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RAMANUJAN CONGRUENCES FOR SIEGEL MODULAR FORMS

MICHAEL DEWAR*,‡ and OLAV K. RICHTER†,§

*Department of Mathematics and Statistics, Queen's University
Kingston, ON K7L 3N6, Canada

†Department of Mathematics, University of North Texas
Denton TX 76203, USA

‡mdewar@mast.queensu.ca

§richter@unt.edu

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We determine conditions for the existence and non-existence of Ramanujan-type congruences for Jacobi forms. We extend these results to Siegel modular forms of degree 2 and as an application, we establish Ramanujan-type congruences for explicit examples of Siegel modular forms.

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1. Introduction and Statement of Results

Congruences in the coefficients of automorphic forms have been the subject of much study. A famous early example involves the partition function $p(n)$ which counts the number of ways of writing n as a sum of non-increasing positive integers. Ramanujan established

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned} \tag{1.1}$$

which are now simply called *Ramanujan congruences*. More generally, an elliptic modular form with Fourier coefficients $a(n)$ is said to have a *Ramanujan-type congruence at* $b \pmod{p}$ if $a(pn+b) \equiv 0 \pmod{p}$, where p is a prime. Ahlgren and Boylan ([1]) build on work by Kiming and Olsson ([9]) to prove that (1.1) are the only such congruences for the partition function. Nevertheless, congruences of non-Ramanujan-type also exist, as Ono ([13]) demonstrates. (See also [14, Chap. 5] for an account of congruences for the partition function.) The existence and non-existence of Ramanujan-type congruences for elliptic modular forms have recently been studied by Cooper, Wage, and Wang ([4]) and Sinick ([20]). See also [5], which generalizes [1] to provide a method to find all Ramanujan-type congruences in certain weakly holomorphic modular forms.

In this paper, we investigate Ramanujan-type congruences for Siegel modular forms of degree 2. Throughout, $Z := \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ is a variable in the Siegel upper half-space of degree 2, $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi i z}$, $q' := e^{2\pi i\tau'}$, and $\mathbb{D} := (2\pi, i)^{-2}\left(4\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'} - \frac{\partial^2}{\partial z^2}\right)$ is the generalized theta operator, which acts on Fourier expansions of Siegel modular forms as follows:

$$\mathbb{D}\left(\sum_{\substack{T={}^tT\geq 0\\ T\text{ even}}} a(T)e^{\pi i\,\mathrm{tr}(TZ)}\right) = \sum_{\substack{T={}^tT\geq 0\\ T\text{ even}}} \det(T)a(T)e^{\pi i\,\mathrm{tr}(TZ)},$$

where tr denotes the trace, and where the sum is over all symmetric, semi-positive definite, integral, and even 2×2 matrices. Additionally, we always let $p \geq 5$ be a prime and (for simplicity) we always assume that the weight k is an even integer.

Definition 1.1. A Siegel modular form $F = \sum a(T)e^{\pi i\,\mathrm{tr}(TZ)}$ with p-integral rational coefficients has a Ramanujan-type congruence at $b \pmod p$ if $a(T) \equiv 0 \pmod p$ for all T with $\det T \equiv b \pmod p$.

Note that such congruences at $0 \pmod p$ have already been studied in [3] and our main result in this paper complements [3] by giving the case $b \not\equiv 0 \pmod p$.

Theorem 1.2. *Let*

$$F(Z) = \sum_{\substack{n,r,m\in\mathbb{Z}\\ n,m,4nm-r^2\geq 0}} A(n,r,m)q^n\zeta^r q'^m$$

be a Siegel modular form of degree 2 *and even weight* k *with* p*-integral rational coefficients and let* $b \not\equiv 0 \pmod p$. *Then* F *has a Ramanujan-type congruence at* $b \pmod p$ *if and only if*

$$\mathbb{D}^{\frac{p+1}{2}}(F) \equiv -\left(\frac{b}{p}\right)\mathbb{D}(F) \pmod p, \tag{1.2}$$

where $\left(\frac{\cdot}{p}\right)$ *is the Legendre symbol. Moreover, if* $p > k$, $p \neq 2k-1$, *and there exists an* $A(n,r,m)$ *with* $p \nmid \gcd(n,m)$ *such that* $A(n,r,m) \not\equiv 0 \pmod p$, *then* F *does not have a Ramanujan-type congruence at* $b \pmod p$.

