Ergod. Th. & Dynam. Sys. (1992), **12**, 441–458
Printed in Great Britain

Random circle homeomorphisms

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(*Received* 13 *November* 1990 *and revised* 15 *August* 1991)

Abstract. We investigate the behaviour of random homeomorphisms of the circle induced by composing a random homeomorphism of the interval with a randomly chosen rotation. These maps and their iterates are a.s. singular and for each rational number r in $[0, 1)$ it is shown that there is a positive probability of obtaining a map with rotation number r. For a 'canonical' method of producing these maps, bounds on the probability of obtaining a fixed point are obtained. We estimate this probability via computer simulations in three different ways. Simulations are also carried out for two periods. It remains unknown for this method whether a rational rotation number is obtained a.s.

1. *Introduction*

In this paper, we investigate the behaviour of random homeomorphisms of the unit circle. The idea is to produce a homeomorphism of the unit interval lift it to a homeomorphism of the circle and compose it with a rotation of the circle also chosen at random. Such a homeomorphism, F, of the circle may be expressed as $F(e^{2\pi it}) = e^{2\pi ig(t)}$, where g is a lift of F. We will take g to be a homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ satisfying $g(t+1) = g(t)+1$, for all t and $g(t) = f(t)+c$, for $0 \le t \le 1$, where f is a strictly increasing homeomorphism of $[0, 1]$ chosen with respect to some probability measure on the space of autohomeomorphisms of the unit interval and c is chosen independently and uniformly with respect to Lebesgue measure. By and large the behaviour of the circle homeomorphism is governed by the homeomorphism of the unit interval employed. Thus, most of this paper is devoted to properties of the interval homeomorphism which reflect to the circle map. For example, in Theorem 10, from the a.s. singularity of the interval homeomorphism, we derive the a.s. singularity of the iterates of the circle map F. Our main example of this procedure is the 'canonical' method, described in § 4, of producing a homeomorphism of $[0, 1]$. This method was first given by Dubins and Freedman for randomly generating distributions on $[0, 1]$ [**2**]. It was studied from the viewpoint of a random

† Supported by National Science Foundation grant DMS 9007035.

homeomorphism of $[0,1]$ by Graf *et al.* [**3**]. We will rely on several results from these works. We present some partial results concerning the still open problem of whether these circle homeomorphisms have, with probability one, a periodic point, or, equivalently, whether with probability one these homeomorphisms have rational rotation number. At least, we show that for each rational number in $[0,1]$ there is a positive probability of obtaining a map with this rotation number. We also give some bounds on the probability of obtaining a fixed point and some computer studies of the probability of obtaining a fixed point or a point of period two. There are some interesting issues here concerning these simulations which we discuss in § 9. Some related numerical problems have been discussed in [**4**] and [**5**].

2. *Random homeomorphisms or distributions by scaling*

Let $\{h_d : d \in D\}$ be a family of independent Borel measurable random variables indexed by dyadic rationals from $(0,1)$. The distribution of h_d will be denoted by τ_d and the system of Borel measures $\tau = \{\tau_d : d \in D\}$ will be called a dyadic transition kernel. The random non-decreasing function $Y_\omega : D \to [0,1]$ is constructed inductively by scaling:

$$Y_\omega(0)=0, \quad Y_\omega(1)=1;$$

If $Y_\omega(d)$ is already defined for all d of the form $d=i/2^{n-1}$ $(0 \le i \le 2^{n-1})$ then, for every odd i, $1 \le i \le 2^n - 1$, we let

$$Y_\omega(i/2^n) = (\alpha-\beta)h_{i/2^n}(\omega)+\beta,$$

where $\alpha = Y_\omega((i+1)/2^n)$, $\beta = Y_\omega((i-1)/2^n)$, which have already been defined in the previous steps. Clearly, each Y_ω is strictly increasing on D provided for each d, $\tau_d(\{0,1\})=0$.

In the sequel we will assume that almost every Y_ω can be extended to a homeomorphism of the unit interval. A condition sufficient for that is given in [**1**, cor. 5.1], and we will quote it here in the current notation without a proof. This condition is similar to the 'uniformly centred' condition given in [**3**, p. 256].

THEOREM 1. *If for every $\varepsilon > 0$ there exists a $\delta \in (0, 1/2)$, such that $\tau_d((\delta, 1-\delta)) > 1-\varepsilon$ for every $d \in D$, then almost every Y_ω can be extended to a homeomorphism $Y_\omega : [0,1] \to [0,1]$.*

The distribution on the space of all homeomorphisms of the unit interval generated by the above scheme is determined by the transition kernel τ and it will be denoted by Q_τ. It is not hard to see that Q_τ is a Borel measure when the space of homeomorphisms is endowed with uniform topology. Of course, this space of autohomeomorphisms of $[0,1]$ is precisely the space of probability measures on $[0,1]$ which are topologically equivalent to Lebesgue measure.

3. *Marginal distributions of Q_τ*

The techniques used in this section are similar to those used in [**3**] and they are based on the observation that for a given value z of $Y_\omega(1/2)$, the conditional distribution of $Y_\omega(x)/z$ with $x < 1/2$ is the same as that of $\bar{Y}_\omega(2x)$ in the scheme with the new kernel $\bar{\tau}_d = \tau_{d/2}$, and hence it is independent of z. For $x > 1/2$ we then apply the reversed scheme $\tilde{Y}_\omega(x) = 1 - Y_\omega(1-x)$.

THEOREM 2. *If $\tau_{1/2}$ is non-atomic, so is the marginal distribution of $Y_\omega(x)$ for every $x \in (0,1)$.*

Proof. Note that $Y_\omega(1/2)$ has the distribution $\tau_{1/2}$. If $0 < x < 1/2$ we have

$$P(Y_\omega(x) = y) = \int_y^1 P(Y_\omega(x) = y) \mid Y_\omega(1/2) = z)\, d\tau_{1/2}(z)$$

$$= \int_y^1 P([Y_\omega(x)/Y_\omega(1/2)] = y/z)\, d\tau_{1/2}(z),$$

since, as previously observed $Y_\omega(x)/Y_\omega(1/2)$ is independent of $Y_\omega(1/2)$. The integrand equals zero, except at at most countably many points, so if $\tau_{1/2}$ is non-atomic, so is the distribution of $Y_\omega(x)$. The proof for $1/2 < x < 1$ now follows by applying the above to the reversed scheme $\tilde{Y}_\omega$. □

In Theorem 6.6 of [3] it was shown that for the canonical scheme, the marginals, F_d, are absolutely continuous provided d is a dyadic rational. The next theorem extends this result to more general schemes and for all values of d.

THEOREM 3. *If $\tau_{1/2}$, $\tau_{1/4}$ and $\tau_{3/4}$ are non-atomic and the distribution function, $F_{1/2}(y)$, has a continuous and bounded derivative on $(0,1)$, then for each $x \in (0,1)$, F_x, the distribution of $Y_\omega(x)$ has a continuous derivative on the open interval $(0,1)$ and therefore, is absolutely continuous with respect to Lebesgue measure.*

