DIRECTIONALLY REINFORCED RANDOM WALKS

R. Daniel Mauldin †‡

Michael Monticino †

University of North Texas
Department of Mathematics, P.O.
Box 5116, Denton, TX 76203

Heinrich v. Weizsäcker

Universität Kaiserslautern
Fachbereich Mathematik, Universität
Kaiserslautern, 67653 Kaiserslautern, Germany

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Mailing address for proofs: Drs. Mauldin and Monticino, University of North Texas, Department of Mathematics, P.O. Box 5116, Denton, TX 76203.

Abstract. This paper introduces and analyzes a class of directionally reinforced random walks. The work is motivated by an elementary model for time and space correlations in ocean surface wave fields. We develop some basic properties of these walks. For instance, we investigate recurrence properties and give conditions under which the limiting continuous versions of the walks are Gaussian diffusion processes.

1. Introduction. This paper introduces a class of directionally reinforced random walks. The motivation for our work was to develop and analyze an elementary model which simulates some of the time and space correlations observed in ocean surface wave fields. In particular, as discussed by West (1986), the direction of motion of a field tends to be reinforced so that at any time a field is more likely to continue moving in its current direction than it is to change direction. We give an elementary mathematical model of this reinforcement and develop some basic properties of the resulting stochastic processes.

Directionaly reinforced random walks in $\mathbb{Z}^d$ are defined in the next section. In section 3, we investigate the recurrence properties of these walks. For instance, it is shown that a walk in $\mathbb{Z}$ is recurrent if and only if it changes direction, at some time, with probability one. An interesting example is given for which a walk in $\mathbb{Z}$ is recurrent, yet changes direction only a finite number of times within any bounded spatial interval. Thus, eventually the walk visits 0 only during fantastically long runs in a particular direction. Under moment conditions on the time until the walk changes direction, we show that the walk is recurrent when $d = 2$ and is transient for $d \geq 3$. The analysis is facilitated by defining a related stopping time process which is a random walk with $i.i.d.$ increments and then applying classic results.

In section 4, we consider the limiting continuous time version of a directionally reinforced walk. Conditions on the reinforcement are presented under which the limiting version is a Gaussian diffusion process. We calculate the diffusion coefficient for a specific example. The greater the directional reinforcement in this example the larger the diffusion coefficient. On the other hand, an example shows that the limiting process is not, in general, necessarily Brownian motion. In the last section, we interpret some of the above results in terms of wave fields and raise some further questions.

Note that the random walks considered here have a different type of reinforcement than that considered in other works on reinforced random walks such as Davis (1990), Pemantle (1988), and Mauldin and Williams (1991). There, broadly speaking, the path crossed by the walk is reinforced in some permanent way. Here the reinforcement is on the current direction that the walk is moving. Once the walk changes direction, the previous reinforcement is forgotten and the new direction of motion is reinforced. This lack of memory approximates the relaxation of reinforcement in a previous direction of motion.

2. Directionally Reinforced Walks in $\mathbb{Z}^d$. For $k \geq 1$, let $0 \leq g(k) \leq 1$. The $g(k)'s$ will characterize the directional reinforcement of the walks defined below.

Denote the set of unit vectors in the d-dimensional lattice by U. And let

$u_0, \ldots, u_{2d-1}$ be an arbitrary enumeration of the $2d$ vectors in U. Now define a sequence of U-valued random vectors, $X_1, X_2, \ldots$, such that, for each $u_i \in U$,

$$P[X_1 = u_i] = \frac{1}{2d}. \tag{2.1}$$

For any $u_i \in U$ and for all $n \geq 0$ and $k \geq 1$,

(2.2)
$$\begin{aligned} &P[X_{n+k+1} = u_i | X_{n+k} = u_i, \ldots, X_{n+1} = u_i, X_n \neq u_i, X_{n-1} = u_{j_{n-1}}, \ldots, X_1 = u_{j_1}] \\ &= P[X_{n+k+1} = u_i | X_{n+k} = u_i, \ldots, X_{n+1} = u_i, X_n \neq u_i] \\ &= g(k). \end{aligned}$$

