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MEASURE AND DIMENSION FUNCTIONS: MEASURABILITY AND DENSITIES

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1. Introduction

During the past several years, a new type of geometric measure and dimension have been introduced, the packing measure and dimension, see [Su], [Tr] and [TT1]. These notions are playing an increasingly prevalent role in various aspects of dynamics and measure theory. Packing measure is a sort of dual of Hausdorff measure in that it is defined in terms of packings rather than coverings. However, in contrast to Hausdorff measure, the usual definition of packing measure requires two limiting procedures, first the construction of a premeasure and then a second standard limiting process to obtain the measure. This makes packing measure somewhat delicate to deal with. The question arises as to whether there is some simpler method for defining packing measure and dimension. In this paper, we find a basic limitation on this possibility. We do this by determining the descriptive set theoretic complexity of the packing functions. Whereas the Hausdorff dimension function on the space of compact sets is Borel measurable, the packing dimension function is not. On the other hand, we show that the packing measure and dimension functions are measurable with respect to the σ-algebra generated by the analytic sets. Thus, the usual sorts of measurability properties used in connection with Hausdorff measure, *e.g.*, measures of sections and projections, remain true for packing measure.

We now give a somewhat more detailed description of our results and we introduce some notation. Throughout this paper (X, d) will be a Polish space, that is, a complete separable metric space. We equip the space $\mathcal{K}(X)$ of non-empty compact subsets of X with the Hausdorff distance ϱ;

$$\varrho(K, L) = \sup\{\mathrm{dist}(x, L), \mathrm{dist}(y, K) : x \in K, y \in L\}.$$

Then $(\mathcal{K}(X), \varrho)$ is a complete separable metric space, see *e.g.*, [R]. We denote by $\mathcal{H}_g$ and $\mathcal{P}_g$ the Hausdorff and packing measures generated by a non-negative non-decreasing function g on the positive reals; their definitions will be given later. We shall study the measurability properties of the functions $\mathcal{H}_g$ and $\mathcal{P}_g$ on $\mathcal{K}(X)$. The Hausdorff measure $\mathcal{H}_g$ is rather simple in this respect; it is in Baire's class 2, and in particular a Borel function. The packing measure is much more complicated,

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as can be expected from its definition. Assuming that g satisfies the doubling condition; $g(2r) \leq kg(r)$, we show that $\mathcal{P}_g$ is measurable with respect to the σ-algebra generated by the analytic sets, but it is not Borel measurable even when X is the unit interval.

From these results on measures we obtain that on $\mathcal{K}(X)$ the Hausdorff dimension is a Borel function and the packing dimension is measurable with respect to the σ-algebra generated by the analytic sets. Again, the packing dimension is not a Borel function. We shall actually prove the measurability of the packing dimension in two ways: via the packing measures and via the box counting dimension (also known as the Minkowski dimension).

We shall mainly work with packing measures whose definition is based on the radii of balls. In Section 5 we briefly discuss the measurability questions related to diameter-based packing measures.

By composing suitable functions, we can use the above results to deduce measurability of various functions. For example, we show in Section 6 that if K is a compact subset of a product space $T \times X$, where T is another Polish space, then the Hausdorff measures and dimensions of the sections $\{x \in X : (t,x) \in K\}$ are Borel functions of t, and their packing measures and dimensions are measurable with respect to the σ-algebra generated by analytic sets. Our example in Section 7 shows that these latter functions need not be Borel measurable.

The last section is independent of the others. There we study the lower density

$$d_g(A,x) = \liminf_{r \to 0} \frac{\mathcal{P}_g\big(A \cap B(x,r)\big)}{g(2r)},$$

where g satisfies the doubling condition. For packing measures this is more natural than the corresponding upper density because even for compact subsets K of $\mathbb{R}^n$ with positive and finite $\mathcal{P}^s$ measure the upper density may be infinite everywhere in K. For the lower density it was shown in [TT2] and [ST] that if $A \subset \mathbb{R}^n$ and $\mathcal{P}^s(A) < \infty$, then $d_g(A,x) = 1$ for $\mathcal{P}^s$ almost all $x \in A$. In Example 8.9 we shall show that this does not extend to any infinite-dimensional inner product space. However, we shall prove for $g(r) = r^s$ the optimal inequalities $2^{-s} \leq d_g(A,x) \leq 1$ for $\mathcal{P}_g$ almost $x \in A$ if A is a subset of an arbitrary Polish space with $\mathcal{P}_g(A) < \infty$, see Theorem 8.3. We shall also prove in Theorem 8.5 that $d_g(A,x)$ does equal one almost everywhere in A, but almost everywhere with respect to $\mathcal{H}_g$ and not $\mathcal{P}_g$. This has the following consequence: if $A \subset X$ with $\mathcal{P}_g(A) < \infty$, then $\mathcal{H}_g(A) = \mathcal{P}_g(A)$ if and only if

$$\lim_{r \to 0} \frac{\mathcal{H}_g\big(A \cap B(x,r)\big)}{g(2r)} = 1 \quad \text{for } \mathcal{P}_g \text{ almost } x \in A. \tag{1.1}$$

In $\mathbb{R}^n$ this has been proved in [TT2] and [ST] (see also [M]) with the help of the aforementioned fact that $d_g(A,x) = 1$ for $\mathcal{P}_g$ almost all $x \in A$. Still in $\mathbb{R}^n$, for $g(r) = r^s$ (1) implies that s must be an integer and A rather regular, i.e. rectifiable with respect to $\mathcal{P}_g$. This gives for sets $A \subset \mathbb{R}^n$ with $g(r) = r^s$ and $\mathcal{P}_g(A) < \infty$ that $\mathcal{H}_g(A) = \mathcal{P}_g(A)$ if and only if A is such a rectifiable set, see [ST] or [M]. We shall extend this result to separable Hilbert spaces in Theorem 8.8.

We shall denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^n$ the Euclidean n-space, by $\mathbb{Q}$ the set of rational numbers, and by $\mathbb{N}$ the set of positive integers. For $A \subset X$, the closure of A will be $\bar{A}$. The σ-algebra generated by analytic sets will be denoted by $\mathcal{B}(\mathcal{A})$. In a product space, proj_1 stands for the projection onto the first factor.

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2. Hausdorff measure and dimension functions

Here and later $g\colon [0,\infty) \to [0,\infty)$ will be a non-decreasing function with $g(0) = 0$. We denote by $d(A)$ the diameter of a set $A \subset X$ with $d(\emptyset) = 0$. Let $0 < \delta \leq \infty$ and $A \subset X$. The approximating Hausdorff g-measure $\mathcal{H}_{g,\delta}(A)$ of A is defined by

$$\mathcal{H}_{g,\delta}(A) = \inf\left\{ \sum_{i=1}^{\infty} g(d(U_i)) : A \subset \bigcup_{i=1}^{\infty} U_i, \text{ each } U_i \text{ is open with } d(U_i) < \delta \right\}$$

and the Hausdorff g-measure $\mathcal{H}_g(A)$ by

$$\mathcal{H}_g(A) = \lim_{\delta \to 0} \mathcal{H}_{g,\delta}(A).$$

We write $\mathcal{H}_{g,\delta} = \mathcal{H}^s_\delta$ and $\mathcal{H}_g = \mathcal{H}^s$ when $0 < s < \infty$ and $g(r) = r^s$ for $r \geq 0$. The Hausdorff dimension $\dim_H A$ of A is defined by

$$\dim_H A = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

We shall use the usual notion of Baire's classes for functions between metric spaces. Thus Baire's class 0 consists of all continuous functions and Baire's class $n+1$ consists of all pointwise limits of sequences of functions in Baire's class n. In particular, the upper and lower semicontinuous functions belong to the Baire's class 1.

2.1. Theorem. a) *For $0 < \delta \leq \infty$, the function*

$$\mathcal{H}_{g,\delta}\colon \mathcal{K}(X) \to [0,\infty]$$

is upper semicontinuous.

b) *The functions*

$$\mathcal{H}_g \quad \textit{and} \quad \dim_H\colon \mathcal{K}(X) \to [0,\infty]$$

are of Baire's class 2. Moreover, in general, these functions are not of Baire's class 1.

