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Zagier-type dualities and lifting maps for harmonic Maass–Jacobi forms ☆

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Abstract

The real-analytic Jacobi forms of Zwegers' PhD thesis play an important role in the study of mock theta functions and related topics, but have not been part of a rigorous theory yet. In this paper, we introduce harmonic Maass–Jacobi forms, which include the classical Jacobi forms as well as Zwegers' functions as examples. Maass–Jacobi–Poincaré series also provide prime examples. We compute their Fourier expansions, which yield Zagier-type dualities and also yield a lift to skew-holomorphic Jacobi–Poincaré series. Finally, we link harmonic Maass–Jacobi forms to different kinds of automorphic forms via a commutative diagram.
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1. Introduction and statement of results

Ramanujan's last letter to Hardy in 1920 (see [19]) features a list of 17 functions such as

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

This was the birth of mock theta functions, which have been the source of many important works since then. However, for a long time, the subject remained quite mysterious, since no rigorous definition of mock theta functions was known. In 2002, Zwegers [26] succeeded in giving such a definition by discovering a crucial link between mock theta functions and real-analytic vector-valued modular forms, which are now part of the theory of harmonic Maass forms. His significant discovery has led to major applications in different areas of mathematics and physics, such as Bringmann and Ono's [9,6] solutions of well-known conjectures in combinatorics and the theory of q-series.

Zwegers [26] also explored certain real-analytic Jacobi forms, which are valuable tools in understanding mock theta functions. Coefficients of such Jacobi forms encode combinatorial statistics such as Dyson's [12] famous rank of partitions. Moreover, such Jacobi forms are vital to the theory of higher weight harmonic Maass forms. For example, the functions in Bringmann and Lovejoy [5], which associate overpartitions to class numbers, may be viewed as derivatives of Jacobi forms with respect to the Jacobi variable. Bringmann [3] and Bringmann, Garvan, and Mahlburg [4] examined quasiharmonic Maass forms, i.e., linear combinations of derivatives of harmonic Maass forms, which are also closely related to derivatives of Jacobi forms. A main tool in understanding these higher weight Maass forms is a certain partial differential equation connecting the rank and the crank of partitions (see Atkin and Garvan [1]). This differential equation may be regarded as the action of the heat operator on Jacobi forms in this context (see Bringmann and Zwegers [10]). Furthermore, the Jacobi forms in [26] appear also as key players in the recent paper of Malmendier and Ono [16], where the authors confirmed an important conjecture by Moore and Witten on $SO(3)$-Donaldson invariants of $\mathbb{CP}^2$.

In this paper, we seek a better understanding of real-analytic Jacobi forms, which should provide new insight on mock theta functions and related topics. More precisely, we propose the study of harmonic Maass–Jacobi forms (see Definition 3). In addition to the usual Jacobi forms, our theory includes the real-analytic Jacobi forms in [26] as well as certain Poincaré series, the main focus of our work here. We introduce a differential operator $\xi_{k,m}$ (see (15)), whose action on harmonic Maass–Jacobi forms is central to our work. The operator $\xi_{k,m}$ is an analog of Bruinier and Funke's [11] operator ξ_k, which maps harmonic Maass forms of weight k to weakly holomorphic modular forms of weight $2 - k$, and which has played a major role in the development of harmonic Maass forms. However, in contrast to the action of ξ_k on harmonic Maass forms, the image of $\xi_{k,m}$ does not consist of holomorphic functions. Specifically, we prove in this paper:

Proposition 1. *We have*

$$\xi_{k,m} : \widehat{\mathbb{J}}_{k,m} \to J_{3-k,m}^{sk!}.$$

Here $\widehat{\mathbb{J}}_{k,m}$ denotes the subspace of harmonic Maass–Jacobi forms of weight k and index m which are holomorphic in the Jacobi variable $z \in \mathbb{C}$ (see Section 4 for details), and $J_{k,m}^{sk!}$ stands

for the space of Skoruppa's [22,23] (weak) skew-holomorphic Jacobi forms of weight k and index m, which are reviewed in Section 3. Moreover, we write $\widehat{\mathbb{J}}_{k,m}^{cusp}$ for the pre-image of $J_{3-k,m}^{sk,cusp}$ under $\xi_{k,m}$, where $J_{k,m}^{sk,cusp}$ denotes the space of cusp forms in $J_{k,m}^{sk!}$.

We now turn our attention to Maass–Jacobi–Poincaré series, which are key examples of $\widehat{\mathbb{J}}_{k,m}^{cusp}$. We consider normalized Maass–Jacobi–Poincaré series $\mathcal{P}_{k,m}^{(n,r)}$ (see (22)), which have Fourier expansions of the form

$$\mathcal{P}_{k,m}^{(n,r)}(\tau,z) = q^n \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi Dy}{m}\right) e\left(\frac{iDy}{4m}\right) \vartheta_{k,m}^{(r)}(\tau,z) + c(\tau,z)$$
$$+ \sum_{n',r' \in \mathbb{Z}} c_{n,r}^{(k)}(n',r') e\left(\frac{iD'y}{4m}\right) \mathcal{W}_{s,k-\frac{1}{2}}\left(-\frac{\pi D'y}{m}\right) q^{n'} \zeta^{r'},$$

where here and throughout the paper $\tau = x + iy \in \mathbb{H}$ (the usual complex upper half plane), $z = u + iv \in \mathbb{C}$, $e(w) := e^{2\pi i w}$, $q := e(\tau)$, $\zeta := e(z)$, $D := r^2 - 4nm$, $D' := r'^2 - 4n'm$, and where $\mathcal{M}_{s,\kappa}$ and $\mathcal{W}_{s,\kappa}$ are modified Whittaker functions defined in (17) and (19), respectively, and where $s \in \{\frac{k}{2} - \frac{1}{4}, \frac{5}{4} - \frac{k}{2}\}$. Moreover, $c(\tau,z)$ is a sum over n', r' with $D' = 0$ defined in (21) and $\vartheta_{k,m}^{(r)}$ is a theta function defined in (10). One finds that (see Corollary 1) the coefficients $c_{n,r}^{(k)}(n',r')$ decompose as

$$c_{n,r}^{(k)}(n',r') = b_{n,r}^{(k)}(n',r') + (-1)^k b_{n,r}^{(k)}(n',-r'). \tag{1}$$

Zagier [24] established a striking duality for Fourier coefficients of weakly holomorphic modular forms. Bringmann and Ono [7] generalized Zagier's results and showed that the duality arises from properties of Fourier coefficients of Maass–Poincaré series. Such a duality cannot hold for the coefficients $c_{n,r}^{(k)}(n',r')$ of $\mathcal{P}_{k,m}^{(n,r)}$ due to the appearance of $(-1)^k$ in (1) and the fact that the weights k and $3 - k$ are "dual" under the action of $\xi_{k,m}$ (see Proposition 1). Nevertheless, our first theorem gives Zagier-type dualities for the coefficients $b_{n,r}^{(k)}(n',r')$. The half-integral weight Maass–Poincaré series in [7] depend also on some integer and the duality in [7] involves only such series attached to negative integers. The situation here is much more complicated than in the modular case: The Maass–Jacobi–Poincaré series $\mathcal{P}_{k,m}^{(n,r)}$ depend on the discriminant D and one might expect that a duality for $\mathcal{P}_{k,m}^{(n,r)}$ would involve only series corresponding to negative discriminants. However, this is not the case, and our first theorem shows that the dualities for the $b_{n,r}^{(k)}(n',r')$ feature Poincaré series with positive and negative discriminants.

