}&\theta^{(\kappa,\lambda)}_{F,H,\zeta_+,\zeta_-}(M\circ Z)\\&\quad=\phi(M,Z)\sum_{j=0}^{[\kappa/2]}\sum_{g=0}^{[\lambda/2]}\kappa!\lambda!\left(\frac{n!}{2}\right)^{j+g}\frac{\det\left(\frac{F[\zeta_+]C}{2\pi i}\right)^j}{j!}\frac{\det\left(\frac{H[\zeta_-]C}{2\pi i}\right)^g}{g!}\\&\qquad\times\frac{\det(CZ+D)^{\kappa-j}}{(\kappa-2j)!}\frac{\det(C\overline{Z}+D)^{\lambda-g}}{(\lambda-2g)!}\theta^{(\kappa-2j,\lambda-2g)}_{F,H,\zeta_+\zeta_-}(Z),\end{aligned}\tag{12}$$

where $\phi(M,Z)$ is given by (9). □

Remark. In the special case when $F[\zeta_+] = H[\zeta_-] = 0$, our theorem reduces to [4, Theorem 2].

Proof. Let $\xi = \left(\begin{smallmatrix}\xi_+\\ \xi_-\end{smallmatrix}\right) \in M_{m,n}(\mathbb{C})$, where $\xi_+ \in M_{k,n}(\mathbb{C})$ and $\xi_- \in M_{l,n}(\mathbb{C})$, and let ∂_+ and ∂_- be the corresponding matrices of differential operators. Set $\widetilde{\eta}_+ = S^{-1}\zeta_+$ and $\widetilde{\eta}_- = S^{-1}\zeta_-$, where $S \in GL_n(\mathbb{R})$ such that $F[S] = E_{k,l}$ and $H[S] = I_m$ and where $E_{k,l} = \left(\begin{smallmatrix}I_k & \\ & -I_l\end{smallmatrix}\right)$. Note that $\widetilde{\eta}_+ = \left(\begin{smallmatrix}\eta_+\\ 0\end{smallmatrix}\right)$ and $\widetilde{\eta}_- = \left(\begin{smallmatrix}0\\ \eta_-\end{smallmatrix}\right)$, for $\eta_+ \in M_{k,n}$ and $\eta_- \in M_{l,n}$, since $(F-H)[S]\widetilde{\eta}_+ = -2\left(\begin{smallmatrix}0 & \\ & I_l\end{smallmatrix}\right)\widetilde{\eta}_+ = 0$ and $(F+H)[S]\widetilde{\eta}_+ = -2\left(\begin{smallmatrix}I_k & \\ & 0\end{smallmatrix}\right)\widetilde{\eta}_- = 0$. Thus [4, (4.4)] can be stated as follows:

$$\begin{aligned}
&\sum_N \mathbf{e}\Big\{\sigma\big(P_+\,{}^t\xi_+\xi_+ + 2\,{}^tQ_+\xi_+ + R_+\big)\Big\}\,\mathbf{e}\,\Big\{\sigma\big(P_-\,{}^t\xi_-\xi_- + 2\,{}^tQ_-\xi_- + R_-\big)\Big\}\\
&\quad = \phi(M,Z)\sum_N \mathbf{e}\Big\{\sigma\big(P'_+\,{}^t\xi_+\xi_+ + 2\,{}^tQ'_+\xi_+ + R'_+\big)\Big\}\\
&\qquad \times \mathbf{e}\Big\{\sigma\big(P'_-\,{}^t\xi_-\xi_- + 2\,{}^tQ'_-\xi_- + R'_-\big)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
&P_+ = Z(CZ+D)^{-1}D, \qquad P_- = -\overline{Z}\big(C\overline{Z}+D\big)^{-1}D, \qquad P'_+ = Z, \qquad P'_- = -\overline{Z},\\
&Q_+ = {}^t\Big(-Z(CZ+D)^{-1}\,{}^tN\,{}^tS^{-1}\left(\begin{smallmatrix}I_k\\ 0\end{smallmatrix}\right)\Big), \qquad Q_- = {}^t\Big(\overline{Z}(C\overline{Z}+D)^{-1}\,{}^tN\,{}^tS^{-1}\left(\begin{smallmatrix}0\\ I_l\end{smallmatrix}\right)\Big),\\
&Q'_+ = {}^t\Big(-Z\,{}^tN\,{}^tS^{-1}\left(\begin{smallmatrix}I_k\\ 0\end{smallmatrix}\right)\Big), \qquad Q'_- = {}^t\Big(\overline{Z}\,{}^tN\,{}^tS^{-1}\left(\begin{smallmatrix}0\\ I_l\end{smallmatrix}\right)\Big),\\
&R_+ = \mathrm{Re}\big(M\circ Z\big)F[N], \qquad R_- = i\,\mathrm{Im}\big(M\circ Z\big)H[N],\\
&R'_+ = \mathrm{Re}(Z)F[N], \qquad R'_- = i\,\mathrm{Im}(Z)H[N]
\end{aligned}$$