Remarks. (1) If F in Theorem 1.2 has a Ramanujan-type congruence at $b \not\equiv 0 \pmod p$, then it also has such congruences at $b' \pmod p$ whenever $\left(\frac{b}{p}\right) = \left(\frac{b'}{p}\right)$, i.e. there are $\frac{p-1}{2}$ or $p-1$ such congruences.

(2) The condition $p \neq 2k-1$ in the second part of Theorem 1.2 is necessary since there are Siegel modular forms F of weight $\frac{p+1}{2}$ such that $F \not\equiv 0 \pmod{p}$ and $\mathbb{D}(F) \equiv 0 \pmod{p}$. For example, let F be the Siegel Eisenstein series of weight 4 normalized by $a\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1$ and take $p = 7$. Such Siegel modular forms satisfy (1.2) for any b and hence have Ramanujan-type congruences at all $b \not\equiv 0 \pmod{p}$. The condition that there exists an $A(n,r,m) \not\equiv 0 \pmod{p}$ where $p \nmid \gcd(n,m)$ is also necessary since there exist Siegel modular forms F of weight $p-1$ such that $F \equiv 1 \pmod{p}$ (see [12, Theorem 4.5]). Such forms have Ramanujan-type congruences at all $b \not\equiv 0 \pmod{p}$.

In Sec. 2, we investigate congruences of Jacobi forms and, in particular, we establish criteria for the existence and non-existence of Ramanujan-type congruences for Jacobi forms. In Sec. 3, we use such congruences for Jacobi forms to prove Theorem 1.2. Using our results, it is now a finite computation to find Ramanujan-type congruences at all $b \not\equiv 0 \pmod{p}$ for any Siegel modular form. We give several explicit examples. Finally, we present a construction of Siegel modular forms that have Ramanujan-type congruences at $b \pmod{p}$ for arbitrary primes $p \geq 5$.

2. Congruences and Filtrations of Jacobi Forms

Let $J_{k,m}$ be the vector space of Jacobi forms of even weight k and index m (for details on Jacobi forms, see [6]). The heat operator $L_m := (2\pi i)^{-2}\left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}\right)$ is a natural tool in the theory of Jacobi forms and plays an important role in this section. In particular, if $\phi = \sum c(n,r) q^n \zeta^r$, then

$$L_m \phi := L_m(\phi) = \sum (4nm - r^2) c(n,r) q^n \zeta^r. \tag{2.1}$$

Set

$$\widetilde{J}_{k,m} := \{\phi \pmod{p} : \phi(\tau,z) \in J_{k,m} \cap \mathbb{Z}_{(p)}[\zeta,\zeta^{-1}][[q]]\},$$

where $\mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$ denotes the local ring of p-integral rational numbers. If $\phi \in \widetilde{J}_{k,m}$, then we denote its filtration modulo p by

$$\Omega(\phi) := \inf\{k : \phi \pmod{p} \in \widetilde{J}_{k,m}\}.$$

Recall the following facts on Jacobi forms modulo p:

Proposition 2.1 ([21]). *Let $\phi(\tau,z) \in J_{k,m} \cap \mathbb{Z}[\zeta,\zeta^{-1}][[q]]$ and $\psi(\tau,z) \in J_{k',m'} \cap \mathbb{Z}[\zeta,\zeta^{-1}][[q]]$ such that $0 \not\equiv \phi \equiv \psi \pmod{p}$. Then $k \equiv k' \pmod{p-1}$ and $m = m'$.*

Proposition 2.2 ([18]). *If $\phi(\tau,z) \in J_{k,m} \cap \mathbb{Z}[\zeta,\zeta^{-1}][[q]]$, then $L_m\phi \pmod{p} \in \widetilde{J}_{k+p+1,m}$. Moreover, we have*

$$\Omega(L_m\phi) \leq \Omega(\phi) + p + 1,$$

with equality if and only if $p \nmid (2\Omega(\phi) - 1)m$.

We will now explore Ramanujan-type congruences for Jacobi forms.