Theorem 1. *The following dualities hold for the coefficients $b_{n,r}^{(k)}(n',r')$ of $\mathcal{P}_{k,m}^{(n,r)}$:*

(1) *If $D = r^2 - 4nm < 0$ and $D' = r'^2 - 4n'm < 0$, then*

$$b_{n,r}^{(k)}(n',r') = b_{n',r'}^{(3-k)}(n,r).$$

(2) *If $D = r^2 - 4nm > 0$, $D' = r'^2 - 4n'm < 0$, and $k > 3$, then*

$$b_{n,r}^{(k)}(n',r') = b_{n',r'}^{(3-k)}(n,r).$$

Remark. Statement (1) in Theorem 1 is a duality between holomorphic parts of Jacobi–Poincaré series analogous to the duality in [7]. However, statement (2) establishes a duality between holomorphic and non-holomorphic parts: $b_{n,r}^{(k)}(n', r')$ belongs to the holomorphic part of $\mathcal{P}_{k,m}^{(n,r)}$, while $b_{n',r'}^{(3-k)}(n, r)$ belongs to the non-holomorphic part of $\mathcal{P}_{3-k,m}^{(n',r')}$.

Our second theorem asserts that $\xi_{k,m}$ maps the Maass–Jacobi–Poincaré series $\mathcal{P}_{k,m}^{(n,r)}$ (with $D > 0$) to the skew-holomorphic Jacobi–Poincaré series $P_{3-k,m}^{(n,r)sk}$, which is defined in (8).

Theorem 2. *If $D = r^2 - 4nm > 0$, then we have*

$$\xi_{k,m}\big(\mathcal{P}_{k,m}^{(n,r)}\big) = P_{3-k,m}^{(n,r)sk}.$$

Remark. The skew-holomorphic Jacobi–Poincaré series $P_{k,m}^{(n,r)sk}$ form a basis of $J_{k,m}^{sk,cusp}$. With the help of Theorem 2, we see that the map $\xi_{k,m} : \widehat{\mathbb{J}}_{k,m}^{cusp} \to J_{3-k,m}^{sk,cusp}$ is surjective.

Finally, we explore lifts between different spaces of automorphic forms. Let $S_{\frac{5}{2}-k}^+$ denote the usual plus-space of cuspidal holomorphic modular forms of weight $\frac{5}{2} - k$ and write $\widehat{\mathbb{S}}_{k-\frac{1}{2}}^+$ for its pre-image under $\xi_{k-\frac{1}{2}}$. If k is even, then we prove that the following diagram is commutative:

$$
\begin{array}{ccc}
\widehat{\mathbb{S}}_{k-\frac{1}{2}}^+ & \xrightarrow{\ \xi_{k-\frac{1}{2}}\ } & S_{\frac{5}{2}-k}^+ \\
{\scriptstyle \mathbb{F}_\theta}\Big\downarrow & & \Big\downarrow{\scriptstyle F_\theta} \\
\widehat{\mathbb{J}}_{k,1}^{cusp} & \xrightarrow{\ \xi_{k,1}\ } & J_{3-k,1}^{sk,cusp}
\end{array}
\tag{2}
$$

where $\mathbb{F}_\theta$ and F_θ are lifts given in terms of theta functions (see Section 6 for details).

The paper is organized as follows. In Section 2, we recall the notion of harmonic Maass forms. In Section 3, we briefly discuss skew-holomorphic Jacobi forms. In Section 4, we present harmonic Maass–Jacobi forms and we prove Proposition 1. In Section 5, we come to the heart of the paper. Here we determine the Fourier expansions of Maass–Jacobi–Poincaré series, which allow us to prove Theorem 1 and Theorem 2. In Section 6, we show that diagram (2) is commutative.

2. Harmonic Maass forms

Zwegers showed in his PhD thesis [26] that mock theta functions appear as holomorphic parts of harmonic Maass forms of weights $1/2$, a fact that has inspired many recent results. We will now introduce some standard notation to briefly review the definition of half-integral weight harmonic Maass forms. For more details, see Fay [14], Bruinier and Funke [11], and also Ono [17], who gives a good overview of the recent development of harmonic Maass forms and its applications to number theory. Throughout the paper we write $\partial_\tau := \frac{\partial}{\partial \tau} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\partial_{\bar\tau} := \frac{\partial}{\partial \bar\tau} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, $\partial_z := \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$, and $\partial_{\bar z} := \frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$. Then

$$\Delta_k := (\tau - \bar{\tau})^2 \partial_{\tau\bar{\tau}} + k(\tau - \bar{\tau})\partial_{\bar{\tau}}$$

is the weight k hyperbolic Laplacian. Let $\Gamma_0(4) := \{ \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod 4 \}$.

Definition 1. A harmonic Maass form of weight $k \in \frac{1}{2} + \mathbb{Z}$ on $\Gamma_0(4)$ is a smooth function $g : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(4)$, we have

$$g\left(\frac{a\tau + b}{c\tau + d} \right) = \left(\frac{c}{d} \right)^{2k} \epsilon_d^{-2k} (c\tau + d)^k g(\tau).$$

Here $\left(\frac{c}{d} \right)$ denotes the Jacobi symbol, $\epsilon_d = 1$ for $d \equiv 1 \pmod 4$ and $\epsilon_d = i$ for $d \equiv 3 \pmod 4$, and $\sqrt{\tau}$ is the principal branch of the holomorphic square root.
(2) We have that $\Delta_k(g) = 0$.
(3) The function g has at most linear exponential growth at all the cusps of $\Gamma_0(4)$.

Let $\widehat{\mathbb{M}}_k$ denote the space of harmonic Maass forms of weight k.

The above definition can be extended to other groups in the usual way.
Note that harmonic Maass forms have Fourier expansions of the form

$$g(\tau) = c_g^- y^{1-k} + \sum_{n \gg -\infty} c_g^+(n) q^n + \sum_{n \ll \infty} c_g^-(n) H(2\pi n y) e(nx). \tag{3}$$

Here $c_g^+ + \sum_{n \gg -\infty} c_g^+(n) q^n$ is the holomorphic part of g, $c_g^- y^{1-k} + \sum_{n \ll \infty} c_g^-(n) H(2\pi n y) e(nx)$ is the non-holomorphic part of g, and the function H (defined on page 55 of [11]) is a solution to the second-order linear differential equation

$$\frac{\partial^2}{\partial w^2} f(w) - f(w) + \frac{k}{w} \left(\frac{\partial}{\partial w} f(w) + f(w) \right) = 0.$$

The function $H(t)$ has the asymptotic behavior

$$H(t) \sim \begin{cases} (2|t|)^{-k} e^{-|t|} & \text{for } t \to -\infty, \\ (-2t)^{-k} e^t & \text{for } t \to \infty. \end{cases}$$

Moreover, in the case that $t < 0$, we have

$$H(t) = e^{-t} \Gamma(1 - k, -2t), \tag{4}$$

where $\Gamma(\alpha, t) := \int_t^\infty e^{-w} w^{\alpha - 1} \, dw$ is the incomplete Gamma-function. Let $\widehat{\mathbb{M}}_k^+$ be the plus-space of harmonic Maass forms, i.e., the space of forms in $\widehat{\mathbb{M}}_k$ whose Fourier expansions in (3) are only over integers n satisfying $(-1)^{k-\frac{1}{2}} n \equiv 0, 1 \pmod 4$.
Furthermore, if $g \in \widehat{\mathbb{M}}_k$ is holomorphic on $\mathbb{H}$, then g is a weakly holomorphic modular form of weight k, i.e., a meromorphic modular form of weight k whose poles (if there are any) are

supported at the cusps. We write $M_k^!$ for the space of weakly holomorphic modular forms of weight k and $M_k^{!+}$ for its plus-space.

