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Multifractal Decompositions of Moran Fractals

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We present a rigorous construction and generalization of the multifractal decomposition for Moran fractals with infinite product measure. The generalization is specified by a system of nonnegative weights in the partition sum. All the usual (smooth) properties of the $f(\alpha)$ theory are recovered for the case that the weights are equal to unity. The generalized spectrum, $f(\alpha, w)$, is invariant to a group of gauge transformations of the weights, and, in addition, need no longer be concave. In case the fractal is a Cantor set generated by an iterated function system of similarities, α is the pointwise dimension of the measure. We discuss properties of some examples.

INTRODUCTION

We analyze multifractal structures of a particular type of fractal which we call a Moran fractal. In so doing, we also present a generalization of the multifractal theory. Moran fractals are constructed by an iterative procedure using a given fixed number of similarity ratios. But, unlike objects constructed by using specified similarity maps, Moran fractals need not be themselves "self-similar" [Fa, Hu]. Thus, Moran fractals encompass a wide class of geometric objects, including, but not limited to, the family of self-similar sets as well as the attractors generated by iterated function systems (IFS) popularized by Barnsley and Demko [Ba]. First, we set the mathematical stage, introduce the notion of a multifractal decomposition, and give a brief historical sketch of the origins of multifractal ideas.

Let J be a nonempty compact subset of m-dimensional euclidean space, R^m, let n be a positive integer, $n \geqslant 2$, and let $t_1, \ldots, t_n$ be fixed numbers between 0 and 1. We also assume J is regular: $J = \mathrm{cl}(\mathrm{int}\, J)$. A Moran

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fractal based on seed set J and similarity ratios $t_1, ..., t_n$ is a set K which can be expressed as

$$K = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in S_k} J(\sigma), \tag{1}$$

where $S_k = \{1, ..., n\}^k$ and the sets $J(\sigma)$ are given recursively by the conditions that $J = J(\emptyset)$ and if $J(\sigma)$, for $\sigma \in S_k$, has been determined, then the sets $J(\sigma*1), ..., J(\sigma*n)$, on the $(k+1)$th level, are nonoverlapping subsets of $J(\sigma)$ such that for each i, $J(\sigma*i)$ is geometrically similar to $J(\sigma)$ via a similarity map with reduction ratio t_i. If $\sigma = (\sigma(1), ..., \sigma(k))$, then by $\sigma*i$, the concatenation of σ and i, we mean $\sigma*i = (\sigma(1), ..., \sigma(k), i)$.

We denote the diameter of a set E by $|E|$. Thus, if $\sigma \in S_k$, then $|J(\sigma)| = |J| \cdot t(\sigma)$, where $\prod_{i=1}^{k} t_{\sigma(i)} = t(\sigma)$. The geometric measure theoretic structure of K was determined by Moran [Mo]. Let d be the unique solution of

$$\sum_{i=1}^{n} t_i^d = 1. \tag{2}$$

Moran showed that the Hausdorff dimension of K is $d : \dim(K) = d$. Moreover, as is also shown in Section 2, if $\mathscr{H}^d$ is the corresponding d-dimensional Hausdorff measure, $0 < \mathscr{H}^d(K) \leqslant |J|^d < \infty$. For convenience, we assume $|J| = 1$. For the record, we note that Spear [Sp1] and, independently, Haase [Haa] have shown that for these constructions and, more generally, for an object generated from directed graph constructions [Mau], the packing dimension and the Hausdorff dimension of the object agree.[1] Spear has shown that Hausdorff and Packing measure differ by a fixed constant multiple over subsets of K [Sp2].

The proof that $0 < \mathscr{H}^d(K)$, as given in Section 2, and our analysis, is made easier with the aid of a natural coding space. The coding space is $\Omega = \{1, ..., n\}^N$, where $N = \{1, 2, 3, ...\}$. For each $\sigma \in \Omega$ and $k \in N$, let $\sigma | k = (\sigma(1), ..., \sigma(k))$. There is a natural coding map g of Ω onto K defined by the condition

$$\{g(\sigma)\} = \bigcap_{k=1}^{\infty} J(\sigma | k). \tag{3}$$

It is easily seen that g is a continuous map of Ω onto K. For each $\sigma \in \{1, ..., n\}^* = \bigcup_{k=1}^{\infty} S_k$, let $|\sigma|$ denote the length of σ and let $C(\sigma)$ be the cylinder set in Ω determined by σ. Thus,

$$C(\sigma) = \{\tau \in \Omega : \tau | k = \sigma, \text{ where } |\sigma| = k\}. \tag{4}$$

[1] S. J. Taylor has suggested the term fractal be reserved for sets for which the packing dimension coincides with the Hausdorff dimension [Ta].

The continuity of g follows from the inclusion $g(C(\sigma)) \subset J(\sigma)$. The coding map g is not necessarily a one-to-one map of Ω onto K. However, in the important case of "pairwise disjoint" Moran constructions, the coding map is a homeomorphism of Ω onto K. A Moran construction is said to be pairwise disjoint provided for each positive integer k, the sets $J(\sigma)$, $\sigma \in S_k$, are pairwise disjoint. Under this assumption, the map g is a homeomorphism and the set K is a topological Cantor set: a compact, dense-in-itself, zero-dimensional subset of R^m. Thus, K is a fractal Cantor set.

A subclass of Moran constructions is the "map specified" case or iterated function systems [Ba]. Here it is assumed that n contracting similarity maps $T_1, ..., T_n$ of R^m are given with similarity ratios $t_1, ..., t_n$, respectively, such that for $\sigma \in S_k$,

$$J(\sigma) = T_{\sigma(1)} \circ T_{\sigma(2)} \circ \cdots \circ T_{\sigma(k)}(J);$$

i.e., K is constructed by iterating these specific maps applied to J.

Our interest in this paper focuses on "multifractal" decompositions of K characterized in terms of the local behaviour of a probability measure induced on K by a product measure on Ω. This is described as follows.

Fix a probability vector $(p_1, ..., p_n)$ with each p_i positive and let $\hat{\rho}$ be the corresponding infinite product measure on Ω. Let ρ be the image measure on K induced by g. So $\rho(E) = \hat{\rho}(g^{-1}(E))$, for $E \subset R^m$. For each α, let

$$\hat{K}_\alpha = \{\sigma \in \Omega : \lim_{k \to \infty} \log p(\sigma | k)/\log t(\sigma | k) = \alpha\} \tag{5}$$

and

$$K_\alpha = g(\hat{K}_\alpha). \tag{6}$$

The disadvantage of the definition of K_α given by Eq. (6) is that the determination of when x is in K_α depends on knowing some sequence of sets $J(\sigma)$ of the construction closing down on x. In Section 3, we eliminate this problem for the case of pairwise disjoint, map specified Moran constructions. We prove that the sets K_α may be characterized in terms of the local behaviour of the measure ρ. In particular, in this case, each set K_α comprises the points of K where the pointwise dimension of ρ is α. In other words, a point x of K is in K_α if and only if

$$\lim_{\varepsilon \to 0} \log \rho(B(x, \varepsilon))/\log |B(x, \varepsilon)| = \alpha, \tag{7}$$

where $B(x, \varepsilon)$ is a ball of diameter ε centered at x. Thus, for the pairwise disjoint, map specified case, the sets K_α are determined by the local external geometric behaviour of ρ, and one does not need to be given explicitly the sets $J(\sigma)$—the method of construction of K. In the framework of Halsey *et al.* [Ha], K_α is specified by Eq. (7) where the limit is assumed always to exist. We verify this complete framework in the case of a Cantor set generated by an iterated function system. Here a generating partition is available analogous to a dynamical system where Markov partitions are available to organize the dynamics on the manifold. It is frustrating that we do not know what the situation is for the general Moran construction.

The collection of sets K_α form a multifractal decomposition of K corresponding to the measure ρ. (For the record, we have shown that each set K_α is a fractal in the sense of Taylor.[2]) In Section 2, we determine the function $f(\alpha) = \dim K_\alpha$ for a class of multifractal decompositions of general Moran fractals K based on ρ; while in Section 4, we produce a new generalization of the earlier multifractal scheme, along with a generalized formula for f, now based on weights as well as measures (see below for the motivation for introduction of the weights.) In Section 4, we prove the corresponding Hausdorff dimension results for the new sets issuing from the generalized multifractal decomposition. In Section 1, we develop some preliminary properties of the usual function f and some associated auxiliary functions. In particular, we show that $f(\alpha)$ is smooth. Properties of the generalized f, which is also smooth, are developed in Sections 4 and 5.

The smoothness of f has been one of the reasons for the popularity and usefulness of the broad circle of ideas of multifractal analysis in physical applications. The first expositions of the idea of representing a strange or fractal set as a decomposition of this sort may have their genesis in an early paper of Mandelbrot [Man] where it was proposed that the bulk of intermittent dissipation of energy in highly turbulent fluid flow ocurs over a subset $S \subset R^3$ of fractional dimension. An independent circle of ideas based on the Renyi entropies [Re, Appendix] of order $q \geqslant 0$ was presented in several papers introducing "higher order dimensions", by Grassberger and Procaccia [Gr1], Heschel and Procaccia [He], and Grassberger [Gr2]. An extension to $q \in R$, and a slightly sharper notion of generalized dimension which seems to have anticipated the later partition function formalism (*vid.* our measure ν_q in Section 2, and Eq. (1.1) upon which it is based) was discussed by Grassberger [Gr3].

Subsequently, in a now famous appendix to an article of U. Frisch, Frisch and Parisi [Fr] proposed a "multifractal" picture which was more restrictive than the complicated intuitive model elaborated previously in the paper of Mandelbrot. Based on an invariance argument, it was suggested in [Fr] that solutions of the three-dimensional Navier–Stokes equations in a zero-viscosity limit might exhibit singularities of order h, on fractal sets $S(h)$ where the velocity field just fails to be a Holder function of order h, and where $S(h) \subset S(h')$ if $h < h'$. The function $\dim_H S(h)$ was computed from assumptions concerning the local behaviour of moments of the fluid

[2] Edgar and Mauldin [Ed] have extended the results of Sections 1–3 to a more general setting and have shown the sets K_α are fractals in the sense of Taylor.

relative velocity. A wider generality of the multifractal decomposition idea was recognized by Benzi *et al.* [Ben], who extended its application to dynamical systems. In the work of Halsey *et al.* [Ha], a general formulation of the scenarios of multifractal theory was elaborated in which there were strong hints at parallels to the theory of statistical mechanics.

Our own generalization of multifractal theory presented herein involves the introduction of weights w_i to the terms $p_i^q t_i^\beta$ of the generating partition for the measures $\nu_{q,w}$ (Section 4—in particular, Eq. (4.1)). We have two kinds of motivation for this. The first is purely mathematical. In a rigorous construction of the multifractal theory, a key ingredient is the availability of a shift-invariant measure ergodic on the coding space, a role played on the image set in R^d by ν_q. There are infinitely many such measures. The measure used for the standard multifractal theory is a product measure (Section 2). We hoped that by exploring some of the others we would gain new insight into the nature of the multifractal formalism. A search for a clean example that would either destroy the multifractal formalism entirely or generate a nontrivial extension of it led us to the introduction of the weights. This simple model has the property that the resulting multifractal "$f(\alpha, w)$-curves" are no longer necessarily concave down and the maximum value of $f(\alpha, w)$ is not necessarily the Hausdorff dimension. On the other hand, some properties of the $f(\alpha, w)$ curves are the same, such as the maximum and minimum values taken by α when all the weights are positive. In addition, there is an interesting new invariance feature to the $f(\alpha, w)$ curve, namely to gauge transformations of the weights (Section 5). (There are many more possibilities here. For example, we have not carried out the corresponding analysis for Markov chains.) Our second motivation is physical, and is inspired by a spin-glass theory analogy [Me1, 2, 3]. Here a large collection of pure equilibrium (Gibbs) clustering states $a = 1, 2, \ldots, N$, are supposed to be present as components of a mixed state, with weighting probabilities P_a, $\Sigma P_a = 1$. The spin glass order parameter is then defined in terms of a distribution over random couplings J_{ij} between spins of the probability for two pure states to have a specified "overlap," where the overlap is a quantity defined in terms of the P_a [Me1, 2]. Macroscopic averaging in spin-glass theory includes an additional statistical feature with respect to the mixture of local pure states. The inclusion of an independent set of weights, w_i, in Eq. (4.1) provides for such an additional statistical feature and may be thought of as simulating the P_a's. (Although we do not require our weights to be a probability vector, they can easily be normalized by a gauge transformation (see below).) We hasten to add that spin glass theory possesses considerable structure and we have not tried to establish real correspondences or close analogies to the generalized multifractal formalism.

Finally, the first rigorous results of multifractal theory besides those presented herein have been due to Collet *et al.* [Co], Bohr and Rand

[Bo], and Rand [Ra], where the context was that of one-dimensional maps, of the unit interval and of the circle, and to Lopez [L.1, L.2], for rational maps of the complex plane.