Proof. As in the proof of Theorem 2, for $0 < x < 1/2$, we have

$$F_x(y) = P(Y_\omega(x) \le y) = F_{1/2}(y) + \int_y^1 P([Y_\omega(x)/Y_\omega(1/2)] \le y/z)\, d\tau_{1/2}(z)$$

$$= F_{1/2}(y) + \int_y^1 \bar{F}_{2x}(y/z) f_{1/2}(z)\, dz,$$

where $f_{1/2}$ is the density of $\tau_{1/2}$, and $\bar{F}_{2x}$ denotes the distribution function of $\bar{Y}_\omega(2x)$ in the scheme with the kernel $\bar{\tau}_d = \tau_{d/2}$. Note that by Theorem 2 and our assumptions, $\bar{F}_{2x}$ is continuous. Thus, for $0 < y < y + h \le 1$,

$$\begin{aligned}
[F_x(y+h) - F_x(y)]/h &= [F_{1/2}(y+h) - F_{1/2}(y)]/h \\
&\quad + h^{-1} \int_{y+h}^1 \bar{F}_{2x}((y+h)/z) f_{1/2}(z)\, dz \\
&\quad - h^{-1} \int_y^1 \bar{F}_{2x}(y/z) f_{1/2}(z)\, dz. \\
&= [F_{1/2}(y+h) - F_{1/2}(y)]/h \\
&\quad + h^{-1} \int_{y+h}^1 [\bar{F}_{2x}((y+h)/z) - \bar{F}_{2x}(y/z)] f_{1/2}(z)\, dz \\
&\quad - h^{-1} \int_y^{y+h} \bar{F}_{2x}(y/z) f_{1/2}(z)\, dz. \\
&= A(h) + B(h) - C(h).
\end{aligned}$$

If we let $h \to 0$, then both $A(h)$ and $C(h)$ converge to $f_{1/2}(y)$, and they cancel out. The term $B(h)$ can be transformed to

$$B(h) = h^{-1} \int_{y+h}^{1} \int_{\bar{F}_{2x}(y/z)}^{\bar{F}_{2x}((y+h)/z)} f_{1/2}(z)\, dt\, dz$$
$$= h^{-1} \int_{\bar{F}_{2x}(y+h)}^{1} \int_{y/\bar{F}_{2x}^{-1}(t)}^{(y+h)/\bar{F}_{2x}^{-1}(t)} f_{1/2}(z)\, dz\, dt - \left(\frac{C_1}{h} - \frac{C_2}{h}\right) \max f_{1/2},$$

where $C_1 \leq h \cdot (1 - \bar{F}_{2x}(y/(y+h))$ and $C_2 \leq (h/(h+y))(\bar{F}_{2x}(y+h) - \bar{F}_{2x}(y))$ (see Figure 1).

Note. If $\bar{F}_{2x}(z)$ fails to be invertible, we assign $\bar{F}_{2x}^{-1}(t)$ to be the supremum of the inverse image set $\bar{F}_{2x}^{-1}(t)$. This can only happen for countably many t's.

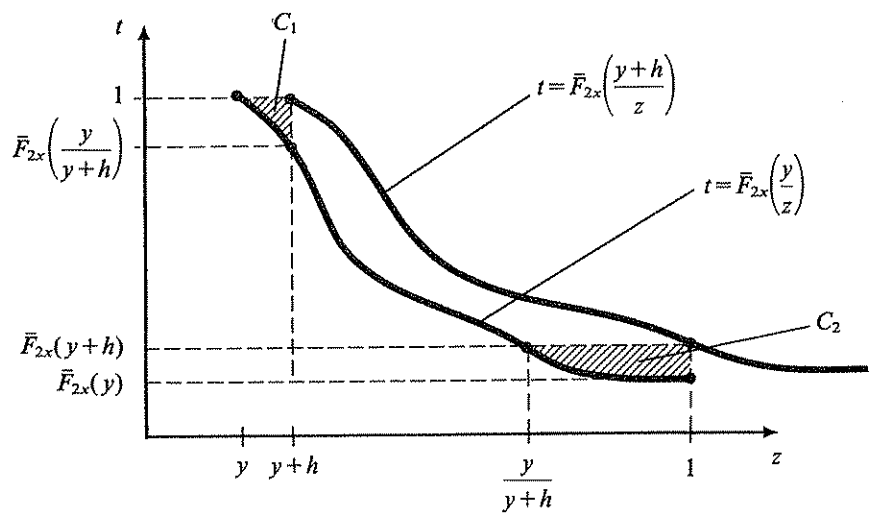

FIGURE 1

Now, both C_1/h and C_2/h go to zero as $h \to 0$, hence we only need to consider the double integral, which can be written as

$$\int_{\bar{F}_{2x}(y+h)}^{1} \frac{1}{\bar{F}_{2x}^{-1}(t)} \left[\frac{\bar{F}_{2x}^{-1}(t)}{h} \int_{y/\bar{F}_{2x}^{-1}(t)}^{y/\bar{F}_{2x}^{-1}(t)+h/\bar{F}_{2x}^{-1}(t)} f_{1/2}(z)\, dz\right] dt.$$

For every fixed t, the expression in square brackets converges to $f_{1/2}(y/\bar{F}_{2x}^{-1}(t))$ (fundamental theorem of calculus), which is bounded, so the convergence holds when integrated with respect to t. Thus, we finally obtain

$$f_x(y) = \int_{\bar{F}_{2x}(y)}^{1} \left[\frac{1}{\bar{F}_{2x}^{-1}(t)} f_{1/2}\left(\frac{y}{\bar{F}_{2x}^{-1}(t)}\right)\right] dt. \tag{1}$$

Clearly, f_x is continuous on $(0, 1]$. For $1 > x > 1/2$ we apply the reversed scheme $\tilde{Y}_\omega$ and find that f_x is continuous on $[0, 1)$. □

Remark 1. Some condition is needed for $F_x'(0)$ to exist. For example, if each $\tau_d = \lambda$, Lebesgue measure on $[0, 1]$, then $F_{1/4}(y) = y - y \log y$.

Remark 2. That some condition is needed on $\tau_{1/4}$ and $\tau_{3/4}$ to insure $F'_{1/4}$ is continuous, can be seen by considering the example where $\tau_{1/2}$ is Lebesgue measure on $[0, 1]$ and $\tau_{1/4}$ is pointmass at $1/2$. In this case, $F'_{1/4}$ is 2 on $(0, 1/2)$ and 0 on $(1/2, 1)$.

Remark 3. Using the same techniques as in the proof of Theorem 3, it can be shown that F_x is absolutely continuous with respect to Lebesgue measure assuming only that $\tau_{1/2}$ is non-atomic. However, in this case, F'_x may not exist at countably many points.

THEOREM 4. *Let $n \geq 1$. If, for each $1 \leq k \leq n+1$ and $1 \leq i \leq 2^n - 1$, the distribution function of $\tau_{i/2^k}$ is differentiable at least $(n-k+1)$ times on $(0, 1)$ with the last derivative continuous and bounded, then for each $x \in (0, 1)$, $F_x \in C^{(n)}((0, 1))$. (By convention, the 0th derivative of F is F.)*

Proof. As before, we first consider $0 < x < 1/2$, then for $1 > x > 1/2$ the reversed scheme will be employed. For $n = 1$, Theorem 3 applies. Suppose then that $n \geq 2$ and the hypotheses are satisfied. We start our calculations with the observation that, since $\tau_{1/8}$, $\tau_{1/4}$ and $\tau_{3/8}$ satisfy the assumptions given in Theorem 2 for $\tau_{1/4}, \tau_{1/2}$ and $\tau_{3/4}$, the existence of the densities $\bar{f}_{2x}(y)$ is guaranteed. Thus,

$$f_x(y) = \int_1^y \frac{1}{z} \bar{f}_{2x}\left(\frac{y}{z}\right) f_{1/2}(z)\, dz.$$

This can be obtained, e.g., from formula (1) by substituting $z = y/\bar{F}_{2x}^{-1}(t)$. Further,

$$f_x(y) = \int_1^y \left[\frac{-y}{z^2} \bar{f}_{2x}\left(\frac{y}{z}\right)\right]\left[\frac{z}{y} f_{1/2}(z)\right] dz.$$

Integrating by parts, we obtain

$$\begin{aligned} f_x(y) = f_{1/2}(y) - \frac{\bar{F}_{2x}(y) f_{1/2}(1)}{y} + y^{-1} \int_y^1 \bar{F}_{2x}\left(\frac{y}{z}\right) f_{1/2}(z)\, dz \\ + y^{-1} \int_y^1 \bar{F}_{2x}\left(\frac{y}{z}\right) f'_{1/2}(z) z\, dz. \end{aligned} \tag{2}$$