Proof. Let $\mathcal{V}$ be a countable basis for the topology of X and let $\{W_n\}_{n=1}^{\infty}$ be an enumeration of all finite unions of the sets of $\mathcal{V}$. Let $c \in \mathbb{R}$ and $K \in \mathcal{K}(X)$. Using the definition of $\mathcal{H}_{g,\delta}$ and the compactness of K, we see that $\mathcal{H}_{g,\delta}(K) < c$ if and only if there are finitely many open sets $U_1, \ldots, U_k$ such that

$$\sum_{i=1}^{k} g(d(U_i)) < c, \quad K \subset \bigcup_{i=1}^{k} U_i \text{ and } d(U_i) < \delta \text{ for } i = 1, \ldots, k.$$

This holds if and only if there are $n_1, \ldots, n_m \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} g(d(W_{n_i})) < c, \quad K \subset \bigcup_{i=1}^{m} W_{n_i} \text{ and } d(W_{n_i}) < \delta \text{ for } i = 1, \ldots, m. \tag{2.1}$$

Since the set of $K \in \mathcal{K}(X)$ for which there exist $n_1, \ldots, n_m$ satisfying (2.1) is clearly open in the exponential or Vietoris topology[K2], we conclude that the set $\{K \in \mathcal{K}(X) : \mathcal{H}_{g,\delta}(K) < c\}$ is open, and a) follows. Letting $\mathbb{Q}^+$ be the set of positive rationals, it then follows that for each α,

$$\{F \in \mathcal{K}(X) : \dim_H F \leq \alpha\} = \bigcap_{\substack{s>\alpha \\ s\in\mathbb{Q}^+}} \bigcap_{n} \{F : \mathcal{H}^s_{1/n}(F) < 1\} \tag{2.2}$$

is a G_δ-set. From this we get $\{F : \beta < \dim_H F < \alpha\}$ is a $G_{\delta\sigma}$-set, for all $0 < \beta < \alpha$. Thus, $\dim_H$ is a Baire class 2 function. As $\mathcal{H}_g = \lim_{n\to\infty} \mathcal{H}_{g,1/n}$, part a) completes the proof of the first part of b).

The following example shows that, in general $\mathcal{H}_g$ is not a Baire class one function. Consider $X = [0,1] \times [0,1]$ and $\mathcal{H}^1 : \mathcal{K}(X) \to [0, +\infty]$. Since $\mathcal{H}^1(F) = 0$ for all finite F, $\mathcal{H}^1$ has value 0 on a dense subset of $\mathcal{K}(X)$. On the other hand, it is easy to show that each open set of the form

$$P(U_1, \ldots, U_n) = \{F : F \subset \cup U_i \text{ and } F \cap U_i \neq \emptyset,\ i = 1, \ldots, n\},$$

where each U_i is a nonempty open subset of X contains an element F such that $\mathcal{H}^1(F) = 1$. Since these sets form a basis for the topology of $\mathcal{K}(X)$, $\mathcal{H}^1$ has value 1 on a dense set. Thus, $\mathcal{H}^1$ cannot have a point of continuity, whereas a Baire class one function has a dense set of points of continuity. This also shows that $\dim_H$ has value 0 on a dense set and also has value 1 on a dense set and is therefore also not a Baire class 1 function.

3. Packing and box dimension functions

The packing dimension can be defined either via the upper box counting (*i.e.*, Minkowski) dimension or via the packing measures. In this section we use the first approach and in the next section the second.

We begin with the definitions and simple measurability properties of the box counting dimensions. Let $K \in \mathcal{K}(X)$. For $\delta > 0$ let $N_\delta(K)$ be the smallest number of open balls of radius δ that are needed to cover K. The upper and lower box counting dimensions $\overline{\dim}_B K$ and $\underline{\dim}_B K$ of K are defined by

$$\begin{aligned}\overline{\dim}_B K &= \limsup_{\delta\to 0} \log N_\delta(K)/(-\log\delta) \\ &= \limsup_{j\to\infty} \log N_{2^{-j}}(K)/(j\log 2),\end{aligned}$$

and

$$\begin{aligned}\underline{\dim}_B K &= \liminf_{\delta\to 0} \log N_\delta(K)/(-\log\delta) \\ &= \liminf_{j\to\infty} \log N_{2^{-j}}(K)/(j\log 2),\end{aligned}$$

where the second equalities are easily seen to hold.

Instead of covering with δ-balls we can also use packings with δ-balls. Let $P_\delta(K)$ be the largest integer k such that there are points $x_1, \ldots, x_k \in K$ with $d(x_i, x_j) > \delta$ for $i \neq j$. Then one verifies easily (or see [F] or [M]) that

$$\begin{aligned}\overline{\dim}_B K &= \limsup_{\delta\to 0} P_\delta(K)/(-\log\delta)\\ &= \limsup_{j\to\infty} P_{2^{-j}}(K)/(j\log 2),\end{aligned}$$

and similarly for $\underline{\dim}_B K$.

3.1. Lemma. *The functions* $\underline{\dim}_B$ *and* $\overline{\dim}_B\colon \mathcal{K}(X) \to [0,\infty]$ *are of Baire's class* 2*.*

Proof. Evidently $\{K \in \mathcal{K}(X) : N_\delta(K) < c\}$ is open for $c \in \mathbb{R}$ and $\delta > 0$, whence N_δ is upper semicontinuous. Thus the functions $g_j : K \mapsto \log N_{2^{-j}}(K)/(j\log 2)$ and $h_k = \inf_{j\geq k} g_j$ are also upper semicontinuous. Hence $\underline{\dim}_B = \lim_{k\to\infty} h_k$ is of Baire's class 2.

For the upper box counting dimension we use the packing function P_δ. It is easy to verify that it is lower semicontinuous. Hence it follows as above that $\overline{\dim}_B$ is of Baire's class 2.

We define the packing dimension $\dim_p A$ for $A \subset X$ by

$$\dim_p A = \inf\left\{\sup_i \overline{\dim}_B K_i : A \subset \bigcup_{i=1}^\infty K_i,\ K_i \in \mathcal{K}(X)\right\}.$$

Then the packing dimension has the countable stability property, which the box counting dimension lacks:

$$\dim_p\left(\bigcup_{i=1}^\infty A_i\right) = \sup_i \dim_p A_i \quad \text{for } A_i \subset X.$$

This leads to the following lemma. It was also essentially proved by Falconer and Howroyd in [FH].

3.2. Lemma. *Suppose $d \in \mathbb{R}$, $K \in \mathcal{K}(X)$ and $\dim_p K > d$. Then there is a non-empty compact set $M \subset K$ such that $\dim_p(M \cap V) \geq d$ for all open sets V with $M \cap V \neq \emptyset$.*

Proof. Let $M_0 = K$. Define a transfinite sequence of compact subsets of K by recursion as follows. For each ordinal α, let $M_{\alpha+1} = \{x \in M_\alpha : dim_p(M_\alpha \cap V) \geq d,$ for all neighborhoods V of x$\}$. For each limit ordinal λ, let $M_\lambda = \cap_{\beta<\lambda} M_\beta$. Since $(M_\alpha)_{\alpha\in\mathrm{Ord}}$ is a descending transfinite sequence of compact sets, there is a countable ordinal γ such that $M_\gamma = M_{\gamma+1}$. Note that for each ordinal α, $\dim_p(M_\alpha \setminus M_{\alpha+1}) \leq d$. We claim that for each countable ordinal α, $dim_p M_\alpha > d$. This is certainly true for $\alpha = 0$. Suppose this claim holds for all $\beta < \alpha$. If $\alpha = \tau + 1$, then since $M_\tau = M_{\tau+1} \cup (M_\tau \setminus M_{\tau+1})$, $dim_p M_{\tau+1} = dim_p M_\alpha > d$. If α is a countable limit ordinal, then since $M_0 = \cup_{\beta<\alpha}(M_\beta \setminus M_{\beta+1}) \cup M_\alpha$, again we have $dim_p M_\alpha > d$. Set $M = M_\gamma$. Then $M \neq \emptyset$. If there were some open set V such that $dim_p(M \cap V) < d$ and $M \cap V \neq \emptyset$, then $M_{\gamma+1} \neq M_\gamma$, a contradiction.

3.3. Lemma. *Let $c \in \mathbb{R}$ and $K \in \mathcal{K}(X)$. Then $\dim_p K \geq c$ if and only if for every $d < c$ there is a non-empty compact set $M \subset K$ such that $\overline{\dim}_B(M \cap \overline{V}) \geq d$ for all open sets V with $M \cap V \neq \emptyset$.*

Proof. The "only if" part follows immediately from Lemma 3.2. To verify the "if" part suppose that the stated condition holds and $\dim_p K < c$, contrary to what is asserted. Choose $d < c$ for which $\dim_p K < d$. Then there are compact sets $K_1, K_2, \dots$ such that $K \subset \bigcup_{i=1}^{\infty} K_i$ and $\overline{\dim}_B K_i < d$ for all i. Let $M \subset K$ be non-empty, compact and such that $\overline{\dim}_B(M \cap \overline{V}) \geq d$, for all open sets V with $M \cap V \neq \emptyset$. Since $M = \bigcup_{i=1}^{\infty}(M \cap K_i)$, the Baire category theorem implies that $M \cap K_i$ has non-empty interior relative to M, for some i . Thus there is an open set V with $\emptyset \neq M \cap V \subset M \cap \overline{V} \subset M \cap K_i$, whence

$$\overline{\dim}_B(M \cap \overline{V}) \leq \overline{\dim}_B(M \cap K_i) < d,$$

which is a contradiction and completes the proof of the lemma.