Bruinier and Funke [11] introduced the differential operator

$$\xi_k := 2i\left(\frac{\tau - \bar{\tau}}{2i}\right)^k \overline{\partial_{\bar{\tau}}} \tag{5}$$

and showed that

$$\xi_k : \widehat{\mathbb{M}}_k \to M_{2-k}^!.$$

This map plays a significant role in theory of harmonic Maass forms and has led to important applications; see for example the work of Bringmann and Ono on Maass–Poincaré series [7,8]. Finally, note that if $g \in \widehat{\mathbb{S}}_k$, the pre-image of S_{2-k} (the space of cusp forms of weight $2 - k$) under ξ_k, then g has a Fourier expansion of the form

$$g(\tau) = \sum_{n \gg -\infty} c_g^+(n) q^n + \sum_{n < 0} c_g^-(n) \Gamma\big(1 - k, 4\pi |n| y\big) q^n. \tag{6}$$

3. Skew-holomorphic Jacobi forms

In 1985, Eichler and Zagier [13] systematically developed a theory of (holomorphic) Jacobi forms. Skoruppa [22,23] introduced skew-holomorphic Jacobi forms, which play a crucial role in understanding liftings of modular forms and Jacobi forms. The theory of Jacobi forms has grown enormously since then with deep connections to modular forms and many other areas of mathematics and physics, for example, the theory of Heegner points, the theory of elliptic genera, string theory, and more recently, mock theta functions.

We will now briefly discuss the definition of skew-holomorphic Jacobi forms. Let $\Gamma^J :=$ $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ be the Jacobi group. For fixed integers k and m, define the following slash operator on functions $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$:

$$\left(\phi|_{k,m}^{sk} A\right)(\tau, z) := \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right)(c\bar{\tau} + d)^{1-k} |c\tau + d|^{-1} e^{2\pi i m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z\right)}$$

for all $A = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)] \in \Gamma^J$. The following definition of weak skew-holomorphic Jacobi forms (slightly) extends the definitions in [22,23].

Definition 2. A function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is a weak skew-holomorphic Jacobi form of weight k and index m if ϕ is real-analytic in $\tau \in \mathbb{H}$, is holomorphic in $z \in \mathbb{C}$, and satisfies the following conditions:

(1) For all $A \in \Gamma^J$, $(\phi|_{k,m}^{sk} A) = \phi$.
(2) The Fourier expansion of ϕ is of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ D \gg -\infty}} c(n, r) e\left(\frac{iDy}{2m}\right) q^n \zeta^r. \tag{7}$$

If the Fourier expansion in (7) is only over $D \geqslant 0$, then ϕ is a skew-holomorphic Jacobi form of weight k and index m as in [22,23]. If the Fourier expansion in (7) is only over $D > 0$, then ϕ is a skew-holomorphic Jacobi cusp form of weight k and index m. We denote the spaces of weak skew-holomorphic Jacobi forms, skew-holomorphic Jacobi forms, and skew-holomorphic Jacobi cusp forms, each of weight k and index m, by $J_{k,m}^{sk!}$, $J_{k,m}^{sk}$, and $J_{k,m}^{sk,cusp}$, respectively.

Remark. Note that the Fourier expansion (7) implies that $L_m(\phi) = 0$, where $L_m := 8\pi i m \partial_\tau - \partial_{zz}$ is the heat operator.

We will next recall the skew-holomorphic Jacobi–Poincaré series in Skoruppa [21]. Let $D = r^2 - 4nm > 0$ with $r, n \in \mathbb{Z}$. Set

$$\Psi_{k,m}^{n,r}(\tau, z) := e(n\tau + rz)e\left(\frac{iDy}{2m}\right),$$

and for $k \geqslant 3$, define

$$P_{k,m}^{(n,r)sk}(\tau, z) := \sum_{A \in \Gamma_\infty^J \backslash \Gamma^J} \left(\Psi_{k,m}^{n,r}\big|_{k,m}^{sk} A\right)(\tau, z), \tag{8}$$

where $\Gamma_\infty^J := \{[\left(\begin{smallmatrix} 1 & \eta \\ 0 & 1 \end{smallmatrix}\right), (0, n)] \mid \eta, n \in \mathbb{Z}\}$. The Fourier expansion of $P_{k,m}^{(n,r)sk}$ features the Kloosterman sum $K_c(n, r, n', r')$ and certain theta series $\vartheta_{\kappa,m}^{(r)}$, which we now define: Let

$$K_c(n, r, n', r') := e_{2mc}(-rr') \sum_{\substack{d \ (\mathrm{mod} \ c)^* \\ \lambda \ (\mathrm{mod} \ c)}} e_c\left(\bar{d}m\lambda^2 + n'd - r'\lambda + \bar{d}n + \bar{d}r\lambda\right), \tag{9}$$

where $e_c(x) := e^{\frac{2\pi i x}{c}}$, the sum over d runs through the primitive residue classes modulo c, and $\bar{d}$ is the inverse of d modulo c. Finally, set

$$\vartheta_{\kappa,m}^{(r)}(\tau, z) := \sum_{\lambda \in \mathbb{Z}} q^{\lambda^2 m} \zeta^{2m\lambda}\left(q^{r\lambda}\zeta^r + (-1)^\kappa q^{-r\lambda}\zeta^{-r}\right). \tag{10}$$

The following theorem of [21] states that $P_{k,m}^{(n,r)sk}$ is a skew-holomorphic Jacobi cusp form of weight k and index m and, in particular, gives the Fourier expansion of $P_{k,m}^{(n,r)sk}$.

Theorem 3. *The Poincaré series $P_{k,m}^{(n,r)sk}$ are elements of $J_{k,m}^{sk,cusp}$. Moreover,*

$$P_{k,m}^{(n,r)sk}(\tau, z) = q^n e\left(\frac{iDy}{2m}\right)\vartheta_{k-1,m}^{(r)}(\tau, z) + \sum_{\substack{n',r' \in \mathbb{Z} \\ D' > 0}} c(n', r')e\left(\frac{iD'y}{2m}\right)q^{n'}\zeta^{r'}$$

(recall $D' = r'^2 - 4n'm$), where

$$c(n', r') := b(n', r') + (-1)^{k+1}b(n', -r').$$

Here

$$b(n', r') = \sqrt{2} \pi i^{-k+1} \left(\frac{D'}{D} \right)^{\frac{k}{2} - \frac{3}{4}} m^{-\frac{1}{2}} \sum_{c>0} c^{-\frac{3}{2}} K_c(n, r, n', -r') J_{k - \frac{3}{2}} \left(\frac{\pi \sqrt{DD'}}{mc} \right),$$

where J is the usual J-Bessel function.