And, for $u_l \neq u_i$,

(2.3)
$$\begin{aligned} &P[X_{n+k+1} = u_l | X_{n+k} = u_i, \ldots, X_{n+1} = u_i, X_n \neq u_i, X_{n-1} = u_{j_{n-1}}, \ldots, X_1 = u_{j_1}] \\ &= P[X_{n+k+1} = u_l | X_{n+k} = u_i, \ldots, X_{n+1} = u_i, X_n \neq u_i] \\ &= \frac{1 - g(k)}{2d-1}. \end{aligned}$$

Define the *directionally reinforced random walk* S_n in $\mathbb{Z}^d$ by

$$S_n = \sum_{i=0}^{n} X_i \ ,$$

where $X_0 \equiv \mathbf{0} = (0, \ldots, 0)$.

Condition (2.1) states that the first step of the walk is equally likely to be any of the possible $2d$ directions. Condition (2.2) states that the probability of a step in a given direction only depends on the number of steps which have been taken in that direction since the last change in direction. So the reinforcement is transitory in the sense that once a change in direction occurs, the previous reinforcement is forgotten. Moreover, the reinforcement is symmetric with respect to direction. Condition (2.3) indicates that when the walk changes direction it is equally likely to move in each of the other $2d - 1$ directions.

Remark 2.1. A directionally reinforced random walk provides an elementary model for certain aspects of ocean surface wave fields in the following way. If a wave field is moving in a given direction then it is more likely to continue moving in that direction. The value $g(k)$ can be interpreted as the probability that a field will continue moving in its present direction given it has moved k units in that direction. The lack of memory once a walk changes direction approximates the relaxation of reinforcement in a previous direction of motion.

Remark 2.2. An intuitive view of the walk can be given as follows. Let T be a random variable with distribution $P[T = k] = g(1)\cdots g(k-1)(1-g(k))$ for $k = 1, 2, \ldots$. The walk starts at the origin and chooses an initial direction at random. It moves in this direction one unit per unit of time for a length of time which has the same distribution as T. It then changes direction. The new direction of motion is chosen uniformly from the $2d-1$ other possibilities. The procedure is repeated independently forever. This alternative viewpoint turns out to be a key part of the analysis of the properties of the walk and will be made more precise in the next section.

Remark 2.3. Note that with (2.1) the X_n are identically distributed. To see this first note an easy induction argument shows that, for any n, $(u_{i_1}, \ldots, u_{i_n}) \in U^n$, and permutation f of U,

$$P[X_1 = u_{i_1}, \ldots, X_n = u_{i_n}] = P[X_1 = f(u_{i_1}), \ldots, X_n = f(u_{i_n})]. \tag{2.4}$$

Hence, for any $n > 1$,

$$\begin{aligned} P[X_n = u_i] &= \sum_{u_j \in U} P[X_n = u_i,\ X_1 = u_j] \\ &= \sum_{u_j \in U} P[X_n = u_j,\ X_1 = u_i] \\ &= P[X_1 = u_i] \\ &= \frac{1}{2d}. \quad \square \end{aligned}$$

As indicated, $g(k)$ denotes the probability that the next step of the walk is taken in a certain direction given that exactly k steps have been taken in that direction since the last change in direction. In the following sections, we investigate the relationship between the $g(k)'s$ and the properties of the walks. Our main motivation was to study walks where the current direction of motion was indeed reinforced - that is, where $g(k)$ is nondecreasing in k. Note, however, that the results given below do not depend on this.