Notice that by the first formula in the proof of Theorem 3

$$y^{-1} \int_y^1 \bar{F}_{2x}\left(\frac{y}{z}\right) f_{1/2}(z)\, dz = y^{-1}[F_x(y) - F_{1/2}(y)].$$

Since $f'_{1/2}$ is bounded, we can set

$$f'_{1/2}(z) z = a f^*(z) - b,$$

for some density function f^* and constants a, b. Thus,

$$y^{-1} \int_y^1 \bar{F}_{2x}\left(\frac{y}{z}\right) f'_{1/2}(z) z\, dz = a y^{-1}[F^*_x(y) - F^*_{1/2}(y)] - b y^{-1}[F^{**}_x(y) - y],$$

where F^*_x and F^{**}_x are the marginal distribution functions in the schemes in which $\tau_{1/2}$ is replaced by the distribution with density f^* and by Lebesgue measure, respectively. Applying the preceding two integral formulas, formula (2) takes the form

$$\begin{aligned} f_x(y) = f_{1/2}(y) + y^{-1}[F_x(y) - F_{1/2}(y) - \bar{F}_{2x}(y) f_{1/2}(1) + a(F^*_x(y) - F^*_{1/2}(y)) \\ - b(F^{**}_x(y) - y)]. \end{aligned} \tag{3}$$

Suppose that the theorem has been proved for $n-1$. It can be seen, by the induction hypothesis, that each of the terms on the right-hand side of the formula (3) is in $C^{(n-1)}((0,1))$, as desired. □

In the sequel we shall need the joint continuity of both $F_x(y)$ and $f_x(y)$ as functions of two variables x, y. A sufficient condition for this follows.

THEOREM 5. *Let $\tau_{1/2}$ be non-atomic. Then $F_x(y)$ is jointly continuous on $(0,1)\times[0,1]$. If, in addition, the distribution function of $\tau_{1/2}$ has a bounded derivative and $\tau_{1/4}$ and $\tau_{3/4}$ are non-atomic, then also $f_x(y)$ is jointly continuous on $(0,1)\times(0,1)$.*

Proof. For $x_1, x_2 \in (0,1)$, $x_1 < x_2$, we have

$$F_{x_1}(y) - F_{x_2}(y) = P(\{\omega\colon Y_\omega(x_1) < y,\ Y_\omega(x_2) \geq y\}),$$

with the set on the right decreasing as either x_1 increases or x_2 decreases. Let x_1, x_2 converge monotonically to a common limit x. We obtain

$$F_{x-}(y) - F_{x+}(y) = P(\{\omega\colon Y_\omega(x-) \leq y,\ Y_\omega(x+) \geq y\}) = P(\{Y_\omega(x) = y\}).$$

By continuity of almost every Y_ω, the last equals zero. Thus, $F_x(y)$ is continuous for x, and also, by Theorem 2, for y. But $F_x(y)$ is monotonic in each direction, and so, it is jointly continuous. Now, if the other assumptions on $\tau_{1/4}$, $\tau_{1/2}$ and $\tau_{3/4}$ hold, we can use the formula (1) for $f_x(y)$, every term of which is seen to be jointly continuous. □

4. *Remarks on the canonical scheme*

Let us consider the canonical example, when $\tau_d = \lambda$, Lebesgue measure, for each $d \in D$ (see [3], the ditribution P_λ). First of all observe that the scheme is symmetric, i.e., the distribution of $\tilde{Y}_\omega$ is the same as that of Y_ω. Also, if the value z of $Y_\omega(1/2)$ is given, the conditional distribution of $Y_\omega(x)/z$ for $x<1/2$ is the same as the marginal distribution of $Y_\omega(2x)$, hence we can write $\bar{F}_{2x} = F_{2x}$. These properties have been discussed in [3] as 'time reversal invariance' and 'amalgamation invariance'. Furthermore, we have $f'_{1/2}(z) = 0$, so $a = b = 0$, and formula (3) reduces to

$$f_x(y) = y^{-1}[F_{1/2}(y) - F_{2x}(y)] \quad (x < 1/2,\ y \in (0,1)).$$

Also, by Theorem 4, we can see that F_x is in $C^\infty((0,1))$ for each $x \in (0,1)$.

Let $d \in D$. So, $d = \sum_{i=1}^{n+1} d_i/2^i$, with $d_{n+1} = 1$. We also set $d_0 = 0$ and we denote by W_d the 0–1 word for d: $W_d = (d_0, d_1, \ldots, d_n)$. Also, we let $\bar{d}_i = |d_i - d_{i-1}|$ for each $i = 1, 2, \ldots, n$, and we denote by $\bar{W}_d$ the associated 0–1 word $(\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_n)$.

THEOREM 6. *Let d be a dyadic rational in $(0,1)$ and $y_0 \in (0,1)$. Then*

$$f_d(y_0) = \int_{y_0^{(\bar{d}_1)}}^1 \frac{1}{y_1} \int_{y_1^{(\bar{d}_2)}}^1 \frac{1}{y_2} \cdots \int_{y_{n-1}^{(\bar{d}_n)}}^1 \frac{1}{y_n}\, dy_n \ldots dy_2\, dy_1, \tag{4}$$

where $(\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_n) = \bar{W}_d$ and $y_i^{(0)} = y_i$, $y_i^{(1)} = 1 - y_i$.

Proof. We see that for $n=0$ the right-hand side of (4) equals 1, so the formula holds for $d = 1/2$. Let d be given, $W_d = (d_0, \ldots, d_n)$ and suppose the theorem has already been proved for all dyadic rationals whose words are shorter than W_d. First, assume

$d<1/2$, so that $d_0=d_1=0$, and hence $\bar{d}_1=0$. By the amalgamation invariance, we have

$$f_d(y_0)=\int_{y_0}^{1} z^{-1}f_{2d}(y_0/z)\,dz=\int_{y_0}^{1} y_1^{-1}f_{2d}(y_1)\,dy_1.$$

The desired formula now follows by noticing that $W_{2d}=(d_1, d_2,\dots, d_n)$ and hence $\bar{W}_{2d}=(\bar{d}_2, \bar{d}_3,\dots,\bar{d}_n)$. Now suppose that $d>1/2$. Thus, $d_1=1$ and $\bar{d}_1=1$. By the time reversal invariance of the scheme, we have

$$f_d(y_0)=f_{(1-d)}(1-y_0).$$

But, $(1-d)<1/2$ and $W_{(1-d)}=(0, 1-d_1, 1-d_2,\dots,1-d_n)$. Thus, $W_{(1-d)}=(0, \bar{d}_2, \bar{d}_3,\dots,\bar{d}_n)$ and it differs from $\bar{W}_d$ only at the first position. Since the integral formula (4) has already been proved for $f_{(1-d)}(y_0)$, we now produce one for $f_d(y_0)$ by substituting y_0 with $1-y_0$, which corresponds precisely to the change of the index $\bar{d}_1$. □

COROLLARY. *By a straightforward integration we obtain*

$$f_{2^{-n}}(y)=(-\ln y)^{n-1}/(n-1)! \quad\text{and}\quad f_{1-2^{-n}}(y)=(-\ln(1-y))^{n-1}/(n-1)!.$$

For other dyadic rationals the explicit formula for $f_d(x)$ does not exist, since it involves integrands of the form $\ln(1-z)/z$, *e.g.*

$$f_{3/8}(y)=\int_y^1 -\ln(1-z)/z\,dz,$$

which is not a standard function.

Remark. Since, by Theorem 5, $f_x(y)$ is jointly continuous, formula (4) can be extended to all $x\in(0,1)$, if we agree to understand the corresponding infinite composition of integrals as the limit of the appropriate multiple integrals.