1. Auxiliary Functions and Their Properties

Our notation for these functions conforms for the most part to that of [Ha] except for our use of $\beta(q)$ in place of their $-\tau(q)$. The first auxiliary function is defined as follows. For each $q \in R$, there is a unique number, $\beta(q)$, such that

$$\sum_{i=1}^{n} p_i^q t_i^{\beta(q)} = 1. \tag{1.1}$$

Clearly, $\beta(1) = 0$ and $\beta(0) = d$. Also, by implicit differentiation

$$\beta'(q) = -\sum_{i=1}^{n} (\log p_i) p_i^q t_i^{\beta(q)} \Big/ \sum_{i=1}^{n} (\log t_i) p_i^q t_i^{\beta(q)} \tag{1.2}$$

and

$$\beta''(q) = -\sum_{i=1}^{n} (\log p_i + \beta'(q) \log t_i)^2 p_i^q t_i^{\beta(q)} \Big/ \sum_{i=1}^{n} (\log t_i) p_i^q t_i^{\beta(q)}. \tag{1.3}$$

Thus, $\beta'(q) < 0$, for all q, so that β is a strictly decreasing function. Also, note that $\beta''(q) \geqslant 0$. As a matter of fact, either $\beta''(q) > 0$, for all q, or else $p_i = t_i^d$ and $\beta(q) = -dq + d$, for all q. This follows from the fact that if $\beta''(q_0) = 0$, for some q_0, then from Eq. (1.3), $\beta'(q_0) = -\log p_1/\log t_1 = \cdots = -\log p_n/\log t_n$. Therefore, $p_i = t_i^{\beta'(q_0)}$, $i = 1, ..., n$. This implies $p_i = t_i^d$. In this case, and only in this case, $\beta(q) = -dq + d$, for all q.

Theorem 1.1. *The function β is strictly decreasing, $\beta(0) = d$, and $\beta(1) = 0$. Either $p_i = t_i^d$, $i = 1, ..., n$, and $\beta(q) = -qd + d$, or $\beta''(q) > 0$, for all q.*

Our second auxiliary function is

$$\alpha(q) = -\beta'(q) = \sum_{i=1}^{n} (\log p_i) p_i^q t_i^{\beta(q)} \Big/ \sum_{i=1}^{n} (\log t_i) p_i^q t_i^{\beta(q)}. \tag{1.4}$$

Theorem 1.2. *The positive function α either is constantly equal to d or else is strictly decreasing.*

Finally, let

$$f(q) = q\alpha(q) + \beta(q). \tag{1.5}$$

We have $f(0)=d$ and $f'(q)=-q\beta''(q)$. So either f is constantly equal to d or else f is strictly increasing from $-\infty$ to 0 and strictly decreasing from 0 to ∞.

Unless we have the exceptional case of $p_i=t_i^d$, the function $q\to\alpha(q)$ is one-to-one and we can express f as a function of α; i.e., for each α between $\alpha(\infty)$ and $\alpha(-\infty)$, set $f(\alpha)=f(q)$, where $\alpha(q)=\alpha$. The graph of f as a function of α is smooth and everywhere concave downward, and has several distinctive features to be derived presently. It is commonly called "the spectrum of scaling indices" or "$f(\alpha)$."

Let us determine the asymptotic behaviour of these auxiliary functions. From

$$1=\sum_{i=1}^{n} p_i^q t_i^{\beta(q)}, \tag{1.1}$$

we immediately have $\lim_{q\to\infty}\beta(q)=-\infty$ and $\lim_{q\to-\infty}\beta(q)=\infty$. Set

$$\lambda_i=\log p_i/\log t_i, \qquad \lambda=\min\lambda_i, \qquad \text{and} \qquad \bar{\lambda}=\max\lambda_i. \tag{1.6}$$

We claim that

$$\lim_{q\to\infty}\beta(q)+\lambda q=e, \tag{1.7}$$

where

$$1=\sum_{\lambda_i=\lambda} t_i^e. \tag{1.8}$$

This may be seen as follows. We have

$$\begin{aligned}1&=\sum_{\lambda_i=\lambda} p_i^q t_i^{\beta(q)}+\sum_{\lambda_i>\lambda} p_i^q t_i^{\beta(q)}\\&=\sum_{\lambda_i=\lambda}(p_i t_i^{-\lambda})^q t_i^{\beta(q)+\lambda q}+\sum_{\lambda_i>\lambda}(p_i t_i^{-\lambda})^q t_i^{\beta(q)+\lambda q},\end{aligned} \tag{1.9}$$

or,

$$1=\sum_{\lambda_i=\lambda} t_i^{\beta(q)+\lambda q}+\sum_{\lambda_i>\lambda}(p_i t_i^{-\lambda})^q t_i^{\beta(q)+\lambda q}. \tag{1.10}$$

Note that $\beta(q)+\lambda q$ is nonincreasing, since from (1.4) and (1.6), we find

$$\beta'(q)+\lambda=\sum_{i=1}^{n}(\lambda-\lambda_i)(\log t_i)\,p_i^q t_i^{\beta(q)}\Big/\sum_{i=1}^{n}(\log t_i)\,p_i^q t_i^{\beta(q)}\leqslant 0.$$

By the monotonicity of $\beta(q)+\lambda q$, there are two cases: (a) $\lim_{q\to\infty} \beta(q)+\lambda q = -\infty$, and (b) $\lim_{q\to\infty} \beta(q)+\lambda q = e$, for some real number e. If case (a) were to hold, then taking the limit as q goes to ∞ in (1.10) and noting that if $\lambda_i > \lambda$, then $p_i t_i^{-\lambda} < 1$, we would have $1=\infty$. Taking limits in case (b), we obtain Eq. (1.8). What we have shown, in other words, is that the line $-\lambda q + e$ is asumptotic to $\beta(q)$ as q goes to ∞.

Similarly, the line $-\bar{\lambda} q + \bar{e}$ is asymptotic to $\beta(q)$ as q goes to $-\infty$, where $\bar{\lambda} = \max \lambda_i$ and $\bar{e}$ is defined by

$$1 = \sum_{\lambda_i = \bar{\lambda}} t_i^{\bar{e}}. \tag{1.11}$$

Remark 1.3. If $\sum_{\lambda_i=\lambda} t_i^e = t_{i_0}^e$, then $e=0$. Similarly, if $\lambda_i = \bar{\lambda}$ for only one value of i, $\bar{e}=0$.

To find the asymptotic behaviour of α, express, with some algebra,

$$\alpha(q) = \frac{\left[\sum_{\lambda_i=\lambda} (\log p_i)\cdot t_i^{\beta(q)+\lambda q} + \sum_{\lambda_i>\lambda} (\log p_i)\cdot t_i^{\beta(q)+\lambda_i q}\right]}{\left[\sum_{\lambda_i=\lambda} (\log t_i)\cdot t_i^{\beta(q)+\lambda q} + \sum_{\lambda_i>\lambda} (\log t_i)\cdot t_i^{\beta(q)+\lambda_i q}\right]}.$$

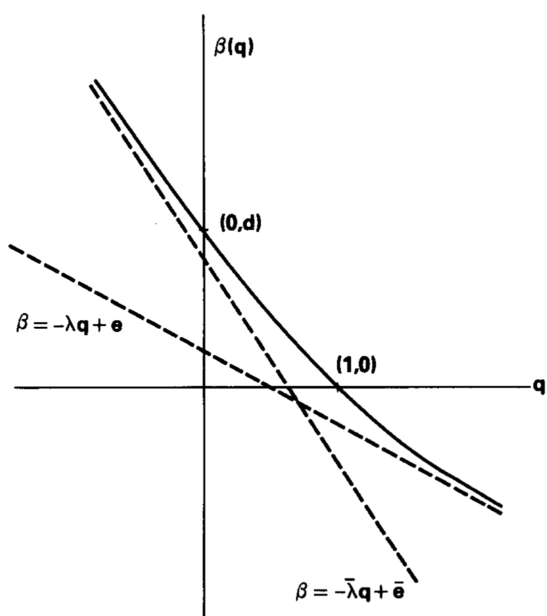

FIG. 1.1. $\beta(q)$: smooth, concave upwards, strictly decreasing, and infinite at $q = \pm\infty$. The straight lines are asymptotes, with slopes and intercepts as shown. The graph of $\beta(q)$ is itself a straight line for the case $p_i = t_i^d$, $i = i, \ldots, n$.

Taking the limit as q goes to ∞ in this last expression, we get

$$\alpha(\infty) = \lim_{q \to \infty} \alpha(q) = \lambda = \min(\log p_i/\log t_i). \tag{1.12}$$

Mutatis mutandis,

$$\alpha(-\infty) = \lim_{q \to -\infty} \alpha(q) = \bar{\lambda} = \max(\log p_i/\log t_i). \tag{1.13}$$

Figures 1.1 and 1.2 are sketches of the graphs of β and α as functions of q illustrating their principal features.

THEOREM 1.4. *$\lim_{q \to \infty} f(q) = e$ and $\lim_{q \to -\infty} f(q) = \bar{e}$.*

Proof. Express

$$f(q) = q\alpha(q) + \beta(q) = q(\alpha(q) - \lambda) + e + \delta(q), \tag{1.14}$$

where $\lim_{q \to \infty} \delta(q) = 0$. Setting

$$S(q) = \sum_{\lambda_i = \lambda} (\log t_i) t_i^{\beta(q) + \lambda q}, \tag{1.15}$$

we obtain, after some manipulations,

$$q(\alpha(q) - \lambda) = \frac{q(1/S(q)) \sum_{\lambda_i > \lambda} (\lambda_i - \lambda)(\log t_i) \cdot t_i^{\beta(q) + \lambda_i q}}{[1 + (1/S(q)) \sum_{\lambda_i > \lambda} (\log t_i) \cdot t_i^{\beta(q) + \lambda_i q}]}. \tag{1.16}$$

Since $S(q) \to \sum_{\lambda_i = \lambda} (\log t_i) t_i^e \neq 0$ and $\lambda_i > \lambda$ for each i, in the sums, the denominator of this last expression converges to one as $q \to \infty$, while the numerator converges to zero. *Mutatis mutandis,*

$$\lim_{q \to -\infty} f(q) = \bar{e}. \tag{1.17}$$

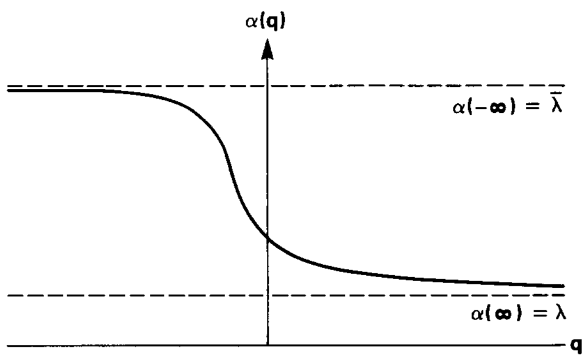

FIG. 1.2. $\alpha(q) = -\beta'(q)$: strictly decreasing in q and bounded between positive finite values. For $p_i = t_i^d$, $i = 1, \ldots, n$, $\alpha(q) = \text{constant} = d$.

Remark 1.5. If the sums in Eqs. (1.8) and (1.11) contain only one term, then $e=\bar{e}=0$ and $f(\alpha)$ vanishes at its endpoints. But neither $f(\alpha(\infty))$ nor $f(\alpha(-\infty))$ need be zero, as the following example indicates.

EXAMPLE 1.6. Let $n=4$ and $J=[0,1]\times[0,1]$, the unit square in R^2 with $t_1=t_2=t_3=t_4=T$, $p_1=p_2=p$, and $p_3=p_4=P>p$. Thus, $\lambda=\log P/\log T$ and $\bar{\lambda}=\log p/\log T$. Then $e=\log(\frac{1}{2})/\log T\neq 0$ and $\bar{e}=\log(\frac{1}{2})/\log T\neq 0$.

Remark 1.7. Regardless of the values of f at $\alpha(-\infty)$ or $\alpha(\infty)$, the limiting slopes of the $f(\alpha)$ curve are infinite. Since $df/d\alpha=(df/dq)\cdot(dq/d\alpha)=q$,

$$\lim_{\alpha\to\alpha(+\infty)}\frac{df}{d\alpha}=+\infty \qquad \text{and} \qquad \lim_{\alpha\to\alpha(-\infty)}\frac{df}{d\alpha}=-\infty.$$

Also, since $\beta(1)=0$, we have $f(\alpha(1))=\alpha(1)$. And since $df/d\alpha|_{\alpha(1)}=1$, the graph of f as a function of α is tangent to the 45° line at $\alpha=\alpha(1)$.