4. Harmonic Maass–Jacobi forms

Maass–Jacobi forms were first introduced by Berndt and Schmidt [2]. Recently, Pitale [18] has used ideas of [2] to give a new and more thorough approach to Maass–Jacobi forms. Nevertheless, there are important types of real-analytic Jacobi forms — such as the Jacobi forms in Section 1.4 of Zwegers [26] as well as the Maass–Jacobi–Poincaré series studied in Section 5 — which have not been part of a theory yet and which do not fit into the framework of [18]. In this section, we suggest a theory of harmonic Maass–Jacobi forms which includes the holomorphic Jacobi forms of Eichler and Zagier [13] as well as the real-analytic Jacobi forms in Section 1.4 of [26] and the Poincaré series in Section 5 as explicit examples.

For fixed integers k and m, define the following slash operator on functions $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$:

$$(\phi|_{k,m} A)(\tau, z) := \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) (c\tau + d)^{-k} e^{2\pi i m \left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2 \tau + 2\lambda z \right)} \quad (11)$$

for all $A = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), (\lambda, \mu)] \in \Gamma^J$. It is well known that (11) can be extended to an action $|_{k,m}^{\mathbb{R}}$ of the real Jacobi group on $C^\infty(\mathbb{H} \times \mathbb{C})$. The center of the universal enveloping algebra of the real Jacobi group is generated by a linear element and a cubic element, the *Casimir element*. The linear element acts by scalars under $|_{k,m}^{\mathbb{R}}$ and the action of the Casimir element under $|_{k,m}^{\mathbb{R}}$ is given (up to the constant $\frac{5}{8} + \frac{3k - k^2}{2}$) by the following differential operator:

$$C^{k,m} := -2(\tau - \bar{\tau})^2 \partial_{\tau\bar{\tau}} - (2k-1)(\tau - \bar{\tau})\partial_{\bar{\tau}} + \frac{(\tau - \bar{\tau})^2}{4\pi i m} \partial_{\bar{\tau}zz}$$

$$+ \frac{k(\tau - \bar{\tau})}{4\pi i m} \partial_{zz} + \frac{(\tau - \bar{\tau})(z - \bar{z})}{4\pi i m} \partial_{zz\bar{z}} - 2(\tau - \bar{\tau})(z - \bar{z}) \partial_{\tau\bar{z}} + k(z - \bar{z})\partial_{\bar{z}}$$

$$+ \frac{(\tau - \bar{\tau})^2}{4\pi i m} \partial_{\tau\bar{z}\bar{z}} + \left(\frac{(z - \bar{z})^2}{2} + \frac{k(\tau - \bar{\tau})}{4\pi i m} \right) \partial_{\bar{z}\bar{z}} + \frac{(\tau - \bar{\tau})(z - \bar{z})}{4\pi i m} \partial_{z\bar{z}\bar{z}}.$$

In particular, $C^{k,m}$ commutes with the action in (11), i.e., if $A \in \Gamma^J$, then

$$\left(C^{k,m} \phi \right) \big|_{k,m} A = C^{k,m} (\phi|_{k,m} A). \quad (12)$$

Definition 3. A function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is a harmonic Maass–Jacobi form of weight k and index m if ϕ is real-analytic in $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$, and satisfies the following conditions:

(1) For all $A \in \Gamma^J$, $(\phi|_{k,m} A) = \phi$.
(2) We have that $C^{k,m}(\phi) = 0$.
(3) We have that $\phi(\tau, z) = O(e^{ay} e^{2\pi m v^2 / y})$ as $y \to \infty$ for some $a > 0$.

We are particularly interested in harmonic Maass–Jacobi forms, which are holomorphic in z; we denote the space of such forms by $\widehat{\mathbb{J}}_{k,m}$.

Remarks. (1) It is not hard to see that every $\phi \in \widehat{\mathbb{J}}_{k,m}$ has a Fourier expansion of the form

$$y^{\frac{3}{2}-k} \sum_{\substack{n,r \in \mathbb{Z} \\ D=0}} c^0(n,r)q^n\zeta^r + \sum_{\substack{n,r \in \mathbb{Z} \\ D \ll \infty}} c^+(n,r)q^n\zeta^r$$

$$+ \sum_{\substack{n,r \in \mathbb{Z} \\ D \gg -\infty}} c^-(n,r)H\left(-\frac{\pi D y}{2m}\right)e\left(\frac{iDy}{4m}\right)q^n\zeta^r, \tag{13}$$

where the H here differs by the H defined on page 55 of [11] in that k is replaced by $k - \frac{1}{2}$. We call $\sum_{D \gg -\infty} c^-(n,r)H(-\frac{\pi D y}{2m})e(\frac{iDy}{2m})q^n\zeta^r$ the non-holomorphic part of ϕ and $\sum_{D \ll \infty} c^+(n,r)q^n\zeta^r$ the holomorphic part of ϕ.

(2) If ϕ is a holomorphic Jacobi form of weight k and index m, then $\phi \in \widehat{\mathbb{J}}_{k,m}$. The definition of harmonic Maass–Jacobi forms can easily be extended to forms of half-integral weights and indices. Each real-analytic Jacobi form $\hat{\mu}$ in Section 1.4 of Zwegers [26] has a decomposition of the form $\hat{\mu} = \mu_1 + \hat{\mu}_2$, where μ_1 is a meromorphic Jacobi form on $\mathbb{H} \times \mathbb{C}^2$ and where $\hat{\mu}_2$ is a real-analytic Jacobi form on $\mathbb{H} \times \mathbb{C}$ (see also the footnote (1) on page 7 of Zagier [25]). It can be verified using MAPLE that $\hat{\mu}_2$ is annihilated by $C^{1/2,-1/2}$ and hence is a harmonic Maass–Jacobi form of weight $1/2$ and index $-1/2$. Further examples of harmonic Maass–Jacobi forms and their properties are discussed in Section 5.

(3) The Maass–Jacobi forms in [18] are real-analytic functions $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ which are eigenfunctions of $\widetilde{C}^{k,m} := y^{k/2}C^{k,m}y^{-k/2} + \frac{5}{8} + \frac{3k-k^2}{2}$, invariant under a slash-operator as in (11), except that $(c\tau + d)^{-k}$ in (11) is replaced by $(\frac{c\tau+d}{|c\tau+d|})^{-k}$, and which satisfy the growth condition $\phi(\tau,z) = O(y^N)$ as $y \to \infty$ for some $N > 0$. Note that the choice of this growth condition is somewhat unfortunate, since, in general, even a holomorphic Jacobi form ϕ does not satisfy $\phi(\tau,z) = O(y^N)$ as $y \to \infty$ (independently of z) for some $N > 0$. In fact, the identity (see Skoruppa [20])

$$\left|q^n\zeta^r\right|e^{-\frac{2\pi mv^2}{y}} = e^{-\frac{\pi y}{2m}((r+\frac{2mv}{y})^2 - D)}$$

applied to the Fourier expansion of a holomorphic Jacobi form shows that such a Jacobi form satisfies condition (3) of Definition 3.