5. *Connections to Markov processes*

Consider another way of producing random homeomorphisms. Namely, let $\{Z_t\}_{t\geq 0}$ be a stochastic process with trajectories continuously increasing from 0 to ∞ almost surely. A random homeomorphism of the unit interval is obtained from a trajectory $Z_\omega(t)$ by switching to the exponential scale for both time and position: $Y_\omega(x)=\exp(-Z_\omega(-\log_2 x))$. If $\{Z_t\}$ is a Markov process (with independent increments), this property now turns to independent quotients for $Y(x)$, i.e., $Y_\omega(x_2)/Y_\omega(x_1)$ is independent of $Y_\omega(x_1)$, whenever $x_1>x_2$. Furthermore, if the Markov process is uniform, then $Y(x)$ also has uniform quotients, i.e. $Y_\omega(x_2)/Y_\omega(x_1)$ has the same distribution as $Y_\omega(x_2/x_1)$. For example, the classical Γ-process (with marginal densities $\gamma_t(z)=(\Gamma(t))^{-1}e^{-z}z^{t-1}$) is easily seen (by the substitution $x=2^{-t}$, $y=e^{-z}$) to correspond to the random homeomorphism with marginal densities

$$f_x(y)=[(-\ln y)^{-\log_2 x-1}]/\Gamma(-\log_2 x).$$

As we recognize, for $x=2^{-n}$ it is the same formula, as we have obtained for the canonical scheme generated by the Lebesgue kernel. Morover, the canonical scheme observed only at the points $x=2^{-n}$ does have independent and uniform quotients, so both examples generate the same joint distribution for $\{Y_{2^{-n}}: n=0,1,\dots\}$.

As a matter of fact, the entire distribution Q_τ generated by any dyadic transition kernel can be obtained by an inductive construction in every step of which we independently pick a trajectory of a certain discrete time Markov process (possibly different in each step), rescale it by first switching to exponential scales and then transforming linearly to make it 'fit' in between two already constructed dots on the graph of Y_ω. Such constructions with the same uniform Markov process used in each step correspond to uniform kernels. Random homeomorphisms generated by dyadic kernels usually do not have independent quotients at points other than 2^{-n}, thus they cannot be obtained from continuous time Markov processes, as described in the beginning of this section.

6. *The iterates of a random homeomorphism*

Throughout this section we will study the iterates Y_ω^n of the random homeomorphism Y_ω generated by a dyadic transition kernel τ.

THEOREM 7. *Let $n \geq 1$ and $x \in (0,1)$. If τ_d is non-atomic for each $d \in D$, then the distribution of the nth iterate $Y_\omega^n(x)$ is non-atomic.*

Proof. For $n=1$, Theorem 2 applies. Suppose the statement holds for each iterate of order less than some $n>1$. Consider the initial trajectory $T=\{x, Y_\omega(x), Y_\omega^2(x), \ldots, Y_\omega^{n-1}(x)\}$. By the induction hypothesis, with probability one, none of the elements of T equals x (except the first one) and hence T consists of n distinct elements, and $Y_\omega^{n-1}(x)$ is not a dyadic rational. Thus, if we denote

$$A_{i,k}=\{\omega\colon T\cap(i/2^k,(i+1)/2^k)=\{Y_\omega^{n-1}(x)\}\},$$

we have

$$P\left(\bigcup_{k\geq 1, 0\leq i<2^k} A_{i,k}\right)=1.$$

Now, it suffices to show non-atomicity of the distribution of $Y_\omega^n(x)$ conditioned with respect to each $A_{i,k}$. So, fix some k and i. It follows from the construction of Y_ω, that the two random functions

$$Y_\omega^{\text{in}}=(Y_\omega-\alpha)/(\beta-\alpha)$$

defined on $[i/2^k,(i+1)/2^k]$, where $\alpha=Y_\omega(i/2^k)$, $\beta=Y_\omega((i+1)/2^k)$ and

$$Y_\omega^{\text{out}}=Y_\omega \text{ restricted to } [0,i/2^k]\cup[(i+1)/2^k,1]$$

are independent. Notice that whether $\omega\in A_{i,k}$ can be verified by only looking at Y_ω^{out}, and because of this, Y_ω^{in} and Y_ω^{out} remain independent with respect to the conditional probability given $A_{i,k}$. Also, notice that then the value z of $Y_\omega^{n-1}(x)$, as well as that of α and β also depend only on Y_ω^{out}. Thus, for each $y\in[0,1]$,

$$P(Y_\omega^n(x)=y\,|\,A_{i,k})=\int P(Y_\omega^{\text{in}}(z)=(y-\alpha)/(\beta-\alpha))\,dP(z,\alpha,\beta\,|\,A_{i,k}).$$

Next, since each τ_d is non-atomic, so is τ_{d_0}, where d_0 is the midpoint of the interval $[i/2^k,(i+1)/2^k]$, hence, by Theorem 2, the integrand equals zero for every triple (z,α,β). □

COROLLARY. *With the assumptions of Theorem 7 satisfied, the point x is periodic for Y_ω with probability zero.*

The singularity of Y_ω has been studied in [3] for uniform kernels. In Theorem 5.20 there, a necessary and sufficient condition for $Y'_\omega(x)$ to equal zero almost surely for each $x \in [0, 1]$ is given. By the same argument as used in the proof Theorem 5.20, one can verify the following in the case of a general kernel τ.

LEMMA 1. *If there exists an $a, b > 0$ such that for every $d \in D$*

$$\text{(i)} \qquad \int \ln y \, d\tau_d(y) < -\ln 2 - a, \int \ln (1-y) \, d\tau_d(y) < -\ln 2 - a,$$

and

$$\text{(ii)} \qquad \int (\ln y)^4 \, d\tau_d(y) < b, \int (\ln (1-y))^4 \, d\tau_d(y) < b,$$

then, for every $x \in [0, 1]$, $Y'_\omega(x) = 0$ with probability one.

THEOREM 8. *If, for each $x \in [0, 1]$, $Y'_\omega(x) = 0$ with probability one, then for every $n \geq 1$ and $x \in [0, 1]$*

$$(Y^n_\omega)'(x) = 0 \text{ almost surely.}$$

Proof. Fix some $n \geq 1$ and $x \in [0, 1]$ and assume the statement has been verified for all iterates of order less than n. Let $B_m = \{Y_\omega : Y^m_\omega(x) = x\}$, for $1 \leq m < n$, and let B_n be the complement to the sum of the B_m's. We will prove the assertion relatively to each B_m and B_n, consecutively. For $m < n$ we have, by the chain rule

$$(Y^n_\omega)'(x) = (Y^{n-m}_\omega)'(x) \cdot (Y^m_\omega)'(x)$$

with both the terms on the right almost surely equal to zero. Within B_n we first partition it into the subsets $A_{i,k}$ and define Y^{in}_ω and Y^{out}_ω in the same manner as in the proof of Theorem 7. Now apply

$$(Y^n_\omega)'(x) = (Y^{n-1}_\omega)'(x) \cdot Y'_\omega(z),$$

where $z = Y^{n-1}_\omega(x)$. The first factor equals zero almost surely by the inductive assumption. Thus, we only need to check that $Y'_\omega(z)$ exists (as a finite number) with probability one. Again, within each $A_{i,k}$ the value of z depends only on Y^{out}_ω, while the existence of Y'_ω at a given z is determined by Y^{in}_ω, and it exists with probability one (in fact $Y'_\omega(z)$ equals zero almost surely). The application of the independence of Y^{in}_ω and Y^{out}_ω completes the proof. □

Consider an arbitrary Borel probability measure μ on $[0, 1]$. The subset $\{(x, Y_\omega) : Y'_\omega(x) = 0\}$ of the appropriate product space is (by the nature of its defining condition) Borel measurable. Thus, by applying Fubini's theorem, we easily obtain

COROLLARY. *With the assumptions of Theorem 8 fulfilled every iterate Y^n_ω of almost every random homeomorphism Y_ω is singular with respect to a given Borel measure μ on $[0, 1]$.*

7. *Random rotation numbers*

A random homeomorphism of the circle, Z_ω, is obtained from a random homeomorphism of the interval, Y_ω, by identifying the endpoints of the interval and then by shifting Y_ω about a randomly chosen number $c_\omega \in [0, 1)$:

$$Z_\omega(x) = (Y_\omega(x) + c_\omega) \bmod 1.$$

We will assume that Y_ω is chosen with accordance to the distribution Q_τ generated by a dyadic transition kernel τ, as described in § 2, and c_ω is selected independently of Y_ω, with some distribution μ on $[0, 1)$.