Remark 1.8. From $df/d\alpha=q(\alpha)$, we have $d^2f/d\alpha^2=\alpha'(q)^{-1}=-\beta''(q)^{-1}<0$. That is, $f(\alpha)$ is everywhere concave downward. For the single value of α for which q vanishes, f has an absolute maximum, and $df/d\alpha$ cannot vanish anywhere else. The value taken by f at its maximum is $f_{\max}=$

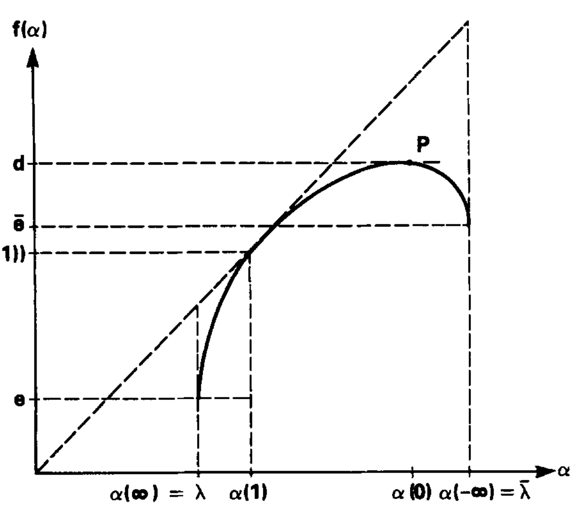

FIG. 1.3. Spectrum of scaling indices, $f(\alpha)$: The slopes are infinite at $\alpha=\lambda$ and $\alpha=\bar{\lambda}$. The dimension values, e and $\bar{e}$, vanish in the nondegenerate case, where $\lambda_i=\log p_i/\log t_i$ takes its maximum and minimum for only single values of i. The straight line with slope one passing through the origin is tangent to the graph of f at $(\alpha(1),f(\alpha(1)))$. For $p_i=t_i^d$, $i=1,\ldots,n$, the otherwise smooth and concave graph of $f(\alpha)$ instead becomes a single point, $P=P(\alpha(0),f(\alpha(0)))=P(d,d)$.

$\beta(0)=d$. Finally, as we shall see in Section 2 (Remark 2.12), the value $f(\alpha(1))=\alpha(1)$ is the Hausdorff dimension of the infinite product measure ρ generated by the probability vector $(p_1, ..., p_n)$. Thus, $f(q)|_{q=1}$ is the information dimension of ρ. Figure 1.3 shows a sketch of $f(\alpha)$ illustrating its principal features.

Remark 1.9. Excluding $p_i=t_i^d$, the concavity of $f(\alpha)$ and the monotone property of $\alpha(q)$ imply the following inequalities:

$$f(\alpha(+\infty))=e<f(\alpha(1))=\alpha(1)<f(\alpha(0))=d \quad \text{and} \quad d>f(\alpha(-\infty))=\bar{e}.$$

2. The Hausdorff Dimension of K_α.

Let us consider in more detail the multifractal decomposition of K. Again, $\lambda_i=\lambda(i)=\log p_i/\log t_i$, $i=1, ..., n$, $\lambda=\min \lambda_i$ and $\bar{\lambda}=\max \lambda_i$. For each $\sigma\in\Omega$ and $i\in N$, express $p_{\sigma(i)}=t_{\sigma(i)}^{\lambda(\sigma(i))}$. Then

$$\log p(\sigma\,|\,k)/\log t(\sigma\,|\,k)=\sum_{i=1}^{k} \lambda(\sigma(i)) \log t_{\sigma(i)} \Big/ \sum_{i=1}^{k} \log t_{\sigma(i)}.$$

Thus, for $\sigma\in\Omega$,

$$\begin{aligned}\lambda=\alpha(\infty) &\leqslant \liminf_{k\to\infty} \log p(\sigma\,|\,k)/\log t(\sigma\,|\,k)\\ &\leqslant \limsup_{k\to\infty} \log p(\sigma\,|\,k)/\log t(\sigma\,|\,k) \leqslant \bar{\lambda}=\alpha(-\infty).\end{aligned}$$

We begin by analyzing K_α, the set of points of K coded by symbol sequences σ such that the limit of the log ratios is some number between λ and $\bar{\lambda}$. From the preceding inequality, $\alpha=\alpha(q)$ for some q and

$$K_{\alpha(q)}=\{g(\sigma) : \lim_{k\to\infty} \log p(\sigma\,|\,k)/\log t(\sigma\,|\,k)=\alpha(q)\}.$$

In fact, we construct a probability measure μ_q supported on $K_{\alpha(q)}$, and show that the dimension of the μ_q is $f(q)$.

Theorem 2.1. *For each* $\alpha(\infty)<\alpha<\alpha(-\infty)$, $\dim(K_\alpha)=f(\alpha)$. *In other words, for* $q\in R$, $\dim(K_{\alpha(q)})=f(q)=q\alpha(q)+\beta(q)$. *Moreover, the dimension of* $\mu_q=f(q)$.

The proof of this theorem is broken into two main parts. First, using the Vitali covering theorem, we show $\dim(K_{\alpha(q)})\leqslant f(q)$. Then, using a geometric lemma specific to these constructions, we show $\dim(K_{\alpha(q)})\geqslant f(q)$. In this part we also prove that the dimension of the measure μ_q is $f(q)$.

Each of these proofs is further subdivided according to whether $q=0$, $q>0$, or $q<0$. The first and easiest case we commented on the introduction.

THEOREM 2.2. $\dim(K_{\alpha(0)}) \leqslant f(0) = d$.

Proof. Since $\dim(K_{\alpha(0)}) \leqslant \dim(K)$, it suffices to show that $\mathscr{H}^d(K) < \infty$. This is easily seen from the estimates obtained from the covers consisting of the sets $J(\sigma)$ constructed on level k. Using the multinomial expansion,

$$\sum_{\sigma \in S_k} |J(\sigma)|^\eta = |J|^\eta \sum_{\sigma \in S_k} t_\sigma^\eta = |J|^\eta (t_1^\eta + \cdots + t_n^\eta)^k. \tag{2.1}$$

Thus, if $\eta = d$,

$$\sum_{\sigma \in S_k} |J(\sigma)|^d = |J|^d \tag{2.2}$$

and $\mathscr{H}^d(K) \leqslant |J|^d = 1$.

For $q>0$, let

$$\hat{U}_q = \{\sigma \in \Omega : \limsup_{k \to \infty} \log p(\sigma \mid k)/\log t(\sigma \mid k) \leqslant \alpha(q)\}$$

and set

$$U_q = g(\hat{U}_q).$$

THEOREM 2.3. *For each* $q>0$, $\dim(K_{\alpha(q)}) \leqslant \dim U_q \leqslant f(q)$. *For each* $\delta > 0$, $\mathscr{H}^{f(q)+\delta}(U_q) = 0$.

The proof of this theorem is based upon the following lemma.

LEMMA 2.4. *Let q and δ be positive. For each positive integer m, there is a collection $\mathscr{G}_m$ of pairwise disjoint sets each with diameter less than $1/m$ such that*

$$(1) \qquad \mathscr{H}^{f(q)+\delta}\left(U_q \Big\backslash \bigcup \mathscr{G}_m\right) = 0 \tag{2.3}$$

and

$$(2) \qquad \sum_{G \in \mathscr{G}_m} |G|^{f(q)+\delta} \leqslant 1. \tag{2.4}$$

Proof of Lemma 2.4. For each $\sigma \in \hat{U}_q$, let M_σ be a positive integer such that if $k > M_\sigma$, then

$$\log p(\sigma \mid k)/\log t(\sigma \mid k) < \alpha(q) + \delta/q \tag{2.5}$$

and

$$t(\sigma \mid k) < 1/m. \tag{2.6}$$

Set $\mathscr{V}_m = \{g(C(\sigma \mid k)) : \sigma \in \hat{U}_q \text{ and } k \geqslant M_\sigma\}$. Clearly, $\mathscr{V}_m$ is a Vitali class for U_q. Therefore, by the Vitali covering lemma [Fa, p. 11], there is a pairwise disjoint subcollection $\mathscr{G}_m$ of $\mathscr{V}_m$ such that either

$$\text{(a)} \qquad \sum_{G \in \mathscr{G}_m} |G|^{f(q)+\delta} = \infty \tag{2.7}$$

or

$$\text{(b)} \qquad \mathscr{H}^{f(q)+\delta}\left(U_q \Big\backslash \bigcup \mathscr{G}_m\right) = 0. \tag{2.8}$$

However, (2.7) does not hold. To see this, suppose the sets G are $g(C(\sigma_i \mid k_i)) \in \mathscr{G}_m$, $i = 1, ..., j$. From (2.5), we have

$$|g(C(\sigma_i \mid k_i))|^{q\alpha(q)+\delta} \leqslant t(\sigma_i \mid k_i)^{q\alpha(q)+\delta} < p(\sigma_i \mid k_i)^q \tag{2.9}$$

for each i. Thus,

$$\sum_{i=1}^{j} |g(C(\sigma_i \mid k_i))|^{f(q)+\delta} < \sum_{i=1}^{j} p(\sigma_i \mid k_i)^q t(\sigma_i \mid k_i)^{\beta(q)}. \tag{2.10}$$

But, since

$$\sum_{i=1}^{n} p_i^q t_i^{\beta(q)} = 1, \tag{1.1}$$

it follows that if $\mathscr{D} \subset \{1, ..., n\}^*$ is such that no two sequences in $\mathscr{D}$ have a common extension, then

$$\sum_{\sigma \in \mathscr{D}} p(\sigma)^q t(\sigma)^{\beta(q)} \leqslant 1. \tag{2.11}$$

Since the sets $g(C(\sigma_i \mid k_i))$ are pairwise disjoint, no two $\sigma_i \mid k_i$ in fact, do have a common extension. Therefore,

$$\sum_{G \in \mathscr{G}_m} |G|^{f(q)+\delta} \leqslant 1. \tag{2.12}$$

Proof of Theorem 2.3. Since Condition (2.3) ensures that $\mathscr{H}^{f(q)+\delta}(U_q \backslash \bigcap_{m=1}^{\infty} \cup \mathscr{G}_m) = 0$, and Condition (2.4) ensures that $\mathscr{H}^{f(q)+\delta}(U_q \cap [\bigcap_{m=1}^{\infty} \cup \mathscr{G}_m]) \leqslant 1$, we have $\mathscr{H}^{f(q)+\delta}(U_q) \leqslant 1$.

For $q < 0$, set

$$\hat{L}_q = \{\sigma \in \Omega : \liminf_{k \to \infty} \log p(\sigma \mid k)/\log t(\sigma \mid k) \geqslant \alpha(q)\}$$

and set

$$L_q = g(\hat{L}_q).$$

One argues as in the case $q > 0$:

THEOREM 2.5. *For each $q < 0$ and $\delta > 0$, $\mathscr{H}^{f(q)+\delta}(L_q) \leqslant 1$. Thus, $\dim(K_{\alpha(q)}) \leqslant f(q)$.*

We turn now to the proofs that $\dim(K_{\alpha(q)}) \geqslant f(q)$. These are based upon the following geometric lemma and examination of the images on K of measures on the coding space. This lemma is proved in a more general setting in [Mau].

LEMMA 2.6. *There is a number $c > 0$ such that if $E \subset R^m$ and $|E| < t_{\min} = \min\{t_i : 1 \leqslant i \leqslant n\}$, then the cardinality of H is $\leqslant c$, where*

$$H = \{\sigma \in \{1, ..., n\}^* : |J(\sigma)| < |E| \leqslant |J(\sigma \mid |\sigma| - 1)| \text{ and } J(\sigma) \cap E \neq \varnothing\}. \tag{2.13}$$

From this point on we fix $c \geqslant 1$ such that Lemma 2.6 holds.

We now introduce the auxiliary measures μ_q supported on $K_{\alpha(q)}$. These are the image under the coding map of the infinite product measure $\hat{\mu}_q$, on Ω, based on the probability vector $(p_1^q t_1^{\beta(q)}, ..., p_n^q t_n^{\beta(q)})$, where

$$\sum_{i=1}^{n} p_i^q t_i^{\beta(q)} = 1. \tag{1.1}$$

Note that $\hat{\mu}_q(\hat{K}_{\alpha(q)}) = 1$. This follows from Birkhoff's individual ergodic theorem applied to the shift transformation, the measure $\hat{\mu}_q$, and the functions $X(\sigma) = \log p_{\sigma(1)}$ and $Y(\sigma) = \log t_{\sigma(1)}$. Thus, we find that for $\hat{\mu}_q$ almost all σ,

$$\lim_{k \to \infty} \left(\frac{1}{k}\right) \log p(\sigma \mid k) = E[X] = \sum_{i=1}^{n} (\log p_i)\, p_i^q t_i^{\beta(q)}. \tag{2.14}$$

Similarly, for μ_q-a.e. σ,

$$\lim_{k \to \infty} (1/k) \log t(\sigma \mid k) = \sum_{i=1}^{n} (\log t_i)\, p_i^q t_i^{\beta(q)}. \tag{2.15}$$

Taking ratios and using Eq. (1.4), we have that for $\hat{\mu}_q$-a.e. σ,

$$\lim_{k \to \infty} \log p(\sigma \mid k)/\log t(\sigma \mid k) = \sum_{i=1}^{n} (\log p_i) p_i^q t_i^{\beta(q)} \Big/ \sum_{i=1}^{n} (\log t_i) p_i^q t_i^{\beta(q)} = \alpha(q). \tag{2.16}$$

Thus, $\hat{\mu}_q(\hat{K}_{\alpha(q)}) = 1 = \mu_q(K_{\alpha(q)})$.