3. Recurrence Properties of S_n. This section examines the recurrence properties of directionally reinforced walks.

The walk S_n is said to be *recurrent* if

$$P[S_n = \mathbf{0} \quad i.o.] = 1;$$

it is *transient* otherwise. The usual recurrence result that S_n is recurrent if and only if $P[S_n = z \quad i.o.] = 1$ for all (possible) $z \in \mathbb{Z}^d$ will hold here. Also, it will follow that S_n is transient if and only if it is strongly transient. We say S_n is *strongly transient* if for all $\epsilon, M > 0$ there exists an $N(M,\epsilon)$ such that

$$P[|S_n| > M \quad for\ all\ n \geq N(M,\epsilon)] > 1 - \epsilon.$$

To avoid some anomalous special cases assume

$$0 < g(k) < 1, \tag{3.1}$$

for every k. This assumption ensures that each $z \in \mathbb{Z}^d$ is possible with respect to S_n. A $z \in \mathbb{Z}^d$ is *possible* if there exists an n such that $P[S_n = z] > 0$.

One case in which it is clear that S_n is strongly transient is when there is a positive probability that the walk will not change direction — i.e., when

$$g(1)g(2)\cdots > 0.$$

In fact, as formally stated in Theorem 3.1, for $d = 1$ this is precisely when S_n is transient. To show this, we define a related stopping time process. This process will be a random walk with symmetric increments.

Recurrence in $\mathbb{Z}$. Until noted assume $d = 1$ and let

$$T_0 = min\{k : \quad X_k = +1, X_{k+1} = -1\}.$$

So S_{T_0} is the position of the walk S_n just before it changes direction to go to the left after its first run of steps to the right. Let T_1 be the time of the first change in direction after T_0. And in general for $i \geq 2$, let T_i be the time of the i^{th} change in direction after T_0. Specifically,

$$T_1 = min\{k > T_0 : \quad X_k = -1, X_{k+1} = +1\}$$

and, for $i \geq 2$,

$$T_i = min\{k > T_{i-1} : \quad X_k = x,\ X_{k+1} = -x\}.$$

Assuming $g(1)g(2)\cdots = 0$, each T_i is well defined and finite almost surely. So without loss of generality assume they are defined and finite everywhere. The differences $T_i - T_{i-1}$ are *i.i.d.* with distribution

$$P[T_i - T_{i-1} = k] = g(1)\cdots g(k-1)(1 - g(k)).$$

Now consider the stopping time process $\{S_{T_{2n}}\}_{n\geq 0}$, where $S_{T_{2n}}$ is the position of the walk S_n just before it changes direction to go to the left after its $(n+1)^{st}$

run of steps to the right. Because the walk S_n takes unit steps, the increments $S_{T_{2n}} - S_{T_{2(n-1)}}$ have the same distribution as $(T_{2n} - T_{2n-1}) - (T_{2n-1} - T_{2(n-1)})$. Thus, these increments are nondegenerate, symmetric, identically distributed random variables. Moreover, they are independent. Hence, it follows (see, for instance Chung (Theorem 8.2.5, 1968)) that

$$-\infty = \liminf_{n\to\infty} S_{T_{2n}} < \limsup_{n\to\infty} S_{T_{2n}} = +\infty.$$

Therefore, we get the following.

Theorem 3.1. *S_n is recurrent if and only if $g(1)g(2)\cdots = 0$. Moreover, S_n is recurrent if and only if $P[S_n = z \ \ i.o.] = 1$ for all $z \in \mathbb{Z}$.*

It is tempting to conclude that the recurrence properties of S_n and $S_{T_{2n}}$ are the same. However, as Example 3.2 shows, S_n may be recurrent while $S_{T_{2n}}$ is strongly transient (by symmetry, the process $\{S_{T_{2n+1}}\}_{n\geq 0}$ also is strongly transient). This is interesting. In this case, S_n almost surely changes direction only a finite number of times within any bounded spatial interval. And so, the walk eventually passes through 0 only as part of extremely long runs in a particular direction.