It can easily be verified that all the theorems of § 6 hold, by the same proofs, if Y_ω is replaced by Z_ω, for some fixed c. Thus, by integrating with respect to μ, we obtain the following statements.

THEOREM 9. *Let $n \geq 1$, $x \in (0, 1)$. If τ_d is non-atomic for each $d \in D$, then for an arbitrary distribution μ of c_ω, the* n*th iterate, $Z^n_\omega(x)$, has a non-atomic distribution. If $n > 1$ and μ does not have an atom at zero, the same also holds for $x = 0$.*

COROLLARY. *With the assumptions of Theorem 9, for each x, x is almost surely not a periodic point for Z_ω.*

THEOREM 10. *If, for each $x \in [0, 1]$, $Y'_\omega(x) = 0$ with probability one, then for every $n \geq 1$ and $x \in [0, 1)$*

$$(Z^n_\omega)'(x) = 0 \text{ almost surely.}$$

The rotation number $\rho(Z_\omega)$ will be considered (see [6] for the definition of the rotation number). In fact, we will view the rotation number modulo one, so by saying, for example, that $\rho(Z)$ is continuous and non-decreasing we admit the jump from values close to one down to zero. We will need the following properties of the rotation number:

(a) $\rho(Z)$ is rational ($\rho = m/n$) if and only if Z has a periodic point with period n.

(b) The map $Z \to \rho(Z)$ is continuous with respect to the uniform topology.

(c) For a fixed homeomorphism Z, the map $\rho_Z(c) = \rho(Z + c)$ is a continuous, non-decreasing transformation of $[0, 1)$ onto $[0, 1)$.

THEOREM 11. *Suppose that μ, the distribution of c_ω, is non-atomic and with full support on $[0, 1)$, and that for each d from an infinite set $D' \subset D$ of the dyadic rationals, τ_d is not a point mass, then the distribution of $\rho(Z_\omega)$ has an atom at each rational and it has no irrational atoms.*

For the proof of this theorem we will need several simple observations. Recall that the random homeomorphism of the interval, Y_ω, was obtained by scaling from a family of independent variables h_d. Every realization of Y_ω corresponds to a realization $h_d(\omega)$ of these variables. It can be seen that $Y_\omega \in \operatorname{supp} Q_\tau$ (the topological support) if and only if $h_d(\omega) \in \operatorname{supp} \tau_d$ for each $d \in D$. We can now prove the following.

LEMMA 2. *For each rational m/n, we can select a homeomorphism Y_0 of the interval and $c_0 \in [0, 1)$ such that $Y_0 \in \operatorname{supp} Q_\tau$, $\rho(Y_0+c_0)=m/n$ and $(Y_0+c_0)^n$ is not the identity.*

Proof. Choose an arbitrary $Y_\omega \in \operatorname{supp} Q_\tau$. By (c) we can find $c_0 \in [0, 1)$, so that $\rho(Y_\omega+c_0)=m/n$. If it happens that $(Y_\omega+c_0)^n$ is the identity, we have to change Y_ω slightly. Let x_0 be a cluster point of D'. We choose an x not on the (finite) orbit of x_0, and find an interval $(i/2^k, (i+1)/2^k)$ disjoint from the orbit of x and containing some $d \in D'$. Now, from the scaling scheme it is seen that altering the value of h_d from $h_d(\omega)$ to $h_d(\omega)+\varepsilon$ changes Y_ω to Y_ε only on the interval $(i/2^k, (i+1)/2^k)$, and hence it does not affect the trajectory of x. So, $(Y_\varepsilon+c)^n x=x$ and, by (a) $\rho(Y_\varepsilon+c)=m_\varepsilon/n$. But, by (b), $\rho(Y_\varepsilon+c)$ is a continuous function of ε and since it only assumes rational values, it must be constant. Thus, $\rho(Y_\varepsilon+c)=\rho$ for each ε. Finally, we can choose $\varepsilon \neq 0$, so that $h_d(\omega)+\varepsilon$ still belongs to $\operatorname{supp} \tau_d$. If, for example $\varepsilon>0$, then $(Y_\varepsilon+c)^n d>(Y_\omega+c)^n d=d$, hence $(Y_\varepsilon+c)^n$ is not the identity. □

Proof of Theorem 11. Let $\rho=m/n$, and let Y_0+c_0 be as in Lemma 2. We have $(Y_0+c_0)^n x_0=x_0$ and $(Y_0+c_0)^n x_1 \neq x_1$ for some x_0, $x_1 \in [0, 1)$. For example, let $(Y_0+c_0)^n x_1<x_1$. Since both $(Y_0+c_0+\varepsilon)^n x_0$ and $(Y_0+c_0+\varepsilon)^n x_1$ are continuous increasing functions of ε, for sufficiently small ε_0 we have

$$(Y_0+c_0+\varepsilon_0)^n x_0>x_0, \quad (Y_0+c_0+\varepsilon_0)^n x_1<x_1.$$

Thus, $(Y_0+c_0+\varepsilon_0)^n$ has a fix point and hence, by continuity, $\rho(Y_0+c_0+\varepsilon_0)$ is rational. But the same holds for every $0 \leq \varepsilon \leq \varepsilon_0$, hence, by continuity, $\rho(Y_0+c_0+\varepsilon_0)=\rho$. Now for some open neighbourhood U of Y_0 and an open interval V around $c_0+\varepsilon_0$ we have

$$(Y+c)^n x_0>x_0, \quad (Y+c)^n x_1<x_1$$

for each $Y \in U$, $c \in V$. Again by continuity, we conclude that $\rho(Y+c)=\rho$ for every such Y and c. Since $Y_0 \in \operatorname{supp} Q_\tau$ and μ has full support, we conclude $(Q_\tau \times \mu) \times (U \times V)>0$, hence $\rho(Z_\omega)$ has an atom at ρ. Now let ρ be irrational. For Y fixed let $c \in \rho_Y^{-1}(\rho)$ (i.e. $\rho(Y+c)=\rho$). Let x be a recurrent point for $Y+c$, i.e. $(Y+c)^{n_k}(x) \to x$ for some sequence n_k. By choosing a subsequence, we can make this convergence monotonic from one side. So, suppose for example that $(Y+c)^{n_k} x \uparrow x$. Then also $(Y+c)^{n_k-1} x \uparrow (Y+c)^{-1} x$. If we increase the value of c, then also the values of $(Y+c)^{n_k-1} x$ will all increase, while that of $(Y+c)^{-1} x$ will decrease. Thus, by the continuity of the above as functions of c, we can find a sequence $c_k \downarrow c$ such that

$$(Y+c_k)^{n_k-1} x=(Y+c_k)^{-1} x$$

which implies that each $\rho(Y+c_k)$ is rational. Since $\rho_Y(c)$ is non-decreasing, $\rho_Y^{-1}(\rho)$ is either a single point or an interval. We have just eliminated the interval, since the above would fail for its interior point c. Finally, since μ is non-atomic, we conclude that

$$P(\rho(Y_\omega+c_\omega)=\rho \mid Y_\omega=Y)=0.$$

Integrating with respect to Q_τ we complete the proof. □

8. *Estimation of the probability of a fixed point*

Throughout this section, we consider the canonical scheme. Let r be the probability that a circle homeomorphism has a fixed point: $r = \text{Prob}$ (Z_ω has a fixed point).