(Many authors have used the ergodic theorem in a similar fashion, e.g., Billingsley [Bi]. The first occurrence of this use is unknown to us.)

It is shown in [Mau] that Lemma 2.6 implies the following lemma.

LEMMA 2.7. *If $E \subset R^m$ and $|E| < t_{\min}$, then*

$$\mu_0(E) < c\,|E|^d. \tag{2.17}$$

Proof. Let

$$H = \{\sigma \in \{1, ..., n\}^* : |J(\sigma)| < |E| \leqslant |J(\sigma \mid |\sigma| - 1)| \text{ and } J(\sigma) \cap E \neq \emptyset\}.$$

Then

$$\mu_0(E) = \mu_0(E \cap K) \leqslant \sum_{\sigma \in H} \mu_0(E \cap J(\sigma)) \leqslant \sum_{\sigma \in H} |J(\sigma)|^d$$

$$\leqslant \sum_{\sigma \in H} |E|^d \leqslant c\,|E|^d. \tag{2.18}$$

If μ is a measure on a metric space X, then $\dim \mu$, the dimension of μ, is defined by $\dim \mu = \min\{\gamma : \exists S \subset X \text{ with } \dim S = \gamma \text{ and } \mu(X \setminus S) = 0\}$.

THEOREM 2.8. $\dim(K_{\alpha(0)}) \geqslant f(0) = d$. *In fact,* $\mathscr{H}^{f(0)}(K_{\alpha(0)}) > 0$ *and* μ_0 *which is supported on* $K_{\alpha(0)}$ *has dimension* $f(0)$.

Proof. Suppose $\mathscr{H}^d(K_{\alpha(0)}) < 1/c$. Then there would be a collection $\mathscr{E}$ of sets each with diameter less than $t_{\min}$ and covering $K_{\alpha(0)}$ such that

$$\sum_{E \in \mathscr{E}} |E|^d < 1/c. \tag{2.19}$$

But then we would also have the contradiction

$$1 > \sum_{E \in \mathscr{E}} c\,|E|^d > \sum_{E \in \mathscr{E}} \mu_0(E) \geqslant \mu_0(K_{\alpha(0)}) = 1. \tag{2.20}$$

Since μ_0 is supported on $K_{\alpha(0)}$, $\dim \mu_0 \leqslant f(0)$. Assume there is some set $S \subset R^m$ with $\mu_0(S) = 1$ and $\dim S < f(0)$. Since $\mathscr{H}^d(S) = 0$, there would be a cover $\mathscr{E}$ of S by sets with diameter $< t_{\min}$ such that $\sum |E|^d < 1/c$. This leads again to the contradiction (2.20).

THEOREM 2.9. *For each* $q > 0$, $\dim(K_{\alpha(q)}) \geqslant f(q)$. *In fact,* μ_q *which is supported on* $K_{\alpha(q)}$ *has dimension* $f(q)$.

Proof. Let $q > 0$. It suffices to show that $\dim \mu_q \geqslant f(q)$. Assume there is some set $S \subset R^m$ with $\mu_q(S) = 1$ and $\mathscr{H}^{f(q)-\delta}(S) = 0$, for some $0 < \delta < f(q)$. For each $\sigma \in \hat{K}_{\alpha(q)}$, let N_σ be a positive integer such that if $k \geqslant N_\sigma$, then

$$\log p(\sigma | k)/\log t(\sigma | k) > \alpha(q) - \delta/q. \tag{2.21}$$

For each M, let $\hat{K}_{\alpha(q),M} = \{\sigma \in \hat{K}_{\alpha(q)} : N_\sigma = M\}$.

Fix M so that $\hat{\mu}_q(\hat{K}_{\alpha(q),M} \cap g^{-1}(S)) > 0$. Set $K_{\alpha(q),M} = g(\hat{K}_{\alpha(q),M})$ and define the measure ν, supported on $K_{\alpha(q),M}$, by

$$\nu(A) = \hat{\mu}_q(g^{-1}(A) \cap \hat{K}_{\alpha(q),M}), \qquad A \subset R^m. \tag{2.22}$$

We need the following lemma which is similar to Lemma 2.7.

LEMMA 2.10. *If* $E \subset R^m$ *and* $|E| < t_{\min}^M$, *then*

$$\nu(E) = \nu(K_{\alpha(q),M} \cap E) \leqslant c\, |E|^{f(q)-\delta}. \tag{2.23}$$

Proof of Lemma 2.10. For each $\sigma \in g^{-1}(E \cap K_{\alpha(q),M}) \cap \hat{K}_{\alpha(q),M}$ choose $m(\sigma)$ such that

$$|J(\sigma | m(\sigma))| < |E| \leqslant |J(\sigma | m(\sigma) - 1)|. \tag{2.24}$$

Evidently, $m(\sigma) > M$, and since $g(\sigma) \in J(\sigma | m(\sigma)) \cap E$, $J(\sigma | m(\sigma)) \cap E \neq \varnothing$.

According to Lemma 2.6, the cardinality of the set of $\sigma | m(\sigma)$ is no more than c. Now,

$$\nu(E \cap K_{\alpha(q),M}) = \hat{\mu}_q(g^{-1}(E \cap K_{\alpha(q),M}) \cap \hat{K}_{\alpha(q),M}). \tag{2.25}$$

Since the cylinder sets $C(\sigma | m(\sigma))$ are pairwise disjoint and cover $g^{-1}(E \cap K_{\alpha(q),M}) \cap \hat{K}_{\alpha(q),M}$,

$$\nu(E \cap K_{\alpha(q),M}) \leqslant \sum \hat{\mu}_q(C(\sigma | m(\sigma)), \tag{2.26}$$

or

$$\leqslant \sum p(\sigma | m(\sigma))^q \cdot t(\sigma | m(\sigma))^{\beta(q)}. \tag{2.27}$$

From (2.21), it follows that $p(\sigma | m(\sigma))^q \leqslant t(\sigma | m(\sigma))^{q\alpha(q)-\delta}$. Thus,

$$\nu(E \cap K_{\alpha(q),M}) \leqslant \sum t(\sigma | m(\sigma))^{q\alpha(q)+\beta(q)-\delta}. \tag{2.28}$$

But, $t(\sigma \mid m(\sigma)) = |J(\sigma \mid m(\sigma))| < |E|$. So,

$$\nu(E) = \nu(E \cap K_{\alpha(q),M}) \leqslant c\,|E|^{f(q)-\delta}. \tag{2.29}$$

Completion of the Proof of Theorem 2.9. The same reasoning as that used for the proof of Theorem 2.8 together with the help of Lemma 2.10 now can be used to prove the inequality

$$\nu(A) \leqslant c\mathscr{H}^{f(q)-\delta}(A), \qquad \text{for} \quad A \subset R^m. \tag{2.30}$$

But this is a contradiction since $\nu(S) = \hat{\mu}_q(g^{-1}(S) \cap \hat{K}_{\alpha(q),M}) > 0$.

THEOREM 2.11. *For each $q < 0$, $\dim(K_{\alpha(q)}) \geqslant f(q)$. In fact, μ_q, which is supported on $K_{\alpha(q)}$, has dimension $f(q)$.*

There is a proof of this theorem very similar to the one just given.

Remark 2.12. In particular, since $\mu_1 = \rho$, $f(\alpha(1)) = \alpha(1) = \dim \rho$, as remarked in Section 1 (see Remark 1.8). In addition, we have the following completeness property:

COROLLARY 2.13. $\rho(\bigcup_{\alpha \in [\underline{\lambda}, \bar{\lambda}]} K_\alpha) = \rho(K) = 1$.

Finally, we analyze the limiting behaviour as $q \to \infty$ and $q \to -\infty$. Recall

$$1 = \sum_{\lambda_i = \underline{\lambda}} t_i^e \tag{1.8}$$

and

$$1 = \sum_{\lambda_i = \bar{\lambda}} t_i^{\bar{e}}. \tag{1.11}$$

THEOREM 2.14. $\dim K_{\alpha(\infty)} = \dim K_{\underline{\lambda}} = f(\alpha(\infty)) = e$ *and* $\dim K_{\alpha(-\infty)} = \dim K_{\bar{\lambda}} = f(\alpha(-\infty)) = \bar{e}$.

Proof. Let $M = \{\sigma \in \Omega : \forall i \log p_{\sigma(i)}/\log t_{\sigma(i)} = \underline{\lambda}\}$. Then $g(M)$ is a subset of $K_{\underline{\lambda}}$ and $g(M)$ is given by a Moran construction with reduction ratios given by the t_i's for which $\log p_i/\log t_i = \underline{\lambda}$. So the Hausdorff dimension of $g(M)$ is e. Thus, $\dim K_{\underline{\lambda}} \geqslant e$. On the other hand, for each $q > 0$ and $\delta > 0$, we have from Theorem 2.3, $\mathscr{H}^{f(q)+\delta}(U_q) \leqslant 1$. Since $K_{\underline{\lambda}} \subset U_q$, $\mathscr{H}^{f(q)+\delta}(K_{\underline{\lambda}}) \leqslant 1$. Thus, $\dim K_{\underline{\lambda}} \leqslant f(q)$, $q > 0$. Since $f(q)$ decreases to e at ∞, $\dim K_{\underline{\lambda}} \leqslant e$.

The proof that $\dim K_{\bar{\lambda}} = \bar{e}$ is similar.

Remark 2.15. Recall as $q \to \infty$, $p_i^q t_i^{\beta(q)} \to 0$ if $\lambda_i > \underline{\lambda}$, and $p_i^q t_i^{\beta(q)} \to t_i^e$ if $\lambda_i = \underline{\lambda}$. Set $G(\underline{\lambda}) = \{i : \lambda_i = \underline{\lambda}\}$. We define $\hat{\mu}_\infty$ to be infinite product measure on $\Omega_\lambda = \{\sigma \in \Omega : \forall i \sigma(i) \in G(\lambda)\}$ generated by the probability vector $(t_i^e)_{i \in G(\lambda)}$. As the proof of Theorem 2.12 shows, μ_∞, the image of $\hat{\mu}_\infty$ under g, has dimension $f(\infty) = f(\alpha(\infty)) = e$. In a similar way, one can define $\mu_{-\infty}$ and obtain the analogous result, $\dim \mu_{-\infty} = f(-\infty) = \bar{e}$.

Remark 2.16. Since $p_i = t_i^\lambda$ and $\sum_{\lambda_i = \lambda} p_i < 1$, Eq. (1.8) gives $e < \lambda$. Similarly, from Eq. (1.11), we have $\bar{e} < \bar{\lambda}$. These relations, together with $f(\alpha(1)) = \alpha(1)$, reflect the concavity of the $f(\alpha)$ vs. α curve (see also Remark 1.9).

Remark 2.17. Only in the cases $q = 0$, ∞, and $-\infty$ have we actually obtained information about the measure of $K_{\alpha(q)}$ with respect to $\mathscr{H}^{f(q)}: 0 < \mathscr{H}^{f(0)}(K_{\alpha(0)}) < 1$, $0 < \mathscr{H}^{f(\infty)}(K_{\alpha(\infty)})$, and $0 < \mathscr{H}^{f(-\infty)}(K_{\alpha(-\infty)})$. In case $e(\bar{e})$ is zero, then $K_{\alpha(\infty)}(K_{\alpha(-\infty)})$ is uncountable and $\mathscr{H}^{f(\infty)}(K_{\alpha(\infty)}) = \infty$ $(\mathscr{H}^{f(-\infty)}(K_{\alpha(-\infty)}) = \infty)$.

QUESTIONS 2.18. *If $q \neq 0$, is it true that $0 < \mathscr{H}^{f(q)}(K_q) < \infty$? (This seems unlikely in general.) If this is not so, is there a slowly varying function $L(t)$ such that $0 < \mathscr{H}^h(K_q) < \infty$, where $h(t) = t^{f(q)}L(t)$?*

The following classical example illustrates our results and our questions.

EXAMPLE 2.19. The fundamental strong law of large numbers as given by Borel yields that for Lebesgue measure almost all x in $[0, 1]$,

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} x_i = \frac{1}{2},$$

where x has its binary expansion: $x = \cdots x_1 x_2 x_3 \cdots$.