Definition 2.3. For $\phi(\tau, z) = \sum c(n,r) q^n \zeta^r \in \widetilde{J}_{k,m}$, we say that ϕ has a Ramanujan-type congruence at $b \pmod{p}$ if $c(n,r) \equiv 0 \pmod{p}$ whenever $4nm - r^2 \equiv b \pmod{p}$.

Equation (2.1) implies that a Jacobi form ϕ has a Ramanujan-type congruence at $0 \pmod{p}$ if and only if $L_m^{p-1}\phi \equiv \phi \pmod{p}$. More generally, ϕ has a Ramanujan-type congruence at $b \pmod{p}$ if and only if

$$L_m^{p-1}(q^{-\frac{b}{4m}}\phi) \equiv q^{-\frac{b}{4m}}\phi \pmod{p}.$$

Ramanujan-type congruences at $0 \pmod{p}$ for Jacobi forms have been considered in [17, 18]. The following proposition determines when Ramanujan-type congruences at $b \not\equiv 0 \pmod{p}$ for Jacobi forms exist.

Proposition 2.4. *Let $\phi \in \widetilde{J}_{k,m}$ and $b \not\equiv 0 \pmod{p}$. Then ϕ has a Ramanujan-type congruence at $b \pmod{p}$ if and only if $L_m^{\frac{p+1}{2}}\phi \equiv -\left(\frac{b}{p}\right) L_m \phi \pmod{p}$.*

Proof. If $\phi \in \mathbb{Z}_{(p)}[\zeta, \zeta^{-1}][[q]]$ and $f \in \mathbb{Z}_{(p)}[[q]]$, then $L_m(f\phi) = L_m(f)\phi + f L_m(\phi)$. This implies

$$\begin{aligned}
L_m^{p-1}(q^{-\frac{b}{4m}}\phi) &= \sum_{i=0}^{p-1} \binom{p-1}{i} L_m^{p-1-i}(q^{-\frac{b}{4m}}) L_m^i \phi \\
&= \sum_{i=0}^{p-1} \binom{p-1}{i} (-b)^{p-1-i} q^{-\frac{b}{4m}} L_m^i \phi \\
&\equiv q^{-\frac{b}{4m}} \sum_{i=0}^{p-1} b^{p-1-i} L_m^i \phi \pmod{p}.
\end{aligned}$$

In particular, ϕ has a Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ if and only if

$$0 \equiv \sum_{i=1}^{p-1} b^{p-1-i} L_m^i \phi \pmod{p}. \tag{2.2}$$

We now rewrite the $L_m^i \phi$ appearing in (2.2) using a standard decomposition of even weight Jacobi forms. See [6, §§8 and 9] for full details and also for the corresponding result for Jacobi forms of odd weight. Let $M_k^{(1)}$ denote the space of elliptic modular forms of weight k. Every even weight $\phi \in J_{k,m}$ can be written as

$$\phi = \sum_{j=0}^{m} f_j (\phi_{-2,1})^j (\phi_{0,1})^{m-j}, \tag{2.3}$$

where

$$\phi_{-2,1}(\tau, z) := (\zeta - 2 + \zeta^{-1}) + (-2\zeta^2 + 8\zeta - 12 + 8\zeta^{-1} - 2\zeta^{-2})q + \cdots$$

and

$$\phi_{0,1}(\tau, z) := (\zeta + 10 + \zeta^{-1}) + (10\zeta^2 - 64\zeta + 108 - 64\zeta^{-1} + 10\zeta^{-2})q + \cdots$$

are weak Jacobi forms with integer coefficients of index 1 and weights -2 and 0, respectively, and where each $f_j \in M^{(1)}_{k+2j}$ is uniquely determined. For any $m \geq 1$, the set $\mathcal{T} := \{\phi^j_{-2,1}\phi^{m-j}_{0,1}\}^m_{j=0}$ is linearly independent over $\mathbb{F}_p$. In fact, the coefficients of q^0 of the elements of $\mathcal{T}$ are linearly independent for the following reason: Let $X := \zeta - 2 + \zeta^{-1}$. It suffices to show that $\mathcal{S} := \{X^{m-j}(X + 12)^j\}^m_{j=0}$ is linearly independent over $\mathbb{F}_p$. But $X^{m-j}(X + 12)^j = X^m + \cdots + 12^j X^{m-j}$, and one finds that $\mathcal{S}$ is linearly independent over $\mathbb{F}_p$ since 12 is invertible. Returning to (2.3), if ϕ has p-integral rational coefficients, then so do all of the f_j's, since otherwise there is some $t \geq 1$ such that $0 \equiv p^t\phi \equiv \sum^m_{j=0}(p^t f_j)(\phi_{-2,1})^j(\phi_{0,1})^{m-j} \pmod{p}$ is a non-trivial linear independence relation for $\mathcal{T}$, contrary to what we have just shown.