THEOREM 12. *The probability r of a fixed point for the canonical scheme is given by*

$$r = E[\max\{Y_\omega(x) - x : x \in [0, 1]\} + \max\{x - Y_\omega(x) : x \in [0, 1]\}].$$

Moreover, if such a circle homeomorphism has a fixed point, then it has a finite even number of fixed points which alternate between attracting and repelling, almost surely.

Proof. Temporarily fix Y_ω. For each c in $[0, 1]$, consider the circle homeomorphism Z_c, which has lift f_c determined by $f_c(x) = Y_\omega(x) + c$, for $x \in [0, 1)$ and $f_c(x+1) = f_c(x) + 1$, for all x. Z_c has a fixed point if and only if there is some number x in $[0, 1]$ such that $f_c(x) = x$ or $f_c(x) = x + 1$. Thus, Z_c has a fixed point if and only if $\exists x[c = x - Y_\omega(x)$ or $c = x + 1 - Y_\omega(x)]$. Or, Z_c has a fixed point if and only if $c \in [0, \max(x - Y_\omega(x)] \cup [1 - \max(Y_\omega(x) - x), 1]$. Note that for a.e. ω, these two intervals are disjoint. Thus, for each ω, the probability that Z_c has a fixed point is $\max\{Y_\omega(x) - x : x \in [0, 1]\} + \max\{x - Y_\omega(x) : x \in [0, 1]\}$. The probability r of a fixed point is now obtained by integration.

Finally, to see that it has a finite even number of fixed points which alternate between attracting and repelling, recall that Dubins and Freedman proved that $Y_\omega(x) + c$ is strictly singular, almost surely. Thus, $Y_\omega + c$ does not have a finite positive derivative anywhere. The remainder of the proof is essentially a repetition of Theorem 6.11 of [3]. □

To estimate r, note that

$$r \leq E[\max\{Y_\omega(x) - x : x \in [0, 1]\}] + E[\max\{x - Y_\omega(x) : x \in [0, 1]\}].$$

But, since the canonical scheme is invariant under 'time reversal' (see [3, Theorem 4.30]), $E[\max\{x - Y_\omega(x) : x \in [0, 1]\}] = E[-\min\{Y_\omega(x) - x : x \in [0, 1]\}] = E[-\min\{1 - Y_\omega(1-x) - x : x \in [0, 1]\}] = E[\max\{Y_\omega(1-x) - (1-x) : x \in [0, 1]\}]$. Set $p(\omega) = p(Y(\omega)) = \max\{Y_\omega(x) - x : x \in [0, 1]\}$. Thus, $r \leq 2\bar{p}$, where $\bar{p} = E[p(\omega)]$.

THEOREM 13. $0.3527\ldots < r < 0.732\ldots = \sqrt{3} - 1$.

First, let us obtain the upper bound. By the amalgamation formula (see [3, § 3]).

$$(*) \qquad \bar{p} = \int \max(Y_\omega(x) - x)\, dP(\omega) = \int\int\int_0^1 p_y(\omega_1, \omega_2)\, dy\, dP(\omega_1)\, dP(\omega_2),$$

where $p_y(\omega_1, w_2) = p([Y(\omega_1), Y(\omega_2)]_y)$ and $[Y(\omega_1), Y(\omega_2)]_y$ is the amalgamation of $Y(\omega_1)$ and $Y(\omega_2)$ at $(1/2, y)$ (see [3, p. 272]). In general, $[f, g]_y$ is obtained by linearly scaling the homeomorphism f to a homeomorphism of the interval $[0, 1/2]$ to $[0, y]$ and scaling g to a homeomorphism of $[1/2, 1]$ onto $[y, 1]$. Formally,

$$[f, g]_y(t) = \begin{cases} yf(2t), & \text{if } 0 \leq t \leq 1/2 \\ y + (1-y)g(2t-1), & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Figure 2 illustrates our considerations in case $1/2 \leq y \leq 1$. In this figure, $y - (1 - p_1)/2$ is the y-intercept of the line with slope 1 passing through B and $y + p_2(1-y) - 1/2$ is the y-intercept of the line with slope 1 passing through A. (The

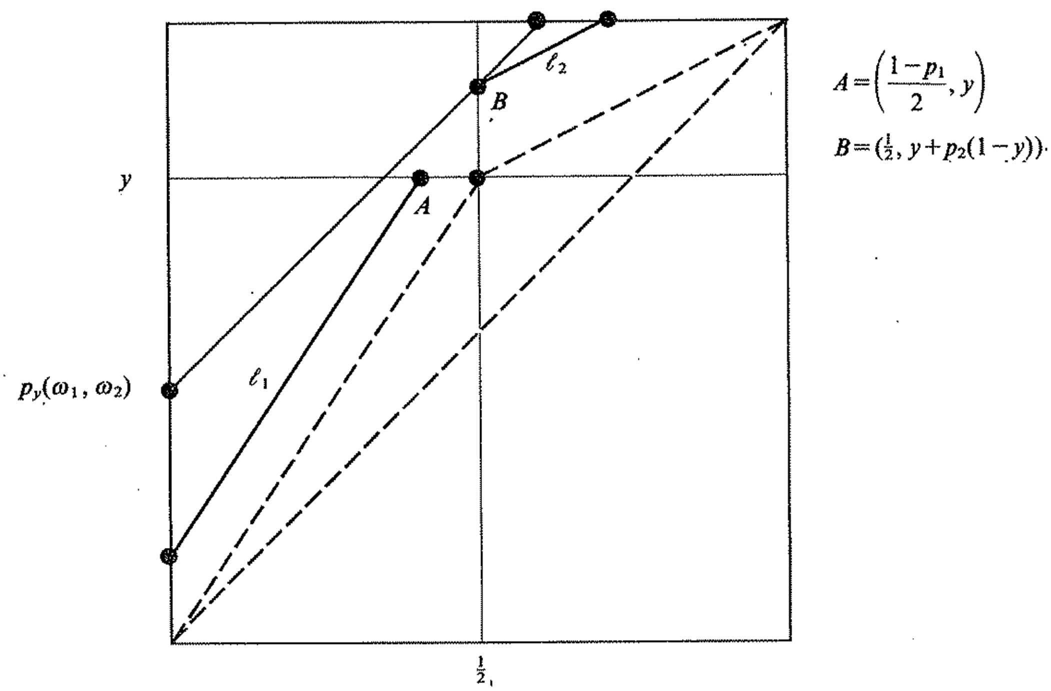

FIGURE 2

point A is determined as follows. Find the line with slope 1 which passes through $(x_1, Y_{\omega_1}(x_1))$, where $Y_{\omega_1}(x_1)-x_1$ is maximized and then scale this line into the box $[0, 1/2]\times[0, y]$ using the same linear transformation as is used to scale Y_{ω_1} into this box by amalgamation. This line is denoted by ℓ_1 in Figure 2 and is parallel to the line through $(0,0)$ and $(1/2, y)$. The point A is the intersection of ℓ_1 with the horizontal line of height y. The point B is found in an analogous fashion.) Clearly, $p_y(\omega_1, \omega_2)$ is dominated by the maximum of these two intercepts. Thus,

$$p_y(\omega_1, \omega_2)\le \max\left(y-\frac{1-p_1}{2}, y+p_2(1-y)-\tfrac{1}{2}\right)=y-\tfrac{1}{2}+\max\left(\frac{p_1}{2}, p_2(1-y)\right),$$

where $p_1=p(Y(\omega_1))$ and $p_2=p(Y(\omega_1))$.