Picking up on earlier partial results of Besicovitch [Bes], Eggleston [Eg] extended this in a certain direction by showing that for each m, $0 \leqslant m \leqslant 1$, the Hausdorff dimension of the set X_m consisting of those x for which the arithmetic density is m, i.e., for which

$$\bar{x} = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} x_i = m, \tag{2.31}$$

is $[m \log m + (1 - m) \log(1 - m)]/\log \frac{1}{2}$, where $0 \log 0 = 0$. We obtain this result as follows. Let $J = [0, 1]$, $t_1 = t_2 = \frac{1}{2}$. Then the sets $J(\sigma)$, $\sigma \in S_k$, are simply the dyadic subintervals of J with length $(\frac{1}{2})^k$. The Moran fractal is a "pathological" fractal, it is the unit interval itself. For each x in $[0, 1]$ which is not a dyadic rational there is a unique $\sigma \in S_k$ such that $x \in J(\sigma)$, namely $\sigma = (x_1, \ldots, x_k)$. For each p, $0 < p < 1$, consider the probability vector $(1-p=p_1, p=p_2)$ and the corresponding ρ measure on $[0, 1]$. If $\sigma \in S_k$, then

$$\rho(J(\sigma)) = p^{\sum_{i=1}^k x_i}(1-p)^{k-\sum_{i=1}^k x_i} \qquad \text{and} \qquad |J(\sigma)| = 2^{-|\sigma|} = 2^{-k},$$

whence

$$K_{\alpha(q)} = \left\{ x \in [0, 1] : \frac{(1-\bar{x})\log(1-p) + \bar{x}\log p}{\log(1/2)} = \alpha(q) \right\},$$

with

$$\alpha(q) = \frac{(1-p)^q \log(1-p) + p^q \log p}{[(1-p)^q + p^q] \cdot \log(1/2)}.$$

The value of $\bar{x} = \bar{x}(q)$ for points $x \in K_{\alpha(q)}$ is given uniquely, for $p \neq \frac{1}{2}$, by

$$\bar{x}(q) = \frac{\alpha(q)\log(1/2) - \log(1-p)}{\log p - \log(1-p)}.$$

For each $m \in [0, 1]$, the monotone property of $\alpha(q)$ guarantees a unique $q = q(m)$ for which $\bar{x} = \bar{x}(q) = m$. Recall $X_m = \{x \in [0, 1] : \bar{x} = m\}$. For $m \neq \frac{1}{2}$, 0, or 1, and $p = m$, $X_m = K_{\alpha(1)}$, while for $m = \frac{1}{2}$, we have $X_{1/2} = K_{\alpha(0)}$, independently of the value of p, $p \neq \frac{1}{2}$. Assume $p > \frac{1}{2}$ for simplicity. Then $q(0) = -\infty$ and $X_0 = K_{\alpha(-\infty)}$. Also, $q(1) = +\infty$ and $X_1 = K_{\alpha(+\infty)}$. Theorem 2.1 now gives

$$\dim X_m = [m \log m + (1-m)\log(1-m)]/\log \tfrac{1}{2}, \qquad m \in [0, 1].$$

This confirms Eggleston's result. In fact, however, Eggleston's results are more general in that they encompass n-ary expansions for all $n \geqslant 2$. The latter more general results also can be obtained from multifractal theory in the same way as for $n = 2$ given above. Smorodinsky [Sm] has shown for the binary case that for $m \neq 0$, $\frac{1}{2}$, or 1, if h is a Hausdorff gauge function concave near 0, then $\mathscr{H}^h(X_m)$ is either 0 or ∞.

3. Pointwise Dimension of ρ

In this section, we relate, for each $\sigma \in \Omega$, the asymptotic behaviour of $\log p(\sigma \mid k)/\log t(\sigma \mid k)$ to the local behaviour of ρ at the point $g(\sigma)$. In particular, we show that for map specified, pairwise disjoint constructions, $\sigma \in \hat{K}_\alpha$ if and only if the pointwise dimension of ρ at $g(\sigma)$ is α. Thus, for these constructions, the asymptotic behaviour of ratios on the abstract symbol space Ω can be transferred to the geometric behaviour of a measure on a geometric realization of the symbol space. When the pairwise disjoint conditions are not met, we are unable to achieve the same result. We may be overlooking something simple, but we do not have the estimates to push the proof through. On the other hand, we do not have a counterexample. Let $B(x, \varepsilon)$ be the ball with center x and diameter ε. Note that we are not following the usual convention that $B(x, \varepsilon)$ is the ball with radius ε.

THEOREM 3.1. *Let K be a Moran fractal and let $g(\sigma) = x$. Then*

$$\begin{aligned}\limsup_{\varepsilon \to 0} \log \rho(B(g(\sigma), \varepsilon))/\log \varepsilon \\ &\leqslant \limsup_{k \to \infty} \log \rho(J(\sigma \mid k))/\log |J(\sigma \mid k)| \\ &= \limsup\nolimits_{k \to \infty} \log p(\sigma \mid k)/\log t(\sigma \mid k). \qquad (3.1)\end{aligned}$$

Proof. Temporarily fix ε with $\varepsilon/2 < \min t_i = 2M$. Choose k such that $J(\sigma \mid k) \subset B(x, \varepsilon)$ and $J(\sigma \mid k-1)$ is not a subset of $B(x, \varepsilon)$. Note that $k \geqslant 2$ and

$$\log \rho(J(\sigma \mid k)) \leqslant \log \rho(B(x, \varepsilon)). \qquad (3.2)$$

Also, $|J(\sigma \mid k)| \leqslant \varepsilon$ and $|J(\sigma \mid k)|/t_{\sigma(k)} = |J(\sigma \mid k-1)| \geqslant \epsilon/2$, whence

$$\log \varepsilon + \log M \leqslant \log |J(\sigma \mid k)|. \qquad (3.3)$$

From inequalities (3.2), (3.3), and some algebra,

$$\begin{aligned}\log \rho(B(x, \varepsilon))/\log \varepsilon \leqslant [\log \rho(J(\sigma \mid k))/\log |J(\sigma \mid k)|] \\ \cdot [(\log M/\log \varepsilon) + 1].\end{aligned}$$

One obtains (3.1) by letting $\varepsilon \to 0$ in this last inequality.

THEOREM 3.2. *Let K be the Moran fractal generated by a pairwise disjoint and map specified construction. In other words, K is a Cantor set generated by an iterated function system of similarities. Let $\sigma \in \Omega$ and let $x = g(\sigma)$. Then*

$$\begin{aligned}\liminf_{\varepsilon \to 0} \log \rho(B(g(\sigma), \varepsilon))/\log \varepsilon \\ &\geqslant \liminf_{k \to \infty} \log \rho(J(\sigma / k))/\log |J(\sigma \mid k)| \\ &= \liminf_{k \to \infty} \log p(\sigma \mid k)/\log t(\sigma \mid k). \qquad (3.4)\end{aligned}$$

Proof. Let $D = \min\{d(u, v) : u \in J(i), v \in J(j), \text{ and } i \neq j\}$. Temporarily fix $\varepsilon > 0$ with $\varepsilon < D$. Let $h_\varepsilon(x) = \max\{i : B(x, \varepsilon) \cap K \subset J(\sigma \mid i)\}$.

We have

$$\log \rho(B(x, \varepsilon)) \leqslant \log \rho(J(\sigma \mid h_\varepsilon(x))). \tag{3.5}$$

Also, there is some $s \neq \sigma(h_\varepsilon(x)+1)$ and point $y \in J((\sigma \mid h_\varepsilon(x))^* s)) \cap K \cap \mathscr{B}(x, \varepsilon)$. Since the construction is map specified, this implies that

$$\varepsilon/2 > d(x, y) > D \cdot t_{\sigma \mid h_\varepsilon(x)}. \tag{3.6}$$

Thus, there is a constant U, independent of ε and x, such that

$$U \cdot \varepsilon > |J(\sigma \mid h_\varepsilon(x))|. \tag{3.7}$$

Using (3.5), (3.7), and some algebra,

$$\log \rho(B(x, \varepsilon))/\log \varepsilon \geqslant [\log \rho(J(\sigma \mid h_\varepsilon(x)))/\log |J(\sigma \mid h_\varepsilon(x))|] \cdot [(\log U/\log \varepsilon) + 1].$$

Letting $\varepsilon \to 0$ in this last inequality, one obtains (3.4).

Putting the last two theorems together, we have

THEOREM 3.3. *Let K be the Moran fractal generated by a pairwise disjoint and map specified construction. Then $x \in K_\alpha$ if and only if the pointwise dimension of ρ at x is α:*

$$\lim_{\varepsilon \to 0} \log \rho(B(x, \varepsilon))/\log \varepsilon = \alpha. \tag{3.8}$$

Remark 3.4. Young [Yo] showed that for measures μ on compact Riemannian manifolds, Eq. (3.8) implies $\dim \mu = \alpha$, which is the content of the second part of Theorem 2.1.

4. GENERALIZED MULTIFRACTALS AND $f(\alpha)$ CURVES WITH WEIGHTS

The infinite product measure $\hat{\mu}_q$, introduced in Section 2 as an auxiliary measure to mediate the multifractal decomposition of K, is only one of many shift-invariant ergodic measures on Ω. Any one of these measures might serve as a mediating measure to induce possibly other multifractal decompositions. We exhibit a class of mediating measures which includes $\hat{\mu}_q$ as a special case and we display some properties of the induced, generalized multifractal decompositions. The basic calculations involve some rather messy algebra which we have concensed as much as possible. Our motivation for bringing this out is explained in the introduction.

We begin by considering a second class of auxiliary functions analogous to those defined in Section 1. We modify Eq. 1.1, which expressed the normalization of the probability vector for $\hat{\mu}_q$, by appending a system of nonnegative weights w,

$$1 = \sum_{i=1}^{n} w_i p_i^q t_i^{\beta(q, w)}, \qquad w_i \geqslant 0, \quad i = 1, ..., n, \tag{4.1}$$

where $w = (w_1, ..., w_n)$ and where $I_w = \{i : w_i > 0\} \neq \varnothing$. For a given vector of weights, w, and for each $q \in R$, there is a unique number $\beta(q, w)$ such that (4.1) holds. When $w_1 = \cdots = w_n = 1$, $\beta(q, w) = \beta(q, 1) = \beta(q)$, introduced in (1.1). As before

$$\lim_{q \to \infty} \beta(q, w) = -\infty \qquad \text{and} \qquad \lim_{q \to -\infty} \beta(q, w) = +\infty.$$

Also,

$$\beta'(q, w) = -\sum_{i=1}^{n} (\log p_i) \cdot w_i p_i^q t_i^{\beta(q, w)} \Big/ \sum_{i=1}^{n} (\log t_i) \cdot w_i p_i^q t_i^{\beta(q, w)} \tag{4.2}$$

and

$$\beta''(q, w) = -\sum_{i=1}^{n} (\log p_i + \beta'(q, w) \cdot \log t_i)^2 w_i p_i^q t_i^{\beta(q, w)} \Big/ \sum_{i=1}^{n} (\log t_i) \cdot w_i p_i^q t_i^{\beta(q, w)}, \tag{4.3}$$

where the prime denotes partial differentiation by q. Again, $\beta'(q, w) < 0$, for all q, so that $\beta(q, w)$ is a strictly decreasing function of q. In addition, if there is no c such that $p_i = t_i^c$, for all $i \in I_w$, we have $\beta''(q, w) > 0$ for all q, so that $\alpha(q, w) = -\beta'(q, w)$ is a strictly decreasing function of q. If $p_i = t_i^c$, for $i \in I_w$, then we have the degenerate case: $\beta(q, w) = -qc + \beta(0, w)$ and $\alpha(q, w) = c$. Note this is automatically the case when I_w is a singleton. Only if $I_w = \{1, ..., n\}$ does $c = d$ necessaily hold.

In place of Eq. (1.5) for $f(q)$, we introduce a new auxiliary function,

$$f(q, w) = \beta(q, w) + q\alpha(q, w) + \gamma(q, w), \tag{4.4}$$

where

$$\gamma(q, w) = \sum_{i=1}^{n} (\log w_i) \cdot w_i p_i^q t_i^{\beta(q, w)} \Big/ \sum_{i=1}^{n} (\log t_i) \cdot w_i p_i^q t_i^{\beta(q, w)}. \tag{4.5}$$

Evidently, $f(q, 1) = f(q)$. However, $f(q, w)$ does not share some other properties of $f(q)$; e.g., monotonicity on $(-\infty, 0)$ or $(0, +\infty)$. If $p_i = t_i^c$, $i \in I_w$, then $f(q, w)$ is a constant, independent of q. If I_w is a singleton, α is a positive constant and $f = 0$. Owing to the monotone behaviour of $\alpha(q, w)$, for any w, we can solve for f as a function of α and w, except of course, the degenerate case $p_i = t_i^c$, $i \in I_w$. That is, for each α between $\alpha(+\infty, w)$ and $\alpha(-\infty, w)$, $f(\alpha, w) = f((q(\alpha, w), w)$, where $q = q(\alpha, w)$. This generalized multifractal spectrum $f(\alpha, w)$ is no longer necessarily everywhere concave downward. A computer calculation of $f(\alpha, w)$ is displayed at the end of this section in Fig. 4.4.

