Calculus Math 1710.006 Spring 2012 (Cohen) Lecture Notes

Developed in the late 17th century, independently, by Isaac Newton and Gottfried Wilhelm Leibniz. We now regard calculus as the study of *change* in functions, especially *continuous* ones. Newton, Leibniz, and their contemporaries, however, originally conceived of "The Calculus" as a computational method for solving problems in *physics*. This motivates our first problem in the course.

Example 1. *A ball is launched into the air at* 96 *ft/s. Gravity accelerates the ball downward at a rate of* 32 *ft/s. The height of the ball in feet after* t *seconds is given by the function:*

$$h(t) = -16t^2 + 96t$$

Question 1: How long until the ball lands?

By setting $h(t) = 0$ and solving for t, we see that the height of the ball is 0 feet after exactly 6 seconds.

Question 2: How high does the ball go?

The graph of $h(t)$ is a parabola and hence symmetrical about some vertex. Since the ball has height 0 at $t = 0$ and $t = 6$, its vertex must lie directly between, at $t = 3$. Then its maximum height is given by $h(3) = 144$ feet.

Question 3: What is the average velocity of the ball in the first 3 seconds?

The ball goes up 144 feet in 3 seconds, so its average velocity is $\frac{144}{3} = 48$ ft/s.

Question 4: What is the average velocity of the ball from $t = 1$ to $t = 3$ seconds?

Velocity should be given by distance/time. The ball's height changes from $h(1)$ feet to $h(3)$ feet between $t = 1$ and $t = 3$. So its average velocity over this interval is given by:

$$\frac{h(3) - h(1)}{3 - 1} = \frac{144 - 80}{3 - 1} = 32 \text{ ft/s}.$$

The formula above should recall the *slope formula* $m = \frac{y_2 - y_1}{x_2 - x_1}$ learned in College Algebra. Indeed, the average velocity from any $t = 1$ to $t = 3$ is exactly the slope of the line passing through the points $(1, h(1))$ and $(3, h(3))$.

Question 5: Exactly how fast is the ball going at $t = 1$ second? (Our first calculus question!)

You can get good estimates of this *instantaneous* velocity by computing the *average* velocity over very small intervals around $t = 1$. For example, we can compute the average velocities for .1, then .01, then .00001 seconds after $t = 1$, as below:

$$\frac{h(1.1) - h(1)}{1.1 - 1} \approx 62.4 \text{ ft/s}$$
$$\frac{h(1.01) - h(1)}{1.01 - 1} \approx 63.8 \text{ ft/s}$$

$$\frac{h(1.00001) - h(1)}{1.00001 - 1} \approx 63.9998 \text{ ft/s}$$

The problem is that each of the above is just an *approximation* of the actual velocity at $t = 1$... but as mathematicians we are truly interested in the exact value! We can get better and better approximations by examining smaller and smaller intervals, but for our purposes, any positive interval at all will be "too large."

To solve this problem, we wish to look at "arbitrarily small distances" on the real number line. Newton/Leibniz referred to such distances as *infinitesimals*, or "infinitely small values," and built their system on this notion. Much of our modern notation is still influenced by the classical concept of infinitesimals. Modern mathematicians, however, regard the notion of an "infinitely small" value as nebulous in meaning, and we abhor the violation of the Archimedean principle! So, in order to solve calculus problems rigorously, we need to develop some new technology, in particular the notion of *limits.*

Limits. (Sec 2.1-2.3)

Definition 1 (Informal Definition)**.** *Let $f(x)$ be a function and a be a real number. If there exists some number L such that $f(x)$ is arbitrarily close to L whenever x is sufficiently close (but not equal) to a, then we write:*

$$\lim_{x\to a} f(x) = L$$

and say "the limit of $f(x)$ as x approaches a is L."

The student may note that the terms "arbitrarily close" and "sufficiently close" are somewhat vague. Rest assured, the notion of a limit has a precise and clearly stated mathematical definition! However, this definition is more easily understood after some intuition about limits is already developed, so we will omit this formal definition here and present it later after we have done a few exercises.

Example 2. *Let $f(x) = x + 2$. Find* $\lim_{x\to 1} f(x)$.

Example 3. *Graphical example: Section 2.2 Example 1.*

Example 4. *Let $f(x) = \frac{\sqrt{x}-1}{x-1}$. Find* $\lim_{x\to 1} f(x)$.

Solution. We can try to solve the above problem by making a table of function values where x is very close to 1.

$x =$	.9	.09	.009	1.001	1.01	1.1
$f(x) =$	.5131670	.5012563	.5001251	.4998750	.4987562	.4880885

It appears that as x gets very close to 1, $f(x)$ gets close to .5, and we guess that $\lim_{x\to 1} f(x) = .5$. This will turn out to be correct, but note that at this point in our course we are merely making a *conjecture*, or educated guess! (How do you know the limit is .5 and not .500000001 or .499999999?)

□

Example 5. *Let $f(x) = \frac{x^2-5x+6}{x-2}$. Find* $\lim_{x\to 2} f(x)$.

Solution. To solve this, we first note that $x^2 - 5x + 6 = (x - 2)(x - 3)$ in the numerator of $f(x)$. Our gut says to divide out the $(x - 2)$ from top and bottom and be done with it! However, it is important to note that the following equation is **NOT** true:

$$\frac{(x-2)(x-3)}{x-2} = x-3$$

The reason the above equation is false is because x could take on the value of any real number. In particular, if $x = 2$, then the left side of the equation is undefined, while the right side is equal to -1; so we don't have true equality here.

HOWEVER, when we are computing the limit of $f(x)$ as x approaches 2, we restrict our attention to all possible values of x which are near, but NOT EQUAL TO the value 2. This means that, since the above equation holds for all $x \neq 2$, the following equation makes perfect sense:

$$\lim_{x\to 2} \frac{(x-2)(x-3)}{x-2} = \lim_{x\to 2} x-3$$

Since the latter limit is -1, we have $\lim_{x\to 2} f(x) = -1$. $\square$

Now we will go back and cover our tracks by including the formal definition of a limit.

Definition 2 (Formal Definition)**.** *Let $f(x)$ be a function and a be a real number. Suppose that $f(x)$ exists for all x in some open interval containing a. We say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, written $\lim_{x\to a} f(x) = L$, if the following statement holds: For **any** positive number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ (depending on ϵ) such that*

$$|f(x) - L| < \epsilon \textit{ whenever } |x - a| < \delta.$$

Why does the above definition make sense? The definition above should give the student confidence that the intuitive notion of a limit may be made mathematically precise. Proceeding with such assurances, all future limit definitions will be presented in the informal style.

One-Sided Limits.

Definition 3. *The **left-hand limit:** If $f(x)$ is arbitrarily close to L for all x sufficiently close to a, **with** $x < a$, then we say "the limit of $f(x)$ as x approaches a from the left is L," and write:* $\lim_{x\to a^-} f(x) = L$.

Definition 4. *The **right-hand limit:** If $f(x)$ is arbitrarily close to L for all x sufficiently close to a, **with** $x > a$, then we say "the limit of $f(x)$ as x approaches a from the right is L," and write:* $\lim_{x\to a^+} f(x) = L$.

Example 6. *Graphical example where left- and right-hand limits are not equal.*

Note: $\lim_{x\to} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} = L$.

Continuity. (Sec 2.6)

Definition 5. *We say that a function f is* **continuous** *at a point a if the following all hold:*

1. *$f(a)$ is defined.*
2. *$\lim_{x\to a} f(x)$ exists.*
3. *$\lim_{x\to a} f(x) = f(a)$.*

If f is continuous at every real number, then we say that f is continuous.

Example 7.

1. *Linear functions ($y = mx + b$) are continuous.*
2. *Polynomials are continuous.*
3. *Rational functions, or functions of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, are continuous at every point except where $q(x) = 0$.*
4. *Sine and cosine are continuous everywhere. The tangent function is continuous everywhere except points of the form $\frac{\pi}{2} + k\pi$, where k is any integer.*

Note: (Important) If a function is continuous at a point a, then by definition $\lim_{x\to a} f(x) = f(a)$. This means to evaluate a continuous function's limit at a, you can just plug a into the function!

Example 8. *Find:*

1. $\lim_{x\to 3} f(x)$, *where* $f(x) = \frac{1}{2}x - 7$
2. $\lim_{x\to -1}(5x^4 - \pi x^3 - 1)$
3. $\lim_{x\to 5} g(x)$, *where* $g(x) = 6$
4. $\lim_{x\to 1} \dfrac{x^2 - 6x + 8}{x^2 - 9}$

Solution. For part (1), simply observe that $f(x)$ is a linear function and hence continuous. In that case, $\lim_{x\to 3} f(x) = f(3) = \frac{1}{2} \cdot 3 - 7 = -\frac{11}{2}$. The remaining problems are solved similarly. □

Other Rules for Computing Limits. (Sec 2.3)

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Suppose c is a real number, and $m, n > 0$ are integers. The following all hold:

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[cf(x)] = c[\lim_{x\to a} f(x)]$
3. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$
4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.

(5) $\lim_{x\to a}[f(x)]^{\frac{n}{m}} = [\lim_{x\to a} f(x)]^{\frac{n}{m}}$, provided $f(x) \geq 0$ for x near a if m is even and $\frac{n}{m}$ is in reduced form.

HOMEWORK: Section 2.2 #7-11, 17, 19; Section 2.3 #11-25 odd, 31-35 odd, 39, 51 (Due 2/1/12)

Example 9. *Compute* $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$.

Solution. Recall: when we tackled this problem earlier using a table, we conjectured that the limit is $\frac{1}{2} = .5$. We will now show that our conjecture was correct by observing the following:

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1}$$

(Note that we may cancel to obtain the second equality, since we are taking a limit.) Now since $\dfrac{1}{\sqrt{x}+1}$ is continuous at $x = 1$, we may substitute to finish the problem.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}. \qquad \square$$

Infinite Limits. (Sec 2.4)

Example 10. *Find* $\lim_{x\to 0} \frac{1}{x^2}$.

Solution. Note that if we take x to be a very small number (i.e. close to 0), then x^2 will be an even smaller POSITIVE number. Then its reciprocal $\frac{1}{x^2}$ will be a very large positive number. In fact, the smaller we take x to be, the larger the value of $\frac{1}{x^2}$ will be, i.e. the function $\frac{1}{x^2}$ becomes *arbitrarily large* as x approaches 0.

By our previous definition of limits, there is no number L which $\frac{1}{x^2}$ approaches when $x \to 0$, so we say that the limit does not exist. However, we wish to describe this kind of phenomenon in functions, so we will now expand our definition suitably. $\square$

Definition 6. *Let $f(x)$ be a function and a a real number. If $f(x)$ grows arbitrarily large for x sufficiently close to a, then we write* $\lim_{x\to a} f(x) = \infty$.

Similarly, if $f(x)$ grows arbitrarily large in magnitude but in the negative direction, then we write $\lim_{x\to a} f(x) = -\infty$.

We can also define the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ to be ∞ or $-\infty$, in the analogous way.