Similarly, if $0\le y\le 1/2$ (see Figure 3)

$$p_y(\omega_1, \omega_2)\le \max\left(p_1 y, \frac{p_2}{2}\right).$$

Thus,

$$\begin{aligned}\int_0^1 p_y(\omega_1, \omega_2)\,dy &\le \tfrac{1}{8}+\int_{1/2}^1 \max\left(\frac{p_1}{2}, p_2(1-y)\right)dy+\int_0^{1/2}\max\left(p_1 y, \frac{p_2}{2}\right)dy\\
&\le \tfrac{1}{8}+\int_0^{1/2}\max\left(p_2 y, \frac{p_1}{2}\right)dy+\int_0^{1/2}\max\left(p_1 y, \frac{p_2}{2}\right)dy,\\
&= \tfrac{1}{8}+\int_0^{(p_1/2p_2)\wedge(1/2)}\frac{p_1}{2}\,dy+\int_{(p_1/2p_2)\wedge(1/2)}^{1/2} p_2 y\,dy\\
&\quad+\int_0^{(p_2/2p_1)\wedge(1/2)}\left(\frac{p_2}{2}\right)dy+\int_{(p_2/2p_1)\wedge(1/2)}^{1/2} p_1 y\,dy.\end{aligned}$$

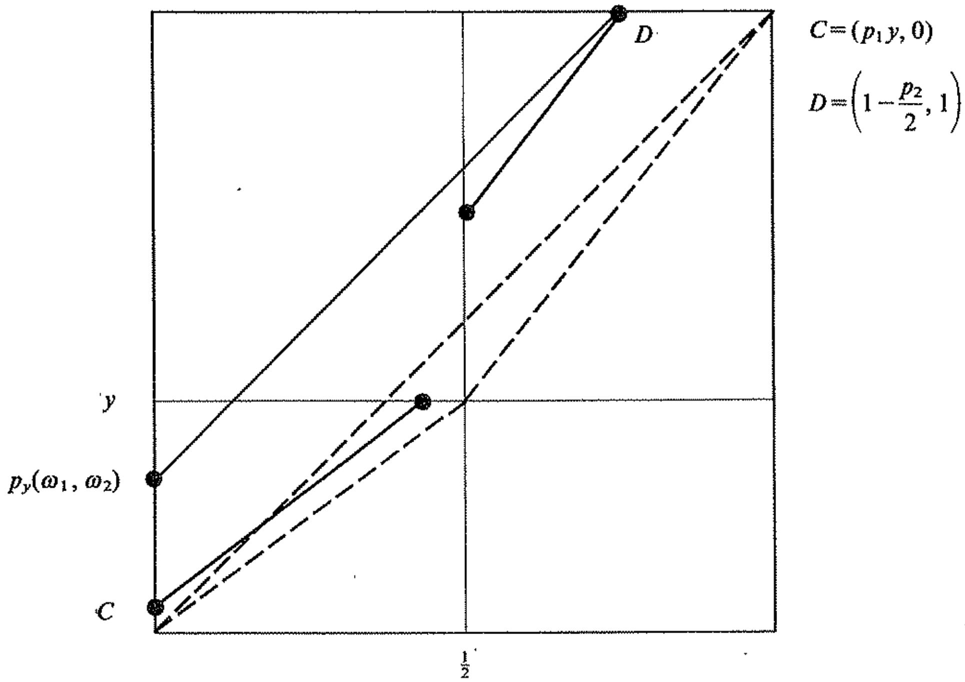

FIGURE 3

So, if $p_1 \le p_2$, it follows that

$$\int_0^1 p_y(\omega_1, \omega_2)\, dy \le \tfrac{1}{8}+\tfrac{3}{8}p_2+\tfrac{1}{8}\frac{p_1^2}{p_2} \le \tfrac{1}{8}+\tfrac{3}{8}p_2+\tfrac{1}{8}p_1.$$

Similarly, if $p_2 \le p_1$,

$$\int_0^1 p_y(\omega_1, \omega_2)\, dy \le \tfrac{1}{8}+\tfrac{3}{8}p_1+\tfrac{1}{8}p_2.$$

In either case, since p_1 and p_2 are less than one,

$$\int_0^1 p_y(\omega_1, \omega_2)\, dy \le \tfrac{1}{8}(1+p_1+p_2)+\tfrac{2}{8}\max(p_1, p_2) \le \tfrac{1}{8}(1+p_1+p_2)+\tfrac{2}{8}(p_1+p_2-p_1p_2)$$

$$\le \tfrac{1}{8}+\tfrac{3}{8}(p_1+p_2)-(p_1p_2)/4.$$

Now, integrating with respect to p_1 and p_2 and using the fact p_1 and p_2 are independent and distributed as p,

$$\bar{p} \le \tfrac{1}{8}+\tfrac{3}{4}\bar{p}-\tfrac{1}{4}\bar{p}^2.$$

Thus, $\bar{p} \le (\sqrt{3}-1)/2$ and $r \le \sqrt{3}-1 = 0.732\ldots$.

Remark. This is obviously a rather crude upper bound. However, we have only been able to improve our upper bound by a few thousandths.

Next, we obtain a lower bound. For each $c \in [0, 1]$, let $f(c) = P(\{\omega: Y_\omega - c$ evaluated at $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ does not guarantee a fixed point$\})$. Using the invariance of the canonical scheme under time reversal,

$$r \ge 1-\int_0^1 f(c)\, dc = 1-2\int_0^{1/2} f(c)\, dc.$$

Now,

$$\int_0^{\frac{1}{4}} f(c)\,dc = \int_0^{\frac{1}{4}}\left[\int_0^{c+\frac{1}{4}} \frac{c+\frac{3}{4}-y}{1-y}\,dy + \int_{c+\frac{1}{4}}^{c+\frac{1}{2}} \left(\frac{c+\frac{1}{4}}{y}\right)\left(\frac{c+\frac{3}{4}-y}{1-y}\right) dy\right] dc.$$

Figure 4 illustrates the considerations involved in these integrals. For example, if $0 \le y = Y_\omega(\frac{1}{2}) \le c+\frac{1}{4}$, then $Y_\omega(\frac{1}{4})$ is automatically less than $c+\frac{1}{4}$, and cannot indicate a fixed point. But, for $Y_\omega(\frac{3}{4})$ not to indicate a fixed point, $Y_\omega(\frac{3}{4})$ must be $<c+\frac{3}{4}$. The first integral comes from these observations. If $c+\frac{1}{4}<y<c+\frac{1}{2}$, then $Y_\omega(\frac{1}{4})$ must be below $c+\frac{1}{4}$ and $Y_\omega(\frac{3}{4})$ must be below $c+\frac{3}{4}$ in order not to indicate a fixed point. Since these events are independent given y, we obtain the second integral. Of course, if $y \ge c+\frac{1}{2}$, a fixed point must exist.

Similar considerations based on Figure 5, yield,

$$\int_{\frac{1}{4}}^{\frac{1}{2}} f(c)\,dc = \int_{\frac{1}{4}}^{\frac{1}{2}}\left[\int_0^{c-\frac{1}{4}} \frac{\frac{5}{4}-c}{1-y}\,dy + \int_{c-\frac{1}{4}}^{c+\frac{1}{4}} 1\,dy + \int_{c+\frac{1}{4}}^{c+\frac{1}{2}} \frac{c+\frac{1}{4}}{y}\,dy\right] dc.$$

Evaluating these integrals, we find $r \ge 0.3527\ldots$

9. *Computer simulations*

First we give three different methods of estimating the probability of a fixed point for the canonical scheme. For each method, one first generates some number of pairs, (h, c), where h is a homeomorphism of $[0, 1]$ and c is in $[0, 1]$. The first method, denoted by Ave, uses Theorem 12: estimate the probability of a fixed point by estimating $E[1 \wedge (\max\{h(x)-x: x \in [0, 1]\} + \max\{x-h(x): x \in [0, 1]\}]$. In other words, for each h, we estimate the probability that composition of h with a rotation by amount c has a fixed point. The second method, denoted by Cross, is to take each pair; generate the lift $f: \mathbb{R} \to \mathbb{R}$; then determine the proportion which crosses