The asymptotic behaviour of the new auxiliary functions is determined in most instances by the same kinds of analyses as before, merely appending the w-dependences. Excluding now the case $p_i = t_i^c$, $i \in I_w$, *mutatis mutandis*, we find

$$\lim_{q \to \infty} \beta(q, w) + \lambda_w q = e(w), \tag{4.6}$$

$$\lim_{q \to -\infty} \beta(q, w) + \bar{\lambda}_w q = \bar{e}(w), \tag{4.7}$$

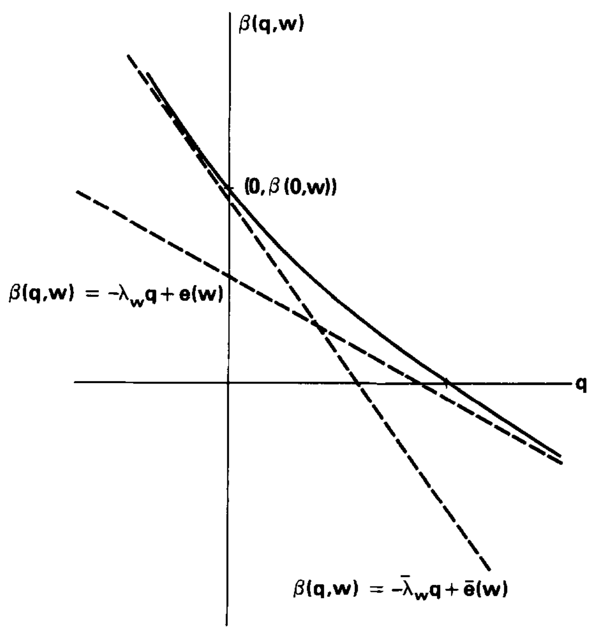

FIG. 4.1. $\beta(q, w)$: smooth, concave upward and strictly decreasing for any choice of non-negative weights with at least two positive; a straight line with negative slope results if only one weight is positive and the others are zero. Slopes and intercepts of the asymptotes are as shown. When all the weights are unity the intercepts are $q = 1$ and $\beta(0, w) = \beta(0, 1) = d$, as in Fig. 1.1. The case $p_i = t_i^c$, $i \in I_w$, again gives a straight line. Graphs of $\beta(q, w)$ against q for gauge equivalent weights are translates of one another, viz. $\beta \to \beta - b$, $q \to q + a$ (see Section 5).

where

$$1 = \sum_{\lambda_i = \lambda_w} w_i t_i^{e(w)} \quad \text{and} \quad 1 = \sum_{\lambda_i = \bar{\lambda}_w} w_i t_i^{\bar{e}(w)} \tag{4.8}$$

in which

$$\lambda_w = \min(\lambda_i : i \in I_w) \quad \text{and} \quad \bar{\lambda}_w = \max(\lambda_i : i \in I_w)$$

and

$$\lim_{q \to +\infty} \alpha(q, w) = \lambda_w \tag{4.9}$$

and

$$\lim_{q \to -\infty} \alpha(q, w) = \bar{\lambda}_w. \tag{4.10}$$

We note that for given p_i's and t_i's, λ_w and $\bar{\lambda}_w$ depend only on I_w. In particular, we note that if the weights are all positive $\alpha(+\infty, w) = \lambda$ and $\alpha(-\infty, w) = \bar{\lambda}$, so that $\alpha(+\infty, w)$ and $\alpha(-\infty, w)$ are independent of w.

Figures 4.1, 4.2, and 4.3 are sketches of the graphs of $\beta(q, w)$, $\alpha(q, w)$, and $\gamma(q, w)$ illustrating their principal features.

In addition and in the same way as before (see (1.16)), $q \cdot (\alpha(q, w) - \lambda_w) \to 0$ as $q \to +\infty$ and $q \cdot (\alpha(q, w) - \bar{\lambda}_w) \to 0$ as $q \to -\infty$. To obtain the asymptotic forms of $\gamma(q, w)$, one proceeds as in the analysis surrounding Eqs. (1.15) and (1.16). We get

$$\lim_{q \to +\infty} \gamma(q, w) = g(w) = T(w)/S(w), \tag{4.11}$$

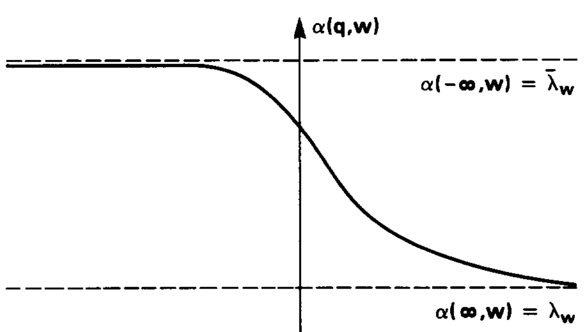

FIG. 4.2. $\alpha(q, w) = -\beta'(q, w)$: strictly decreasing in q for any choice of nonnegative weights with at least two positive; a single constant value results if only one weight is positive. The limits λ_w and $\bar{\lambda}_w$, at $q = \infty$ and $q = -\infty$, respectively, depend only on I_w; in particular, if all the weights are positive, $\lambda_w = \lambda$ and $\bar{\lambda}_w = \bar{\lambda}$. Gauge transformed weights result in graphs of $\alpha(q, w)$ against q that are horizontal translates of one another, viz. $\alpha \to \alpha$, $q \to q + a$ (see Section 5).

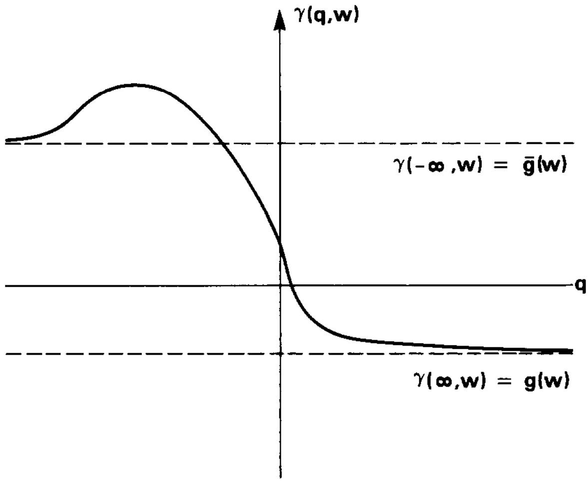

FIG. 4.3. $\gamma(q, w)$; finite and smooth for all $q \in R$. For any choice of nonnegative weights $\gamma(q, w)$ tends to finite limits, $g(w)$ at $q = \infty$ and $\bar{g}(w)$ at $q = -\infty$. If $0 < w_i < 1$, $i \in I_w$, then $\gamma(q, w) > 0$, while if $w_i > 1$, $i \in I_w$, $\gamma(q, w) < 0$; if all the weights are one, then γ vanishes for all q. When $p_i = t_i^c$, $i \in I_w$, $\gamma(q, w)$ is constant, independent of q. Gauge transformation properties of $\gamma(q, w)$ depend on $\alpha(q, w)$ (see Section 5).

where

$$S(w) = \sum_{\lambda_i = \lambda_w} \log t_i \cdot w_i t_i^{e(w)} \quad \text{and} \quad T(w) = \sum_{\lambda_i = \lambda_w} \log w_i \cdot w_i t_i^{e(w)}. \tag{4.12}$$

Also,

$$\lim_{q \to -\infty} \gamma(q, w) = \bar{g}(w) = \bar{T}(w)/\bar{S}(w), \tag{4.13}$$

where

$$\bar{S}(w) = \sum_{\lambda_i = \bar{\lambda}_w} \log t_i \cdot w_i t_i^{\bar{e}(w)} \quad \text{and} \quad \bar{T}(w) = \sum_{\lambda_i = \bar{\lambda}_w} \log w_i \cdot w_i t_i^{\bar{e}(w)}. \tag{4.14}$$

These results supply us with

THEOREM 4.1. *The following asymptotic limits hold*:

$$\lim_{q \to \infty} f(q, w) = e(w) + g(w) \quad \textit{and} \quad \lim_{q \to -\infty} f(q, w) = \bar{e}(w) + \bar{g}(w). \tag{4.15}$$

Finally, there is the somewhat more involved matter of the slopes of

$f(\alpha, w)$ at its endpoints, corresponding to $q \to \infty$ and $q \to -\infty$; that is, we seek the asymptotic behaviour of

$$df(\alpha, w)/d\alpha = f'(q, w)/\alpha'(q, w) = q + \gamma'(q, w)/\alpha'(q, w). \tag{4.16}$$

THEOREM 4.2. *Except for the case* $p_i = t_i^c$, $i \in I_w$,

$$\lim_{\alpha \to \alpha(\infty)} df(\alpha, w)/d\alpha = \infty \qquad \textit{and} \qquad \lim_{\alpha \to \alpha(-\infty)} df(\alpha, w)/d\alpha = -\infty.$$

Proof. It suffices to establish for any w that

$$\left| \lim_{|q| \to \infty} \gamma'(q, w)/\alpha'(q, w) \right| < \infty. \tag{4.17}$$

We first need more precise information concerning the asymptotic form of $\beta(q, w)$. For the $q \to \infty$ behaviour, we have

$$\beta(q, w) = -\lambda_w q + e(w) + \delta(q, w),$$

where $\delta(q, w)$ is finite for all q, and as already noted, goes to zero as $q \to \infty$. Re-express Eq. (4.1) after the method for Eq. (1.15) *et seq.*,

$$\begin{aligned} 1 &= \sum_{\lambda_i = \lambda_w} w_i \cdot t_i^{\beta(q,w) + \lambda_i q} + \sum_{\lambda_i > \lambda_w} w_i \cdot t_i^{\beta(q,w) + \lambda_i q} \\ &= \sum_{\lambda_i = \lambda_w} w_i \cdot t_i^{e(w)} \cdot \exp[\log t_i \cdot \delta(q, w)] \\ &\quad + \sum_{\lambda_i > \lambda_w} w_i \cdot t_i^{e(w)} \cdot \exp[(\lambda_i - \lambda_w) \cdot \log t_i \cdot q + \log t_i \cdot \delta(q, w)]. \end{aligned} \tag{4.18}$$

We regard Eq. (4.18) as an equation for $\delta(q, w)$ which we will solve to leading order asymptotically. To organize the problem we note first that, since, by the same argument as that previously adduced beneath Eq. (1.10) to prove Eq. (1.7), $\delta(q, w)$ tends to zero as $q \to \infty$, each term of the second sum is exponentially decreasing. The second sum will be dominated by the subset of terms for which that rate of decrease is smallest. If we set $I_0 = \{i \mid \lambda_i > \lambda_w,\ i \in I_w\}$, then that dominating subset is (indexed by) I_{k_0}, namely,

$$I_{k_0} = \{i_k, k \in I_0 \mid (\lambda_{i_k} - \lambda_w) \log t_{i_k} = \max_{i \in I_w} (\lambda_i - \lambda_w) \log t_i \equiv -\mu_0\},$$

which is necessarily nonempty since the case $p_i = t_i^c$, $i \in I_w$, has been excluded here and in particular, also, I_w has at least two elements. Taking note of Eq. (4.8), we rewrite Eq. (4.18) as

$$\sum_{\lambda_i=\lambda_w} w_i t_i^{e(w)}\{1-\exp[\log t_i\cdot\delta(q,w)]\}$$
$$=\left[\sum_{k=1}^{k_0} w_{i_k} t_{i_k}^{e(w)+\delta(q,w)}\right]e^{-\mu_0 q}$$
$$+\sum_{\lambda_i>\lambda_w}{}' w_i t_i^{e(w)}\exp[(\lambda_i-\lambda_w)\log t_i\cdot q+\log t_i\cdot\delta(q,w)], \tag{4.19}$$

where the prime on the last sum denotes omission of the terms indexed by members of I_{k_0}. The last sum can be organized in the same way, as it, too, is dominated asymptotically by the subset of terms for which the rate of exponential decrease is minimized. To iterate this procedure, we introduce the indexing sets $I_{r+1}=I_r\backslash I_{k_r}$, $r=0, ..., R$, with

$$I_{k_r}=\{i_k, k=1, ..., k_r \mid (\lambda_{i_k}-\lambda_w)\log t_{i_k}=\max_{i\in I_r}(\lambda_i-\lambda_w)\log t_i\equiv -\mu_r\}.$$

We note that $0<\mu_0<\mu_1<\cdots<\mu_R$. Equation (4.19) becomes

$$\sum_{\lambda_i=\lambda_w} w_i t_i^{e(w)}\{1-\exp[\log t_i\cdot\delta(q,w)]\}$$
$$=\sum_{r=0}^{R}\left[\sum_{k=1}^{k_r} w_{i_k} t_{i_k}^{e(w)+\delta(q,w)}\right]e^{-\mu_r q}. \tag{4.20}$$

Rearranging,

$$\left[-\sum_{\lambda_i=\lambda_w}\log t_i\cdot w_i t_i^{e(w)}\right]\delta(q,w)$$
$$=\sum_{r=0}^{R}\left[\sum_{k=1}^{k_r} w_{i_k} t_{i_k}^{e(w)+\delta(q,w)}\right]e^{-\mu_r q}$$
$$+\sum_{\lambda_i=\lambda_w} w_i t_i^{e(w)}\{\exp[\log t_i\cdot\delta(q,w)]-1-\log t_i\cdot\delta(q,w)\},$$

whence

$$\delta(q,w)=\Delta_0(w,\delta)e^{-\mu_0 q}+\sum_{r=1}^{R}\Delta_r(w,\delta)e^{-\mu_r q}$$
$$+\sum_{\lambda_i=\lambda_w} C_i(w)\{\exp[\log t_i\cdot\delta(q,w)]-1-\log t_i\cdot\delta(q,w)\}, \tag{4.21}$$

where

$$\Delta_r(w, \delta(q, w)) = -\frac{\sum_{k=1}^{k_r} w_{i_k} t_{i_k}^{e(w)+\delta(q,w)}}{\sum_{\lambda_i=\lambda_w} \log t_i \cdot w_i t_i^{e(w)}}, \qquad r = 0, 1, ..., R,$$

$$C_i(w) = \frac{-w_i t_i^{e(w)}}{\sum_{\lambda_i=\lambda_w} \log t_i \cdot w_i t_i^{e(w)}}, \qquad \lambda_i = \lambda_w.$$