By Proposition 2.2, for every i there exists $\psi_i \in J_{k+i(p+1),m}$ such that $L^i_m\phi \equiv \psi_i \pmod{p}$. Hence there exist $F_{i,j} \in M^{(1)}_{k+i(p+1)+2j}$ with p-integral rational coefficients such that

$$L^i_m\phi \equiv \psi_i \equiv \sum^m_{j=0} F_{i,j}(\phi_{-2,1})^j(\phi_{0,1})^{m-j} \pmod{p}$$

and hence (2.2) is equivalent to

$$0 \equiv \sum^m_{j=0}\left(\sum^{p-1}_{i=1} b^{p-1-i}F_{i,j}\right)(\phi_{-2,1})^j(\phi_{0,1})^{m-j} \pmod{p}.$$

By the linear independence of the elements of $\mathcal{T}$, we deduce that (2.2) is equivalent to $\sum^{p-1}_{i=1} b^{p-1-i}F_{i,j} \equiv 0 \pmod{p}$ for every j. Elliptic modular forms modulo p have a natural direct sum decomposition (see [22, Sec. 3] or [19, Theorem 2]) graded by their weights modulo $p - 1$. Thus (2.2) is equivalent to

$$0 \equiv b^{p-1-i}F_{i,j} + b^{(p-1)/2-i}F_{i+(p-1)/2,j} \pmod{p}$$

and hence also

$$F_{i+(p-1)/2,j} \equiv -\left(\frac{b}{p}\right)F_{i,j} \pmod{p}$$

for all $0 \leq j \leq m$ and $1 \leq i \leq \frac{p-1}{2}$. This implies, for all $1 \leq i \leq \frac{p-1}{2}$,

$$\begin{aligned} L^{i+\frac{p-1}{2}}_m\phi &\equiv \sum^m_{j=0} F_{i+\frac{p-1}{2},j}(\phi_{-2,1})^j(\phi_{0,1})^{m-j} \\ &\equiv \sum^m_{j=0} -\left(\frac{b}{p}\right)F_{i,j}(\phi_{-2,1})^j(\phi_{0,1})^{m-j} \\ &\equiv -\left(\frac{b}{p}\right)L^i_m\phi \pmod{p}. \end{aligned}$$

We conclude that

$$L_m^{\frac{p+1}{2}}\phi \equiv -\left(\frac{b}{p}\right) L_m\phi \pmod{p},$$

which completes the proof. □

By (2.1), $L_m^p\phi \equiv L_m\phi \pmod{p}$. We call $L_m\phi, L_m^2\phi, \ldots, L_m^{p-1}\phi$ the *heat cycle* of ϕ and we say that ϕ is in its own heat cycle whenever $L_m^{p-1}\phi \equiv \phi \pmod{p}$. Assume $L_m\phi \not\equiv 0 \pmod{p}$ and $p \nmid m$. By Proposition 2.2, applying L_m to ϕ increases the filtration of ϕ by $p+1$ except when $\Omega(\phi) \equiv \frac{p+1}{2} \pmod{p}$. If $\Omega(L_m^i\phi) \equiv \frac{p+1}{2} \pmod{p}$, then call $L_m^i\phi$ a *high point* and $L_m^{i+1}\phi$ a *low point* of the heat cycle. By Propositions 2.1 and 2.2,

$$\Omega(L_m^{i+1}\phi) = \Omega(L_m^i\phi) + p + 1 - s(p-1) \tag{2.4}$$

where $s \geq 1$ if and only if $L_m^i\phi$ is a high point and $s = 0$ otherwise. The structure of the heat cycle of a Jacobi form is similar to the structure of the theta cycle of a modular form (see [8, §7]). We will now prove a few basic properties:

Lemma 2.5. *Let $\phi \in \widetilde{J}_{k,m}$ with $p \nmid m$ a prime such that $L_m\phi \not\equiv 0 \pmod{p}$.*

(1) *If $j \geq 1$, then $\Omega(L_m^j\phi) \not\equiv \frac{p+3}{2} \pmod{p}$.*
(2) *The heat cycle of ϕ has a single low point if and only if there is some $j \geq 1$ with $\Omega(L_m^j\phi) \equiv \frac{p+5}{2} \pmod{p}$. Furthermore, $L_m^j\phi$ is the low point.*
(3) *If $j \geq 1$, then $\Omega(L_m^{j+1}\phi) \neq \Omega(L_m^j\phi) + 2$.*
(4) *The heat cycle of ϕ either has one or two high points.*

Proof. (1) If $\Omega(L_m^j\phi) \equiv \frac{p+3}{2} \pmod{p}$, then by (2.4) for $1 \leq n \leq p-1$ we have

$$\Omega(L_m^{j+n}\phi) = \Omega(L_m^j\phi) + n(p+1).$$

In particular, $L_m^{j+p-1}\phi \not\equiv L_m^j\phi \pmod{p}$, which is impossible.

(2) If $\Omega(L_m^j\phi) \equiv \frac{p+5}{2} \pmod{p}$, then by (2.4), for $1 \leq n \leq p-2$ we have

$$\Omega(L_m^{j+n}\phi) = \Omega(L_m^j\phi) + n(p+1)$$

and

$$\Omega(L_m^j\phi) = \Omega(L_m^{j+p-1}\phi) = \Omega(L_m^j\phi) + (p-1)(p+1) - s(p-1)$$

where s must be $p+1$ and there can be no other low point. On the other hand, if there is a single low point, then the filtration must increase $p-2$ consecutive times. The only way this is possible is if the low point has filtration $\frac{p+5}{2} \pmod{p}$.

(3) By Proposition 2.2, $\Omega(L_m^{j+1}\phi) = \Omega(L_m^j\phi) + 2$ can only happen when $\Omega(L_m^j\phi) \equiv \frac{p+1}{2} \pmod{p}$. Suppose $\Omega(L_m^{j+1}\phi) = \Omega(L_m^j\phi) + 2 \equiv \frac{p+5}{2} \pmod{p}$. By part (2), this implies that the filtration increases $p-2$ more times before falling. Hence $L_m^{j+p-1}\phi \not\equiv L_m^j\phi \pmod{p}$, which is impossible.

(4) Suppose there are $t \geq 2$ high points $L_m^{i_j}\phi$ where $1 \leq i_1 < \cdots < i_t \leq p-1$. By (2.4) and part (3) above, there are $s_j \geq 2$ such that

$$\Omega(L_m^{i_j+1}\phi) = \Omega(L_m^{i_j}\phi) + p + 1 - s_j(p-1). \tag{2.5}$$

Hence

$$\Omega(L_m\phi) = \Omega(L_m^p\phi) = \Omega(L_m\phi) + (p-1)(p+1) - \sum_{j=1}^{t} s_j(p-1),$$

and so $\sum s_j = p+1$. By (2.5), $\Omega(L_m^{i_j+1}\phi) \equiv \frac{p+1}{2} + 1 + s_j \pmod{p}$ and so there will be $p-1-s_j$ increases before the next fall. That is, for $1 \leq j \leq t$, $i_{j+1} - i_j = p - s_j$ where we take $i_{t+1} = i_1 + p - 1$ for convenience. Thus

$$p - 1 = i_{t+1} - i_1 = \sum_{j=1}^{t}(i_{j+1} - i_j) = \sum_{j=1}^{t}(p - s_j) = tp - (p+1),$$

i.e. $t = 2$. We conclude that the heat cycle of ϕ has at most two (i.e. one or two) high points. □

The following corollary of Proposition 2.4 is a key ingredient in the proof of Proposition 2.7 below.