A direct computation reveals that

$$D_-^{(m)} := \left(\frac{\tau - \bar{\tau}}{2i}\right)\left(-(\tau - \bar{\tau})\partial_{\bar{\tau}} - (z - \bar{z})\partial_{\bar{z}} + \frac{1}{4\pi m}\left(\frac{\tau - \bar{\tau}}{2i}\right)\partial_{\bar{z}\bar{z}}\right)$$

is a "lowering" operator, i.e., if ϕ is a smooth function on $\mathbb{H} \times \mathbb{C}$ and if $A \in \Gamma^J$, then

$$\left(D_-^{(m)}\phi\right)\big|_{k-2,m}A = D_-^{(m)}(\phi|_{k,m}A). \tag{14}$$

In the spirit of the definition of ξ_k in (5), we introduce the differential operator

$$\xi_{k,m} := \left(\frac{\tau - \bar{\tau}}{2i}\right)^{k-5/2} D_-^{(m)}. \tag{15}$$

We will now prove Proposition 1, which gives the action of $\xi_{k,m}$ on $\widehat{\mathbb{J}}_{k,m}$. If $\phi \in \widehat{\mathbb{J}}_{k,m}$, then (14) implies that $(\xi_{k,m}\phi)|_{3-k,m}^{sk} A = \xi_{k,m}\phi$ for all $A \in \Gamma^J$. Moreover, applying $\xi_{k,m}$ to a Fourier expansion of the form (13) yields a Fourier expansion of a weak skew-holomorphic Jacobi form, which completes the proof of Proposition 1.

We end this section with two remarks.

Remarks. (1) In the introduction, we define $\widehat{\mathbb{J}}_{k,m}^{cusp}$ as the pre-image of $J_{3-k,m}^{sk,cusp}$ under $\xi_{k,m}$. Note that elements in $\widehat{\mathbb{J}}_{k,m}^{cusp}$ have a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ D \ll \infty}} c^+(n,r) q^n \zeta^r + \sum_{\substack{n,r \in \mathbb{Z} \\ D > 0}} c^-(n,r) \Gamma\left(\frac{3}{2} - k, \frac{\pi D y}{m}\right) q^n \zeta^r.$$

(2) As in (2) of the previous remarks, we write each function $\hat{\mu}$ in [26] as $\hat{\mu} = \mu_1 + \hat{\mu}_2$. Then $\hat{\mu}_2(\tau, z)$ is a harmonic Maass–Jacobi form of weight $1/2$ and index $-1/2$, which is not holomorphic in z (where $z = u - v$ with the variables u and v of [26]). Nevertheless, one can determine its image under $\xi_{\frac{1}{2}, -\frac{1}{2}}$. One finds that

$$\xi_{\frac{1}{2}, -\frac{1}{2}}(\hat{\mu}_2) = \frac{\sqrt{2}}{\sqrt{y}} e^{-2\pi v^2/y} \sum_{n \in \frac{1}{2} + \mathbb{Z}} (-1)^{n-\frac{1}{2}} \left(n + \frac{v}{y}\right) e^{-\pi i n^2 \bar{\tau}} e^{2\pi i n z},$$

which satisfies the transformation law of a skew-holomorphic Jacobi form of weight $5/2$ and index $-1/2$ and is also in the kernel of the heat operator $4\pi i \partial_\tau + \partial_{zz}$. However, $\xi_{\frac{1}{2}, -\frac{1}{2}}(\hat{\mu}_2)$ is not holomorphic in z and hence it is not a skew-holomorphic Jacobi form in the sense of Definition 2.

5. Maass–Jacobi–Poincaré series and the proofs of Theorem 1 and Theorem 2

In this section, we present Maass–Jacobi–Poincaré series. We determine their Fourier expansions, which allow us to prove Theorem 1 and Theorem 2.

First, we construct an eigenfunction of the differential operator $C^{k,m}$ in order to define the Maass–Jacobi–Poincaré series in (18). Let $M_{\nu,\mu}$ be the usual M-Whittaker function, which is a solution to the differential equation

$$\frac{\partial^2}{\partial w^2} f(w) + \left(-\frac{1}{4} + \frac{\nu}{w} + \frac{\frac{1}{4} - \mu^2}{w^2}\right) f(w) = 0. \tag{16}$$

Let $D = r^2 - 4nm \neq 0$, and for $s \in \mathbb{C}$, $\kappa \in \frac{1}{2}\mathbb{Z}$, and $t \in \mathbb{R} \setminus \{0\}$, define

$$\mathcal{M}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}} M_{\mathrm{sgn}(t)\frac{\kappa}{2}, s-\frac{1}{2}}(|t|) \tag{17}$$

and

$$\phi_{k,m,s}^{(n,r)}(\tau, z) := \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi Dy}{m}\right)e\left(rz + \frac{ir^2 y}{4m} + nx\right).$$

Lemma 1. *The function $\phi_{k,m,s}^{(n,r)}$ is an eigenfunction of the operator $C^{k,m}$ with eigenvalue $-2s(1-s) - \frac{1}{2}(k^2 - 3k + \frac{5}{4})$.*

Proof. If ℓ is an integer, then one can verify that (see also [7])

$$\varphi_{k,-\ell,s}(\tau) := \mathcal{M}_{s,k}(-4\pi\ell y)e(-\ell x)$$

is an eigenfunction of Δ_k with eigenvalue $s(1-s) + \frac{1}{4}(k^2 - 2k)$. It is easy to see that the action of $C^{k,m}$ on functions in $\widehat{\mathbb{J}}_{k,m}$ agrees with that of

$$-2(\tau - \bar{\tau})^2\partial_{\tau\bar{\tau}} - (2k-1)(\tau - \bar{\tau})\partial_{\bar{\tau}} + \frac{(\tau - \bar{\tau})^2}{4\pi im}\partial_{\bar{\tau}zz} = -2 \cdot \Delta_{k-\frac{1}{2}} + \frac{(\tau - \bar{\tau})^2}{4\pi im}\partial_{\bar{\tau}zz}.$$

We write

$$\phi_{k,m,s}^{(n,r)}(\tau, z) = e\left(\frac{r^2}{4m}\tau + rz\right)\varphi_{k,-\frac{D}{4m},s}(\tau)$$

to find that

$$C^{k,m}\left(\phi_{k,m,s}^{(n,r)}\right) = e\left(\frac{r^2}{4m}\tau + rz\right)\left(-2\Delta_{k-\frac{1}{2}}\left(\varphi_{k,-\frac{D}{4},s}\right)\right)$$

$$= \left(-2s(1-s) - \frac{1}{2}\left(k^2 - 3k + \frac{5}{4}\right)\right)\phi_{k,m,s}^{(n,r)}. \quad \square$$

We consider the Poincaré series

$$P_{k,m,s}^{(n,r)}(\tau, z) := \sum_{A\in\Gamma_\infty^J \backslash \Gamma^J}\left(\phi_{k,m,s}^{(n,r)}\big|_{k,m}A\right)(\tau, z). \tag{18}$$

The estimate

$$\mathcal{M}_{s,k-\frac{1}{2}}(y) \ll y^{\text{Re}(s) - \frac{2k-1}{4}} \quad (y \to 0)$$

yields that $P_{k,m,s}^{(n,r)}$ is absolutely and uniformly convergent for $\text{Re}(s) > \frac{5}{4}$. Of particular interest there are the cases $s \in \{\frac{k}{2} - \frac{1}{4}, \frac{5}{4} - \frac{k}{2}\}$, for which the $P_{k,m,s}^{(n,r)}$ are annihilated by $C^{k,m}$ and thus provide elements of $\widehat{\mathbb{J}}_{k,m}$. We give the Fourier expansion of $P_{k,m,s}^{(n,r)}$ in the next theorem after introducing a modified W-Whittaker function. For $s \in \mathbb{C}$, $\kappa \in \frac{1}{2}\mathbb{Z}$, and $t \in \mathbb{R} \setminus \{0\}$, set

$$\mathcal{W}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}}W_{\text{sgn}(t)\frac{\kappa}{2},s-\frac{1}{2}}(|t|), \tag{19}$$

where $W_{\nu,\mu}$ denotes the usual W-Whittaker function, which is also a solution to the differential equation (16).