We note that $\Delta_0(w, \delta(q, w)) > 0$ since for $r = 0$ the sum cannot be empty, while for $r > 0$, $\Delta_r(w, \delta(q, w)) \geqslant 0$. Also, $C_i(w) > 0$, and the quantity in curly brackets is positive for q finite. Thus, every term on the right side of Eq. (4.21) is positive, and hence $\delta(q, w) > 0$, for q finite. In particular, we note also that $\delta(q, w)$ is larger than the first term on the right hand side. From the inequality, $e^{-x} - 1 + x < x^2/2$, $x > 0$, we have

$$\delta(q, w) < \Delta_0(w, \delta(w, q)) e^{-\mu_0 q} + \sum_{r=1}^{R} \Delta_r(w, \delta(q, w)) e^{-\mu_r q} + C(w) \delta(q, w)^2, \tag{4.22}$$

where

$$C(w) = 1/2 \sum_{\lambda_i=\lambda_w} C_i(w)(\log t_i)^2 > 0.$$

Since $\mu_0 < \mu_r$, $r = 1, ..., R$, one of the two terms, the first or the last, on the right side of Eq. (4.22) must dominate, as $q \to \infty$ and $\delta(q, w) \to 0$. But $\delta(q, w) > 0$, and every term on the right hand side is positive, so it cannot be the last, as then there would be a positive constant Q for which $(C(w) \delta(q, w))^2 - C(w) \delta(q, w) > 0$, $q > Q$, which can't happen since $C(w) \delta(q, w) \to 0$ as $q \to \infty$. This implies that $e^{\mu_0 q} \delta(q, w)$ is bounded above as $q \to \infty$. Since also $\Delta_r(w, \delta(q, w)) \leqslant \Delta_r(w, 0) \equiv \Delta_r(w)$, $r = 1, ..., R$, we can now conclude from (4.21) and from (4.22) that

$$0 < e^{\mu_0 q} \delta(q, w) - \Delta_0(w, \delta(q, w)) < \sum_{r=1}^{R} \Delta_r(w) e^{-(\mu_r - \mu_0) q} + e^{-\mu_0 q} C(w)(e^{\mu_0 q} \delta(q, w))^2. \tag{4.23}$$

Thus, as $q \to \infty$, the right side of (4.23) tends to zero and we have

$$\lim_{q \to \infty} [e^{\mu_0 q} \delta(q, w) - \Delta_0(w, \delta(w, q))] = 0. \tag{4.24}$$

Using the explicit form of $\Delta_0(w, \delta(q, w))$, and setting $\Delta_0(w, 0) \equiv \Delta_0(w)$, we have

$$\lim_{q \to \infty} [e^{\mu_0 q} \delta(q, w) - \Delta_0(w)] = 0, \tag{4.25}$$

which gives the asymptotic form of $\delta(q, w)$ as $\delta \sim \Delta_0(w) e^{-\mu_0 q}$.

We use this result to determine the large q behavior of the quantity

$$E_w[\varphi(p, t, w)] = \sum_{i=1}^{n} \varphi_i w_i p_i^q t_i^{\beta(q, w)}, \tag{4.26}$$

where $\varphi_i = \varphi(p_i, t_i, w_i)$, which we need to fix the forms of $\gamma'(q, w)$ and $\alpha'(q, w)$. We have

$$\begin{aligned} E_w[\varphi] &= \sum_{\lambda_i = \lambda_w} \varphi_i w_i t_i^{\beta(q,w) + \lambda_i q} + \sum_{\lambda_i > \lambda_w} \varphi_i w_i t_i^{\beta(q,w) + \lambda_i q} \\ &= \sum_{\lambda_i = \lambda_w} \varphi_i w_i t_i^{e(w)} + e^{-\mu_0 q} \Bigg[\Delta_0(w) \sum_{\lambda_i = \lambda_w} \varphi_i \log t_i \cdot w_i t_i^{e(w)} \\ &\quad + \sum_{k=1}^{k_0} \varphi_{i_k} w_{i_k} t_{i_k}^{e(w)} \Bigg] + \cdots \end{aligned} \tag{4.27}$$

where the neglected terms go to zero faster than $e^{-\mu_0 q}$. After some algebra, we find

$$\alpha(q, w) = \frac{E_w[\log p]}{E_w[\log t]} = \lambda_w + \Lambda(w) e^{-\mu_0 q} + \cdots,$$

where neglected terms tend to zero faster than $e^{-\mu_0 q}$, and where

$$\Lambda(w) = S(w)^{-1} \sum_{k=1}^{k_0} (\lambda_{i_k} - \lambda_w) \log t_{i_k} \cdot w_{i_k} \cdot t_{i_k}^{e(w)}.$$

A similar expression may be seen to result for the asymptotic form of $\gamma(q, w)$,

$$\gamma(q, w) = \frac{E_w[\log w]}{E_w[\log t]} = g(w) + \Gamma(w) e^{-\mu_0 q} + \cdots,$$

where $\Gamma(w)$ is finite for all w. Also, since $\lambda_{i_k} > \lambda_w$, for $i_k \in I_{\kappa_0}$, where I_{k_0} is nonempty, $\Lambda(w)$ cannot vanish, and we have that the limit,

$$\lim_{q \to \infty} \frac{\gamma'(q, w)}{\alpha'(q, w)} = \frac{\Gamma(w)}{\Lambda(w)},$$

is finite.

In the same way we establish the asymptotic forms of $\beta(q, w)$ and $E_w[\varphi]$ for $q \to -\infty$. *Mutatis mutandis*, we find

$$\begin{aligned} \beta(q, w) &= -\bar{\lambda}_w q + \bar{e}(w) + \bar{\delta}(q, w), \\ \bar{\delta}(q, w) &= \bar{\Delta}_0(w) e^{+\bar{\mu}_0 q} + \cdots, \end{aligned}$$

where the dots refer to contributions that go to zero faster than $e^{\bar{\mu}_0 q}$, and where

$$\bar{\Delta}_0(w) = \frac{-\sum_{k=1}^{k_0} w_{i_k} t_{i_k}^{\bar{e}(w)}}{\sum_{\lambda_i = \bar{\lambda}_w} \log t_i \cdot w_i^{\bar{e}(w)}},$$

in which, defining $\bar{I}_0 = \{i \mid \lambda_i < \bar{\lambda}_w,\ i \in I_w\}$, the i_k belong to the indexing set,

$$I_{k_0} = \{i_k, k \in \bar{I}_0 \mid (\lambda_{i_k} - \bar{\lambda}_w) \log t_{i_k} = \min_{i \in \bar{I}_w} (\lambda_i - \bar{\lambda}_w) \log t_i \equiv \bar{\mu}_0\},$$

and wherein $\bar{\mu}_0$ incidentally also has been specified, and may be seen to be positive. Finally, using

$$E_w[\varphi] = \sum_{\lambda_i - \bar{\lambda}_w} \varphi_i w_i t_i^{\bar{e}(w)} + e^{\bar{\mu}_0 q}\left[\bar{\Delta}_0(w) \sum_{\lambda_i = \bar{\lambda}_w} \varphi_i \cdot \log t_i \cdot w_i t_i^{\bar{e}(w)} + \sum_{k=1}^{k_0} \varphi_{i_k} w_{i_k} t_{i_k}^{\bar{e}(w)}\right] + \cdots$$

we find

$$\lim_{q \to -\infty} \frac{\gamma'(q, w)}{\alpha'(q, w)} = \frac{\bar{\Gamma}(w)}{\bar{\Lambda}(w)},$$

where $\bar{\Gamma}(w)$ is finite for all w and $\bar{\Lambda}(w)$ is given by

$$\bar{\Lambda}(w) = \bar{S}(w)^{-1} \sum_{k=1}^{k_0} (\lambda_{i_k} - \bar{\lambda}_w) \log t_{i_k} \cdot w_{i_k} \cdot t_{i_k}^{e(w)},$$

which again cannot vanish, and so we are done.

A sample simulation of $f(\alpha, w)$ is displayed in Fig. 4.4.

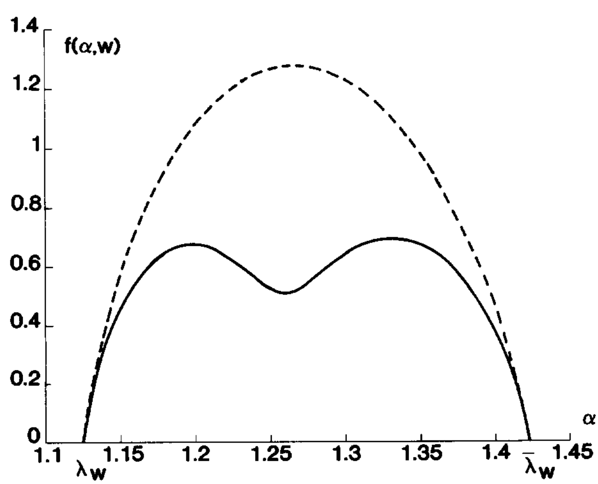

FIG. 4.4. Computer calculation of the generalized spectrum of scaling indices, $f(\alpha, w)$. The example chosen had $n = 4$, $t_1 = t_2 = t_3 = t_4 = \frac{1}{3}$, $p_1 = 0.21$, $p_2 = p_3 = 0.25$, $p_4 = 0.29$, and $w_1 = w_2 = 1$, $w_3 = w_4 = 0.01$. The dashed curve has all the weights equal to unity.

We are now in a position to analyze the multifractal decomposition induced by a system of weights. The decomposition is given in terms of the pointwise behaviour or the measure and the weights. For each w, set $\bar{\Omega} = \{\sigma \in \Omega : \forall i \sigma(i) \in I_w\}$ so that $\bar{\Omega} = \Omega$ if all the weights are positive. For each q and w, let

$$\hat{K}^{q,w} = \{\sigma \in \bar{\Omega} : \lim_{k \to \infty} \log p(\sigma \mid k)/\log t(\sigma \mid k) = \alpha(q, w)$$
$$\text{and } \lim_{k \to \infty} \log w(\sigma \mid k)/\log t(\sigma \mid k) = \gamma(q, w)\}. \tag{4.28}$$

Set

$$K^{q,w} = g(\hat{K}^{q,w}). \tag{4.29}$$

THEOREM 4.3. *Let $w = (w_1, ..., w_n)$ be a system of weights. For each $q \in R$,*

$$\dim K^{q,w} = f(q, w) = \beta(q, w) + q\alpha(q, w) + \gamma(q, w). \tag{4.30}$$

Proof. The proof of this theorem is similar to the proof given in Section 2 for the case $w = 1$. We just indicate the alterations required for the case $q > 0$. For $q > 0$, let

$$\hat{U}_{q,w} = \{\sigma \in \bar{\Omega} : \limsup_{k \to \infty} \log p(\sigma \mid k)/\log t(\sigma \mid k) \leqslant \alpha(q, w)$$
$$\text{and } \limsup_{k \to \infty} \log w(\sigma \mid k)/\log t(\sigma \mid k) \leqslant \gamma(q, w)\}$$

and set

$$U_{q,w} = g(\hat{U}_{q,w}).$$

The altered form of Lemma 2.4 reads:

LEMMA 4.4. *Let q and δ be positive. For each positive integer m, there is a collection $\mathscr{G}_m$ of pairwise disjoint sets each with diameter less than $1/m$ such that*

$$(1) \qquad \mathscr{H}^{f(q,w)+\delta}\left(U_{q,w} \Big\backslash \bigcup \mathscr{G}_m\right) = 0 \tag{4.31}$$

and

$$(2) \qquad \sum_{G \in \mathscr{G}_m} |G|^{f(q,w)+\delta} \leqslant 1. \tag{4.32}$$

Proof of Lemma 4.4. The proof of this lemma follows that of Lemma 2.4 with the alteration that for each $\sigma \in \hat{U}_{q,w}$, let M_σ be a positive integer such that if $k > M_\sigma$, then

$$\begin{aligned} \log p(\sigma \mid k)/\log t(\sigma \mid k) &< \alpha(q, w) + \delta/2q, \\ \log w(\sigma \mid k)/\log t(\sigma \mid k) &< \gamma(q, w) + \delta/2, \end{aligned} \tag{4.33}$$

and

$$t(\sigma \mid k) < 1/m.$$

Set $\mathscr{V}_m = \{ g(C(\sigma \mid k)) : \sigma \in \hat{U}_{q,w} \text{ and } k \geqslant M_\sigma \}$. Clearly again, $\mathscr{V}_m$ is a Vitali class for $U_{q,w}$. As before, there is a pairwise disjoint subcollection $\mathscr{G}_m$ of $\mathscr{V}_m$ such that either

$$\text{(a)} \qquad \sum_{G \in \mathscr{G}_m} |G|^{f(q,w)+\delta} = \infty, \tag{4.34}$$

or

$$\text{(b)} \qquad \mathscr{H}^{f(q,w)+\delta}\left(U_{q,w} \Big\backslash \bigcup \mathscr{G}_m \right) = 0.$$