Corollary 2.6. *If $\phi \in \widetilde{J}_{k,m}$ has a Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ and $L_m\phi \not\equiv 0 \pmod{p}$, then the heat cycle of ϕ has two low points which both have filtration congruent to $2 \pmod{p}$.*

Proof. Since $L_m^{\frac{p+1}{2}}\phi \equiv -\left(\frac{b}{p}\right)L_m\phi \pmod{p}$, we have $\Omega\left(L_m^{\frac{p+1}{2}}\phi\right) = \Omega\left(L_m\phi\right) = \Omega(L_m^p\phi)$. Hence there is a fall in the first half of the heat cycle and in the second half of the heat cycle. Furthermore, after a low point, the filtration increases $\frac{p-3}{2}$ times and then falls once. Thus, the filtration of the low points is $2 \pmod{p}$. □

Our final result in this section gives the non-existence of Ramanujan-type congruences of Jacobi forms.

Proposition 2.7. *Let $\phi \in \widetilde{J}_{k,m}$ where $k \geq 4$, $L_m(\phi) \not\equiv 0 \pmod{p}$ and let $b \not\equiv 0 \pmod{p}$. If $p > k$ and $p \nmid m$, then ϕ does not have a Ramanujan-type congruence at $b \pmod{p}$.*

Proof. Assume that ϕ has a Ramanujan-type congruence at $b \pmod{p}$. First suppose $k = \frac{p+1}{2}$. Then $\Omega(\phi) = \frac{p+1}{2}$ and so we must have $s \geq 1$ in (2.4). Since we need $\Omega(L_m\phi) \geq 0$, we must have $s = 1$ and hence $\Omega(L_m\phi) = \frac{p+5}{2}$. But by Lemma 2.5(2), this implies there is only one low point, contrary to Corollary 2.6.

Now suppose $k \neq \frac{p+1}{2}$. Then $\Omega(L_m\phi) = k + p + 1$. There must be a low point of the heat cycle with filtration either $k + p + 1$ or k. By Corollary 2.6, either $k + 1 \equiv 2 \pmod{p}$ or $k \equiv 2 \pmod{p}$. Both of these alternatives are impossible since $p > k \geq 4$. □

3. Proof of Theorem 1.2 and Examples

We employ the Fourier–Jacobi expansion of a Siegel modular form (as in [3]) to prove Theorem 1.2. Let $M_k^{(2)}$ denote the vector space of Siegel modular forms of degree 2 and even weight k (for details on Siegel modular forms, see, for example, [7, 10]).

Proof of Theorem 1.2. Let $F \in M_k^{(2)}$ be as in Theorem 1.2 with Fourier–Jacobi expansion $F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi i m\tau'}$, i.e. $\phi_m \in J_{k,m}$. Let $b \not\equiv 0 \pmod{p}$. Then F has a Ramanujan-type congruence at $b \pmod{p}$ if and only if for all m, ϕ_m has a Ramanujan-type congruence at b. By Proposition 2.4, it is equivalent that for all m

$$L_m^{\frac{p+1}{2}} \phi_m \equiv -\left(\frac{b}{p}\right) L_m \phi_m \pmod{p},$$

which is equivalent to (1.2), since

$$\mathbb{D}(F) = \sum_{m=0}^{\infty} L_m(\phi_m(\tau, z)) e^{2\pi i m\tau'}.$$

Now we turn to the second part of Theorem 1.2. Here we assume that $p > k$, $p \neq 2k-1$, and that there exists an $A(n, r, m)$ with $p \nmid \gcd(n, m)$ such that $A(n, r, m) \not\equiv 0 \pmod{p}$. Suppose that F has a Ramanujan-type congruence at $b \pmod{p}$. Then all Fourier–Jacobi coefficients ϕ_m have such a congruence at b. We would like to apply Proposition 2.7. First, $k \geq 4$, since F is non-constant and $M_k^{(2)} \subset \mathbb{C}$ if $k < 4$. Moreover, if $\phi_m \not\equiv 0 \pmod{p}$ with $p \nmid m$, then $\Omega(\phi_m) = k$ by Proposition 2.1 (since $p > k$ and F is non-constant modulo p) and $\Omega(L_m\phi_m) = k+p+1$ by Proposition 2.2. In particular, $L_m\phi_m \not\equiv 0 \pmod{p}$ and Proposition 2.7 implies that such a ϕ_m does not have a Ramanujan-type congruence at $b \pmod{p}$. Hence, if $p \nmid m$, then $\phi_m \equiv 0 \pmod{p}$, i.e. $A(n, r, m) \equiv 0 \pmod{p}$. By assumption, there exists an $A(n, r, m) \not\equiv 0 \pmod{p}$ with $p \nmid \gcd(n, m)$, which is only possible if $p \mid m$ and hence $p \nmid n$. However, $F(\tau, z, \tau') = F(\tau', z, \tau)$ and $p \nmid n$ together yield the contradiction $A(n, r, m) = A(m, r, n) \equiv 0 \pmod{p}$. We conclude that F does not have a Ramanujan-type congruence at $b \pmod{p}$. □