Theorem 4. *We have*

$$P_{k,m,s}^{(n,r)}(\tau,z) = q^n \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi D y}{m}\right) e\left(\frac{i D y}{4m}\right) \vartheta_{k,m}^{(r)}(\tau,z) + \sum_{n',r' \in \mathbb{Z}} c_{y,s}(n',r') q^{n'} \zeta^{r'}, \quad (20)$$

where

$$c_{y,s}(n',r') := b_{y,s}(n',r') + (-1)^k b_{y,s}(n',-r')$$

and where $b_{y,s}(n',r')$ is given as follows (recall $D' = r'^2 - 4n'm$):

(1) *If $DD' > 0$, then $b_{y,s}(n',r')$ equals*

$$\sqrt{2}\pi i^{-k} m^{-\frac{1}{2}} \frac{\Gamma(2s)}{\Gamma(s - \operatorname{sgn}(D')\frac{2k-1}{4})}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{3}{4}} e\left(\frac{i D' y}{4m}\right) \mathcal{W}_{s,k-\frac{1}{2}}\left(-\frac{\pi D' y}{m}\right)$$

$$\times \sum_{c>0} c^{-\frac{3}{2}} K_c(n,r,n',r') J_{2s-1}\left(\frac{\pi\sqrt{D'D}}{mc}\right),$$

where Γ is the usual Gamma-function.

(2) *If $D' = 0$, then $b_{y,s}(n',r')$ equals*

$$y^{\frac{5-2k}{4}-s} \frac{1}{\Gamma(s + \frac{2k-1}{4})\Gamma(s - \frac{2k-1}{4})} a_s(n'r'),$$

where $a_s(n',r')$ is holomorphic for $\sigma > \frac{5}{4}$.

(3) *If $DD' < 0$, then $b_{y,s}(n',r')$ equals*

$$\sqrt{2}\pi i^{-k} m^{-\frac{1}{2}} \frac{\Gamma(2s)}{\Gamma(s - \operatorname{sgn}(D')\frac{2k-1}{4})}\left(\frac{|D'|}{|D|}\right)^{\frac{k}{2}-\frac{3}{4}} e\left(\frac{i D' y}{4m}\right) \mathcal{W}_{s,k-\frac{1}{2}}\left(\frac{-\pi D' y}{m}\right)$$

$$\times \sum_{c>0} c^{-\frac{3}{2}} K_c(n,r,n',r') I_{2s-1}\left(\frac{\pi\sqrt{|D'D|}}{mc}\right),$$

where I is the usual I-Bessel-function.

Proof. A set of representatives of $\Gamma_\infty^J \backslash \Gamma^J$ is given by $\{[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (a\lambda, b\lambda)]\}$, where $c, d \in \mathbb{Z}$ with $(c,d) = 1$, $\lambda \in \mathbb{Z}$, and where for each pair (c,d), the integers a, b are chosen such that $ad - bc = 1$. It is easy to see that the contribution from $c = 0$ yields the first term on the right-hand side of (20). We now only consider the contribution from $c > 0$, since the case $c < 0$ is similar. We use the identities

$$\frac{a\tau + b}{c\tau + d} = \frac{a}{c} - \frac{1}{c(c\tau + d)},$$

$$m\lambda^2 \frac{a\tau + b}{c\tau + d} + \frac{2\lambda mz}{c\tau + d} - \frac{cmz^2}{c\tau + d} = -\frac{c}{c\tau + d} m\left(z - \frac{\lambda}{c}\right)^2 + \frac{a}{c} m\lambda^2,$$

$$\frac{z}{c\tau + d} + \lambda \frac{a\tau + b}{c\tau + d} = \frac{z - \frac{\lambda}{c}}{c\tau + d} + \frac{a\lambda}{c}$$

to verify that the contribution from $c > 0$ is given by

$$\sum_{\substack{c>0 \\ d \ (\mathrm{mod} \ c)^* \\ \lambda \ (\mathrm{mod} \ c) \\ \alpha,\beta \in \mathbb{Z}}} c^{-k}\left(\tau + \frac{d}{c} + \alpha\right)^{-k} e\left(-\frac{1}{\tau + \frac{d}{c} + \alpha} m\left(z - \frac{\lambda}{c} - \beta\right)^2 + \frac{a}{c} m\lambda^2\right)$$

$$\times \phi_{k,m,s}^{(n,r)}\left(\frac{a}{c} - \frac{1}{c^2(\tau + \frac{d}{c} + \alpha)}, \frac{z - \frac{\lambda}{c} - \beta}{c(\tau + \frac{d}{c} + \alpha)} + \frac{a\lambda}{c}\right).$$

The next task is to compute the Fourier expansion of

$$F(\tau, z) := \sum_{\alpha,\beta \in \mathbb{Z}} (\tau + \alpha)^{-k} e\left(-\frac{1}{\tau + \alpha} m(z - \beta)^2\right) \phi_{k,m,s}^{(n,r)}\left(\frac{a}{c} - \frac{1}{c^2(\tau + \alpha)}, \frac{z - \beta}{c(\tau + \alpha)} + \frac{a\lambda}{c}\right).$$

Poisson summation shows that

$$F(\tau, z) = \sum_{n',r' \in \mathbb{Z}} a_y(n', r') e(n'x + r'z)$$

with

$$a_y(n', r') = \int_{\mathbb{R}^2} t^{-k} e\left(-\frac{mw^2}{t}\right) \phi_{k,m,s}^{(n,r)}\left(\frac{a}{c} - \frac{1}{c^2 t}, \frac{w}{ct} + \frac{a\lambda}{c}\right) e(-n'x' + r'w) \, dx' \, du',$$

where $w = u' + iv'$ with v' arbitrary and $t = x' + iy$. We employ the identities

$$\mathrm{Re}\left(\frac{a}{c} - \frac{1}{c^2 t}\right) = \frac{a}{c} - \frac{x'}{c^2 |t|^2},$$

$$\mathrm{Im}\left(\frac{a}{c} - \frac{1}{c^2 t}\right) = \frac{y}{c^2 |t|^2}$$

to find that

$$a_y(n', r') = e\left(\frac{na}{c} + \frac{ra\lambda}{c}\right) \int_{\mathbb{R}} t^{-k} \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi Dy}{mc^2|t|^2}\right) e\left(-n'x' + \frac{ir^2 y}{4mc^2|t|^2} - \frac{nx'}{c^2|t|^2}\right)$$

$$\times \int_{\mathbb{R}} e\left(-r'w - \frac{w^2 m}{t} + \frac{rw}{ct}\right) du' \, dx'$$

$$= e\left(\frac{na}{c} + \frac{ra\lambda}{c} - \frac{rr'}{2mc}\right) m^{-\frac{1}{2}} (2i)^{-\frac{1}{2}} e\left(\frac{ir'^2 y}{4m}\right)$$

$$\times \int_{\mathbb{R}} t^{-(k-\frac{1}{2})} \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi Dy}{mc^2|t|^2}\right) e\left(-\frac{D'x'}{4m} + \frac{Dx'}{4mc^2|t|^2}\right) dx'.$$

We omit further details and only point out that the evaluations of the Bessel function integral on page 176 of Fay [14] will finish the proof. □

We now restrict to the cases $s \in \{\frac{k}{2} - \frac{1}{4}, \frac{5}{4} - \frac{k}{2}\}$. We observe that the Gamma-function has poles at non-positive integers to see that Theorem 4 reduces to the following corollary.