Example 3.2. For each $k \geq 1$, suppose $g(k) = \sqrt{\frac{k}{k+1}}$. Then $g(1)g(2)\cdots = 0$ and, by Theorem 3.1, S_n is recurrent. However, $S_{T_{2n}}$ is strongly transient. To see this, let F be the common distribution function of the increments $S_{T_{2i}} - S_{T_{2(i-1)}}$ and suppose that $m > 0$. Then

$$\begin{aligned} 1 - F(m-1) &= \sum_{n=m}^{\infty} P[(T_2 - T_1) - (T_1 - T_0) = n] \\ &= \frac{1}{\sqrt{m}} \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) \sqrt{\frac{m}{m+k}}. \end{aligned}$$

Thus, for any $\epsilon > 0$, there exists an $M(\epsilon)$ such that, for all $m \geq M(\epsilon)$,

$$\frac{1}{\sqrt{m}}(1-\epsilon) \leq 1 - F(m-1) \leq \frac{1}{\sqrt{m}}.$$

Now with this estimate on the tail of F apply a discrete version of the argument given by Chung and Fuchs (1951, Example (4.4)) to complete the demonstration. □

Recurrence in higher dimensions. Now consider the recurrence properties of S_n for $d \geq 2$. Under moment conditions on the time until the walk changes directions, Theorems 3.3 and 3.4 show that the walk is recurrent when $d = 2$ and is transient for $d \geq 3$. Note that the latter result holds when the expected time until a direction change is finite. Heuristically, it might seem that the opposite should be true. For instance, when the expected time is small, it would obstensibly seem that the walk would be forced back to the origin. We conjecture that there is no nontrivial reinforcement scheme which makes the walk recurrent in $\mathbb{Z}^3$.

To begin, we formalize the description of S_n given in Remark 2.2. As in the remark, let T be a random variable with distribution

$$P[T = k] = g(1) \cdots g(k-1)(1 - g(k)).$$

Define the following random variables.

$\{T_j\}_{j \geq 1}$ is a random walk on the positive integers generated by an *i.i.d.* sequence with the same distribution as T.

I_0 is uniformly distributed on the set $\{0, \ldots, 2d-1\}$, independent of the T_n's.

$\{I_j\}_{j \geq 1}$ is a random walk on the cyclic group $\{0, \ldots, 2d-1\}$ generated by an *i.i.d.* sequence of random variables uniformly distributed on the finite set $\{1, \ldots, 2d-1\}$ independent of I_0 and the T_j's. (So I_j is chosen at random from the set $\{i \in \{0, \ldots, 2d-1\} : \ i \neq I_{j-1}\}$.)

We can construct the directionally reinforced random walk S_n — with the desired conditional distributions specified by (2.1)–(2.3) — by first setting $S_0 = (0, \ldots, 0)$. Then, for $1 \leq n \leq T_1$, let $S_n = n u_{I_0}$. And, for $T_j + 1 \leq n \leq T_{j+1}$, set $S_n = S_{T_j} + (n - T_j) u_{I_j}$. With this formulation, we give Theorems 3.3 and 3.4. The proofs of the theorems use similar techniques, so they are given together.

Theorem 3.3. *If $E[T^2] < \infty$ and $d = 2$, then S_n is recurrent.*

Theorem 3.4. *If $E[T] < \infty$ and $d \geq 3$, then S_n is transient.*

Proof of Theorems 3.3 and 3.4. We prove the recurrence and transience under the assumption that the direction of the first step of the walk S_n is deterministic. That is, we work with the conditional probabilities given $I_0 = i_0$ for some $i_0 \in \{0, \ldots, 2d-1\}$. From this case, the results will clearly hold for random I_0.