Example 11. *Graphical example.*

Example 12. *Find (a)* $\lim_{x\to \frac{\pi}{2}^-} \tan x$ *and (b)* $\lim_{x\to \frac{\pi}{2}^+} \tan x$.

Solution. A quick sketch of the graph of the tangent function shows that $\lim_{x\to \frac{\pi}{2}^-} \tan x = \infty$ and $\lim_{x\to \frac{\pi}{2}^+} \tan x = -\infty$. $\square$

Finding Infinite Limits Analytically.

Example 13. *Find* $\lim_{x\to 0} \dfrac{5+x}{x}$.

Solution. Taking x to be a very small *positive* number, we observe that $5+x$ will be very close to 5, and hence we can approximate $\dfrac{5+x}{x} \approx \dfrac{5}{x}$. The latter fraction will grow arbitrarily large as x approaches 0 from the right, so we have $\lim_{x\to 0^+} \dfrac{5+x}{x} = \infty$.

On the other hand, if we take x to be a very small *negative* number, then we still have $5+x \approx 5$ and hence $\dfrac{5+x}{x} \approx \dfrac{5}{x}$. This time, however, the latter fraction will grow arbitrarily large in magnitude in the *negative* direction, since a positive number divided by a negative number is negative. This implies $\lim_{x\to 0^-} \dfrac{5+x}{x} = -\infty$.

Since this function has two *different* one-sided limits, we say that the limit *does not exist.* □

The lesson we take from the above example is that for functions of the form $\dfrac{p(x)}{q(x)}$, if $p(x)$ stays relatively constant but $q(x)$ goes to 0 as $x \to a$, then $\dfrac{p(x)}{q(x)}$ will blow up in either the positive or negative direction. Whether the limit is ∞, $-\infty$, or does not exist will depend on the signs of $p(x)$ and $q(x)$ when x is near a.

Example 14. *Find (a)* $\lim_{x\to 3^+} \dfrac{2-5x}{x-3}$ *and (b)* $\lim_{x\to 3^-} \dfrac{2-5x}{x-3}$.

Limits at Infinity. (Sec 2.5)

Definition 7. *If $f(x)$ becomes arbitrarily close to L for all sufficiently large and positive x, we write* $\lim_{x\to\infty} f(x) = L$*.*

If $f(x)$ becomes arbitrarily close to L for all sufficiently large and negative X, then we write $\lim_{x\to-\infty} f(x) = L$*.*

Example 15. *Let* $f(x) = \dfrac{x}{\sqrt{x^2+1}}$*. Find* $\lim_{x\to\infty} f(x)$*.*

Example 16. *Find* $\lim_{x\to-\infty}(2 + \frac{10}{x^2})$*.*

Example 17. *Find* $\lim_{x\to\infty}(2x+8)$*.*

Note: The previous example shows that it makes sense to combine *infinite limits* and *limits at infinity*, when appropriate.

We can also use limits at infinity to characterize the *end behavior* of functions. The student may recall from a previous algebra course that for polynomial functions $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, the end behavior of $f(x)$ is determined entirely by the first term $a_n x^n$, according to the following rules:

If n is even and a_n is positive, then $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$.

If n is even and a_n is negative, then $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = -\infty$.

If n is odd and a_n is positive, then $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$.

If n is odd and a_n is negative, then $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$.

Now let us consider the limits at infinity of *rational* functions, i.e. fractions of polynomials.

Example 18. *Find the following limits.*

(1) $\lim_{x\to\infty} \dfrac{12x-7}{x^3+1}$

(2) $\lim_{x\to\infty} \dfrac{5x^5-4x^2+2}{2x^4-\pi x^2+6x-1}$

(3) $\lim_{x\to\infty} \dfrac{3x^7-22x^5+19x}{8x^7-100x^6+22}$

The previous example helps us extrapolate the following general rule.

Limits at Infinity for Rational Functions. Suppose $f(x)$ is a rational function, i.e.

$$f(x) = \frac{ax^n + ...}{bx^m + ...}$$

for some nonnegative integers n and m and some non-zero leading coefficients a and b. Then the following hold:

(1) If $n < m$, then $\lim_{x\to\infty} f(x) = 0$.

(2) If $n > m$, then $\lim_{x\to\infty} f(x) = \infty$.

(3) If $n = m$, then $\lim_{x\to\infty} f(x) = \frac{a}{b}$.

The previous rule takes care of limits at *positive* infinity for all rational functions, what about limits at *negative* infinity? A very similar rule should apply for determining $\lim_{x\to-\infty} f(x)$, where f is a rational function, but it needs to depend on whether n and m are even or odd (as an even exponent will flip the sign on any large negative input x). We leave it to the student to develop this analogous rule for limits at minus infinity.

We need one more major limit rule before we turn to officially defining the derivative.

Example 19. *Find* $\lim_{x\to\infty} \cos x$.

Solution. Since $\cos x$ oscillates back and forth between -1 and 1 for arbitrarily large x, this limit does not exist. $\square$

Example 20. *Find* $\lim_{x\to\infty} \frac{\cos x}{x}$.

Solution. First note that $-1 \leq \cos x \leq 1$ for all values of x. It follows that $-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$ for all values. Thus the values of $\frac{\cos x}{x}$ oscillate up and down, as in the previous example, but they never exceed $\frac{1}{x}$ nor go below $-\frac{1}{x}$. Since $\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$ and our function simply wiggles in between, we must have $\lim_{x\to\infty} \frac{\cos x}{x} = 0$ as well.

□

This example suggests the following theorem, which we give without proof but which should seem intuitively obvious to the reader:

Theorem 1 (Squeeze Theorem)**.** *Assume f, g, and h are functions which satisfy $f(x) \leq g(x) \leq h(x)$ for values of x near a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

(The analogous result also holds for limits at infinity.)

Example 21. *Evaluate* $\lim_{x\to\infty}(5 + \frac{\sin x}{\sqrt{x}})$.

HOMEWORK: Sec 2.4 #7, 10, 16, 17-21 odd; Sec 2.5 #9-17 odd, 42, 43 (Due 2/8/12)

Beginning Calculus: The Derivative. (Sec 3.1)

Recall our physics problem from earlier: A ball is launched into the air, and its height in feet after t seconds is given by $f(t) = -16t^2 + 96t$. (Note we have changed the name of our function from h to f; this is because we wish to use the letter h to represent something else a little later on.) We wish to determine exactly how fast the ball is moving at $t = 1$ second.

We already know how to compute the *average velocity* for the ball between any two points $t = a$ and $t = b$; we simply compute the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ as follows:

$$\text{average velocity} = \frac{f(b) - f(a)}{b - a}$$

We had the idea to approximate the instantaneous velocity at $t = 1$ by computing the average velocity over some very short time interval after $t = 1$. For instance, we can let h represent some very small number (as small as you want) and compute the average velocity from $t = 1$ to $t = 1 + h$:

$$\text{approximate instantaneous velocity} = \frac{f(1 + h) - f(1)}{h}, \text{ where } h \text{ is very small.}$$

The stumbling block we ran into earlier was that there was no way to take h small enough to suit our purposes; one may always take smaller values for h and get better approximations. Now, however, we have the notion of a *limit* to account for this problem: We simply take our average velocity formula and let h run down arbitrarily close to 0, i.e. we take the limit as $h \to 0$.

$\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$
$= \lim_{h\to 0} \dfrac{[-16(1+h)^2+96(1+h)]-[80]}{h}$
$= \lim_{h\to 0} \dfrac{-16-32h-16h^2+96+96h-80}{h}$
$= \lim_{h\to 0} \dfrac{-16h^2+64h}{h}$
$= \lim_{h\to 0} -16h+64$
$= -16(0)+64$
$= 64.$

So our conjecture from earlier was indeed the actual instantaneous velocity of the ball at $t=1$. We identify this concept with the *slope of the tangent line* to the graph of f at $t=1$.

Now we make some of the concepts above official:

Definition 8. *Let $f(x)$ be any function, and a be any real number. The* ***average rate of change*** *of f from $x=a$ to $x=a+h$, where h is some real number, is given by:*

$$\frac{f(a+h)-f(a)}{h}$$

This is also the ***slope of the secant line*** *connecting the points $(a, f(a))$ and $(b, f(b))$. We refer to the formula above as the* ***difference quotient.***

The ***instantaneous rate of change*** *of f at $x=a$, or the* ***slope of the tangent line*** *to the graph of f at $x=a$, is given by:*

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Example 22. *Let $f(x)=x^2+4x$.*

(1) *Find the average rate of change (AROC) of f from $x=1$ to $x=4$.*

(2) *Find the instantaneous rate of change (IROC) of f at $x=1$.*

(3) *Find the equation of the line tangent to the graph of f at $x=1$.*

Solution. We will leave the first problem to the reader, and compute the IROC of f at $x=1$ below:

$\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$
$= \lim_{h\to 0} \dfrac{[(1+h)^2+4(1+h)]-[1^2+4(1)]}{h}$
$= \lim_{h\to 0} \dfrac{1+2h+h^2+4+4h-x^2-1-4}{h}$
$= \lim_{h\to 0} \dfrac{h^2+6h}{h}$
$= \lim_{h\to 0}(h+6)$
$= 6.$

So the slope of the tangent line to the graph of f at $x = 1$ is 6. To find the equation of this line, we need only note that the line must meet the graph at the point $(1, f(1)) = (1, -5)$, and apply the point-slope formula:

$$y - y_1 = m(x - x_1)$$
$$y - (-5) = 6(x - 1)$$
$$y = 6x - 11$$ □

The derivative. Now our idea is the following: Given a function f, we wish to define a *new* function f', called *the derivative* of f, which takes a real number x for input, and for output gives *the slope of the tangent line to the graph of f at the point x.*

Definition 9. *The **derivative** of f is the function:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

*wherever this limit exists. If $f'(x)$ exists, we say that f is **differentiable** at the point x.*

Example 23. *Let $f(x) = -16x^2 + 96x$. Find its derivative $f'(x)$. Compute $f'(1)$, $f'(0)$, $f'(3)$, $f'(5)$. Do the values make sense given the context of our physics example?*

Solution. We compute the derivative by the definition below:

$$f'(x)$$
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{[-16(x+h)^2 + 96(x+h)] - [-16x^2 + 96x]}{h}$$
$$= \lim_{h \to 0} \frac{-16x^2 - 32xh - 16h^2 + 96x + 96h + 16x^2 - 96x}{h}$$
$$= \lim_{h \to 0} \frac{-32xh - 16h^2 + 96h}{h}$$
$$= \lim_{h \to 0} -32x - 16h + 96$$
$$= -32x - 16(0) + 96$$
$$= -32x + 96$$

So the derivative is $f'(x) = -32x + 96$.

Now we observe that $f'(1) = -32(1) + 96 = 64$, which is the instantaneous velocity at $x = 1$ we computed earlier.

$f'(0) = -32(0) + 96 = 96$, which is the velocity at which the ball is initially launched according to our original problem. (In other words, the instantaneous velocity at $x = 0$.

$f'(3) = -32(3) + 96 = 0$. So the velocity at $t = 3$, the moment the ball tops out its arc and begins to fall back to the earth, is exactly 0, as expected.