3.4. Theorem. *For $c \in \mathbb{R}$ the set*

$$A = \{K \in \mathcal{K}(X) : \dim_p K \geq c\}$$

is analytic. In particular, the function $\dim_p \colon \mathcal{K}(X) \to [0, \infty]$ is measurable with respect to $\mathcal{B}(\mathcal{A})$.

Proof. Let $\{V_1, V_2, \dots\}$ be a basis for the topology of X. For m, $n \in \mathbb{N}$ define

$$\begin{aligned} B_{m,n} = \{(K, M) \in \mathcal{K}(X) \times \mathcal{K}(X) : M \subset K \text{ and either } M \cap V_n = \emptyset \\ \text{or } \overline{\dim}_B(M \cap \overline{V_n}) \geq c - 1/m\}. \end{aligned}$$

Then by Lemma 3.3,

$$A = \bigcap_{m=1}^{\infty} \operatorname{proj}_1 \Big(\bigcap_{n=1}^{\infty} B_{m,n} \Big).$$

Thus it suffices to show that each $B_{m,n}$ is a Borel set.

The sets $\{(K, M) : M \subset K\}$ and $\{M : M \cap V_n = \emptyset\}$ are closed. The function $M \mapsto N_\delta(M \cap \overline{V_n})$ is easily seen to be upper semicontinuous for all $\delta > 0$, whence $M \mapsto \overline{\dim}_B(M \cap \overline{V_n})$ is Borel measurable. Thus every $B_{m,n}$ is a Borel set.

4. Packing measure function

Let $A \subset X$ and $\delta > 0$. We say that $\{(x_i, r_i)\}_{i=1}^n$ is a δ-packing of A if $x_i \in A$, $\delta \geq 2r_i > 0$ and $r_i + r_j < d(x_i, x_j)$ for i, $j = 1, \dots, n$, $i \neq j$. Then the closed balls $B(x_i, r_i)$ are disjoint. We first define the prepacking measures $P_{g,\delta}$ and P_g by

$$\begin{aligned} P_{g,\delta}(A) &= \sup \Big\{ \sum_{i=1}^n g(2r_i) : \{(x_i, r_i)\}_{i=1}^n \text{ is a } \delta\text{-packing of } A \Big\}, \\ P_g(A) &= \lim_{\delta \to 0} P_{g,\delta}(A). \end{aligned}$$

If g is continuous we can replace the condition $r_i + r_j < d(x_i, x_j)$ by $r_i + r_j \leq d(x_i, x_j)$ without changing the definitions of $P_{g,\delta}$ and P_g. The following simple lemma is given only for completeness; it will not be needed later on.

4.1. Lemma. *The function* $P_{g,\delta}\colon \mathcal{K}(X) \to [0,\infty]$ *is lower semicontinuous and the function* $P_g\colon \mathcal{K}(X) \to [0,\infty]$ *is of Baire's class* 2*.*

Proof. For $c \in \mathbb{R}$,

$$\{K \in \mathcal{K}(X) : P_{g,\delta}(K) > c\} = \bigcup_{n=1}^{\infty} G_n$$

where $K \in G_n$ if and only if there exists a δ-packing $\{(x_i, r_i)\}_{i=1}^n$ of K such that $\sum_{i=1}^n g(2r_i) > c$. Each G_n is open, consequently $P_{g,\delta}$ is lower semicontinuous. Hence $P_g = \lim_{n\to\infty} P_{g,1/n}$ is of Baire's class 2.

Since P_g is not countably subadditive one needs a standard modification to get an outer measure out of it. Thus we define the packing g-measure for $A \subset X$ by

$$\mathcal{P}_g(A) = \inf\left\{\sum_{i=1}^{\infty} P_g(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i\right\}.$$

If A is compact, the sets A_i can also be taken compact. Then $\mathcal{P}_g$ is a Borel regular outer measure. When $g(r) = r^s$ we denote $\mathcal{P}_g = \mathcal{P}^s$. The packing dimension, which was introduced in the previous section, can also be defined in terms of the packing measures:

$$\dim_p A = \inf\{s : \mathcal{P}^s(A) = 0\}.$$

For these relations, see *e.g.*, [TT1], [F] or [M]. The proofs there are in $\mathbb{R}^n$ but they generalize without changes.

Let us indicate why dealing with packing measure is somewhat delicate. This can be seen by carrying out a straightforward logical analysis of its definition. Let

$$E = \{K \in \mathcal{K}(X) : \mathcal{P}_g(K) < c\}.$$

Then

$$\begin{aligned} E &= \operatorname{proj}_1\left(\left\{(K, K_1, K_2, \dots) \in \mathcal{K}(X)^\infty : K \subset \bigcup_{i=1}^{\infty} K_i \text{ and } \sum_{i=1}^{\infty} P_g(K_i) < c\right\}\right) \\ &= \operatorname{proj}_1(C \cap D), \end{aligned}$$

where

$$C = \left\{(K, K_1, K_2, \dots) \in \mathcal{K}(X)^\infty : K \subset \bigcup_{i=1}^{\infty} K_i\right\}$$

and

$$D = \left\{(K, K_1, K_2, \dots) \in \mathcal{K}(X)^\infty : \sum_{i=1}^{\infty} P_g(K_i) < c\right\}.$$

It follows from Lemma 4.1 that D is a Borel set. But, from its definition, the set C is a coanalytic set. Also, it is not a Borel set. (If it were, then consider $C \cap F$, where F is the Borel set of all $(K, K_1, K_2, \dots)$ such that each K_i is a singleton. The projection of this set on its first coordinate would be an analytic set. But this is the set of all countable closed sets, a classic example proven by Hurewicz not to be analytic, see [H]). Thus, this analysis only yields that E is a so-called PCA (or $\sum_2^1$) set and whether these types of sets are measurable is independent of the Zermelo–Fraenkel axioms of set theory, see [J, pp. 528 and 563].

In the following $\Pr(X)$ stands for the set of all Borel probability measures on X. We equip $\Pr(X)$ with the topology of weak convergence. Then it is a complete separable metrizable space. The support of a measure $\mu \in \Pr(X)$ is denoted by $\operatorname{spt}\mu$.

4.2. Theorem. *Suppose there is $k < \infty$ such that $g(2r) \leq kg(r)$ for $r > 0$. Then for all $c \in \mathbb{R}$ the set*

$$\{K \in \mathcal{K}(X) : \mathcal{P}_g(K) \geq c\}$$

is analytic. In particular, the function $\mathcal{P}_g : \mathcal{K}(X) \to [0, \infty]$ is measurable with respect to $\mathcal{B}(\mathcal{A})$.

Proof. We may assume $c > 0$. We shall make use of the theorem of Joyce and Preiss [JP] according to which for any compact K with $\mathcal{P}_g(K) \geq c$ there exists a compact $M \subset K$ such that $c \leq \mathcal{P}_g(M) < \infty$. Defining $\mu \in \Pr(X)$ by $\mu(B) = \mathcal{P}_g(M \cap B)/\mathcal{P}_g(M)$ we have $\operatorname{spt}\mu \subset K$ and $c\mu(L) \leq \mathcal{P}_g(L)$ for all $L \in \mathcal{K}(X)$. The converse of this holds trivially: if there exists $\mu \in \Pr(X)$ such that $\operatorname{spt}\mu \subset K$ and $c\mu(L) \leq \mathcal{P}_g(L)$ for all $L \in \mathcal{K}(X)$, then $\mathcal{P}_g(K) \geq c$. It follows that

$$\{K \in \mathcal{K}(X) : \mathcal{P}_g(K) \geq c\} = \operatorname{proj}_1 A$$

where

$$A = \{(K, \mu) \in \mathcal{K}(X) \times \Pr(X) : \operatorname{spt}\mu \subset K,\ c\mu(L) \leq \mathcal{P}_g(L) \text{ for } L \in \mathcal{K}(X)\}.$$

Thus it suffices to show that A is a Borel set. We have

$$A = S \cap \bigcap_{m=1}^{\infty} A_m$$

where

$$S = \{(K, \mu) \in \mathcal{K}(X) \times \Pr(X) : \operatorname{spt}\mu \subset K\}$$

and

$$A_m = \{(K, \mu) \in \mathcal{K}(X) \times \Pr(X) : c\mu(L) \leq P_{g,1/m}(L) \text{ for compact sets } L \subset K\}.$$

First, the set S is clearly closed. Secondly, $(K, \mu) \in A_m$ if and only if for every $j \in \mathbb{N}$ and every compact set $L \subset K$ there is a $(1/m)$-packing $\{(x_i, r_i)\}_{i=1}^n$ of L such that

$$(c - 1/j)\,\mu(L) < \sum_{i=1}^{n} g(2r_i).$$

For a fixed j the set of all such pairs $(K, \mu) \in \mathcal{K}(X) \times \Pr(X)$ is open. To see this note that its complement consists of those (K, μ) for which there exists a compact set $L \subset K$ such that for every $(1/m)$-packing $\{(x_i, r_i)\}_{i=1}^n$ of L we have

$$(c - 1/j)\,\mu(L) \geq \sum_{i=1}^{n} g(2r_i),$$

and this set is easily seen to be closed. Thus each A_m is a G_δ-set, and so is A.