Let $j_0^* = 0$ and, for $m \geq 1$, let j_m^* be a random index given by

$$j_m^* = \min\{j > j_{m-1}^* : \quad I_j = I_0\}.$$

Set $\tau_m = T_{j_m^*}$. Then the process $\{S_{\tau_m}\}_{m \geq 1}$ is a d-dimensional random walk. To see this, first note that $I_{j_m^*} = I_0 = i_0$ a.s. and hence the random segment $I_{j_m^*}, \ldots, I^*_{j_{m+1}}$

has the same distribution as $I_0, \ldots, I_{j_1^*}$, independent of $S_{\tau_1}, \ldots, S_{\tau_m}$. Further, $\{T_{j_m^*+j} - T_{j_m^*}\}_{j\geq 1}$ is independent of $S_{\tau_1}, \ldots, S_{\tau_m}$ with the same distribution as $\{T_j\}_{j\geq 1}$. And thus, $P[S_{\tau_{m+1}} - S_{\tau_m} = z | S_{\tau_1}, \ldots, S_{\tau_m}] = P[S_{\tau_1} = z]$.

Now, by construction, $I_1 \neq i_0$ and $P[I_{j+1} = i_0 | I_j \neq i_0] = \frac{1}{2d-1}$. Thus, the random index $j_1^* - 1$, being the waiting time for a success in an *i.i.d.* sequence of trials with success probability $\frac{1}{2d-1}$, has a geometric distribution. Hence, $E[j_1^*] = 2d$. The random time τ_1 is the sum of j_1^* *i.i.d* copies with the same distribution as T. Thus, by independence and Wald's identity, we have $E[\tau_1] = 2dE[T]$ and $Var[\tau_1] = 2dVar[T] + Var[j_1^*](E[T])^2$. In particular, under the hypotheses of Theorem 3.4, $E[\tau_1] < \infty$; and, under those of Theorem 3.3, $E[\tau_1^2] < \infty$.

Since the process S_n takes unit steps, we have that the random walk S_{τ_m} has finite second moments in the case of Theorem 3.3. Moreover, it has zero expectations. To see this, notice that

$$E[S_{\tau_1}] = E[T]u_{i_0} + \sum_{i\neq i_0} a_i E[T]u_i,$$

where $a_i = E[\#\{j : \; 1 \leq j < j_1^*, \; I_j = i\}]$. By symmetry, all a_i, $i \neq i_0$, are equal. Since they add up to $E[j_1^* - 1] = 2d - 1$, they must all be equal to 1. Therefore,

$$E[S_{\tau_1}] = E[T] \sum_{i=0}^{2d-1} u_i = 0.$$

And thus, by Chung and Fuchs (1951, Theorem 5), the random walk S_{τ_m} is recurrent for $d = 2$. Theorem 3.3 now clearly follows.

In the case $d \geq 3$, Chung and Fuchs (1951, Theorem 6) also gives that S_{τ_m} is transient (even without the moment conditions). It remains to show that this implies the transience of S_n. For this, note

$$\begin{aligned}
\sum_{n=0}^{\infty} P[S_n = 0] &= \sum_{m=0}^{\infty} E[\#\{n : \; \tau_m \leq n < \tau_{m+1}, \; S_n = 0\}] \\
&= \sum_{m=0}^{\infty} \sum_{z\in\mathbb{Z}^d} E[\#\{n : \; \tau_m \leq n < \tau_{m+1}, \; S_n = 0\} | S_{\tau_m} = -z] P[S_{\tau_m} = -z] \\
&= \sum_{z\in\mathbb{Z}^d} f(z) E[\#\{n : \; 0 \leq n < \tau_1, \; S_n = z\}],
\end{aligned}$$

where $f(z) = \sum_{m=0}^{\infty} P[S_{\tau_m} = -z]$ is uniformly bounded by $1 + \sum_{m=0}^{\infty} P[S_{\tau_m} = 0]$. But, by the hypothesis of Theorem 3.4,

$$\sum_{z\in\mathbb{Z}^d} E[\#\{n : \; 0 \leq n < \tau_1, \; S_n = z\}] = E[\tau_1] = 2dE[T] < \infty.$$

Therefore, S_n is transient. $\square$

Note the above proof that the transience of S_{τ_n} implies the transience of S_n when $E[T] < \infty$ also works in two dimensions. So, if $d = 2$ and $E[T] < \infty$, then S_n is recurrent if and only if the random walk S_{τ_n} is recurrent. In particular, for any $\epsilon > 0$, there are cases for which $E[T^{2-\epsilon}] < \infty$ but where S_n is transient (cf. Chung and Lindvall (1980)). Furthermore, a straightforward argument can be given to show that whenever T is such that the associated d dimensional walk S_{τ_n} is transient then the reinforced walk S_n in dimension $d+1$ is transient. Hence, by Chung and Fuchs (1951, Theorem 6), Theorem 3.4 holds without the moment condition for $d \geq 4$.