and where $\phi(M,Z)$ is defined in (7) and given explicitly in (9). We apply Lemma 1 with $L = L_+L_- = \det({}^t\eta_+\partial_+)^\kappa \det({}^t\eta_-\partial_-)^\lambda$, and we set $\xi_+ = \xi_- = 0$. Then (12) follows from observing that ${}^t\widetilde{\eta}_+\widetilde{\eta}_+ = {}^t\eta_+\eta_+ = F[\zeta_+]$ and ${}^t\widetilde{\eta}_-\widetilde{\eta}_- = {}^t\eta_-\eta_- = H[\zeta_-]$. ■

The transformation formula (3) is an immediate consequence of Theorem 3:

$$\begin{aligned}
&\Theta_{F,H,\zeta_+,\zeta_-}\big(M\circ Z, (CZ+D)^{-2}X\big)\\
&\quad = \phi(M,Z)\sum_{\kappa\ge 0}\sum_{j=0}^{\kappa}\left(\frac{2}{n!}\right)^{\kappa-j}\frac{\det(2\pi i X)^{\kappa-j}}{(2\kappa-2j)!}\,\frac{\left(\dfrac{\det\big(F[\zeta_+]CX\big)}{\det(CZ+D)}\right)^j}{j!}\\
&\qquad \times\sum_{\lambda\ge 0}\sum_{g=0}^{\lambda}\left(\frac{2}{n!}\right)^{\lambda-g}\frac{\det\big(2\pi i\overline{X}\big)^{\lambda-g}}{(2\lambda-2g)!}\,\frac{\left(\dfrac{\det\big(H[\zeta_-]C\overline{X}\big)}{\det\big(C\overline{Z}+D\big)}\right)^g}{g!}\,\theta^{(2\kappa-2j,2\lambda-2g)}_{F,H,\zeta_+,\zeta_-}(Z)\\
&\quad = \phi(M,Z)\exp\left\{\frac{\det\big(F[\zeta_+]CX\big)}{\det(CZ+D)} + \frac{\det\big(H[\zeta_-]C\overline{X}\big)}{\det\big(C\overline{Z}+D\big)}\right\}\Theta_{F,H,\zeta_+,\zeta_-}(Z,X).
\end{aligned}$$

4 Conclusion

It would be very interesting to find other examples that satisfy the transformation property (3). We now explain how functions that satisfy a special case of (3) allow us to construct Siegel modular forms.