4.3. Question. *Is $\mathcal{P}_g$ $\mathcal{B}(\mathcal{A})$-measurable without the doubling condition $g(2r) \leq kg(r)$?*

In the proof we needed the doubling condition since Joyce and Preiss proved their result using it.

5. Diameter-based packing measure and dimension

The definitions for the upper box counting dimension and packing dimension of Section 3 can also be given in terms of the diameters of balls instead of their radii. More precisely, for $K \in \mathcal{K}(X)$ let $\widetilde{N}_\delta(K)$ be the smallest number of open balls of diameter at most δ that are needed to cover K. If we replace N_δ by $\widetilde{N}_\delta$ in the definitions of $\overline{\dim}_B$ and $\dim_p$, we get the same dimensions. For packing measures and the dimensions induced by them the situation is different in general metric spaces, and we look at that in some detail. We restrict here to the gauge functions $g(t) = t^s$.

Let $A \subset X$ and $\delta > 0$. Define

$$\widetilde{P}^s_\delta(A) = \sup \sum_{i=1}^{n} d(B_i)^s$$

where the supremum is taken over all finite sequences $B_1, \ldots, B_n$ of disjoint open balls centered at A and of diameter at most δ. As before, we then define

$$\widetilde{P}^s(A) = \lim_{\delta \to 0} \widetilde{P}^s_\delta(A),$$
$$\widetilde{\mathcal{P}}^s(A) = \inf \left\{ \sum_{i=1}^{\infty} \widetilde{P}^s(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i \right\},$$

and also

$$\widetilde{\dim}_p A = \inf\{s : \widetilde{\mathcal{P}}^s(A) = 0\}.$$

Clearly, $\widetilde{\mathcal{P}}^s \leq \mathcal{P}^s$ and $\widetilde{\dim}_p \leq \dim_p$, and it is easy to give examples of metric spaces, even compact subspaces of $\mathbb{R}$, where both of these inequalities are strict.

We would now like to modify the method of Section 3 to prove the measurability of $\widetilde{\dim}_p$. For that we need a formula giving $\widetilde{\dim}_p$ in terms of a box counting-type dimension. For $K \in \mathcal{K}(X)$ and $j = 1, 2, \ldots$, let $p_j(K)$ be the largest number n of disjoint open balls $B_1, \ldots, B_n$ centered at K with $2^{-j-1} < d(B_i) \leq 2^{-j}$. Of course, $p_j(K)$ may be zero for some j's. Set

$$\widetilde{\dim}_B K = \limsup_{j \to \infty} \frac{\log p_j(K)}{j \log 2}$$

(with $\log 0 = -\infty$). Then p_j is lower semicontinuous and so $\widetilde{\dim}_B$ is of Baire's class 2.

5.1. Lemma. *For $K \in \mathcal{K}(X)$,*

$$\widetilde{\dim}_p K = \inf \left\{ \sup_i \widetilde{\dim}_B K_i : K \subset \bigcup_{i=1}^{\infty} K_i,\ K_i \in \mathcal{K}(X) \right\}.$$

Only very minor modifications are needed to carry out the argument of Tricot from [T] for the radius packing measures and dimensions, it is given also in [F] and [M]. We leave the details to the reader. Using Lemma 5.1 one can now argue as before to prove the following theorem.

5.2. Theorem. *The function $\widetilde{\dim}_p : \mathcal{K}(X) \to [0, \infty]$ is $\mathcal{B}(\mathcal{A})$-measurable.*

5.3. Question. *Is $\widetilde{\mathcal{P}}^s$ $\mathcal{B}(\mathcal{A})$-measurable?*

The problem here is that in Section 4 we used the theorem of Joyce and Preiss [JP] which does not hold for the measures $\widetilde{\mathcal{P}}^s$; a counterexample has been constructed by Joyce [Jo].

6. Measurability of sections

We shall now study the measurability of the Hausdorff and packing measures and dimensions of sections in a product space. Let T be a Polish space.

If $E \subset T \times X$, and $t \in T$, then let $E_t = \{x \in X : (t, x) \in E\}$.

6.1. Theorem. *Let $B \subset T \times X$ be a Borel set. If B_t is σ-compact for all $t \in T$, then the functions $t \mapsto \mathcal{H}_g(B_t)$ and $t \mapsto \dim_H B_t$, $t \in T$, are Borel measurable.*

Proof. Assume first that each B_t is compact. Then the map $t \mapsto B_t$ is Borel measurable (even upper semicontinuous, see [K2, p. 58]) from T into $\mathcal{K}(X)$. Thus, the assertions follow from Theorem 2.1.

If each B_t is σ-compact we use a result of Saint Raymond [S] to find Borel sets $B_1 \subset B_2 \subset \dots$ such that $B = \bigcup_{n=1}^{\infty} B_n$ and $(B_n)_t$ is compact for all $t \in T$, $n = 1, 2, \dots$. As $\mathcal{H}_g(B_t) = \lim_{n\to\infty} \mathcal{H}_g((B_n)_t)$ and $\dim_H B_t = \lim_{n\to\infty} \dim_H (B_n)_t$, the theorem follows.

6.2. Remark. Dellacherie proved in [D] that $t \mapsto \mathcal{H}^s(B_t)$, and hence also $t \mapsto \dim_H B_t$, is $\mathcal{B}(\mathcal{A})$-measurable provided B is an analytic subset of $T \times X$ and T and X are compact metric spaces; his proof easily extends to the case where T and X are complete and separable. We shall now show that these functions need not be Borel measurable even when B is a Borel subset of $[0, 1] \times [0, 1]$.

6.3. Example. *There exists a Borel set $B \subset [0, 1] \times [0, 1]$ such that the function $t \mapsto \dim_H B_t$, $t \in [0, 1]$, is not Borel measurable.*

Proof. Let $E = \{K_\alpha : 0 \le \alpha \le 1\}$ be a copy of the Baire space (that is, a space homeomorphic to the irrationals) in $\mathcal{K}([0, 1])$ consisting of pairwise disjoint elements and such that $\dim_H K_\alpha > 0$ for all $\alpha \in [0, 1]$. To see that such a set E exists, consider f, a typical continuous function with respect to the Wiener measure. For almost all y in the range of f, the level set $f^{-1}\{y\}$ has Hausdorff dimension $1/2$, see [T]. The map $y \mapsto f^{-1}\{y\}$ is one-to-one and Borel measurable. Choose a copy of the Baire space $D \subset [0, 1]$ on which this map is a homeomorphism, see [K1]. Then we can take as E the image of D.

Let C be a coanalytic subset of $[0,1]$ which is not a Borel set. By [K1] there exists a continuous map φ of E onto $A = [0,1] \setminus C$.

Define

$$B = \{(x,y) \in [0,1] \times [0,1] : x = \varphi(K) \text{ and } y \in K \text{ for some } K \in E\}.$$

Then B is a Borel set. To see this let $S = \{(x,y,K) : x = \varphi(K)$ and $y \in K\}$. Then S is a Borel set and the projection map onto the first two coordinates is one-to-one and maps S onto B whence S is a Borel set by [K1, p. 487]. Also the projection of B into the x-axis is A and for $x \in A$, B_x contains some $K \in E$, whence $\dim_H B_x > 0$. If $x \in C$, then $B_x = \emptyset$ and $\dim_H B_x = 0$. Thus for the map g, $g(x) = \dim_H B_x$, the set $g^{-1}\{0\} = C$ is not a Borel set and so g is not a Borel function.

We now turn to the measurability of the packing dimension and measure of the sections. Applying Theorems 3.4, 5.2 and 4.2 we can use exactly the same argument as in 6.1 to prove the following theorem.