$f'(5) = -32(5) + 96 = -64$. So the velocity at $t = 5$, i.e. 2 seconds after the ball tops out its arc, is -64, i.e. the ball is falling toward the earth at 64 ft/s. Considering at $t = 1$ the ball was flying UP at 64 ft/s, this should not be surprising! $\square$

Notation: (Mostly Old-Timey) The Greek letter Δ is often used to represent change. So instead of h one may write Δx, the "change in x". Moreover instead of writing $f(x+h) - f(x)$, one may regard this as the "change in y", and write Δy. So we have

$$\frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}.$$

Then when taking the derivative, i.e. taking the limit as $\Delta x \to 0$, we use the notation dx to represent the notion of "*infinitesimal*" change in x, and dy to represent the corresponding "infinitesimal" change in y. So:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y} = \frac{dy}{dx}.$$

Recall that the notion of an infinitesimal is a bit nebulous in our context and we prefer the more concrete notion of a limit; hence this notation is somewhat old-fashioned. However, it is **VERY** common, and so we will use it in this course interchangeably with $f'(x)$.

Some other notations which also stand for the derivative are the following: $\frac{df}{dx}$, $\frac{d}{dx}(f(x))$, y', $y'(x)$.

Example 24. *Let $y = f(x) = \sqrt{x}$.*

(1) *Find $\frac{dy}{dx}$.*
(2) *Find the equation of the line tangent to the graph of f at $(4, 2)$.*

Example 25. *Let $g(t) = \frac{1}{t^2}$. Find $g'(t)$.*

Example 26. *Graphical example: Sec 3.1 Example 6.*

Example 27. *Graphical example: Sec 3.1 Example 7.*

Note that if a function is differentiable at a point, then it must be continuous at that point. The converse, however, is not true: A function may be continuous at a point but not differentiable there.

A function is *not* differentiable at a point a if at least one of the following holds:

(1) f is not continuous at a.

(2) f has a corner at a. (Example: $f(x) = |x|$.)

(3) f has a vertical tangent at a. (Example: $f(x) = x^{\frac{1}{3}}$.)

Rules of Differentiation. (Sec 3.2)

Many of our derivative calculations in the previous section were long, tedious, and/or computationally difficult. We wish to develop shortcuts by which one may rapidly compute the derivatives of familiar functions. We'll begin by computing a few the old-fashioned way.

Example 28. *Let c be any real number, and let $f(x) = c$. Find $f'(x)$.*

Solution. Since the graph of f is a horizontal line, every tangent line to the graph should have slope 0. Thus we conjecture that $f'(x) = 0$. Let's see if this intuition jibes with our definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0. \qquad \square$$

Constant Rule: If c is a real number, then $\frac{d}{dx}(c) = 0$.

Example 29. *Find the following:*

(1) $\frac{d}{dx}(x)$
(2) $\frac{d}{dx}(x^2)$
(3) $\frac{d}{dx}(x^3)$

Solution. Some basic computations will show that the three derivatives are 1, $2x$, and $3x^2$, respectively. It is also easy to show that $\frac{d}{dx}(x^4) = 4x^3$, $\frac{d}{dx}(x^5) = 5x^4$, etc., which leads us to the following rule. $\square$

Power Rule: If n is a positive integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Example 30. *Evaluate the following derivatives:*

(1) $\frac{d}{dx}(x^9)$
(2) $\frac{d}{dx}(x^{275})$
(3) $\frac{d}{dx}(2^8)$

Constant Multiple Rule: If f is differentiable at x and c is a constant, then $\frac{d}{dx}[cf(x)] = cf'(x)$.

Sum Rule: If f and g are differentiable at x, then $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$.

The two rules above say that taking derivatives "distributes over" sums, differences, and multiples of functions.

Example 31. *Find $\frac{d}{dx}(2x^3 + 9x^2 - 6x + 4)$.*

Example 32. *Let $f(x) = 2x^3 - 15x^2 + 24x$.*

(1) *Find an equation of the line tangent to the graph of f at the point $(2, 4)$.*
(2) *At what points on the graph of f is the tangent line horizontal?*
(3) *For what values of x does the tangent line have a slope of 6?*

HOMEWORK: Sec 3.1 #17, 20, 26, 29, 33, 35, 37, 39-41; Sec 3.2#7-23 odd, 39, 40 (Due 2/15/12)

The Product Rule and Quotient Rule (Section 3.3)

The "sum rule" above tells us that derivatives "distribute over addition", i.e. $(f+g)' = f' + g'$ for any two functions f and g. We may be tempted to derive a similar conclusion for multiplication, i.e. that the derivative of a product of two functions is the product of the two derivatives. Unfortunately this is not the case, as the following example illustrates:

Example 33. *Let $f(x) = x^2$ and $g(x) = x^3$.*

(1) *Find $f'(x)$, $g'(x)$, and $f'(x) \cdot g'(x)$.*

(2) *Find $(fg)'(x)$.*

Solution. By the power rule we have $f'(x) = 2x$ and $g'(x) = 3x^2$, so $f'(x) \cdot g'(x) = 6x^3$. On the other hand, $(fg)(x) = x^2 \cdot x^3 = x^5$, and hence $(fg)'(x) = 5x^4$. So clearly we have $(fg)'(x) \neq f'(x)g'(x)$. □

So this overly simply idea of a "product rule" is clearly false. However, we can obtain a rule which is almost as simple and nice:

Product Rule: If f and g are differentiable at x, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$.

Proof. To show the product rule, we simply compute the derivative $\frac{d}{dx}[f(x)g(x)]$ below. Note in our second line, we use the trick of adding and subtracting $\frac{f(x)g(x+h)}{h}$ to the equation (which is the same as adding 0).

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h} \right] \\
&= \lim_{x\to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right] \\
&= \left[\lim_{x\to 0} \frac{f(x+h) - f(x)}{h} \right] (\lim_{h\to 0} g(x+h)) + (\lim_{h\to 0} f(x)) \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

□

Example 34. *Use the product rule to find the following derivatives:*

(1) $\frac{d}{dx}[x^2 \cdot x^3]$

(2) $\frac{d}{dx}[(x^3-8)(x^2+4)]$

(3) $\frac{d}{dx}[(7x^5-4x^2+8x)(14x^3-19)]$

We now wish to provide a "quotient rule" for derivatives as well; to emphasize the importance of the order of the terms in this rule, we will change our function names from f and g to N and D (for "numerator" and "denominator".

Quotient Rule: If N and D are differentiable at x and $D(x) \neq 0$, then $\frac{d}{dx}[\frac{N(x)}{D(x)} = \frac{D(x)N'(x)-N(x)D'(x)}{[D(x)]^2}$.

Proof. This rule should follow easily from the product rule, if we set things up the right way. Set $q(x) = \frac{N(x)}{D(x)}$; we wish to find $q'(x)$.

First note that $N(x) = q(x)D(x)$, so the product rule tells us that

$$N'(x) = q'(x)D(x) + q(x)D'(x)$$

In that case, solving for $q'(x)$, we have $q'(x) = \frac{N'(x)-q(x)D'(x)}{D(x)}$. In order to simplify this fraction, we multiply by $\frac{D(x)}{D(x)}$, and observe that $q(x)D(x) = \frac{N(x)}{D(x)}D(x) = N(x)$ below:

$$\begin{aligned} q'(x) &= \frac{N'(x)-q(x)D'(x)}{D(x)} \cdot \frac{D(x)}{D(x)} \\ &= \frac{D(x)N'(x)-q(x)D(x)D'(x)}{[D(x)]^2} \\ &= \frac{D(x)N'(x)-N(x)D'(x)}{[D(x)]^2} \end{aligned}$$

So our quotient rule holds. $\square$

Example 35. *Find and simplify the following derivatives.*

(1) $\frac{d}{dx}[\frac{x^2+3x+4}{x^2-1}]$

(2) $\frac{d}{dx}(2x^{-3})$

Example 36. *Find an equation of the line tangent to the graph of* $f(x) = \frac{x^2+1}{x^2-4}$ *at the point* $(3, 2)$.

Example 37. *Let n be any negative integer. Compute* $\frac{d}{dx}x^n$.

Solution. Let $m = -n$, so m is a positive integer. This will allow us to use the power rule we learned earlier (recall it only applies for positive integer powers). Then we begin the derivative calculation, using the quotient rule:

$$
\begin{aligned}
\frac{d}{dx}x^n &= \frac{d}{dx}x^{-m} \\
&= \frac{d}{dx}\frac{1}{x^m} \\
&= \frac{(x^m)(\frac{d}{dx}1) - (1)(\frac{d}{dx}x^m)}{(x^m)^2} \\
&= \frac{-mx^{m-1}}{x^{2m}} \\
&= -m\frac{x^{m-1}x^{-(m-1)}}{x^{2m}x^{-(m-1)}} \\
&= -m\frac{1}{x^{2m-m+1}} \\
&= -m\frac{1}{x^{m+1}} \\
&= -mx^{-m-1} \\
&= nx^{n-1}
\end{aligned}
$$

□

Since this holds for any negative integer n, we have strengthened the power rule to include negative numbers.

(Strengthened) Power Rule: If n is *any* integer, then $\frac{d}{dx}x^n = nx^{n-1}$.

In fact, the power rule holds for all rational exponents as well, i.e. if q is any rational number, then $\frac{d}{dx}x^p = px^{p-1}$. We will prove this fact later when we introduce the concept of implicit differentiation, but the reader will find this fact to be very useful on the homework assignments from this point on.

Derivatives of Trigonometric Functions (Section 3.4)

Example 38. *Sketch a rough graph of the derivative of $f(x) = \sin x$.*

Solution. The graph of $f(x) = \sin x$ has horizontal tangent lines at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...,$ etc., so we know that $f'(x) = 0$ at each of these points.

At $x = 0$, the graph of $f(x) = \sin x$ has some positive slope m, which will recur every 2π units up and down the real line. At $x = \pi$ the graph should have slope $-m$, which will also recur every 2π units.

In general the graph of $f'(x)$ should look like a continuous periodic curve which oscillates back and forth between some values $-m$ and m. In particular, if $m = 1$, then the graph of $f'(x)$ should call to mind the graph of $\cos x$! □

Trigonometric Derivatives: $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$.

Example 39. *Calculate $\frac{dy}{dx}$ for the following functions.*

(1) $y = x^2 \cos x$

(2) $y = \sin x - x \cos x$

(3) $y = \frac{1+\sin x}{1-\sin x}$

Example 40. *Calculate* $\frac{d}{dx} \tan x$.

Other Trigonometric Derivatives: $\frac{d}{dx} \tan x = \sec^2 x$; $\frac{d}{dx} \cot x = -\csc^2 x$; $\frac{d}{dx} \sec x = \sec x \tan x$; and $\frac{d}{dx} \csc x = -\csc x \cot x$.

Example 41. *Let* $y = \sec x \csc x$*, and compute* y'.

Derivatives as Rates of Change

We will briefly segue here to look at some practical applications of the derivative. The most natural application, as we've already seen, is to position functions. Suppose $f(t)$ is a function which describes the position of some object after t seconds have passed. Then its derivative $f'(t)$ is supposed to describe the instantaneous rate of change of the position of the object with respect to time; in other words, $f'(t)$ gives the *instantaneous velocity* of the object at t seconds.