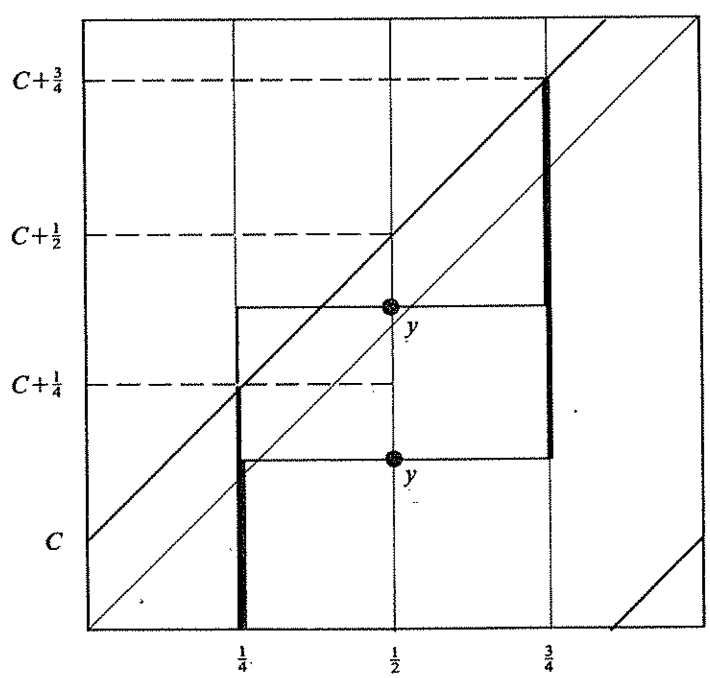

FIGURE 4

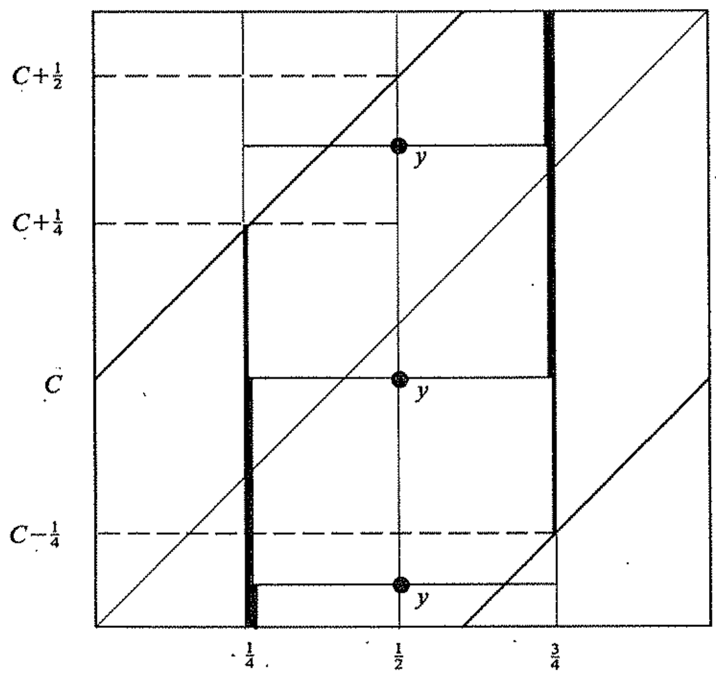

FIGURE 5

the lines $y = x$ or $y = x + 1$. For the third method, denoted by Probe, we look at the circle homeomorphism induced by the pair (h, c) and check how it behaves relative to a large number of sectors on the circle. If we find two points of the circle α and β such that the images of α and β both lie on the same arc determined by cutting the circle at α and β (and preserving order), then we have a fixed point.

Table 1 describes the results obtained from these methods using a Cray at generate 1000 circle homeomorphisms. The first column lists the depth in the dyadic tree to which h is constructed. So, a depth level of 20 means the value of each homeomorphism of the interval $[0, 1]$ is determined for all dyadic rational $i/2^n$ where $n = 20$. Thus, 1,048,576 values were determined for each homeomorphism. The second column indicates the expected value obtained. Notice that the entries in this column stabilize rather quickly compared with the other methods. We guess that the probability of a fixed point is around 0.43.... This quick stabilization is as it should be, since first column is estimating an expected value by measuring the contribution of

TABLE 1

	Fixed point			2 period	
Depth	Ave	Cross	Probe	Cross	Probe
13	0.42871	0.396	0.469	0.120	0.163
14	0.43010	0.419	0.456	0.136	0.159
15	0.42736	0.402	0.442	0.143	0.154
16	0.43075	0.434	0.440	0.122	0.135
20	0.43301	0.417	0.418	0.158	0.156

each homeomorphism whereas for the other two methods, we are in essence throwing darts at each homeomorphism and getting either a 0 or 1. The third column indicates how unstable probing is. Since these homeomorphisms are singular there is a lot of stretching. This also introduces a lot of uncertainty in computing the second iterate. The second and third methods seem to have settled down somewhat by depth 14. The second part of Table 1 concerns the probability of a two period. It is obtained by examining those homeomorphisms which did not have a fixed point. To apply the crossing test to each such homeomorphism, we must construct the second iterate of the lift and determine whether it crosses at least one of the lines $y=x$, $y=x+1$ or $y=x+2$. The probe test is applied as before.

Note. To obtain a good deal of independence between the generation of the values of h, the number c and the probe angles, the following method of generating uniformly distributed numbers in $[0, 1]$ is used—a modification of radical inverse functions. Let p be a prime, $p>2$. The value of the function $\rho_p: \mathbb{N}\to\mathbb{N}$ at n is found as follows. First, write n in base p. Second, reverse the digits in this expansion. Third, modify these digits as follows. Write $p^{1/2}$ in base $p-1$; then add one to these digits (giving p-ary digits which vary from 1 to $p-1$). Now, digitwise multiply together the number obtained in step two and the number just obtained. This completes step three. For the fourth step, take the number with periodic expansion $012\ldots p-1012\ldots p-1$; then add this number digitwise to the number obtained in step three. The number in $[0, 1]$ whose p-ary expansion is this last number is $\rho_p(n)$. The third and fourth steps are put in because even though digit reversal is asymptotically uniform, it is not so good at the beginning. The last two steps smooth this transient behaviour. There are several other reasons for using this scheme. One is that the functions ρ_p become uniformly distributed rather quickly at least at rate $\log n/n$. The functions ρ_p are also asymptotically independent. So, for the nth pair (h_n, c_n) generated $c_n=\rho_3(n)$. Note that c_n is not a dyadic rational. The probing angles are generated using ρ_5 and ρ_7 and are also not dyadic rationals.

Table 2 shows the results of a study of 1000 homeomorphisms carried out on a Compaq 386. About one hour of computer time was required to obtain the results at depth 14. By comparison, it took about one and a half hours on the Cray to generate the results on level 20.

TABLE 2

	Fixed point			2 period	
Depth	Ave	Cross	Probe	Cross	Probe
9	0.43460	0.445	0.949	0.124	0.046
10	0.42336	0.401	0.814	0.147	0.152
11	0.42916	0.446	0.699	0.117	0.176
12	0.43164	0.432	0.554	0.132	0.184
13	0.43021	0.444	0.551	0.120	0.162
14	0.42029	0.423	0.459	0.143	0.169

10. *Problems*

CONJECTURE 1. *For almost every Z_ω, for each positive integer n, Z_ω^n is strictly singular (Z_ω^n does not have a finite positive derivative at any x).*

Dubins and Freedman showed this conjecture is true for $n = 1$.

CONJECTURE 2. *For almost every Z_ω, if Z_ω has a periodic point, then Z_ω has a finite even number of periodic points which alternate between attracting and repelling.*

Note. If Conjecture 1 is true, then Conjecture 2 holds. The proof that this is so is similar to that given in [3] for Theorem 6.11.

The main unanswered problem is

Question. Is it true that almost every Z_ω has a periodic point?

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