6.4. Theorem. *Let $B \subset T \times X$ be a Borel set such that all the sections B_t, $t \in T$, are compact.*

(1) *The functions $t \mapsto \dim_p B_t$ and $t \mapsto \widetilde{\dim}_p B_t$, $t \in T$, are $\mathcal{B}(\mathcal{A})$-measurable.*
(2) *If g is as in Theorem 4.2 the function $t \mapsto \mathcal{P}_g(B_t)$ is $\mathcal{B}(\mathcal{A})$-measurable.*

7. An example

In this section we show that the packing dimension and measure functions need not be Borel measurable. For this we shall use continued fractions, see *e.g.*, [R]. Any $z \in [0,1]$ can be written as

$$z = [b_1, b_2, \dots] = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \dots}}$$

where $b_i \in \mathbb{N}$. The sequence $b_1, b_2, \dots$ is finite if and only if z is rational. We then write

$$[b_1, \dots, b_j] = p_j/q_j$$

where the integers p_j and q_j are as in [R, p. 136]. The following basic relations hold for them, see (3) and (5) on page 136 of [R]: defining $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = b_0$, $q_0 = 1$, we have

$$p_j = b_j p_{j-1} + p_{j-2},\ q_j = b_j q_{j-1} + q_{j-2} \quad \text{for } j = 1, 2, \dots, \tag{7.1}$$

$$p_j q_{j-1} - p_{j-1} q_j = (-1)^{j-1} \quad \text{for } j = 1, 2, \dots. \tag{7.2}$$

For irrationals $z = [b_1, b_2, \dots]$ we use the fact that the even convergents $[b_1, b_2, \dots, b_{2j}]$ increase to z and the odd convergents $[b_1, b_2, \dots, b_{2j+1}]$ decrease to z. Also, $[b_1, \dots, b_j, b]$ is increasing with b if j is odd and decreasing if j is even. For the denominators q_j we shall need the following simple estimates.

7.3. Lemma. *$b_1 \cdot \ldots \cdot b_j \le q_j \le 2^j b_1 \cdot \ldots \cdot b_j$ for $j = 1, 2, \ldots$.*

Proof. Since $q_1 = b_1$ and $q_2 = b_1 b_2 + 1$, this holds for $j = 1, 2$. Suppose the asserted inequalities are valid for $j < m$. Then by (7.1)

$$\begin{aligned} b_1 \cdot \ldots \cdot b_m &\le b_m(b_1 \cdot \ldots \cdot b_{m-1}) + q_{m-2} \le b_m q_{m-1} + q_{m-2} \\ &= q_m \le b_m(2^{m-1} b_1 \cdot \ldots \cdot b_{m-1}) + 2^{m-2} b_1 \cdot \ldots \cdot b_{m-2} \\ &\le 2^m b_1 \cdot \ldots \cdot b_m. \end{aligned}$$

Thus the lemma follows by induction.

We denote by $\mathcal{I} = [0,1] \setminus \mathbb{Q}$ the set of irrationals in $[0,1]$. Let $z = [a_1, a_2, \ldots] \in \mathcal{I}$, and define

$$N(z) = \{[a_1, k_1, a_2, k_2, \ldots] : k_i \in \mathbb{N} \quad \text{for } i = 1, 2, \ldots\}.$$

7.4. Lemma. *For $z \in \mathcal{I}$, $\dim_p N(z) \ge 1/2$.*

Proof. We shall show that $N(z)$ contains a compact set M such that $\overline{\dim}_B(\overline{M \cap V}) \ge 1/2$ whenever V is an open set with $M \cap V \ne \emptyset$. This implies $\dim_p N(z) \ge 1/2$ by Lemma 3.3.

Temporarily fix $n_1, n_2, \ldots \in \mathbb{N}$, $n_i \ge 11$. Let

$$M = \{[a_1, k_1, a_2, k_2, \ldots] : 1 \le k_i \le n_i,\ k_i \in \mathbb{N}, \quad \text{for } i = 1, 2, \ldots\}.$$

Then M is a compact subset of $N(z)$.

Let $J = J(a_1, k_1, \ldots, a_m, k_m)$ be the smallest interval containing all the points of M whose expansions begin with $a_1, k_1, \ldots, a_m, k_m$. Using the above mentioned monotonicity properties of $[b_1, \ldots, b_j, b]$, we see that J contains the interval with left endpoint $A = [a_1, k_1, \ldots, a_m, k_m, a_{m+1}, 2]$ and right endpoint $B = [a_1, k_1, \ldots, a_m, k_m, a_{m+1}, n_{m+1}]$. Let p_i and q_i, $i = 0, 1, \ldots, 2m+1$, be the partial numerators and denominators generated by the sequence $a_1, k_1, \ldots, a_m, k_m, a_{m+1}$ with $p_0 = 0$, $q_0 = 1$. Then by (7.1) $A = u/v$ where $u = 2p_{2m+1} + p_{2m}$ and $v = 2q_{2m+1} + q_{2m}$. Similarly, $B = U/V$ where $U = n_{m+1} p_{2m+1} + p_{2m}$ and $V = n_{m+1} q_{2m+1} + q_{2m}$. Using (7.2) we estimate the length of the interval $[A, B]$:

$$\begin{aligned} B - A &= \frac{(n_{m+1} - 2)(p_{2m+1} q_{2m} - p_{2m} q_{2m+1})}{(n_{m+1} q_{2m+1} + q_{2m})(2q_{2m+1} + q_{2m})} \\ &= \frac{n_{m+1} - 2}{(n_{m+1} q_{2m+1} + q_{2m})(2q_{2m+1} + q_{2m})} \\ &> \frac{n_{m+1} - 2}{(n_{m+1} + 1)\, 3q_{2m+1}^2} \ge \frac{1}{4q_{2m+1}^2}, \end{aligned}$$

as $n_{m+1} \ge 11$.

Hence, recalling Lemma 7.3, the length of J satisfies

$$\begin{aligned} \mathcal{H}^1\big(J(a_1, k_1, \ldots, a_m, k_m)\big) &\ge \frac{1}{4q_{2m+1}^2} \\ &\ge \frac{1}{4^{2m+2}(a_1 \cdot \ldots \cdot a_{m+1})^2 (n_1 \cdot \ldots \cdot n_m)^2} \\ &:= \delta_m. \end{aligned}$$

We want to show now that two different intervals as above are disjoint. By the basic monotonicity properties stated at the beginning of this section, it suffices to consider $J_1 = J(a_1, k_1, \ldots, k_{m-1}, a_m, k)$ and $J_2 = J(a_1, k_1, \ldots, k_{m-1}, a_m, k')$ where $1 \le k < k' \le n_m$. These two sequences generate the same p_i and q_i for $i < 2m$. The interval J_1 lies to the left of the interval J_2. The right hand endpoint of J_1 is at most

$$B_1 = [a_1, k_1, \ldots, a_m, k, a_{m+1}, n_{m+1} + 1].$$

So $B_1 = U_1/V_1$ where by (7.1)

$$\begin{aligned} U_1 &= (n_{m+1} + 1)\big((ka_{m+1} + 1) + k\big)\, p_{2m-1} + a_{m+1} p_{2m-2}\big) + kp_{2m-1} + p_{2m-2} \\ &= \big((n_{m+1} + 1)(ka_{m+1}) + k\big)\, p_{2m-1} + \big((n_{m+1} + 1)\, a_{m+1} + 1\big)\, p_{2m-2} \\ &= \alpha p_{2m-1} + \beta p_{2m-2}, \end{aligned}$$

and similarly, with the same α and β,

$$V_1 = \alpha q_{2m-1} + \beta q_{2m-2}.$$

The left hand endpoint of J_2 is at least

$$A_2 = [a_1, k_1, \ldots, a_m, k', a_{m+1}, 1] = u_2/v_2$$

where, as above

$$u_2 = \alpha' p_{2m-1} + \beta' p_{2m-2}, \quad v_2 = \alpha' q_{2m-1} + \beta' q_{2m-2}$$

with

$$\alpha' = k' a_{m+1} + 1 + k', \quad \beta' = a_{m+1} + 1.$$

Thus by (7.2)

$$\begin{aligned} A_2 - B_1 &= \frac{u_2 V_1 - U_1 v_2}{v_2 V_1} \\ &= \frac{(\alpha'\beta - \alpha\beta')(p_{2m-1} q_{2m-2} - p_{2m-2} q_{2m-1})}{v_2 V_1} \\ &= \frac{\alpha'\beta - \alpha\beta'}{v_2 V_1}. \end{aligned}$$

Now

$$\begin{aligned} \alpha'\beta - \alpha\beta' &= (k' a_{m+1} + k' + 1)\big((n_{m+1} + 1)\, a_{m+1} + 1\big) \\ &\quad - \big((n_{m+1} + 1)(ka_{m+1} + 1) + k\big)(a_{m+1} + 1) \\ &= \big((n_{m+1} + 1)\, a_{m+1}^2 + 1\big)(k' - k) \\ &\quad + (n_{m+1} + 2)\, a_{m+1}(k' - k) - n_{m+1} > 0, \end{aligned}$$

whence $A_2 - B_1 > 0$ so that J_1 and J_2 are indeed disjoint.