We will use Theorem 1.2 to discuss Ramanujan-type congruences for explicit examples of Siegel modular forms after reviewing a few facts on Siegel modular forms modulo p. Set

$$\widetilde{M}_k^{(2)} := \left\{ F \pmod{p} : F(Z) = \sum a(T) e^{\pi i \operatorname{tr}(TZ)} \in M_k^{(2)} \text{ where } a(T) \in \mathbb{Z}_{(p)} \right\}.$$

Recall the following two theorems on Siegel modular forms modulo p:

Theorem 3.1 ([12]). *There exists an $E \in M_{p-1}^{(2)}$ with p-integral rational coefficients such that $E \equiv 1 \pmod{p}$. Furthermore, if $F_1 \in M_{k_1}^{(2)}$ and $F_2 \in M_{k_2}^{(2)}$ have p-integral rational coefficients where $0 \not\equiv F_1 \equiv F_2 \pmod{p}$, then $k_1 \equiv k_2 \pmod{p-1}$.*

Theorem 3.2 ([2]). *If $F \in \widetilde{M}_k^{(2)}$, then $\mathbb{D}(F) \in \widetilde{M}_{k+p+1}^{(2)}$.*

Theorems 3.1 and 3.2 imply that for any $F \in \widetilde{M}_k^{(2)}$, we have

$$G := \mathbb{D}^{\frac{p+1}{2}}(F) + \left(\frac{b}{p}\right)\mathbb{D}(F) \in \widetilde{M}^{(2)}_{k+\frac{(p+1)^2}{2}}. \tag{3.1}$$

Theorem 1.2 states that $F \in \widetilde{M}_k^{(2)}$ has a Ramanujan-type congruence at $b \not\equiv 0 \pmod p$ if and only if $G \equiv 0 \pmod p$ in (3.1). One can apply the following analog of Sturm's theorem for Siegel modular forms of degree 2 to verify that $G \equiv 0 \pmod p$ in (3.1) for concrete examples of Siegel modular forms.

Theorem 3.3 ([15]). *Let $F = \sum a(T)e^{\pi i\,\mathrm{tr}(TZ)} \in M_k^{(2)}$ be such that for all T with dyadic trace $w(T) \le \frac{k}{3}$ one has that $a(T) \in \mathbb{Z}_{(p)}$ and $a(T) \equiv 0 \pmod p$. Then $F \equiv 0 \pmod p$.*

Remark. If $T = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) > 0$ is Minkowski reduced (i.e. $2|b| \le a \le c$), then $w(T) = a + c - |b|$. For more details on the dyadic trace $w(T)$, see [16].

The following table gives all Ramanujan-type congruences at $b \not\equiv 0 \pmod p$ for Siegel cusp forms of weight 20 or less when $p \ge 5$. Let E_4, E_6, χ_{10}, and χ_{12} denote the usual generators of $M_k^{(2)}$ of weights 4, 6, 10, and 12, respectively, where the Eisenstein series E_4 and E_6 are normalized by $a\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = 1$ and where the cusp forms χ_{10} and χ_{12} are normalized by $a\left(\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)\right) = 1$. Poor and Yuen kindly provided Fourier coefficients up to dyadic trace $w(T) = 74$ of the basis vectors for $M_k^{(2)}$ with $k \le 20$. We used Magma to check that $G \equiv 0 \pmod p$ in (3.1) for each of the forms in Table 1 below. It is not difficult to verify that (up to scalar multiplication) no further Ramanujan-type congruences at $b \not\equiv 0 \pmod p$ exist for Siegel cusp forms of weights 20 or less.

