Exercises on Modular Symbols

Day - One

Exercise 1.
- If $\Gamma \leq SL_2(\mathbb{Z})$ has finite index then $\Gamma \setminus \mathbb{P}^1(\mathbb{Q})$ is finite.
- Let $\Gamma = \Gamma_0(p)$, then show that $\Gamma_0(p) \setminus \mathbb{P}^1(\mathbb{Q}) = \{[0], [i\infty]\}$.
- Let $\Gamma = \Gamma_0(N)$ then show that $\frac{p_1}{q_1} \sim \frac{p_2}{q_2}$ if and only if $gcd(q_1, N) = gcd(q_2, N) = t$ and $p_1(\frac{q_1}{t}) \equiv p_2(\frac{q_2}{t})(mod \frac{N}{t})$. Where the condition $p_1(\frac{q_1}{t}) \equiv p_2(\frac{q_2}{t})(mod \frac{N}{t})$ is not required when N is square free.

Exercise 2. Verify the 7 modular symbol relations stated in the lecture. In addition, show that $\{\alpha, g(\alpha)\}_\Gamma = 0$ if g is elliptic or parabolic element of Γ.

Exercise 3. Verify that $B(\Gamma) \leq Z(\Gamma)$ and $B(N) \leq Z(N)$.

Exercise 4. Compute $\mathscr{L}(N)$ for $N = 11, 13, 17, 19, 23$. If keen try for N not prime.

Exercise 5. Determine J s.t. the following sequence is exact.

$$0 \longrightarrow J \hookrightarrow \mathbb{Z}[SL_2(\mathbb{Z})] \longrightarrow \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \longrightarrow 0$$

Exercise 6. Determine I s.t. the following sequence is exact.

$$0 \longrightarrow I \hookrightarrow \mathbb{Z}[SL_2(\mathbb{Z})] \longrightarrow \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 \longrightarrow 0$$

Day - Two

Exercise 1. (a) For a given $\Gamma \leq SL_2(\mathbb{Z})$, compute $dim\mathscr{L}(\Gamma)$.

(Recall: $\mathscr{L}(\Gamma) = \{\lambda : \Gamma \setminus SL_2(\mathbb{Z}) \longrightarrow \mathbb{C} : | : \lambda(x) + \lambda(xS) = \lambda(x) + \lambda(xR) + \lambda(xR^2) = 0, \forall x \in \Gamma \setminus SL_2(\mathbb{Z})\}$).

Hint: Its easier to compute $dim\mathscr{L}(\Gamma)^*$.

(b) Compute g_Γ(using Riemann-Hurwitz Formula) and compare to part(a).

Exercise 2. Determine the Hecke eigenforms in $Hom_{\Gamma_0(N)}(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})], \mathbb{C})$.

Hint:

$$M_2^{Eis}(N) \bigoplus E_2(z) = \bigoplus_{tf^2|N} \bigoplus_{\chi} E_\chi(tz),$$

where the sum is running over the primitive Dirchilet characters χ(mod f) with

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$$

and

$$E_\chi(z) = const. + \sum_{n\geq} \Bigg(\sum_{d|n} d\chi(d)\overline{\chi(\frac{n}{d})} \Bigg) q^n$$

Exercise 3. Let $M_\ell = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2} : | : ad - bc = \ell, a > b \geq 0, d > c \geq 0\}$.

(a) Show that M_ℓ is finite.

(b) Determine M_ℓ for $\ell = 2, 3$.

(c) Find the eigenvalue of $T(2), T(3)$ on $\mathscr{L}^-(11)$.