It follows now that at least $n_1 \dots n_m$ intervals of length δ_m are needed to cover M. Thus

$$\begin{aligned}\overline{\dim}_B M &\geq \limsup_{m\to\infty} \frac{\log N_{\delta_m}(M)}{-\log \delta_m} \\ &\geq \limsup_{m\to\infty} \frac{\log(n_1 \dots n_m)}{(2m+2)\log 4 + 2\log(a_1 \dots a_m) + 2\log(n_1 \dots n_m)}.\end{aligned}$$

Choosing the sequence (n_m) to grow sufficiently fast, the last upper limit is $1/2$ and then $\overline{\dim}_B M \geq 1/2$. The same argument shows that also $\overline{\dim}_B(\overline{M \cap V}) \geq 1/2$ for all open sets V with $M \cap V \neq \emptyset$. Thus $\dim_p M \geq 1/2$ as required.

For $A \subset \mathbb{R}^2$ let

$$A_x = \{y \in \mathbb{R} : (x,y) \in A\}.$$

7.5. Theorem. a) *There exists $M \in \mathcal{K}(\mathbb{R}^2)$ such that $\{x \in \mathbb{R} : \dim_p(M_x) > 0\}$ is an analytic non-Borel set.*

b) *The set $E = \{K \in \mathcal{K}([0,1]) : \dim_p K > 0\}$ is an analytic non-Borel set.*

Indeed, E is a complete $\sum_1^1$ set: if A is an analytic subset of a Polish space X, then there is a Borel measurable map h of X into $\mathcal{K}([0,1])$ such that $h^{-1}(E) = A$.

(Complete sets are discussed in [KL].)

Proof. The results in Sections 3 and 4 show that the sets in question are analytic. For the claim that they are not Borel sets a) clearly implies b), since $x \mapsto M_x$ is Borel measurable. To prove the theorem it suffices to prove E is $\sum_1^1$ complete. To do this we shall make some additional observations to a beautiful technique of Mazurkiewicz and Sierpinski, [MS].

Let A be an analytic subset of a Polish space X. Let f be a continuous map of $\mathcal{I} = \mathbb{N}^{\mathbb{N}}$ onto A. For $z = (a_1, a_2, \dots) \in \mathcal{I}$, let

$$\varphi(z) = (a_1, a_3, a_5, \dots).$$

Then φ is a continuous map of $\mathcal{I}$ onto itself. Let $g = f \circ \varphi \colon \mathcal{I} \to A$, which is also continuous and onto. Moreover

$$g^{-1}\{x\} \supset N(a_1, a_2, \dots)$$

whenever $x \in A$ and $f(a_1, a_2, \dots) = x$. Thus by Lemma 5.1

$$\dim_p g^{-1}\{x\} \geq 1/2 \quad \text{for } x \in A.$$

Let M be the closure of $\{(g(\sigma), \sigma) : \sigma \in \mathcal{I}\}$. Then M is a closed subset of $X \times [0,1]$. If $x \in A$, then $M_x \supset g^{-1}\{x\}$, whence $\dim_p M_x \geq 1/2$. If $x \in [0,1]$ and $y \in M_x$, then there is a sequence $\sigma_i \in \mathcal{I}$ such that $(g(\sigma_i), \sigma_i) \to (x, y)$. If $y \in \mathcal{I}$, then $x = \lim g(\sigma_i) = g(y) \in A$. Thus for $x \in [0,1] \setminus A$, $M_x \subset \mathbb{Q}$ and so $\dim_p M_x = 0$. Hence we have shown that $h^{-1}(E) = A$, where $h(x) = M_x$.

8. Densities

In this section we shall study the lower density

$$d_g(A,x) = \liminf_{r\to 0} \frac{\mathcal{P}_g\big(A\cap B(x,r)\big)}{g(2r)}$$

of packing measures $\mathcal{P}_g$ for $x \in X$ and $A \subset X$. We shall always assume that the gauge function g satisfies the doubling condition

$$g(2r) \le kg(r) \quad \text{for } r > 0 \tag{8.1}$$

with some $k < \infty$. We also denote

$$d^s(A,x) = d_g(A,x) \quad \text{when } g(r) = r^s.$$

If $A \subset \mathbb{R}^n$ and $\mathcal{P}_g(A) < \infty$, then

$$d_g(A,x) = 1 \quad \text{for } \mathcal{P}^s \text{ almost all } x \in A$$

according to [ST, Corollary 7.2]. We shall show in Example 8.9 that this is false even for $g(r) = r^s$ in any infinite-dimensional Hilbert space. However, in Theorem 8.3 we derive the optimal inequalities $2^{-s} \le d^s(A,x) \le 1$. For this we need a modification of well-known covering lemmas.

8.2. Lemma. *Let $0 < \eta < 1/2$, let $\mathcal{B}$ be a collection of closed balls in X such that*

$$\sup\{d(B) : B \in \mathcal{B}\} < \infty,$$

and suppose that $A \subset X$ is such that every $x \in A$ is a center of some $B \in \mathcal{B}$. Then there exist $B(x_i, r_i) \in \mathcal{B}$, $i = 1, 2, \dots,$ such that the balls $B(x_i, \eta r_i)$ are disjoint and

$$A \subset \bigcup_i B(x_i, r_i).$$

Proof. For each $x \in A$ choose some $B(x) = B\big(x, r(x)\big) \in \mathcal{B}$. Set

$$M = \sup\{r(x) : x \in A\}.$$

Let $2\eta < t < 1$ and define

$$A_1 = \{x \in A : tM < r(x) \le M\}.$$

Choose a subset $B_1 \subset A_1$ which is maximal with respect to the property:

$$\text{if } x, y \in B_1 \text{ and } x \ne y, \text{ then } x \notin B(y) \text{ or } y \notin B(x).$$

Then $A_1 \subset \bigcup_{x\in B_1} B(x)$ and the balls $B\big(x, \eta r(x)\big)$, $x \in B_1$, are disjoint. Next let

$$A_2 = \Big\{x \in A \setminus \bigcup_{x\in B_1} B(x) : t^2M < r(x) \le tM\Big\}$$

and choose a maximal subset $B_2 \subset A_2$ such that for any $x, y \in B_2$ with $x \ne y$, $x \notin B(y)$ or $y \notin B(x)$. It follows that

$$A_1 \cup A_2 \subset \bigcup_{x\in B_1\cup B_2} B(x)$$

and the balls $B\big(x, \eta r(x)\big)$, $x \in B_1 \cup B_2$, are disjoint. Continuing in this manner we find the desired balls.

8.3. Theorem. *Let $A \subset X$ and $\mathcal{P}^s(A) < \infty$. Then*

$$2^{-s} \leq d^s(A,x) \leq 1 \quad \textit{for } \mathcal{P}^s \textit{ almost all } x \in A.$$

Proof. The right hand inequality follows with essentially the same proof which was used in [M, 6.10], or one can consult Cutler [Cu] who proved this for $\mathcal{P}_g$ with general gauge functions. To prove the left hand inequality, we may assume that A is a Borel set by the Borel regularity of $\mathcal{P}^s$. Given $\lambda > 2$ it is enough to show that $d^s(A,x) \geq \lambda^{-s}$ for $\mathcal{P}^s$ almost all $x \in A$. Let $t < \lambda^{-s}$ and let

$$B \subset \{x \in A : d^s(A,x) < t\} \equiv A_t.$$

Given $\delta > 0$ we can apply Lemma 8.2 to find disjoint balls $B(x_i, r_i/\lambda)$ such that $x_i \in B$, $2r_i < \delta$,

$$\begin{aligned} &\mathcal{P}^s\big(A \cap B(x_i,r_i)\big) < t(2r_i)^s \quad \text{and} \\ &B \subset \bigcup_{i=1}^{\infty} B(x_i,r_i). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{P}^s(B) &\leq \sum_{i=1}^{\infty} \mathcal{P}^s\big(B \cap B(x_i,r_i)\big) \\ &\leq t\sum_{i=1}^{\infty}(2r_i)^s = t\lambda^s \sum_{i=1}^{\infty}(2r_i/\lambda)^s \leq t\lambda^s P^s_\delta(B). \end{aligned}$$

Letting $\delta \to 0$ we have $\mathcal{P}^s(B) \leq t\lambda^s P^s(B)$ for all $B \subset A_t$, which implies $\mathcal{P}^s(A_t) \leq t\lambda^s \mathcal{P}^s(A_t)$. As $t\lambda^s < 1$, we obtain $\mathcal{P}^s(A_t) = 0$ for all $t < \lambda^{-s}$, and this yields that $d^s(A,x) \geq \lambda^{-s}$ for $\mathcal{P}^s$ almost all $x \in A$ as desired.