Corollary 1. *For* $s \in \{\frac{k}{2} - \frac{1}{4}, \frac{5}{4} - \frac{k}{2}\}$ *the functions* $P_{k,m,s}^{(n,r)}$ *are in* $\widehat{\mathbb{J}}_{k,m}$ *and have Fourier expansions of the form*:

$$P_{k,m,s}^{(n,r)}(\tau, z) = q^n \mathcal{M}_{s,k-\frac{1}{2}}\left(-\frac{\pi Dy}{m}\right) e\left(\frac{iDy}{4m}\right) \vartheta_{k,m}^{(r)}(\tau, z) + c(\tau, z)$$

$$+ \sum_{n',r' \in \mathbb{Z}} c_{n,r}^{(k)}(n', r') e\left(\frac{iD'y}{4m}\right) \mathcal{W}_{s,k-\frac{1}{2}}\left(-\frac{\pi D'y}{m}\right) q^{n'} \zeta^{r'},$$

where

$$c(\tau, z) := \sum_{\substack{n',r' \in \mathbb{Z} \\ D'=0}} c_{n,r}^{(k)}(n', r') q^{n'} \zeta^{r'} \tag{21}$$

only occurs for $k < 0$ *and where*

$$c_{n,r}^{(k)}(n', r') := b_{n,r}^{(k)}(n', r') + (-1)^k b_{n,r}^{(k)}(n', -r').$$

We have (recall $D' = r'^2 - 4n'm$*):*

(1) *If* $D > 0$ *and* $k > 3$, *then* $b_{n,r}^{(k)}(n', r') = 0$ *unless* $D' < 0$, *in which case it equals*

$$\sqrt{2} \pi i^{-k} m^{-\frac{1}{2}} \left(\frac{|D'|}{D}\right)^{\frac{k}{2} - \frac{3}{4}} \sum_{c>0} c^{-\frac{3}{2}} K_c(n, r, n', r') I_{k-\frac{3}{2}}\left(\frac{\pi \sqrt{|D'|D}}{mc}\right).$$

(2) *If $D > 0$ and $k < 0$, then $b_{n,r}^{(k)}(n', r')$ is given by*

$$
\sqrt{2}\pi i^{-k} m^{-\frac{1}{2}} \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{3}{4}} \left(\frac{3}{2}-k\right) \sum_{c>0} c^{-\frac{3}{2}} K_c(n, r, n', r') J_{\frac{3}{2}-k}\left(\frac{\pi\sqrt{D'D}}{mc}\right) \quad \text{if } D' > 0,
$$

$$
\sqrt{2}\pi i^{-k} m^{-\frac{1}{2}} \left(\frac{|D'|}{D}\right)^{\frac{k}{2}-\frac{3}{4}} \Gamma\left(\frac{5}{2}-k\right) \sum_{c>0} c^{-\frac{3}{2}} K_c(n, r, n', r') I_{\frac{3}{2}-k}\left(\frac{\pi\sqrt{|D'|D}}{mc}\right) \quad \text{if } D' < 0.
$$

(3) *If $D < 0$ and $k > 3$, then $b_{n,r}^{(k)}(n', r') = 0$ unless $D' < 0$, in which case it equals*

$$
\sqrt{2}\pi i^{-k} m^{-\frac{1}{2}} \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{3}{4}} \sum_{c>0} c^{-\frac{3}{2}} K_c(n, r, n', r') J_{k-\frac{3}{2}}\left(\frac{\pi\sqrt{D'D}}{mc}\right).
$$

(4) *If $D < 0$ and $k < 0$, then $b_{n,r}^{(k)}(n', r')$ is given by*

$$
\sqrt{2}\pi i^{-k} m^{-\frac{1}{2}} \left(\frac{D'}{|D|}\right)^{\frac{k}{2}-\frac{3}{4}} \left(\frac{3}{2}-k\right) \sum_{c>0} c^{-\frac{3}{2}} K_c(n, r, n', r') I_{\frac{3}{2}-k}\left(\frac{\pi\sqrt{D'|D|}}{mc}\right) \quad \text{if } D' > 0,
$$

$$
\sqrt{2}\pi i^{-k} m^{-\frac{1}{2}} \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{3}{4}} \Gamma\left(\frac{5}{2}-k\right) \sum_{c>0} c^{-\frac{3}{2}} K_c(n, r, n', r') J_{\frac{3}{2}-k}\left(\frac{\pi\sqrt{D'D}}{mc}\right) \quad \text{if } D' < 0.
$$

We normalize the Poincaré series for $s \in \{\frac{k}{2} - \frac{1}{4}, \frac{5}{4} - \frac{k}{2}\}$ as follows:

$$
\mathcal{P}_{k,m}^{(n,r)}(\tau, z) := \begin{cases} i^{2k-1}(\frac{3}{2}-k) P_{k,m,\frac{k}{2}-\frac{1}{4}}^{(n,r)}(\tau, z) & \text{if } D > 0, \ k > 0, \\[2mm] -(\frac{\pi D}{m})^{k-\frac{3}{2}} \frac{1}{k-\frac{3}{2}} P_{k,m,\frac{5}{4}-\frac{k}{2}}^{(n,r)}(\tau, z) & \text{if } D > 0, \ k < 0, \\[2mm] \Gamma(k-\frac{1}{2}) i^{2k-1} P_{k,m,\frac{k}{2}-\frac{1}{4}}^{(n,r)}(\tau, z) & \text{if } D < 0, \ k > 0, \\[2mm] -P_{k,m,\frac{5}{4}-\frac{k}{2}}^{(n,r)}(\tau, z) & \text{if } D < 0, \ k < 0. \end{cases} \tag{22}
$$

An inspection of the Fourier expansions of $\mathcal{P}_{k,m}^{(n,r)}$ reveals that Theorem 1 follows from the identity

$$
K_c(n, r, n', r') = K_c(n', r', n, r), \tag{23}
$$

which is proved by replacing first $d \mapsto \bar{d}$ and then $\lambda \mapsto -\lambda \bar{d}$ in (9).