Now consider for a moment: suppose we have some velocity function $v(t)$, which gives the instantaneous velocity of an object at t seconds. Then its derivative $v'(t)$ should describe the *rate of change of the velocity* of an object at time t; this is precisely the *acceleration* of the object.

Definition 10. *Let* $f(x)$ *be a differentiable function. If the derivative* $f'(x)$ *is also differentiable, then we define the second derivative* $f''(x) = \frac{d}{dx} f'(x)$.

We also write $f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}(\frac{d}{dx} f(x)) = \frac{d^2}{dx^2} f(x)$.

If $f(t)$ is the *position* function of some object, then its *velocity* function is given by $v(t) = f'(t)$, and its *acceleration* function is given by $a(t) = v'(t) = f''(t)$.

Example 42. *Suppose a stone is thrown vertically upward with an initial velocity of* 64 *ft/s from a bridge* 96 *ft above a river. By Newton's laws of motion, the position of the stone (measured as the height above the river) after* t *seconds is* $s(t) = -16t^2 + 64t + 96$*, where* $s = 0$ *is the level of the river.*

(1) *Find the velocity and acceleration functions.*

(2) *What is the highest point above the river reached by the stone?*

(3) *With what velocity will the stone strike the river?*

HOMEWORK: Sec 3.3 #7-21 odd, 31, 33, 41, 43; Sec 3.4 #15-21 odd, 51, 52; Sec 3.5 #17, 28 (Due 2/22/12)

The Chain Rule (Section 3.6)

Recall: If $f(x)$ and $g(x)$ are two functions, then the *composition* of f and g, denoted by $(f \circ g)$, is defined by the rule $(f \circ g)(x) = f(g(x))$. For example, if $f(x) = x^{90}$ and $g(x) = 5x+4$, then $(f \circ g)(x) = (5x+4)^{90}$ and $(g \circ f)(x) = 5x^{90}+4$. We wish to be able to take derivatives of such compositions of functions, so we need a new rule:

Chain Rule. Let $f(x)$ and $g(x)$ be differentiable functions. Then $(f \circ g)(x)$ is differentiable and $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

This rule is easier understood through practice than by staring at the definition.

Example 43. *Let $f(x) = x^{90}$ and $g(x) = 5x+4$. Find the derivative of $(f \circ g)(x) = f(g(x)) = (5x+4)^{90}$.*

Solution. We already know $f'(x) = 90x^{89}$ and $g'(x) = 5$. The chain rule says to multiply the two derivatives, but evaluate $f'(x)$ at the point $g(x)$ as follows:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 90(5x+4)^{89} \cdot 5 = 450(5x+4)^{89}.$$

□

Example 44. *Find derivatives of the following functions.*

(1) $y = (7x^3 - 5x)^8$

(2) $y = (5x^2 + 15)^{-2}$

(3) $y = \sin^3 x$

(4) $y = \sin x^3$

Now we wish to note that there is another elegant way to view the chain rule, as a "substitution" rule. For example, consider the function $y = (5x+4)^{90}$; we wish to find the derivative y'. Set $u = 5x+4$, so $y = u^{90}$. Then $\frac{dy}{du} = 90u^{89}$ and $\frac{du}{dx} = 5$. In this case the chain rule says the following:

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

So we have $y' = \frac{dy}{dx} = [90u^{89}] \cdot [5] = 450u^{89} = 450(5x+4)^{89}$.

Note: Stating the chain rule in the above manner is simple and elegant and makes it very easy to remember, as it jibes with our intuition of "cancelling out" terms in products of fractions. However this intuition is misleading; derivatives are NOT fractions and the terms $\frac{dy}{dx}$, $\frac{dy}{du}$, $\frac{du}{dx}$ are simply notational tools!

Example 45. *Find* $\frac{d}{dx}[\sqrt{5x^2+1}]$.

Example 46. *Find* $\frac{d}{dt}[\left(\frac{5t^2}{3t^2+2}\right)^3]$.

Example 47. *Let $y = \sin(\cos x^2)$. Find y'.*

L'Hospital's Rule.

Now we need to go back and develop a few more limit tools. Recall that if f is a function continuous at a point a, then $\lim_{x\to a} = f(a)$, i.e. we can evaluate limits for continuous functions by just plugging in numbers. However, we have already seen that we often encounter problems when trying to do this. For instance, in our bonus assignment we proved that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

But if you just attempt to plug 0 into $\frac{\sin x}{x}$, you get something like "$\frac{0}{0}$", which doesn't make sense. This is an example of what we will call an **indeterminate form.** Some other examples are the following:

$$\lim_{x\to 0} \frac{5x}{x} = 5$$
$$\lim_{x\to 0} \frac{10x}{x^3}$$

We say that these limits "have the form $\frac{0}{0}$." The examples above illustrate that, in general, a limit of the form $\frac{0}{0}$ may take *any* value, including any real number, ∞, or $-\infty$. (This should jibe with the reader's intuition about the division $\frac{0}{0}$.)

The other indeterminate form we are concerned with is "$\frac{\infty}{\infty}$", for example the following:

$$\lim_{x\to\infty} \frac{4x^3-8}{3x^3} = \frac{4}{3}$$
$$\lim_{x\to\frac{\pi}{2}^-} \frac{1+\tan x}{\sec x} = 1$$

Theorem 2 (L'Hospital's Rule)**.** *Suppose f and g are functions differentiable at $x = a$, and $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of indeterminate form, i.e.*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0, \textit{ or}$$
$$\lim_{x\to a} f(x) = \pm\infty \textit{ and } \lim_{x\to a} g(x) = \pm\infty.$$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ (provided the limit on the right-hand side exists).

We will omit the proof of L'Hospital's rule in this class, but the proof of a special case is included in the Briggs/Cochran text.

Example 48. *Use L'Hospital's rule to verify the limits in the examples above.*

Example 49. *Evaluate the following limits:*

(1) $\lim_{x\to 1} \frac{x^3+x^2-2x}{x-1}$

(2) $\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x}$

(3) $\lim_{x\to 2} \frac{x^3-3x^2+4}{x^4-4x^3+7x^2-12x+12}$

Next we will note that if we are clever, we can apply L'Hospital's rule to evalute limits "of the indeterminate form $0\cdot\infty$."

Example 50. *Find* $\lim_{x\to\infty} x^2 \sin(\frac{1}{4x^2})$.

Maxima and Minima. (Section 4.1)

Definition 11. *Let f be defined on an interval containing c. Then f has an **absolute maximum** value on I at c is $f(c) \geq f(x)$ for every $x \in I$. Similarly, f has an **absolute minimum** value on I at c is $f(c) \leq f(x)$ for every x in I.*

Example 51. *Graphical examples: $f(x) = x^2$ on: $(-\infty, \infty)$, $[0, 2]$, $(0, 2]$, and $(0, 2)$.*

From the above examples we see that determining absolute maxima and minima depends on not only the function f but also the choice of interval I. However, we can guarantee their existence by requiring two things: first, that f be continuous, and second, that I contain its endpoints.

Theorem 3 (Extreme Value Theorem)**.** *A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.*

Definition 12. *Let f be defined at c. If there exists some open interval I containing c, such that f is defined on I, and $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum** of f.*

*We also refer to local maxima (and minima) as **relative** maxima (and relative minima). If $f(c)$ is a maximum (or minimum) then we say f has a maximum at $x = c$ (or a minimum at $x = c$.*

Now, suppose that f is some polynomial function. Polynomials are continuous and differentiable at every point, so its graph should look like some smooth curve, perhaps winding up and down, and eventually going to either ∞ or $-\infty$ in either the left or right direction. How can we compute at which points f has local maxima or local minima?

It should be clear by now that for a smooth curve (like that of a polynomial) the only places x where f can "top out" or "bottom out" (i.e. have a local maximum or minimum) are points where the tangent line to the graph is horizontal, i.e. where $f'(x) = 0$. In fact, this applies to any differentiable function: If c is some point where $f'(c)$ exists, and such that f has a local maximum or local minimum at $x = c$, then $f'(c) = 0$.

On the other hand, we know that it is possible to have a local maximum or minimum at points where no derivative exists. For example, if $f(x) = |x|$, then f has a corner point at $x = 0$, so $f'(0)$ does not exist, but f certainly has a minimum at $x = 0$. It is also possible to have local maxima/minima at points of discontinuity, and if f is discontinuous at a point then it has no derivative there. These cases, however, account for all the possibilities: if f has a local maximum or minimum at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c. This leads us to the following definition:

Definition 13. *Suppose f is defined at the point c. We say c is a **critical point** of f if either $f'(c) = 0$ or if $f'(c)$ does not exist.*

So if f has a local maximum/minimum at $x = c$, then c must be a critical point of f. However, if c is a critical point, then f does NOT necessarily have a local maximum or minimum at c. For example, consider $f(x) = x^3$ or $f(x) = \sqrt[3]{x}$ at $x = 0$.

Example 52. *Find the critical points of* $f(x) = \frac{x}{x^2+1}$.

Example 53. *Find the absolute maximum and minimum values of the following:*

(1) $f(x) = x^4 - 2x^3$ *on the interval* $[-2, 2]$

(2) $g(x) = x^{\frac{2}{3}}(2 - x)$ *on the interval* $[-1, 2]$

Solution. We will solve part (a) and leave part (b) to the reader. By the Extreme Value Theorem we know that an absolute maximum and minimum value for f on $[-2, 2]$ are guaranteed to exist. Such values can only occur at either the critical points of f, or at the endpoints of the interval $[-2, 2]$. So all we need to do is find the critical values of f, and determine at which of the points f takes its highest and lowest values.

We start by finding the derivative $f'(x) = 4x^3 - 6x^2$. Since f' is a polynomial, it is defined everywhere, so there will be no critical values x for which $f'(x)$ does not exist. Then we need only find all points x where $f'(x) = 0$:

$$\begin{aligned} 4x^3 - 6x^2 &= 0 \\ 2x^2(2x - 3) &= 0 \end{aligned}$$

So either $2x^2 = 0$ or $2x - 3 = 0$; this gives us two solutions, $x = 0$ and $x = \frac{3}{2}$. To finish the problem we test f at the two critical values $x = 0$ and $x = \frac{3}{2}$, and also at the endpoints $x = -2$ and $x = 2$.

$$\begin{aligned} f(0) &= 0 \\ f(\tfrac{3}{2}) &= -\tfrac{27}{16} \\ f(-2) &= 32 \\ f(2) &= 0 \end{aligned}$$

So f obtains an absolute maximum of 32 at $x = -2$ and an absolute minimum of $-\frac{27}{16}$ at $x = \frac{3}{2}$. □

HOMEWORK: Sec 3.6 #7-16 (Disregard instructions regarding "version of chain rule"), 38, 39; Sec 4.7 #9, 10, 12, 17, 18; Sec 4.1 #12-14, 22, [31, 33, 36 (ignore part (c) of the instructions for these last three)] (Due 3/7/12)

Increasing and Decreasing Functions.