8.4. Remark. The inequalities

$$c \leq d_g(A,x) \leq 1 \quad \text{for } \mathcal{P}^s \text{ almost all } x \in A$$

hold for any g satisfying (8.1) if $\mathcal{P}_g(A) < \infty$. Here c is a positive constant (not necessarily optimal) depending on g. This was proved by Cutler in [Cu]. The above proof also applies.

Next we show that instead of inequalities we have equality $\mathcal{H}_g$ almost everywhere in A. Recall that we are assuming (8.1).

8.5. Theorem. *If $A \subset X$ with $\mathcal{P}_g(A) < \infty$, then*

$$d_g(A,x) = 1 \quad \textit{for } \mathcal{H}_g \textit{ almost all } x \in A.$$

Proof. By the Borel regularity of $\mathcal{P}_g$ we may assume that A is a Borel set. If our claim is false there are $t < 1$ and a Borel set $C \subset A$ such that $\mathcal{H}_g(C) > 0$ and

$d_g(A,x) < t$ for $x \in C$ (recall Remark 8.4). As $\mathcal{H}_g \leq \mathcal{P}_g$ (see [Cu, 2.6], also the proof of this fact in $\mathbb{R}^n$ for $g(r) = r^s$ in [M, 5.12] generalizes without difficulty), we can use the Radon–Nikodym theorem to find a non-negative Borel function f on C such that

$$\mathcal{H}_g(B) = \int_B f\, d\mathcal{P}_g$$

for Borel sets $B \subset C$. Since $\mathcal{H}_g(C) > 0$, there are $u > 0$ and a Borel set $D \subset C$ such that $\mathcal{H}_g(D) > 0$ and $f \geq u$ on D which yields

$$\mathcal{H}_g(B) \geq u\mathcal{P}_g(B) \quad \text{for } B \subset D. \tag{8.2}$$

Let $\delta > 0$. By the analogue of Theorem 8.3 for $g(r) = r^s$, recall Remark 8.4, $c < d_g(D,x) < t$ for $\mathcal{P}^s$ almost all $x \in D$. For the upper density with respect to the Hausdorff measure we have, *cf.* [Fe, 2.10.18 (3)], also the proof of [M, 6.2] for $g(r) = r^s$ generalizes easily,

$$\limsup_{r\to 0} \mathcal{H}_g\big(D \cap B(x,r)\big)/g(2r) \leq 1 \tag{8.3}$$

for $\mathcal{H}_g$ almost all $x \in X$. From (8.2) we see that (8.3) also holds for $\mathcal{P}_g$ almost all $x \in D$. Let $B \subset D$ be a Borel set. By a standard covering theorem, see *e.g.*, [Fe, 2.8.6], we can find $x_i \in B$ and $0 < r_i < \delta/2$ such that the balls $B_i = B(x_i, r_i)$ are disjoint,

$$\begin{aligned}
&B \subset \bigcup_{i=1}^{n} B(x_i, r_i) \cup \bigcup_{i=n+1}^{\infty} B(x_i, 5r_i) \quad \text{for all } n = 1, 2, \ldots,\\
&cg(2r_i) \leq \mathcal{P}_g(D \cap B_i) \leq tg(2r_i) \quad \text{and}\\
&\mathcal{H}_g\big(D \cap B(x_i, 5r_i)\big) \leq 2g(10r_i).
\end{aligned}$$

Thus recalling also (8.1) and (8.2) we have

$$\begin{aligned}
\mathcal{P}_g(B) &\leq \sum_{i=1}^{n} \mathcal{P}_g(D \cap B_i) + \sum_{i=n+1}^{\infty} \mathcal{P}_g\big(D \cap B(x_i, 5r_i)\big)\\
&\leq t\sum_{i=1}^{n} g(2r_i) + u^{-1} \sum_{i=n+1}^{\infty} \mathcal{H}_g\big(D \cap B(x_i, 5r_i)\big)\\
&\leq tP_{g,\delta}(B) + 2k^3u^{-1} \sum_{i=n+1}^{\infty} g(2r_i)\\
&\leq tP_{g,\delta}(B) + 2k^3u^{-1}c^{-1} \sum_{i=n+1}^{\infty} \mathcal{P}_g(D \cap B_i).
\end{aligned}$$

Here

$$\sum_{i=n+1}^{\infty} \mathcal{P}_g(D \cap B_i) \leq \mathcal{P}_g(A) < \infty.$$

Thus letting first $n \to \infty$ and then $\delta \to 0$, we obtain

$$\mathcal{P}_g(B) \le tP_g(B)$$

for all Borel sets $B \subset D$. This implies

$$0 < \mathcal{H}_g(D) \le \mathcal{P}_g(D) \le t\mathcal{P}_g(D) < \infty$$

which is a contradiction since $t < 1$.

8.6. Corollary. *Let g satisfy the doubling condition. Let $A \subset X$ with $\mathcal{P}_g(A) < \infty$. Then $\mathcal{H}_g(A) = \mathcal{P}_g(A)$ if and only if*

$$\lim_{r \to 0} \frac{\mathcal{H}_g\big(A \cap B(x,r)\big)}{g(2r)} = 1 \quad \textit{for } \mathcal{P}_g \textit{ almost all } x \in A.$$

The proof of Theorem 6.12 of [M] applies without any essential changes when we have Theorem 8.5 at our disposal.

8.7. Remark. An interesting application of Corollary 8.6 in $\mathbb{R}^n$ is the following theorem of Saint Raymond and Tricot. Let $A \subset \mathbb{R}^n$ with $0 < \mathcal{P}^s(A) < \infty$. Then $\mathcal{H}^s(A) = \mathcal{P}^s(A)$ if and only if s is an integer and A is $\mathcal{P}^s$-rectifiable in the sense that $\mathcal{P}^s$ almost all of A can be covered with countably many Lipschitz images of $\mathbb{R}^s$, see [ST] or [M]. Attempts to extend this to more general metric spaces starting from 8.6 boil down to the following: in which metric spaces X is it true that if $A \subset X$ with $\mathcal{H}^s(A) < \infty$, then

$$\lim_{r \to 0} \frac{\mathcal{H}^s\big(A \cap B(x,r)\big)}{(2r)^s} = 1 \quad \text{for } \mathcal{H}^s \text{ almost all } x \in A$$

if and only if s is an integer and $\mathcal{H}^s$ almost all of A can be covered with countably many Lipschitz images of $\mathbb{R}^s$? Kirchheim proved in [K] that the "if" part is valid in any metric space. Chlebík has given in [C] a proof which shows that the "only if" part is valid at least in all infinite-dimensional inner product spaces. Thus combining these results with Corollary 8.6 the argument to prove Theorem 17.11 in [M] shows that the theorem of Saint–Raymond and Tricot is valid in Hilbert spaces.

8.8. Theorem. *Let X be a Hilbert space, $s > 0$ and $A \subset X$ with $0 < \mathcal{P}^s(A) < \infty$. Then $\mathcal{H}^s(A) = \mathcal{P}^s(A)$ if and only if s is an integer and A is $\mathcal{P}^s$-rectifiable.*

Now we construct an example in an infinite-dimensional space to show that the lower bound in Theorem 8.3 is the best possible.

8.9. Example. *Let X be an infinite-dimensional Hilbert space and $s > 0$. Then there is a compact set $K \subset X$ such that $0 < \mathcal{P}^s(K) < \infty$ and*

$$d^s(K,x) \le 2^{-s} \quad \textit{for } x \in K.$$

Proof. Let $e_1, e_2, \dots$ be an orthonormal basis of X. Let (m_k) be a sequence of positive integers such that $m_1 > (1+\sqrt{2})^s$

$$m_k \nearrow \infty \quad \text{and} \quad \sum_{k=1}^{\infty} m_k^{-1} = \infty. \tag{8.4}$$

Define the positive numbers λ_k by

$$m_k \lambda_k^s = \lambda_{k-1}^s \quad \text{with } \lambda_0 = 1. \tag{8.5}$$

Then $\lambda_k/\lambda_{k-1} \to 0$. For $i = 1, \ldots, m_1$ set

$$x(i) = \lambda_1 e_i \quad \text{and} \quad B(i) = U(x_i, \lambda_1/\sqrt{2})$$

where $U(x,r)$ denotes the open ball with center x and radius r. Suppose then that for some $k \in \mathbb{N}$ the points $x(i_1, \ldots, i_k)$ have been defined. We put

$$x(i_1, \ldots, i_k, i) = x(i_1, \ldots, i_k) + \lambda_{k+1} e_i, \quad i = 1, \ldots, m_{k+1}.$$

Set

$$B(i_1, \ldots, i_k) = B\big(x(i_1, \ldots, i_k), \lambda_k/\sqrt{2}\big).$$

Let $\Omega = \prod_{i=1}^{\infty} \{1, \ldots, m_i\}$ have the product topology. The system $\{B(i_1, \ldots, i_k) : 1 \leq i_j \leq m_j, j \leq k, k \in \mathbb{N}\}$ of open balls is a Cantor scheme:
(8.6)
for each k the balls $B(i_1, \ldots, i_k)$, $1 \leq i_j \leq m_j$, $1 \leq j \leq k$, are pairwise disjoint.