Finally, are there any cases analogous to Example 3.2 for $d = 2$ or 3 — that is, where S_n is recurrent while S_{τ_n} is transient? We conjecture that these cases can not occur, but so far we have been unable to show this.

4. Continuous-time Directionally Reinforced Processes. Here we construct continuous-time processes using one–dimensional discrete time directionally reinforced walks and present a case in which these processes turn out to be versions of Brownian motion. The construction proceeds in a standard way using the partial sums S_n. In this section assume $d = 1$.

Specifically, for $t \in [0, 1]$, set

$$W_n(t) = \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} = \frac{\sum_{i=1}^{\lfloor nt \rfloor} X_i}{\sigma\sqrt{n}}, \tag{4.1}$$

where $\lfloor s \rfloor$ is the greatest integer less than or equal to s and $\sigma > 0$. Let $W(t)$ denote the weak limit (assuming it exists) of $\{W_n(t)\}_{n\geq 1}$ in $D[0, 1]$, the set of real–valued functions on $[0, 1]$ which have left hand limits and are continuous from the right, given the Skorohod topology. $W(t)$ represents our continuous version of a directionally reinforced random walk. The natural question is: "what sort of process is $W(t)$?" Proposition 4.1 shows that if $E[T^2] < \infty$, then $W(t)$ is Brownian motion. The effect of the reinforcement, in this case, is seen in the diffusion coefficient σ. For instance, Example 4.2 indicates that strengthening the reinforcement increases the rate of diffusion. In wave field models, a large diffusion coefficient corresponds to a high sea state. Thus, the example indicates that strong directional reinforcement is associated with high sea state.

Let $T_1, T_2, \ldots$ be the stopping times defined in section 3 (T_i gives the time of the i^{th} direction change of S_n) and recall that $\{S_{T_{2n}}\}_{n\geq 0}$ is a standard random walk with $i.i.d.$ increments. Proposition 4.1 is established using these notions and the functional central limit theorem.

Proposition 4.1. *If $E[T^2] < \infty$, then the weak limit of $\{W_n(t)\}_{n\geq 1}$, $W(t)$, is a version of standard Brownian motion with $\sigma^2 = \frac{Var[T]}{E[T]}$.*

Proof. Set $\widehat{\sigma}^2 = E[(S_{T_{2n}} - S_{T_{2(n-1)}})^2] = E[S^2_{T_2}] = 2Var[T]$. Since $E[T^2] < \infty$ and $E[S_{T_{2n}} - S_{T_{2(n-1)}}] = 0$, the functional central limit theorem (see, for instance, Billingsley (1968, Theorem 16.1)) gives

$$\frac{S_{T_{\lfloor 2nt \rfloor}}}{\widehat{\sigma}\sqrt{n}} \Rightarrow W(t),$$

where "$\Rightarrow$" denotes weak convergence. Hence, a change in the time scale by a constant factor yields

$$\frac{S_{T_{\lfloor nt \rfloor}}}{\frac{\widehat{\sigma}}{\sqrt{2}}\sqrt{n}} \Rightarrow W(t). \tag{4.2}$$

The functional central limit theorem applied to the sequence $T_1, T_2, \ldots$ implies that, for each $\epsilon > 0$, there exists a $\lambda(\epsilon)$ and $N(\epsilon)$ such that, for all $n > N(\epsilon)$,

$$P\left[\max_{i\leq n} |T_i - iE[T]| \geq \lambda(\epsilon)\sqrt{n}\right] \leq \epsilon. \tag{4.3}$$