Definition 14. *Suppose f is a function defined on an interval I. We say f is* ***increasing*** *if, whenever x_1 and x_2 are in I and $x_1 < x_2$, then $f(x_1) < f(x_2)$. We* *say f is* ***decreasing*** *if whenever x_1 and x_2 are in I and $x_1 < x_2$, then $f(x_1) > f(x_2)$.*

Now notice: if f is differentiable on I and $f'(x) > 0$ for all x in I, then f's tangent lines must all have a positive slope, and hence f is increasing on I. Likewise if $f'(x) < 0$ for all x in I, then f must be decreasing on I.

Example 54. *Sketch a function f which is continuous on $(-\infty, \infty)$ and satisfies the following:*

(1) $f' > 0$ *on* $(-\infty, 0)$, $(4, 6)$, *and* $(6, \infty)$

(2) $f' < 0$ *on* $(0, 4)$

(3) $f'(0)$ *is undefined*

(4) $f'(4) = f'(6) = 0$

Example 55. *Find the intervals on which $f(x) = 2x^3 + 3x^2 + 1$ is increasing and decreasing.*

Example 56. *Find all local maxima and minima of $f(x) = 2x^3 + 3x^2 + 1$.*

First Derivative Test. Suppose f is continuous on an interval I that contains a point x, and f is differentiable on I (except possibly at the point c).

(1) If f' changes sign from positive to negative as x increases through c, then f has a local maximum at c.

(2) If f' changes sign from negative to positive as x increases through c, then f has a local minimum at c.

(3) If f' does not change sign at c then f has no local extreme value at c.

Example 57. *Let $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. Find all local extrema of f.*

Example 58. *Find all local extrema of $f(x) = x^{\frac{2}{3}}(2 - x)$.*

Example 59. *Find all absolute extrema of $f(x) = \frac{1}{4}x^4 - x^3 + \frac{3}{2}x^2 - 9x + 2$.*

Concavity.

Consider the graph of $f(x) = x^3$. For values of $x > 0$, the curve of the graph bends upward. In other words, as we move from left to right, the tangent lines get steeper. In other words, the*first derivative f'* is *increasing*; since f' is differentiable, we know that the *second derivative f''* must satisfy $f''(x) > 0$ for all $x > 0$. We refer to this portion of the graph as **concave up**.

On the other hand, for all values of $x < 0$, the curve of the graph bends downward, i.e. the graph is **concave down**. As we move left to right on the graph, the tangent lines become less steep, i.e. the *first derivative f'* is *decreasing*, i.e. the *second derivative f''* satisfies $f''(x) < 0$ for all $x < 0$.

Definition 15. *Let f be differentiable on an open interval I. If f' is increasing on I then f is* ***concave up*** *on I. If f' is decreasing on I that f is* ***concave down*** *on I. If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an* ***inflection point*** *at c.*

Notice the following useful characterization of concavity: if a graph is concave up at a given point x, then the graph near the point lies *above* the tangent line at x. Conversely if the graph is concave down at x, then the graph near the point lies *below* the tangent line at x.

Example 60. *Graphical example.*

Notice also that there is no explicit relationship between the concavity of a function and whether it is increasing or decreasing. A function can be increasing and concave down, or decreasing and concave up, or any other combination of these properties.

Example 61. *Let $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. Identify the intervals on which f is concave up or concave down, and find any inflection points.*

Second Derivative Test. Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.

(1) If $f''(c) > 0$ then f has a local maximum at c.

(2) If $f''(c) < 0$ then f has a local minimum at c.

(3) If $f''(c) = 0$, then the test is inconclusive.

Example 62. *Use the Second Derivative Test to find all local extrema of $f(x) = 2x^3 - 3x^2 + 12$.*

HOMEWORK: Section 4.2 #8, 11, 12, 17, 22, 26, 30, 33, 40, 43, 51, 56; Section 4.3 #8, 12, 14 (Due 3/14/12)

Some Optimization Problems.

We will now apply our maximization/minimization techniques to some easily formulated word problems.

Example 63. *Consider all pairs of positive numbers whose product is 10000. Is there a pair with minimal sum? Is there a pair with maximal sum? If so, identify the pairs.*

Solution. Notice that there can be no maximal such pair, for if x is ANY positive real number, then $x \cdot \dfrac{10000}{x} = 10000$, and $x + \dfrac{10000}{x}$ is bigger than x. Since we can choose x as large as we want, we can get sums as large as we want.

On the other hand, we can use calculus to prove that there is a minimal such pair. The first step is to define an *sum function*, which, intuitively speaking, should take a pair of numbers which multiply to 10000 as input and spit out the sum of the pair as output. Then all we need to do is find the absolute minimum of this function using the techniques we have learned.

This function is easy to define but we need to make one change in order to use our techniques. Let x and y be any two numbers whose product is 10000; then their sum is given by:

$$S(x, y) = x + y$$

Our only problem with the above sum function is that it takes two variables for input, and we would like it to be a single-variable function. Note, now, that x and y are not independent pieces of information- that is, if x is any particular positive number, then there is exactly one y for which $xy = 10000$! In fact, we have $y = \dfrac{10000}{x}$ for any chosen x. We can now substitute this equality into our sum function to get a function in one variable:

$$S(x) = x + \frac{10000}{x} = x + 10000x^{-1}$$

Now we maximize $S(x)$. We have $S'(x) = 1 - 10000x^{-2}$. This derivative is 0 if $x = 100$ or $x = -100$, and is undefined at $x = 0$, so we have three critical points. But the problem asks us about *positive* numbers, so we need only consider the single critical point $x = 100$.

Since $S''(x) = 20000x^{-3}$, applying the second derivative test at $x = 100$ yields $S''(100) = \dfrac{1}{50} > 0$. So S is concave up at $x = 100$ and hence S indeed has a minimum value at $x = 100$.

If $x = 100$ then $y = 100$ and $S = 100 + 100 = 200$ is the minimum possible sum. □

In the previous example, we (1) built a general "sum" function $S = x + y$; (2) reduced S to to a function of one variable by observing that $xy = 10000$ and subsituting for y; and (3) minimized S with our calculus techniques. This process should reveal the following general strategy:

Strategy for Optimization Word Problems.

(1) Identify what you are being asked to maximize or minimize, and build a function which expresses this value in terms of any possible inputs.

(2) Use any constraints given in the problem to reduce your function to just one variable input.

(3) Optimize your function using calculus.

Example 64. *Let P be any fixed positive number, and consider all rectangles that have perimeter P. Is there such a rectangle with maximal area? Is there one with minimal area? If so, identify the rectangles.*

In what way are the previous two examples related?

Example 65. *A rancher wishes to build a rectangular corral using* 400 *ft of fencing. One wall of the corral will lie alongside a barn, so the rancher doesn't have to use any fencing on one side. The corral will be split into three congruent rectangular sections, each with one wall alongside the barn. Which dimensions should the rancher choose for his corral in order to maximize its area?*

Implicit Differentiation. (Section 3.7)

Up to this point we have restricted our attention to functions which are defined *explicitly*, e.g. we say $y = f(x)$ where f is some function, and we compute the derivative y'. Now we wish to consider the concept of differentiation where the relationship between variables is defined *implicitly*. For example, the set of all solutions to the equation $x^2 + y^2 = 1$ is all the points on the unit circle. In this case, y is not a function of x (as the vertical line test fails) and x is not a function of y, but it still makes sense to consider tangent lines to the graph of the unit circle.

The chain rule now gives us the tool we need to find a reasonable derivative function y'.

Example 66. *Consider the unit circle $x^2 + y^2 = 1$. Find the slope of the tangent line to the unit circle at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$.*

Solution. We will take the derivative of both sides of the equation $x^2 + y^2 = 1$. The crucial fact we will use is that, by the chain rule, the derivative of y^2 (with respect to x is given by $\frac{d}{dx}y^2 = (\frac{d}{dy}y^2) \cdot \frac{dy}{dx}) = 2y \cdot \frac{dy}{dx}$. We now compute y' below:

$$\begin{aligned}
\frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}(1) \\
\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \\
2x + (\frac{d}{dy}y^2)(\frac{dy}{dx}) &= 0 \\
2x + 2y \cdot \frac{dy}{dx} &= 0 \\
y' = \frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$

Notice that y' depends on both y and x! This should not be surprising, as the x-coordinate alone is not sufficient to pick out a point on the unit circle. We can now compute the appropriate slopes: $y'(\frac{1}{2}, \frac{\sqrt{3}}{2}) = -\frac{1}{\sqrt{3}}$ and $y'(\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{1}{\sqrt{3}}$. $\square$

Example 67. *Find an equation of the line tangent to the curve $x^2 + xy - y^3 = 7$ at $(3, 2)$.*

We mentioned earlier that the power rule holds for rational exponents as well as integer exponents. Now that we have the technology of implicit differentiation available, we wish to prove this fact! Recall: we already know that $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is any integer. (Originally we had the rule for positive integers; then we used the quotient rule to show it held for negative integers as well.) In the below example we will try to show that the power rule holds for all *rational* numbers n.

Example 68. *Suppose $n = \frac{p}{q}$ where p and q are integers, i.e. n is a rational number. Compute $\frac{d}{dx}x^n$.*

Solution. Set $y = x^n = x^{\frac{p}{q}}$. Then we have $y^q = x^p$, where p and q are integers; so we can use implicit differentiation and apply the power rule.

$$\begin{aligned}
\frac{d}{dx}(y^q) &= \frac{d}{dx}x^p \\
qy^{q-1} \cdot \frac{dy}{dx} &= px^{p-1} \\
\frac{dy}{dx} &= \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} \\
&= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{\frac{p}{q}})^{q-1}} \\
&= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{(p-\frac{p}{q})}} \\
&= \frac{p}{q} \cdot x^{(p-1)-(p-\frac{p}{q})} \\
&= \frac{p}{q} \cdot x^{\frac{p}{q}-1} \\
&= nx^{n-1}
\end{aligned}$$

This shows that the power rule also holds for rational exponents. □

Power Rule. If n is any rational number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Example 69. *Calculate $\frac{dy}{dx}$ for the following functions:*

(1) $y = \sqrt{x}$

(2) $y = x^{\frac{3}{2}} - 15x^{\frac{2}{3}}$

(3) $y = (x^6 + 3x)^{\frac{4}{3}}$

Applications of Implicit Differentiation: Related Rates (Section 3.8)

Example 70. *An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of* 30 *m/hr, how fast is the area of the patch increasing when the patch has a radius of* 100 *m?*

solution. Recall that the relationship between the radius r and the area A of a circle is given by the formula $A = \pi r^2$. Now let t represent the time variable; in this case, both A and r will increase as a function of t (since the oil spill is spreading as time passes). We can write $A = A(t)$ and $r = r(t)$ to emphasize that they are both functions of time.