Now we turn to the proof of Theorem 2. Note that the differential operator $\xi_{k,m}$ annihilates meromorphic functions. For $k > 0$ the Poincaré series are meromorphic and hence we may assume that $k < 0$. We decompose $\mathcal{P}_{k,m}^{(n,r)}$ (for $D > 0$) into holomorphic and non-holomorphic parts. The identities

$$\mathcal{W}_{\frac{5}{4}-\frac{k}{2},k-\frac{1}{2}}(y) = \mathcal{W}_{\frac{2k-1}{4},k-\frac{1}{2}}(y) = e^{-\frac{y}{2}},$$

$$\mathcal{W}_{\frac{5}{4}-\frac{k}{2},k-\frac{1}{2}}(-y) = \mathcal{W}_{\frac{2k-1}{4},k-\frac{1}{2}}(-y) = e^{\frac{y}{2}}\Gamma\left(\frac{3}{2}-k,y\right),$$

$$\mathcal{M}_{\frac{2k-1}{4},k-\frac{1}{2}}(-y) = e^{\frac{y}{2}},$$

$$\mathcal{M}_{\frac{5}{4}-\frac{k}{2},k-\frac{1}{2}}(-y) = \left(k-\frac{3}{2}\right)e^{\frac{y}{2}}\Gamma\left(\frac{3}{2}-k,y\right) - \left(k-\frac{3}{2}\right)e^{\frac{y}{2}}\Gamma\left(\frac{3}{2}-k\right)$$

give that

$$\mathcal{P}_{k,m}^{(n,r)}(\tau,z) = -q^n\left(\frac{\pi D}{m}\right)^{k-\frac{3}{2}}\Gamma\left(\frac{3}{2}-k,\frac{\pi Dy}{m}\right)\vartheta_{k,m}^{(r)}(\tau,z) + \sum_{n',r'\in\mathbb{Z}} c_{n,r}^{(k)}(n',r')q^{n'}\zeta^{r'}$$

$$- \frac{1}{k-\frac{3}{2}}\left(\frac{\pi D}{m}\right)^{k-\frac{3}{2}} \sum_{\substack{n',r'\in\mathbb{Z} \\ D'>0}} c_{n,r}^{(k)}(n',r')\Gamma\left(\frac{3}{2}-k,\frac{\pi D'y}{m}\right)q^{n'}\zeta^{r'}.$$

It is easy to check that

$$\xi_{k,m}\left(\Gamma\left(\frac{3}{2}-k,ay\right)\right) = -a^{\frac{3}{2}-k}e^{-ay}, \tag{24}$$

which yields the Fourier expansion of $\xi_{k,m}(\mathcal{P}_{k,m}^{(n,r)})$. A comparison with the Fourier expansion of $P_{3-k,m}^{(n,r)sk}$ in Theorem 3 (where we may replace $r' \mapsto -r'$) leads to Theorem 2. Finally, we remark that Theorem 2 shows that $\mathcal{P}_{k,m}^{(n,r)} \in \widehat{\mathbb{J}}_{k,m}^{cusp}$.

6. Lifting maps

In this section, we will show that diagram (2) is commutative if k is even. (Note also that $J_{3-k,1}^{sk,cusp} = \{0\}$ if k is odd.) First recall that $f \in \widehat{\mathbb{S}}_{k-\frac{1}{2}}^+$ has a Fourier expansion of the form:

$$f(\tau) = \sum_{\substack{n\gg-\infty \\ (-1)^{k-1}n\equiv 0,1 \ (\mathrm{mod}\ 4)}} c^+(n)q^n + \sum_{\substack{n<0 \\ (-1)^{k-1}n\equiv 0,1 \ (\mathrm{mod}\ 4)}} c^-(n)\Gamma\left(\frac{3}{2}-k,4\pi|n|y\right)q^n.$$

A direct computation shows that

$$h(\tau) := \xi_{k-\frac{1}{2}}(f) = -(4\pi)^{\frac{3}{2}-k} \sum_{\substack{n>0 \\ (-1)^k n\equiv 0,1 \ (\mathrm{mod}\ 4)}} \overline{c^-(-n)}n^{\frac{3}{2}-k}q^n \in S_{\frac{5}{2}-k}^+.$$

Set

$$h_0(\tau) := \frac{1}{4}\sum_{j=0}^{3} h\left(\frac{\tau+j}{4}\right), \qquad h_1(\tau) := \frac{1}{4}\sum_{j=0}^{3}(-i)^j h\left(\frac{\tau+j}{4}\right),$$

and for $\mu = 0, 1$,

$$\theta_{1,\mu}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2}}} q^{\frac{r^2}{4}} \zeta^r.$$

It is easy to see that

$$F_\theta(\tau, z) := \overline{h_0(\tau)} \theta_{1,0}(\tau, z) + \overline{h_1(\tau)} \theta_{1,1}(\tau, z)$$

has a Fourier series as in (7) with $D > 0$. Furthermore, Lemma 2.1 of [15] implies that

$$
\begin{aligned}
\overline{h_0(\tau + 1)} &= \overline{h_0(\tau)}, \\
\overline{h_1(\tau + 1)} &= -i\overline{h_1(\tau)}, \\
\overline{h_0\left(\frac{-1}{\tau}\right)} &= \frac{1+i}{2} \overline{\tau}^{\frac{5}{2}-k} \left(\overline{h_0(\tau)} + \overline{h_1(\tau)}\right), \\
\overline{h_1\left(\frac{-1}{\tau}\right)} &= \frac{1+i}{2} \overline{\tau}^{\frac{5}{2}-k} \left(\overline{h_0(\tau)} - \overline{h_1(\tau)}\right)
\end{aligned}
\tag{25}
$$

and (7) and (9) of Section 5 of [13] for $m = 1$ give

$$
\begin{aligned}
\theta_{1,0}(\tau + 1, z) &= \theta_{1,0}(\tau, z), \\
\theta_{1,1}(\tau + 1, z) &= i\theta_{1,1}(\tau, z), \\
\theta_{1,0}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) &= \frac{1-i}{2} \tau^{\frac{1}{2}} e^{\frac{2\pi i z^2}{\tau}} \left(\theta_{1,0}(\tau, z) + \theta_{1,1}(\tau, z)\right), \\
\theta_{1,1}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) &= \frac{1-i}{2} \tau^{\frac{1}{2}} e^{\frac{2\pi i z^2}{\tau}} \left(\theta_{1,0}(\tau, z) - \theta_{1,1}(\tau, z)\right).
\end{aligned}
\tag{26}
$$

The equations in (25) and (26) show that

$$F_\theta|_{3-k,1}^{sk} A = F_\theta \tag{27}$$

for $A = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (0,0)] \in \Gamma^J$, while if $A = [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), (\lambda_1, \lambda_2)] \in \Gamma^J$, then (27) follows directly from the definition of $\theta_{1,\mu}$. Hence (27) holds for all $A \in \Gamma^J$ and $F_\theta \in J_{3-k,1}^{sk,cusp}$.

On the other hand, for $f \in \widehat{\mathbb{S}}_{k-\frac{1}{2}}^+$, set

$$H_0(\tau) := \frac{1}{4} \sum_{j=0}^{3} f\left(\frac{\tau + j}{4}\right) \quad \text{and} \quad H_1(\tau) := \frac{1}{4} \sum_{j=0}^{3} i^j f\left(\frac{\tau + j}{4}\right).$$

Analogous to the proof of Theorem 4.4 of [18] one can show that

$$\mathbb{F}_\theta(\tau, z) := H_0(\tau) \theta_{1,0}(\tau, z) + H_1(\tau) \theta_{1,1}(\tau, z) \in \widehat{\mathbb{J}}_{k,1}^{cusp} \tag{28}$$

and (24) then yields that $\xi_{k,1}(\mathbb{F}_\theta) = F_\theta$. We conclude that diagram (2) is commutative.

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