Fix an $\epsilon > 0$ and $\alpha > 0$. Then, by (4.3),

(4.4)

$$\lim_{n\to\infty} P\left[\max_{i\leq n} \frac{|S_{iE[T]} - S_{T_i}|}{\frac{\widehat{\sigma}}{\sqrt{2}}\sqrt{n}} \geq \alpha\right] \leq \lim_{n\to\infty} P\left[\max_{\substack{i\leq n \\ |i-j|\leq\lambda(\epsilon)\sqrt{n}}} \frac{|S_{T_j} - S_{T_i}|}{\frac{\widehat{\sigma}}{\sqrt{2}}\sqrt{n}} \geq \alpha\right] + \epsilon$$

$$\leq \lim_{\delta\downarrow 0} P\left[\sup_{|t-s|<\delta} |W(t) - W(s)| \geq \alpha\right] + \epsilon$$

$$= \epsilon.$$

The last inequality follows from the functional central limit theorem applied to

$$\left\{\frac{S_{T_{\lfloor nt \rfloor}}}{\frac{\widehat{\sigma}}{\sqrt{2}}\sqrt{n}}\right\}_{n\geq 1}$$

and the last equality by the properties of Brownian motion. Therefore, since ϵ and α were arbitrary, (4.2) and (4.4) imply

$$\frac{S_{\lfloor E[T]nt \rfloor}}{\frac{\widehat{\sigma}}{\sqrt{2}}\sqrt{n}} \Rightarrow W(t).$$

And so, a change in the time scale by a constant factor gives

$$\frac{S_{\lfloor nt \rfloor}}{\frac{\widehat{\sigma}}{\sqrt{2E[T]}}\sqrt{n}} = \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} \Rightarrow W(t). \quad \square$$

In Example 4.2, we apply Proposition 4.1 to the case that the reinforcement only looks back one step.

Example 4.2. Let $0 < p < 1$ and suppose that $g(k) = p$ for all $k \geq 1$. In this case, T is a geometric random variable with parameter $(1 - p) = q$. Therefore, by Proposition 4.1, $W(t)$ is a version of Brownian motion and the diffusion rate is

$$\sigma^2 = \frac{p}{q}.$$

Thus, increasing the magnitude of the reinforcement (i.e., increasing p) increases the rate of diffusion. □

What about other reinforcement sequences? In the case that $g(1)g(2)\cdots > 0$, if the norming factor $\sqrt{n}\sigma$ in (4.1) is replaced by n, then $W(t)$ is a random translation process which moves to the right at unit rate with probability $\frac{1}{2}$ and moves to the left at unit rate with probability $\frac{1}{2}$. On the other hand, it might seem reasonable to conjecture that $W(t)$ is a version of Brownian motion whenever the probability of a change in direction is 1 (i.e., $g(1)g(2)\cdots = 0$). This, however is not necessarily so — as Example 4.3 shows. Basically, what happens is that T can be constructed so that the limiting process would need to give positive probability to paths which are linear over a common nontrivial time interval. It would be interesting to completely characterize the types of limiting processes which could arise. For instance, is the Cauchy process a potential limit (or, perhaps, a Cauchy process with the jumps replaced by straight lines)? This is discussed further in the following section.