In this case $r'(t)$ will be the rate of change of the radius of the spill with respect to time, and $A'(t)$ will be the rate of change of the area with respect to time. We will now use implicit differentiation to reveal the relationship between the two derivatives:

$$\begin{aligned} A'(t) &= \frac{d}{dt}[\pi(r(t))^2] \\ &= \pi\frac{d}{dt}(r(t))^2 \\ &= \pi \cdot 2(r(t))r'(t) \\ &= 2\pi r(t)r'(t) \end{aligned}$$

Now we substitute the values specified in the word problem, i.e. we take $r(t) = 100$ and $r'(t) = 30$. Then the rate of change of the area is $A'(t) = 2\pi(100)(30) = 6000\pi$. (This should be interpreted as square meters per hour.) □

Example 71. *Two airplanes approach an airport, one flying due west at* 120 *mi/hr and the other flying due north at* 150 *mi/hr. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is* 180 *mi from the airport and the northbound plane is* 225 *mi from the airport?*

Example 72. *An observer stands* 200 *m from the launch site of a hot air balloon. The balloon rises vertically at a constant rate of* 4 *m/s. How fast is the angle of elevation of the baloon increasing* 30 *s after the launch?*

HOMEWORK: Section 4.4 #10, 11, 14; Section 3.7 #7-15 odd; Section 3.8 #6, 9, 14, 17 (Due 3/28/12)

Rolle's Theorem and the Mean Value Theorem.

Theorem 4 (Rolle's Theorem)**.** *Let f be continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.*

Proof. Since f is continuous on $[a, b]$, by the Extreme Value Theorem f must obtain some absolute maximum and absolute minimum value on $[a, b]$. Either f obtains both its maximum and minimum at the endpoints of the interval ($x = a$ and $x = b$), or else f obtains one of its extreme inside the open interval (a, b). We will consider both cases.

Case 1: Suppose f has its local maximum and local minimum at either $x = a$ or $x = b$. But $f(a) = f(b)$ by our hypothesis, so if f doesn't go above or below these values, then f must in fact be a constant function. So $f(x) = K$ for some number

K. Then $f'(x) = 0$ for every point in (a, b).

Case 2: Suppose instead that f obtains a local extreme value at some point $x = c$ in the middle interval (a, b). Then f must have a critical point at c, i.e. $f'(c) = 0$ or $f'(c)$ does not exist. But we assumed f is differentiable on (a, b), so $f'(c)$ DOES exist; therefore $f'(c) = 0$.

In either case 1 or case 2 we have produced a point c in (a, b) with $f'(c) = 0$, so the theorem is proved. □

We will now use Rolle's Theorem to prove the Mean Value Theorem, a stronger result.

Theorem 5 (Mean Value Theorem)**.** *Let f be continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

(Stated informally, there is some point c in (a, b) where the slope of the tangent line at $(c, f(c))$ is the same as the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$.

Proof. Let $\ell(x)$ be the linear function whose graph is the line passing through $(a, f(a))$ and $(b, f(b))$. Define a new function $g(x)$ by the rule $g(x) = f(x) - \ell(x)$, for all x in $[a, b]$. The function $g(x)$ is just a difference of continuous differentiable functions and hence it is continuous and differentiable as well. Notice that $g(a) = f(a) = \ell(a) = f(b) - f(b) = 0$ and $g(b) = f(b) - \ell(b) = f(b) - f(b) = 0$, since $\ell(a) = f(a)$ and $\ell(b) = f(b)$. Hence $g(a) = g(b)$, and therefore the function $g(x)$ meets the conditions required in Rolle's Theorem.

In that case, Rolle's Theorem implies that for some c in (a, b), we must have $g'(c) = 0$. But then we have $g'(c) = f'(c) - \ell'(c) = 0$ and hence $f'(c) = \ell'(c)$. Since the graph of $\ell(x)$ is just the line between $(a, f(a))$ and $(b, f(b))$, we have $\ell'(c) = \frac{f(b)-f(a)}{b-a}$; so $f'(c) = \frac{f(b)-f(a)}{b-a}$ as required. □

Notice that the Mean Value Theorem obvious implies Rolle's Theorem, since if $f(a) = f(b)$, then the slope $\frac{f(b)-f(a)}{b-a}$ is just 0, so there is some point c in (a, b) with $f'(c) = 0$. However, since we can use Rolle's Theorem to prove the Mean Value Theorem, the two really have basically the same content; the Mean Value Theorem is just Rolle's Theorem adjusted for alteration by some linear function $\ell(x)$.

Why do we care about the Mean Value Theorem? Let's investigate some immediate consequences.

First suppose $f(x)$ is any continuous function for which $f'(x) = 0$ at every point x. Then given any two numbers a and b (with $a < b$), choosing any point c in (a, b) yields $f'(c) = 0$. Hence for the Mean Value Theorem to be true, we must have that the slope $\frac{f(b)-f(a)}{b-a}$ is just 0. This only happens if $f(b) - f(a) = 0$, i.e. if $f(a) = f(b)$. Since this is true for ANY a and b, f must be a CONSTANT function, i.e. $f(x) = C$ for some number C.

Then what can we say about two functions which have the same derivative? Suppose f and g are continuous functions such that $f'(x) = g'(x)$ for every x. Then $f'(x) - g'(x) = 0$ everywhere, and hence the above paragraph applies that $f(x) - g(x) = C$ for some constant C. Then $f(x) = g(x) + C$, i.e. f and g differ only by some constant.

This is relevant because we wish to define the notion of an *antiderivative* in the near future, i.e. we wish to define a process which is the inverse of differentiation. Now we know that if f is some continuous function, then any two antiderivatives of f must be the same, except for some constant term! We will revisit this concept a few sections later.

Antiderivatives.

We can now take steps toward developing the integral calculus. The goal of the differential calculus is to find the derivative f' of a given function f. The reverse process, called antidifferentiation, is the opposite process: given a function f, we look for some function F whose derivative is f, i.e. $F' = f$.

Definition 16. *A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$ for all x in I.*

Theorem 6. *Let F be any antiderivative of f. Then all antiderivatives of f have the form $F + C$, where C is some constant.*

Proof. We proved this in our discussion of the Mean Value Theorem above. □

Example 73. *Find all antiderivatives of the following functions:*

(1) $f(x) = 1$

(2) $f(x) = x$

(3) $f(x) = x^2$

(4) $f(x) = x^3$

The example above suggests a ”reverse power rule” for antiderivatives:

Power Rule for Antiderivatives. Let p be any rational number (with $p \neq= -1$). Then all antiderivatives of x^p have the form $\frac{1}{p+1}x^{p+1}+C$, where C is some constant.

In addition, we'll observe that since derivatives ”distribute over” sums and constant multiples of functions, antiderivatives do as well, as we can see in the following example:

Example 74. *Find all antiderivatives of $f(x) = 3x^5 + 2 - 5x^{-\frac{3}{2}}$.*

Now we must introduce some notation.

Definition 17. *To denote the operation "find all antiderivatives of f", we write the following:*

$$\int f(x)dx$$

We refer to this string of symbols as the ***indefinite integral*** *of f, and it corresponds to an infinite collection of functions (all of which differ from one another by just a constant). We use this term indefinite integral interchangeably with antiderivative.*

We call the function $f(x)$ the ***integrand****. The term dx indicates that we are antidifferentiating with respect to the variable x.*

This notation, like $\frac{d}{dx}$ and $\frac{dy}{dx}$, is also old-timey and wrapped up in the notion of "infinitesimals." We will simply take the symbols at face value for now, and try to motivate their meaning later on when we start talking about **definite integrals**.

HOMEWORK: Section 4.8 #9-26, 31-34 (Due 4/11/12)

Example 75. *Find $\int \cos(3x)dx$.*

Indefinite Integrals of Trigonometric Functions. The following all hold:

(1) $\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C$

(2) $\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$

(3) $\int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + C$

(4) $\int \csc^2(ax)dx = -\frac{1}{a}\cot(ax) + C$

(5) $\int \sec(ax)\tan(ax)dx = \frac{1}{a}\sec(ax) + C$

(6) $\int \csc(ax)\cot(ax)dx = -\frac{1}{a}\csc(ax) + C$

Example 76. *Find the following:*

(1) $\int \sec^2(7x)dx$

(2) $\int \cos(\frac{x}{2})dx$

Approximating Areas Under Curves Using Riemann Sums.

Consider a car moving at a constant rate of 60 mi/hr. How far does the car move in exactly 2 hours? We know immediately that the answer is (60 mi/hr) $\cdot$ (2 hr) = 120 miles. How can we visualize this solution geometrically? Consider the graph of the velocity equation $v(t) = 60$, i.e. the graph of the horizontal line at height 60. The distance traveled after t seconds is always given by the equation $60 \cdot t$, which is exactly the *area* under the graph between $t = 0$ and $t = 2$. (Recall that the derivative of a distance function gives a velocity function, and in this case the area under the curve of a velocity function gives a distance function; this foreshadows the inverse relationship between the derivative and the integral!)

Now if a graph is given by a straight line, it is easy to compute the area under the curve by a simple geometric formula; however we wish to compute the exact areas under graphs of more complicated functions. We will need to develop some technology to do this. We'll start by doing simple approximations.

Example 77. *Suppose the velocity in m/s of a moving object is given by the equation $v(t) = t^2$, for $0 \leq t \leq 8$. Estimate the displacement of the object after 8 seconds.*

Solution. One approach we may take is to divide the interval $[0, 8]$ into, say, four equal subintervals and try to estimate the displacement in each subinterval, i.e. make an estimate for every 2 seconds that pass and then add them together. For instance, in the first two seconds the velocity of the object increases from $v(0) = 0$ to $v(2) = 4$; we can perhaps get a decent approximation of the object's velocity over this interval by taking $v(1) = 1$ m/s. Then we estimate that the object moves approximately (1 m/s) $\cdot$ (2 s) = 2 m in the first 2 seconds.

Then we can make similar estimates for the remaining intervals: say the object moves about $v(3) = 9$ m/s from $t = 2$ to $t = 4$; about $v(5) = 25$ m/s from $t = 4$ to $t = 6$; and about $v(7) = 49$ m/s from $t = 6$ to $t = 8$. Then we may compute an estimate of the displacement of the object as follows:

$$v(1) \cdot 2 + v(3) \cdot 2 + v(5) \cdot 2 + v(7) \cdot 2 = (1 + 9 + 25 + 49) \cdot 2 = 168 \text{ m}.$$

Notice that we can visualize the product $v(1) \cdot 2$ as the area of the rectangle with base $[0, 2]$ and height $v(1) = 1$. Likewise the next product $v(3) \cdot 2$ corresponds to the area of the rectangle with base $[2, 4]$ and height $v(3)$; and so forth with the others. So what we are really computing is the sum of the areas of these four rectangles; geometrically, we are approximating the area under the curve of the graph.

What happens if we repeat this process by dividing the interval into, say, 8 subintervals instead of 4? What about 16, or 100, or 1000 subintervals? □

The above example motivates the following definitions, to help us estimate the areas under curves of graphs.