(8.7) for each k and $1 \leq i \leq m_{k+1}$,

$$\overline{B(i_1, \ldots, i_k, i)} \subset B(i_1, \ldots, i_k).$$

To verify these, recall that $m_k > (1 + \sqrt{2})^s$. Moreover,

$$d\big(B(i_1, \ldots, i_k, i)\big) = (\sqrt{2}\,\lambda_{k+1})^s \tag{8.8}$$

and

$$\sum_{i=1}^{m_{k+1}} d\big(B(i_1, \ldots, i_k, i)\big)^s = d\big(B(i_1, \ldots, i_k)\big)^s. \tag{8.9}$$

Define

$$K = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \ldots i_k} B(i_1, \ldots, i_k) = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \ldots i_k} \overline{B(i_1, \ldots, i_k)}.$$

Then

$$K \cap \overline{B(i_1, \ldots, i_k)} \subset B\big(x(i_1, \ldots, i_k), 2\lambda_{k+1}\big). \tag{8.10}$$

Let g be the coding map from the coding space Ω onto K. Thus,

$$\{g(\sigma)\} = \bigcap_{k=1}^{\infty} B(\sigma|k)$$

for $\sigma = (i_1, i_2, \dots) \in \Omega$ where $\sigma|k = (i_1, \dots, i_k)$. By properties (8.6) and (8.7), g is a homeomorphism of the Cantor space Ω onto K and g maps the cylinder set $[i_1, \dots, i_k]$ onto $K \cap B(i_1, \dots, i_k)$. Define $\widetilde{\mu}$ on the cylinder sets in Ω by

$$\widetilde{\mu}([i_1, \dots, i_k]) = (\sqrt{2}\,\lambda_k)^s.$$

By properties (8.8) and (8.9), $\widetilde{\mu}$ satisfies Kolmogorov's consistency conditions. Let $\widetilde{\mu}$ also denote the extension of $\widetilde{\mu}$ to a Borel measure on Ω. Let μ be the image measure on K of $\widetilde{\mu}$ via the coding map g. Thus

$$\mu\big(K \cap B(i_1, \dots, i_k)\big) = (\sqrt{2}\,\lambda_k)^s = d\big(B(i_1, \dots, i_k)\big)^s. \tag{8.11}$$

We use μ to show that $\mathcal{P}^s(K) > 0$. Let $A \subset K$ be a Borel set and $\delta > 0$. Then there is a disjoint subfamily $\{B_j = U(x_j, r_j)\}$ of the family $\{B(i_1, \dots, i_k)\}$ such that $d(B_j) < \delta$, $B_j \cap A \neq \emptyset$, say $x_j \in B_j \cap A$, and $A \subset \bigcup_j B_j$. In fact, by choosing for every $x \in B$ the largest $B = B(i_1, \dots, i_k)$ such that $x \in B$ and $d(B) < \delta$, we find the sequence (B_j). Using (7) and the fact that $\lambda_{k+1}/\lambda_k \to 0$ we see that for some r_j' and x_j' with $r_j \leq \big(1 + o(\delta)\big)\, r_j'$, $x_j' \in K$ we have $B_j' = B(x_j, r_j') \subset B_j$. Thus by (8.11),

$$\begin{aligned} \mu(A) &\leq \sum_j \mu(B_j) = \sum_j d(B_j)^s \leq \big(1 + o(\delta)\big) \sum_j d(B_j')^s \\ &\leq \big(1 + o(\delta)\big)\, P^s_\delta(A). \end{aligned}$$

Hence $\mu(A) \leq P^s(A)$ for all Borel sets $A \subset K$, which gives

$$0 < \mu(K) \leq \mathcal{P}^s(K)$$

as desired.

We now show that $\mathcal{P}^s(K) < \infty$ and prove the lower density estimate. Both of these will follow from

$$P^s\big(K \cap B(\sigma|\ell)\big) \leq (\sqrt{2}\,\lambda_\ell)^s \quad \text{for } \sigma \in \Omega,\ \ell = 1, 2, \dots. \tag{8.12}$$

(In fact, it is easy to show that also the converse inequality in (8.12) holds.) Let $\sigma \in \Omega$, $\ell \geq 1$, $0 < \delta < \lambda_{\ell+1}$ and let $D_1, \dots, D_n$ be disjoint closed balls centered at $K \cap B(\sigma|\ell)$ with

$$d_j = d(D_j) < \delta. \tag{8.13}$$

We want to show that

$$\sum_j d_j^s \leq \big(1 + o(\delta)\big)(\sqrt{2}\,\lambda_\ell)^s \tag{8.14}$$

which clearly yields (8.12). For each j let k_j be the unique integer such that

$$2\sqrt{2}\,\lambda_{k_j+1} < d_j \leq 2\sqrt{2}\,\lambda_{k_j}, \tag{8.15}$$

and let $B_j = B(i_1, \ldots, i_{k_j})$ be such that $\sigma|k_j = (i_1, \ldots, i_{k_j})$ and $K \cap B_j \cap D_j \neq \emptyset$. Since $\lambda_{k+1}/\lambda_k \to 0$ as $k \to \infty$, we may assume, in order to prove (8.14), that

$$2\sqrt{2}\,\lambda_{k_j+1} + 4\lambda_{k_j+2} < d_j < 2\sqrt{2}\,\lambda_{k_j} - 4\lambda_{k_j+1}.$$

This has the effect that $K \cap B_j \subset D_j$, whence the balls B_j are disjoint. It follows then from (8.6)–(8.9) that

$$\sum d(B_j)^s \leq d\big(B(\sigma|\ell)\big)^s = (\sqrt{2}\,\lambda_\ell)^s. \tag{8.16}$$

Set

$$\begin{aligned} I &= \{j : d_j \leq \sqrt{2}\,\lambda_{k_j} + 4\lambda_{k_j+1}\} \quad \text{and} \\ J &= \{j : d_j > \sqrt{2}\,\lambda_{k_j} + 4\lambda_{k_j+1}\}. \end{aligned}$$

Then by (8.16)

$$\begin{aligned} \sum_{j\in I} d_j^s &\leq \sum_{j\in I} (\sqrt{2}\,\lambda_{k_j} + 4\lambda_{k_j+1})^s \\ &\leq \big(1 + o(\delta)\big) \sum_{j\in I} (\sqrt{2}\,\lambda_{k_j})^s = \big(1 + o(\delta)\big) \sum_{j\in I} d(B_j)^s \\ &\leq \big(1 + o(\delta)\big)(\sqrt{2}\,\lambda_\ell)^s. \end{aligned} \tag{8.17}$$

For each k every ball $B(i_1, \ldots, i_{k-1})$ can contain at most one D_j with $k_j = k$ and $j \in J$ because of the disjointness. Thus there are at most $m_1, \ldots, m_{k-1}$ indices $j \in J$ with $k_j = k$. Hence by (8.15) and (8.5)

$$\sum_{j\in J, k_j = k} d_j^s \leq m_1 \ldots m_{k-1} (2\sqrt{2}\,\lambda_k)^s = (2\sqrt{2})^s / m_k.$$

So by (8.15) and (8.4),

$$\sum_{j\in J} d_j^s \leq (2\sqrt{2})^s \sum_{\lambda_{k+1} < \delta} m_k^{-1} \to 0. \tag{15}$$

Thus (8.17) and (8.18) yield (8.14) and hence also (8.12).

Let $\sigma \in \Omega$ and $x = g(\sigma) \in K$. By the construction and (8.12),

$$\lim_{k\to\infty} \frac{P^s\big(K \cap B(x, \sqrt{2}\,\lambda_k - 4\lambda_{k+1})\big)}{\big(2(\sqrt{2}\,\lambda_k - 4\lambda_{k+1})\big)^s} = \lim_{k\to\infty} \frac{P^s\big(K \cap B(\sigma|k)\big)}{\big(2(\sqrt{2}\,\lambda_k - 4\lambda_{k+1})\big)^s} \leq 2^{-s},$$

which completes the proof since $\mathcal{P}^s \leq P^s$.

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