Example 4.3. Let F be the distribution function for the number of steps taken between successive changes in direction of the discrete–time walk S_n. Let $T_1, T_2, \ldots$ be an *i.i.d.* sequence with common distribution F. Set

$$C_n = \sum_{i=1}^{n} T_i,$$

$T_0 \equiv 0$. Now redefine the processes $W_n(t)$ somewhat more generally by

$$W_n(t) = \frac{\sum_{i=1}^{\lfloor nt \rfloor} X_i}{\sigma_n},$$

where $\{\sigma_n\}_{n\geq 1}$ is a given norming sequence. For $M \geq 1$,

$$\begin{aligned} P[\exists\ n \leq M \text{ such that } X_n = X_{n+1} = \cdots = X_{n+M}] \\ &= P[\exists\ k \text{ such that } C_k \leq M,\ T_{k+1} > M] \\ &= \sum_{k=0}^{\infty} P[C_k \leq M,\ T_{k+1} > M] \\ &= (1 - F(M)) \sum_{k=0}^{\infty} P[C_k \leq M] \end{aligned}$$

Setting $A(M) = (1 - F(M)) \sum_{k=0}^{\infty} P[C_k \leq M]$, suppose

$$\lim_{M\to\infty} A(M) > 0. \tag{4.5}$$

(A(M) and its limiting value occur frequently in renewal theory. For instance, Feller (1966, Chapter XIV) gives a class of distributions for which (4.5) is satisfied.) Then there exists a constant $K > 0$ and either infinitely many M such that $P[B_M^1] \geq K$ or infinitely many M such that $P[B_M^2] \geq K$, where

$$B_M^i = \left[\exists \lfloor \frac{(i-1)M}{4} \rfloor \leq n \leq \lfloor \frac{iM}{4} \rfloor \text{ such that } X_n = X_{n+1} = \ldots X_{n+\lfloor \frac{M}{2} \rfloor}\right].$$

Without loss of generality assume the former. Hence, for infinitely many M,

$$P[X_{\lfloor \frac{M}{4} \rfloor} = \ldots X_{\lfloor \frac{M}{2} \rfloor}] \geq K.$$

Now, if σ_n is $o(n)$, then, for every $r > 0$,

$$\limsup_{n\to\infty} P[|W_n(\frac{1}{2}) - W_n(\frac{1}{4})| \geq r] \geq K.$$

On the other hand, if σ_n is not $o(n)$, then, for some $r > 0$,

$$\limsup_{n\to\infty} P[|W_n(1)| \leq r] = 1.$$

Either case would contradict W_n converging to Brownian motion. □

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5. Remarks and Further Questions. Here we discuss an application of our work and collect further questions.

As mentioned in the introduction, observation has indicated that wave fields exhibit time and space correlations. A motivation for our work was to analyze an elementary model of these correlations — specifically, directional reinforcement — in order to understand their relationship to other properties of wave fields. For instance, we determined conditions on the reinforcement parameters under which the fields were recurrent. Also, recall Example 4.2, where the continuous time directionally reinforced process was a diffusion process. The greater the directional reinforcement in the example the larger the diffusion coefficient. The diffussion coefficient can be associated with sea state. Thus, the example can be interpreted to indicate that strong reinforcement corresponds to high sea state.

Currently there is a great deal of analysis on the strength required of artificial ocean structures (e.g., oil drilling platforms) to withstand surface wave forces.

In particular, to obtain failure probabilities for different platform designs, structural response models incorporate stochastic surface wave field models. Typical approaches (see, for instance, Longuet-Higgins (1952)) use uncorrelated Gaussian processes to represent these wave fields. Standard techniques are applied to obtain first passage times and other extremal statistics. These results are in turn used by Moe and Crandall (1977), among others, to predict the probability of a platform failure due to surface wave action. Our work represents an initial step in understanding the modeling implications of accounting for wave field correlations.

Some further questions which would be interesting to pursue include the following.

• What are necessary and sufficient conditions for the limiting continuous time process, $W(t)$, to be a version of Brownian motion?

• Under what conditions does the limiting process have stationary independent increments?

• If T has a stable distribution of index $\alpha \in (\frac{1}{2}, 1)$ and $\sigma_n = o(n)$, is the limiting process the stable process of index α or is it some slowed down version of that process.

• Suppose that the directionally reinforced random walk does not necessarily take unit steps. Under what conditions is S_n recurrent? Do the limiting processes still necessarily have continuous sample paths, or are jumps observed?

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