Definition 18. *Suppose $[a, b]$ is a closed interval. We may break up $[a, b]$ into n distinct subintervals $[x_0, x_1]$, $[x_1, x_2]$,...,$[x_{n-1}, x_n]$ of equal length $\Delta x = \frac{b-a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, ..., x_n$ are called* ***grid points*** *and they create a* ***regular partition*** *of $[a, b]$. In general, the k-th grid point is $x_k = a + k\Delta x$, for $k = 0, 1, ..., n$.*

Example 78. (1) *Find a regular partition of $[1, 9]$ into 4 subintervals.*

(2) *Find a regular partition of $[5, 7]$ into 9 subintervals.*

Definition 19. *Suppose f is a function defined on a closed interval $[a, b]$, and let $x_0, ..., x_n$ give a regular partition of $[a, b]$ into n subintervals. Let $\overline{x}_k$ be any point in the k-th subinterval $[x_{k-1}, x_k]$, for each $k = 1, 2, ..., n$. Then the sum*

$$f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + ... + f(\overline{x}_k)\Delta x$$

*is called a **Riemann sum** for f on $[a, b]$.*

*Furthermore, we call this sum a **left** Riemann sum if $\overline{x}_k$ is always the left endpoint of $[x_{k-1}, x_k]$; a **right** Riemann sum of $\overline{x}_k$ is always the right endpoint; and a **midpoint** Riemann sum of $\overline{x}_k$ is always the midpoint.*

Example 79. *Find a regular partition for $[0, 1]$ into 4 subintervals; then compute a left, right, and midpoint Riemann sum for $f(x) = x^3$ using this partition. How do the area estimates compare to the actual area under the curve?*

Sigma Notation for Sums.

Now that we are working with Riemann sums of n terms, where n may be extremely large, it is in our interest to develop some new notation to describe when we are adding together a large finite number of terms.

Suppose we wish to represent the sum of the first 1000 perfect squares, i.e. $1^2 + 2^2 + 3^2 + ... + 999^2 + 1000^2$. We introduce the following string of symbols to represent this sum:

$$\sum_{k=1}^{1000} k^2$$

The Greek letter $\sum$, i.e. the capital *sigma*, is an S for "Sum." The "$k = 1$" below the sigma and the "1000" above the sigma tell us how many terms we wish to add together: in particular, we should start counting terms from $k = 1$ all the way up to $k = 1000$. Now we should interpret the "k^2" term to the right of the capital sigma as a type of function; it means we should take each whole number k (from 1 up to 1000) and spit out its square k^2, and then add them all up.

This type of notation is better learned through practice than through extensive explanation. The following examples illustrate the use of sigma notation to describe large finite sums:

$$\sum_{k=1}^{99} k = 1 + 2 + 3 + ... + 98 + 99 = 4950$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + (n-1) + n$$

$$\sum_{i=0}^{3} i^3 = 0 + 1 + 8 + 27 = 36$$

$$\sum_{j=1}^{4}(2j + 1) = 3 + 5 + 7 + 9 = 24$$

$$\sum_{k=-1}^{2}(k^2 + k) = [(-1)^2 + (-1)] + [0^2 + 0] + [1^2 + 1] + [2^2 + 2] = 8$$

Now, with this new sigma-notation for finite sums, we can rewrite any Riemann sum in a very compact form:

$$f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + ... + f(\overline{x}_n)\Delta x = \sum_{k=1}^{n} f(\overline{x}_k)\Delta x$$

Negative Area

Example 80. *Evaluate and interpret a midpoint Riemann sum for* $f(x) = 1 - x^2$ *on the interval* $[1, 3]$*, using a regular partition into* 4 *subintervals.*

We should know that the idea of **area**, in the geometric sense, is a concrete quantity and can only take nonnegative values. However, for our purposes, when we are working with some function $f(x)$ whose graph lies **below** the x-axis on a closed interval $[a, b]$, then we consider the region bounded by the graph of f and the x-axis, on $[a, b]$, to be a region with "negative area." This way, the Riemann sums make sense as an area approximation.

Let's consider a slightly more complicated example:

Example 81. *Evaluate and interpret a midpoint Riemann sum for* $f(x) = 1 - x^2$ *on the interval* $[0, 3]$*, using a regular partition into* 6 *subintervals.*

Definition 20. *Let R be the region bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the area of the parts of R that lie above the x-axis minus the area of the parts of R that lie below the x-axis on* $[a, b]$.

HOMEWORK: Section 5.1 #5, 7, 10, 16, 33; Section 5.2 #23, 26, 36, 38, 40. Optional Problem: Section 5.2 #45. (Due 4/18/12)

The Definite Integral.

Now we wish to define the definite integral of a function $f(x)$ on an interval $[a, b]$. Recall that the derivative, which was supposed to compute the slope of the tangent line to the graph of f at a point, was defined by looking at all possible *approximations*, i.e. the slopes of all nearby secant lines, and then taking a limit to find an exact value. Now we wish to define the definite integral, which will compute the net area under the graph of $f(x)$ bounded by $[a, b]$. By analogy, we will do so by looking at all possible *approximations*, i.e. all possible Riemann sums on $[a, b]$, and taking the limit to get an exact number.

Recall that not every function is differentiable... also by analogy, we will see that not every function is integrable! We will restrict our attention to situations where taking a Riemann integral makes sense.

Definition 21. *Let f be a function defined on a closed interval $[a, b]$. For any given n, there is a unique regular partition of $[a, b]$ into n subintervals, given by $x_0, x_1, ..., x_n$. If $\lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x$ exists, AND does not depend on the choices of $\overline{x}_k$ in each subinterval, then we say f is **integrable** on $[a, b]$. Furthermore, we define this limit to be the **definite integral** of f on $[a, b]$, and we write:*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^n f(\overline{x}_k)\Delta x$$

Here we will briefly comment on the choice of notation. Recall from when we were working on derivatives, we can use the fraction $\frac{\Delta y}{\Delta x}$ to represent the slope of a secant line determined by some small change Δx in the input variable x. Then when we take the limit to define the derivative, we get $\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$. Here "$dx$" is supposed to represent an "infinitesimal" change in x and "dy" is supposed to represent an "infinitesimal" change in y.

Now consider our definition of the definite integral: as we take $n \to \infty$, i.e. as we let n grow arbitrarily large, the length of each subinterval in the regular partition shrinks down to 0, i.e. $\lim_{n\to\infty} \Delta x = \lim_{n\to\infty} \frac{b-a}{n} = 0$. So as $n \to \infty$, we have $\Delta x \to 0$. Then in our definite integral notation, the symbol "$\int$" is supposed to be an elongated "S" for "sum", which represents the finite sum $\sum_{k=1}^n f(\overline{x}_k)$ taken to an infinite limit, and the symbol "dx" is supposed to be the "infinitesimal" limit of Δx as n tends to infinity, just as in the notation for the derivative.

These notations are old-fashioned but deeply entrenched so we are going to stick with them.

Example 82. *Which of the following functions are integrable on* $[0,1]$*?*

(1) $f(x) = \sin x$
(2) $f(x) = 1$ *if* x *is rational, and* $f(x) = 0$ *if* x *is irrational*
(3) $f(x) = 1 - x^2$ *if* $0 \le x < \frac{1}{2}$*, and* $f(x) = 2x - 1$ *if* $\frac{1}{2} \le x \le 1$

Theorem 7. *If* f *is continuous on* $[a,b]$ *or bounded on* $[a,b]$ *with a finite number of discontinuities, then* f *is integrable on* $[a,b]$.

Now before we attempt to do any limit computations, let's get some practice understanding the basic meaning of the definite integral.

Example 83. *Compute the following definite integrals.*

(1) $\int_2^4 (2x+3)dx$

(2) $\int_1^6 (2x-6)dx$

(3) $\int_3^4 \sqrt{1-(x-3)^2}dx$

Example 84. *Graphical example: Sec 5.2 Ex #4.*

Some Properties of Integrals. Let f and g be integrable functions on $[a,b]$, let c be in $[a,b]$, and let k be any constant. Then:

(1) $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(2) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

In addition, we also adopt the following two conventions:

(4) $\int_a^a f(x)dx = 0$

(5) $\int_b^a f(x)dx = -\int_a^b f(x)dx$

These last two we will just take as a definition!

Example 85. *Suppose* $\int_0^5 f(x)dx = 3$ *and* $\int_0^7 f(x)dx = -10$. *Evaluate the following integrals:*

(1) $\int_0^7 2f(x)dx$

(2) $\int_5^7 f(x)dx$

(3) $\int_5^0 f(x)dx$

(4) $\int_7^0 6f(x)dx$

Now we will attempt an actual explicit limit computation.

Example 86. *Find* $\int_0^2 (x^3 + 1)dx$.

Note: To solve example 82, we will need to make a one-time use of the summation formula $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

The Fundamental Theorem of Calculus.

We have now defined the antiderivative (or indefinite integral) of a function as well as the definite integral of a function, and we have used the same terminology and the same symbols for both concepts, but up to this point in the course we have not made the connection between the two explicit. (Outside of our copious hint-dropping.)

We have also seen in the previous example that although the definition of the definite integral is fairly natural, performing the actual computations involved range from the difficult to the impossible. Thus we desire a much easier process for computing definite integrals. The following theorem shows us the connection between the antiderivative and the definite integral, and as a consequence gives us a powerful computational tool:

Theorem 8 (The Fundamental Theorem of Calculus.)**.** *Let f be a continuous function and let a be any point where f is defined. Define the* ***area*** *function of f, centered at a, as follows:*

$$A(x) = \int_a^x f(t)dt$$

(This definition makes sense whenever A is integrable on the closed interval between a and x.) Also let F be any antiderivative of f. Then the following two statements hold:

(1) $A'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$

(2) $\int_a^b f(x)dx = F(b) - F(a)$

Notice that statement (1) above can be read as "The definite integral of f from a to x is an antiderivative of f," and (2) can be read as "Any antiderivative of f computes the definite integral of f from a to b." In other words, up to a few details, the antiderivative and the definite integral are exactly the same concept. So it makes sense to regard integration as the "inverse process" of differentiation.

Proof of the FTC. Let f be continuous and let $A(x) = \int_a^x f(t)dt$ be the area function determined by f. By the definition of the derivative, we have

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

Now notice that $A(x+h) - A(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$, i.e. $A(x+h) - A(x)$ represents the net area under the graph of f bounded by the interval $[x, x+h]$ (if $h > 0$; otherwise $[x+h, x]$).

Now since f is continuous, by the Extreme Value Theorem f must attain its absolute minimum and its absolute maximum on this interval. Call these values m and M, respectively. Then it follows that $mh \leq A(x+h) - A(x) \leq Mh$. (Draw the picture to see this!) Thus, dividing by h on each side, we get

$$m \leq \frac{A(x+h) - A(x)}{h} \leq M$$

But since f is continuous, we must have $\lim_{h \to 0} m = f(x)$ and $\lim_{h \to 0} M = f(x)$ (since all the function values on $[x, x+h]$ must become close to $f(x)$ as h shrinks to 0). Therefore, applying the Squeeze Theorem, we get

$$\lim_{x \to h} m \leq \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \leq \lim_{h \to 0} M$$

Since the left- and right-hand sides above are both $f(x)$, we have $A'(x) = f(x)$. This proves part (1) of the FTC.