Table 1.

	$b \not\equiv 0 \pmod p$
χ_{12}	$b \equiv 1, 4 \pmod 5$ and $b \equiv 2, 6, 7, 8, 10 \pmod{11}$
$E_4\chi_{12}$	$b \equiv 1, 4 \pmod 5$
$E_4\chi_{12} - E_6\chi_{10}$	$b \equiv 3, 5, 6 \pmod 7$
$E_6\chi_{12}$	$b \equiv 1, 4 \pmod 5$
$E_4^2\chi_{10} + 7E_6\chi_{12}$	$b \equiv 1, 2, 4, 8, 9, 13, 15, 16 \pmod{17}$
$E_4^2\chi_{12}$	$b \equiv 1, 4 \pmod 5$
$\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$	$b \equiv 2, 3, 8, 10, 12, 13, 14, 15, 18 \pmod{19}$

Remarks. (1) For $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10} \pmod{19}$ we have $G \in \widetilde{M}_{220}^{(2)}$ in (3.1) and we really do need Fourier coefficients up to dyadic trace $w(T) = \frac{220}{3}$, i.e., up to 74 in Theorem 3.3 to prove that $G \equiv 0 \pmod{19}$.

(2) For Siegel modular forms in the Maass Spezialschar one could decide the existence and non-existence of their Ramanujan-type congruences also using Propositions 2.4 and 2.7 in combination with Maass' lift [11] (see also [6, §6]). However, Theorem 1.2 is an essential tool in establishing such results for Siegel modular forms that are not in the Maass Spezialschar, such as $E_4^2\chi_{12}$ and $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ for example.

The following construction generates infinitely many Siegel modular forms with Ramanujan-type congruences. Note that this construction also works for elliptic modular forms and for Jacobi forms by replacing $\mathbb{D}$ with $\Theta := \frac{1}{2\pi i}\frac{d}{dz}$ and L_m, respectively. For any $F \in \widetilde{M}_k^{(2)}$ and any prime $p \geq 5$, set

$$F_0 := F - \mathbb{D}^{p-1}F \in \widetilde{M}_{k+p^2-1}^{(2)}$$

$$F_{+1} := \frac{1}{2}(\mathbb{D}^{p-1}F + \mathbb{D}^{\frac{p-1}{2}}F) \in \widetilde{M}_{k+p^2-1}^{(2)}$$

$$F_{-1} := \frac{1}{2}(\mathbb{D}^{p-1}F - \mathbb{D}^{\frac{p-1}{2}}F) \in \widetilde{M}_{k+p^2-1}^{(2)}.$$

Clearly $F = F_0 + F_{+1} + F_{-1}$ and if $F = \sum a(T)e^{\pi i\,\mathrm{tr}(TZ)}$, then for $s = 0, \pm 1$, one finds that

$$F_s = \sum_{\left(\frac{\det(T)}{p}\right)=s} a(T)e^{\pi i\,\mathrm{tr}(TZ)}. \tag{3.2}$$

Hence F_s has Ramanujan-type congruences at all b with $\left(\frac{b}{p}\right) \neq s$. For example, if $F := \chi_{10}^2$, then a computation (in combination with Theorem 3.3) reveals that

$$\begin{aligned} F_0 &\equiv 3E_4^5\chi_{12}^2 + 2E_4^4E_6\chi_{10}\chi_{12} &&\pmod 5\\ F_{+1} &\equiv E_4^6\chi_{10}^2 + 4E_4^3\chi_{10}^2\chi_{12} + 4E_4^5\chi_{12}^2 + 2E_4^4E_6\chi_{10}\chi_{12} + 3E_4^3E_6^2\chi_{10}^2 &&\pmod 5\\ F_{-1} &\equiv E_4^3\chi_{10}^2\chi_{12} + 3E_4^5\chi_{12}^2 + E_4^4E_6\chi_{10}\chi_{12} + 2E_4^3E_6^2\chi_{10}^2 &&\pmod 5. \end{aligned}$$

Since $E_4 \equiv 1 \pmod 5$, we actually have $F_0 \in \widetilde{M}_{28}^{(2)}$ and $F_{\pm 1} \in \widetilde{M}_{32}^{(2)}$.

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