For $j = 1, 2$, let $f_j(Z, X)$ be holomorphic functions on $\mathfrak{H}^{(n)} \times M_{n,n}(\mathbb{C})$ of the form

$$f_j(Z, X) = \sum_{\nu \geq 0} f_\nu^{(j)}(Z) \det(2\pi i X)^\nu. \tag{13}$$

Suppose that, for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(n)}$,

$$f_j\left(M \circ Z, (CZ + D)^{-2} X\right) = \chi_j(M) \det(CZ + D)^{k_j} \exp\left\{\frac{\det(CX)}{\det(CZ + D)}\right\} f_j(Z, X) \tag{14}$$

for some nonnegative integers k_j and characters χ_j. Notice that, when $n = 1$, f_j are Jacobi-like forms in the sense of Zagier [10] and Cohen, Manin, and Zagier [6]. Then

$$F(Z, X) = f_1(Z, X) f_2\left(Z, \mathbf{e}\left\{\frac{1}{n}\right\} X\right) = \sum_{\nu \geq 0} F_\nu(Z) \det(2\pi i X)^\nu,$$

where $F_\nu(Z)$ is a Siegel modular form of weight $k_1 + k_2 + 2\nu$ and character $\chi_1 \chi_2$. Hence, as an application of Theorem 3, we have the following corollary.

Corollary 1. Suppose F is positive definite ($F = H$) such that $\det(F[\zeta_+]) = 1$. Set

$$\Theta_{F,\zeta_+}(Z, X) = \sum_{\kappa \geq 0} \left(\frac{2}{n!}\right)^\kappa \frac{\theta_{F,H,\zeta_+\zeta_-}^{(2\kappa,0)}(Z)}{(2\kappa)!} \det(2\pi i X)^\kappa \tag{15}$$

and

$$F(Z, X) = \Theta_{F,\zeta_+}(Z, X) \Theta_{F,\zeta_+}\left(Z, \mathbf{e}\left\{\frac{1}{n}\right\} X\right) = \sum_{\nu \geq 0} F_\nu(Z) \det(2\pi i X)^\nu. \tag{16}$$

Then

$$F_\nu(Z) = \left(\frac{2}{n!}\right)^\nu \sum_{\kappa=0}^{\nu} (-1)^\kappa \frac{\theta_{F,H\zeta_+,\zeta_-}^{(2\kappa,0)}(Z)}{(2\kappa)!} \frac{\theta_{F,H,\zeta_+,\zeta_-}^{(2\nu-2\kappa,0)}(Z)}{(2\nu - 2\kappa)!} \tag{17}$$

is a Siegel modular form on $\Gamma_0^{(n)}(q)$ of weight $m + 2\nu$ and character $(\chi_F(M))^2$, with $\chi_F(M)$ as in Theorem 1. □

References

[1] A. N. Andrianov, *Spherical theta-series*, Math. USSR-Sb. **62** (1989), 289–304.

[2] ———, *Symmetries of harmonic theta-functions of integral quadratic forms*, Russian Math. Surveys **50** (4) (1995), 661–700.

[3] A. N. Andrianov and G. N. Maloletkin, *Behavior of theta series of degree* N *under modular substitutions*, Math. USSR-Izvestija **9** (2) (1975), 227–241.

[4] ———, *Behavior of theta series of genus* n *of indefinite quadratic forms under modular substitutions*, Proc. Steklov. Inst. Math. **4** (1980), 1–12.

[5] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley-Interscience, New York, 1974.

[6] P. B. Cohen, Y. Manin, and D. Zagier, *Automorphic Pseudodifferential Operators*, Progr. Nonlinear Differential Equations Appl. **26**, Birkhäuser, Boston, 1997, 17–47.

[7] M. Eichler, *Introduction to the Theory of Algebraic Numbers and Functions*, Pure Appl. Math. **23**, Academic Press, New York, 1966.

[8] H. M. Stark, *On the transformation formula for the symplectic theta function and applications*, J. Fac. Sci. Univ. Tokyo, Sect. 1A Math. **29** (1982), 1–12.

[9] R. Styer, *Prime determinant matrices and the symplectic theta function*, Amer. J. Math. **106** (1984), 645–664.

[10] D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), 57–75.

Department of Mathematics, University of California, Santa Cruz, California 95064, USA;
richter@math.ucsc.edu