To see that part (2) is also true, let F be any antiderivative of f. We just finished showing that A is another antiderivative of f, and thus by our previous discussion on the uniqueness of antiderivatives, we know that $F(x) = A(x) + C$ for some constant term C. It follows then that $F(b) - F(a) = [A(b) - C] - [A(a) - C] = A(b) - A(a) = \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt - 0 = \int_a^b f(x)dx$. (We may make the variable switch in the last step because t, x are just dummy variables in this case.) This shows part (2) and the theorem is proved. $\square$

Let's see how we can make use of this incredibly powerful theorem.

Example 87. *Use the FTC to verify that* $\int_0^2 (x^3+1)dx = 6$ *(a la our previous difficult computation).*

Example 88. *Compute the following definite integrals.*

(1) $\int_0^{10}(60x-6x^2)dx$

(2) $\int_0^{2\pi} 3\sin x dx$

(3) $\int_{\frac{1}{16}}^{\frac{1}{4}} \frac{\sqrt{t}-2t}{t}dt$

HOMEWORK: Section 5.3 #23, 24, 26, 34, 35, 36, 37, 42, 49, 50; Section 5.5 #35-38, 52, 54 (Due 4/25/12)

The Substitution Rule.

In general, the chain rule, product rule, and quotient rule are sufficient for us to compute the derivatives of just about any reasonable functions we run across, even if they are very complicated. On the other hand, right now the family of functions which we can antidifferentiate is extremely small. One of the main goals of the reader's upcoming Calculus II course will be to expand this family as much as possible by introducing a large variety of integration techniques.

There will always be many functions for which finding an explicit formula for an antiderivative is impossible- however, we can get a start here by introducing one very powerful integration technique. Consider the following examples.

Example 89. *Find* $\int \cos(2x)dx$.

Example 90. *Find* $\int 5x^4(x^5+6)^9 dx$.

The examples above suggest a "reverse chain rule" for integration.

Substitution Rule. Let $u = g(x)$. Then $\int f(g(x))g'(x)dx = \int f(u)du$.

Procedure for the Substitution rule:

(1) Given an indefinite integral with a composite function $f(g(x))$ appearing in the integrand, identify an "inner function" $g(x)$ whose derivative $g'(x)$ also appears as a factor in the integrand. (i.e. Look for something of the form $f(g(x))g'(x)$.)

(2) Set $u = g(x)$ and compute $du = g'(x)dx$.

(3) Evaluate the integral with respect to u.

(4) Un-substitute $u = g(x)$ to finish the problem.

Example 91. $\int 2(2x+1)^3 dx$.

Example 92. $\int \cos^3 x \sin x dx$.

Example 93. $\int \frac{x}{\sqrt{x+1}} dx$.

Now suppose F is any antiderivative of f. Then for any differentiable function g, we have

$$\tfrac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$$

by the chain rule. In other words, the composite function $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. Then if $[a, b]$ is any closed interval, by the Fundamental Theorem of Calculus, it follows that

$$\begin{aligned}
\int_a^b f(g(x))g'(x)dx &= [F(g(x))]_a^b \\
&= F(g(b)) - F(g(a)) \\
&= [F(u)]_{g(a)}^{g(b)} \\
&= \int_{g(a)}^{g(b)} f(u)du
\end{aligned}$$

Thus the substitution rule applies to definite integrals as well as indefinite ones, as long as apply a vital **change of parameter** from $x = a \to x = b$ to $u = g(a) \to u = g(b)$.

Example 94. *Compute* $\int_0^2 \frac{dx}{(x+3)^3}$.

Example 95. *Compute* $\int_{-1}^2 \frac{x^2 dx}{(x^3+2)^3}$.

Example 96. *Compute* $\int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx$.

Mean Value Theorem for Integrals.

Definition 22. *The* ***average value*** *of a function f on an interval $[a, b]$ is* $\frac{1}{b-a} \int_a^b f(x)dx$.

Theorem 9 (Mean Value Theorem for Integrals)**.** *Let f be continuous on the interval $[a, b]$. Then there exists a point c in $[a, b]$ such that* $f(c) = \frac{1}{b-a} \int_a^b f(x)dx$.

Proof. By the extreme value theorem, f obtains some absolute minimum value m and some absolute maximum value M on the interval $[a, b]$.

Now since $\int_a^b f(x)dx$ gives the net area under the curve of f on $[a, b]$, we can bound this area above and below by the areas of the rectangles with base $[a, b]$ and height M and m, respectively. In other words,

$$(b-a)m \leq \int_a^b f(x)dx \leq (b-a)M$$

And hence:

$$m \leq \tfrac{1}{b-a} \int_a^b f(x)dx \leq M$$

Now because f is continuous on $[a,b]$, by the intermediate value theorem we know that f obtains all values between m and M somewhere on $[a,b]$. In particular, there must be some point c where $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. $\square$

Example 97. *Find the point (or points) on the interval* $[0,1]$ *where* $f(x) = 2x(1-x)$ *attains its average value on* $[0,1]$.

Regions Between Curves.

Example 98. *Find the area of the region bounded by the graphs of* $f(x) = \frac{4}{\sqrt{x+1}}$ *and* $g(x) = x - 1$ *and the* y*-axis.*

Example 99. *Find the area of the region bounded by the graphs of* $f(x) = x + 3$ *and* $g(x) = |2x|$.

3-Dimensional Volume: The Disk and Washer Methods.

Consider a 3-dimensional solid that extends in the x-direction from $x = a$ to $x = b$. To estimate the volume of this solid, we could take a "3-dimensional Riemann sum:" first we subdivide the interval $[a,b]$ into n equal subintervals, thereby slicing the solid into n different 3-dimensional chunks, each of width $\Delta x = \frac{b-a}{n}$. Then we estimate the volume of each of the n chunks by taking the area of one cross-section (call it $A(x_k)$, for $1 \leq k \leq n$), and multiplying it by the width Δx. The sum of these volume estimates gives an estimate for the volume of the entire solid, i.e. $V \approx \sum_{k=1}^n A(x_k)\Delta x$.

Then, as in the case with integration, to find an exact volume all we need to do is take the limit of the above sum as n runs to infinity. So we get

$$V = \lim_{n\to\infty} \sum_{k=1}^n A(x_k)\Delta x = \int_a^b A(x)dx$$

where $A(x)$ is a function which takes a point x in $[a,b]$ for input and returns the area of the cross section of the solid at x.

Example 100. *A solid has a base that is bounded by the curves* $y = x^2$ *and* $y = 2 - x^2$ *in the xy-plane. Cross sections through the solid perpendicular to the* x*-axis are semicircular disks. Find the volume of the solid.*

Example 101. *Let* R *be the region bounded by the curve* $f(x) = (x+1)^2$, *the* x*-axis, and the lines* $x = 0$ *and* $x = 2$. *Find the volume of the* ***solid of revolution*** *obtained by revolving* R *about the* x*-axis.*

The previous example suggests a general method for computing the volumes of solids of revolution.

Disk Method about the x-axis. Let f be continuous with $f(x) \geq 0$ on the interval $[a,b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi(f(x))^2 dx.$$

Example 102. *The region R is bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$ between $x = 0$ and $x = 1$. What is the volume of the solid that results when R is revolved about the x-axis?*

Washer Method about the x-axis. Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R by the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi(f(x)^2 - g(x)^2)dx.$$

HOMEWORK: Section 6.2 # 7, 11, 12, 23, 24; Section 6.3 #10, 15, 18, 23, 26; Section 6.4 #5, 6 (Due 5/2/12)

3-Dimensional Volume: The Shell Method.

Let f be a continuous function, and let R be the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$. The washer method gave us an easy way to compute the volume of the solid generated when R is revolved around the x-axis, but what about when R is revolved about the y-axis instead? We wish to have a convenient method for measuring this kind of solid of revolution.

Let's consider again a type of 3-dimensional "Riemann sum." Divide the interval $[a, b]$ up into n equal subintervals, which divides R up into n chunks. When one of these chunks is revolved about the y-axis, instead of a prism, we get a a cylindrical *shell.* To estimate the volume of this shell, imagine cutting it in one place and stretching it out to look like a rectangular prism. Then its volume will be given by its width, which is the length of the subinterval ΔX, times its height $f(x_k)$ (where x_k is some point in the subinterval), times its length, which is the circumference of the circle of radius $2\pi x_k$. So the volume of each chunk is approximately $2\pi x_k \cdot f(x_k) \cdot \Delta x$.

When we add up the volumes of each of the n chunks, we get an estimate for the volume of the solid, given by $V \approx \sum_{k=1}^{n} 2\pi x_k f(x_k)\Delta x$. Then to obtain the exact volume, we again take the limit is $n \to \infty$, to find that $V = \int_a^b 2\pi x f(x)dx$.

The above gives a computation of the simplest case, where R is bounded by just one function f, and the x-axis. The shell method given below will be true in the more general case, where R is bounded above by a function f and bounded below by a function g.

Shell Method about the y-axis. Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b 2\pi x(f(x) - g(x))dx.$$

Example 103. *Let R be the region bounded by the graph of $f(x) = \sin x^2$, the x-axis, and the vertical line $x = \sqrt{\frac{\pi}{2}}$. Find the volume of the solid generated when R is revolved about the y-axis.*

An Application of Integration to Geometry: Arc Length.

Suppose f is a function with a continuous first derivative on the interval $[a, b]$ (hence f is continuous on $[a, b]$). Then the length of the curve which joins the points $(a, f(a))$ and $(b, f(b))$ on the graph of f is given by

$$L = \int_a^b \sqrt{1 + f'(x)^2}dx.$$

See Section 6.5 in the Briggs/Cochran text for a very clear justification of this fact!

Example 104. *Find the length of the curve $f(x) = x^3 + \frac{1}{12x}$ on the interval $[\frac{1}{2}, 2]$.*

An Application of Integration to Physics: Work.

Recall that if an object moves at a constant velocity v for some interval of time $t = a$ to $t = b$, then the displacement of the object is given by its velocity times the length of time, i.e. $D = v \cdot (b - a)$. More generally, if an object moves at some variable velocity, which can be expressed as a function of time $v(t)$, then the displacement of the object between $t = a$ and $t = b$ is given by

$$D = \int_a^b v(t)dt.$$

(This should jibe with the reader's intuition about the Fundamental Theorem of Calculus, when one considers that velocity functions are derivatives of displacement functions.)

Now we introduce a different, but analogous, concept from physics. Suppose a force F is applied to some object, which causes a displacement of the object in the direction the force was applied. We refer to the change in energy when a force causes a displacement as **work**. In general, if a *constant* force F causes an object to move some distance d, then the work done is exactly the force times the displacement, i.e. $W = F \cdot d$. Forces are measured in **newtons (N)**, which is the amount of force required to give a 1-kg mass an acceleration of 1 m/s^2, and work is measured in **joules**, which are newton-meters, i.e., the amount of work done by a 1-N force over a distance of 1 m.

More generally, suppose a variable force is applied an object over some distance, where the force applied at any given position x is expressed by the function $F(x)$. Then we may compute the work done as follows:

$$W = \int_a^b F(x)dx.$$

Example 105. *Do all the problems on the final exam review sheet.*