However, (4.34) does not hold. To see this, suppose the sets G are $g(C(\sigma_i \mid k_i)) \in \mathscr{G}_m$, $i = 1, ..., j$. From (4.33), we have

$$|g(C(\sigma_i \mid k_i))|^{q\alpha(q,w)+\delta/2} \leqslant t(\sigma_i \mid k_i)^{q\alpha(q,w)+\delta/2} < p(\sigma_i \mid k_i)^q, \tag{4.35}$$

and

$$|g(C(\sigma_i \mid k_i))|^{\gamma(q,w)+\delta/2} \leqslant t(\sigma_i \mid k_i)^{\gamma(q,w)+\delta/2} < w(\sigma_i \mid k_i),$$

for each i. Thus,

$$\sum_{i=1}^{j} |g(C(\sigma_i \mid k_i))|^{f(q,w)+\delta} < \sum_{i=1}^{j} w(\sigma_i \mid k_i)\, p(\sigma_i \mid k_i)^q\, t(\sigma_i / k_i)^{\beta(q,w)}. \tag{4.36}$$

But, since $\sum_{i=1}^{n} w_i p_i^q t_i^{\beta(q,w)} = 1$, it follows as before that

$$\sum_{G \in \mathscr{G}_m} |G|^{f(q,w)+\delta} \leqslant 1. \tag{4.37}$$

Thus, as in Section 2, we have for $q \neq 0$, $\dim(K^{q,w}) \leqslant f(q, w)$.

We turn now to the proof that $\dim(K^{q,w}) \geqslant f(q, w)$. Again, the proof is based upon the geometric Lemma 2.6. We also make use of auxiliary measures $\mu_{q,w}$ supported on $K^{q,w}$ which are the image under the coding map of the infinite product measure $\hat{\mu}_{q,w}$ on $\bar{\Omega}$ based on the probability vector $(w_i p_i^q t_i^{\beta(q,w)})_{i \in I_w}$, where

$$\sum_{i \in I_w} w_i p_i^q t_i^{\beta(q,w)} = 1. \tag{4.1}$$

Note that $\hat{\mu}_{q,w}(\hat{K}^{q,w}) = 1$. This follows from Birkhoff's individual ergodic theorem applied to the shift transformation on $\bar{\Omega}$, $\hat{\mu}_{q,w}$, and the functions $X(\sigma) = \log p_{\sigma(1)}$, $Y(\sigma) = \log w_{\sigma(1)}$, and $Z(\sigma) = \log t_{\sigma(1)}$. Thus, we find that for $\hat{\mu}_{q,w}$ almost all σ,

$$\begin{aligned} \lim_{k \to \infty} (1/k) \log p(\sigma \mid k) &= E[X] = \sum_{i=1}^{n} (\log p_i) \cdot w_i p_i^q t_i^{\beta(q,w)}, \\ \lim_{k \to \infty} (1/k) \log w(\sigma \mid k) &= E[Y] = \sum_{i=1}^{n} (\log w_i) \cdot w_i p_i^q t_i^{\beta(q,w)}, \end{aligned} \tag{4.38}$$

and

$$\lim_{k \to \infty} (1/k) \log t(\sigma \mid k) = E[Z] = \sum_{i=1}^{n} (\log t_i) \cdot w_i p_i^q t_i^{\beta(q,w)}.$$

Taking ratios and using equations (4.2) and (4.5), we have that for $\hat{\mu}_{q,w}$-almost all σ,

$$\begin{aligned} &\lim_{k \to \infty} \log p(\sigma \mid k)/\log t(\sigma \mid k) \\ &\quad = \sum_{i=1}^{n} (\log p_i) \cdot w_i p_i^q t_i^{\beta(q,w)} \Big/ \sum_{i=1}^{n} (\log t_i) \cdot w_i p_i^q t_i^{\beta(q,w)} \\ &\quad - \beta'(q, w) = \alpha(q, w). \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} &\lim_{k \to \infty} \log w(\sigma \mid k) | \log t((\sigma \mid k) \\ &\quad = \sum_{i=1}^{n} (\log w_i) \cdot w_i p_i^q t_i^{\beta(q,w)} \Big/ \sum_{i=1}^{n} (\log t_i) \cdot w_i p_i^q t_i^{\beta(q,w)} = \gamma(q, w). \end{aligned}$$

Thus, $\hat{\mu}_{q,w}(\hat{K}^{q,w}) = 1 = \mu_{q,w}(K^{q,w})$.

The proof for $q < 0$ continues as before. In analogy with Theorems 2.9 and 2.11:

THEOREM 4.5. *For each $q \neq 0$, $\dim(K^{q,w}) \geqslant f(q, w)$. In fact, $\mu_{q,w}$, which is supported on $K^{q,w}$, has dimension $f(q,w)$.*

Proof. We indicate the proof for $q > 0$. It suffices to show $\dim \mu_{q,w}$ is

$f(q, w)$. Assume $\mu_{q,w}(S)=1$ and $\mathscr{H}^{f(q,w)-\delta}(S)=0$, where $0<\delta<f(q, w)$. For each $\sigma \in \hat{K}^{q,w}$, let N_σ be a positive integer such that if $k \geqslant N_\sigma$; then

$$\log p(\sigma \mid k)/\log t(\sigma \mid k) > \alpha(q, w) - \delta/2q \tag{4.40}$$

and

$$\log w(\sigma \mid k)/\log t(\sigma \mid k) > \gamma(q, w) - \delta/2.$$

For each M, let $\hat{K}^{q,w}_M = \{\sigma \in \hat{K}^{q,w} : N_\sigma = M\}$.

Fix M so that $\hat{\mu}_{q,w}(\hat{K}^{q,w}_M \cap g^{-1}(S)) > 0$. Set $K^{q,w}_M = g(\hat{K}^{q,w}_M)$ and define the auxiliary measure ν supported on $K^{q,w}_M$ by

$$\nu(A) = \hat{\mu}_{q,w}(g^{-1}(A) \cap \hat{K}^{q,w}_M), \qquad \text{for} \quad A \subset \mathbb{R}^m. \tag{4.41}$$

As before, in analogy with Lemma 2.10, if $E \subset R^m$ and $|E| < t^M_{\min}$, then

$$\nu(E) = \nu(K^{q,w}_M \cap E) \leqslant c\, |E|^{f(q,w)-\delta}. \tag{4.42}$$

This last inequality implies

$$\nu(A) \leqslant c\mathscr{H}^{f(q,w)-\delta}(A), \qquad \text{for} \quad A \subset R^m. \tag{4.43}$$

But, $\nu(S) > 0$, which is a contradiction.

THEOREM 4.6. $\dim K^{0,w} = f(0, w)$. *In fact, the dimension of* $\mu_{0,w}$ *is* $f(0, w)$.

Proof. For the upper bound, set

$$\hat{U}_{0,w} = \{\sigma \in \bar{\Omega} : \limsup_{k \to \infty} \log w(\sigma \mid k)/\log t(\sigma \mid k) \leqslant \gamma(0, w)\}$$

and set $U_{0,w} = g(\hat{U}_{0,w})$. One proceeds as in the proof of Lemma 4.4 to show that for each $\delta > 0$, $\mathscr{H}^{f(0,w)+\delta}(K^{0,w}) \leqslant \mathscr{H}^{f(0,w)+\delta}(U_{0,w}) \leqslant 1$. Note that in the inequalities obtained in the proof of the lemma, those involving $p(\sigma \mid k)$ are not required, since $f(0, w) = \beta(0, w) + \gamma(0, w)$. Similarly, for the lower bound, one proceeds as in the preceeding theorem.

Remark 4.7. The special case $q=0$ with $w \neq 1$ differs from the case where all the weights are one. For instance, it is no longer necessarily the case that $\dim K^{0,w} = d$; nor is it necessarily so that $0 < \mathscr{H}^{f(0,w)}(K^{0,w}) < \infty$. Cf. Remark 2.17.

We end this section with a brief exploration of properties of $f(\alpha, w)$ as a function of w. When the weights are all equal to unity, so that

$f(\alpha, w) = f(\alpha, 1) = f(\alpha)$, the graph of f vs. α is smooth and everywhere concave downwards, except for the case $p_i = t_i^d$, $i = 1, ..., n$. As noted previously in Section 1, this follows from differentiating the equation

$$\frac{df(\alpha)}{d\alpha} = \frac{f'(q)}{\alpha'(q)} = q, \tag{4.44}$$

where $\alpha = \alpha(q) = -\beta'(q)$; further

$$\frac{d^2 f(\alpha)}{d\alpha^2} = \left[\frac{d\alpha(q)}{dq}\right]^{-1} = -\beta''(q)^{-1} < 0,$$

for all $q \in \mathbb{R}$.

For the general weights case, Eq. (4.44) contains an additional nonzero term $\gamma'(q, w)/\alpha'(q, w)$ added to the right side, and the downward concavity of $f(\alpha, w)$ is no longer guaranteed. If $p_i = t_i^d$, $i = 1, ..., n$, we have again the case where the graph of $f(\alpha, w)$ reduces to a single point. We shall establish the following property of $f(\alpha, w)$.

THEOREM 4.8. *$f(\alpha)$ is a stationary surface of $f(\alpha, w)$ for w in the neighborhood of unity. Specifically,*

$$\left[\frac{\partial f(\alpha, w)}{\partial w_j}\right]_\alpha \bigg|_{w=1} = 0, \qquad j = 1, ..., n. \tag{4.45}$$

The technical problem of evaluating derivatives of f with respect to w at constant α is that of the intermediate q-dependence.

LEMMA 4.9. *Let $g(\alpha, w)$ be a differentiable functiable function of w. Then*

$$\left[\frac{\partial g(\alpha, w)}{\partial w_j}\right]_\alpha = \mathscr{D}_j\, \bar{g}(q, w), \tag{4.46}$$

where $\mathscr{D}_j = \mathscr{D}_j(q, w)$ is the operator,

$$\mathscr{D}_j = \frac{\partial}{\partial w_j} - \frac{1}{\alpha'(q, w)} \frac{\partial \alpha(q, w)}{\partial w_j} \frac{\partial}{\partial q}, \tag{4.47}$$

and where $\bar{g}(q, w) = g(\alpha(q, w), w)$. We prove the lemma later; first we prove the theorem.

From Eq. (4.1), we find

$$\frac{\partial \beta(q, w)}{\partial w_j} = \frac{-p_j^q t_j^{\beta(q,w)}}{E_w[\log t]}; \tag{4.48}$$

from $\alpha(q, w) = E_w[\log p]/E_w[\log t]$,

$$\frac{\partial\alpha(q, w)}{\partial w_j} = \frac{\partial\beta(q, w)}{\partial w_j} \cdot A_j(q, w), \tag{4.49}$$

where

$$A_j(q, w) = \frac{E_w[\log p \log t]}{E_w[\log t]} - \frac{E_w[\log p] \cdot E_w[(\log t)^2]}{(E_w[\log t])^2}$$
$$- \log p_j + \frac{E_w[\log p]}{E_w[\log t]} \log t_j; \tag{4.50}$$

and from $\gamma(q, w) = E_w[\log w]/E_w[\log t]$,

$$\frac{\partial\gamma(q, w)}{\partial w_j} = \frac{\partial\beta(q, w)}{\partial w_j} B_j(q, w), \tag{4.51}$$

where

$$B_j(q, w) = \frac{E_w[\log w \log t]}{E_w[\log t]} + \frac{E_w[\log w] \cdot E_w[(\log t)^2]}{(E_w[\log t])^2}$$
$$- (1 + \log w_j) + \frac{E_w[\log w]}{E_w[\log t]} \log t_j. \tag{4.52}$$

We apply the lemma to $f(\alpha, w)$ using Eq. (4.4). The necessary smoothness of $f(\alpha, w)$ follows from that of $\beta(q, w)$ and $\gamma(q, w)$, together with that of $\alpha(q, w)$ and the smooth uniqueness of its inverse, $q = q(\alpha, w)$. We have

$$\left[\frac{\partial f(\alpha, w)}{\partial w_j}\right]_\alpha = \frac{\partial\beta(q, w)}{\partial w_j} + q\frac{\partial\alpha(q, w)}{\partial w_j} + \frac{\partial\gamma(q, w)}{\partial w_j}$$
$$- \frac{\partial\alpha(q, w)}{\partial w_j}\left[q + \frac{\gamma'(q, w)}{\alpha'(q, w)}\right]$$
$$= \left[1 - \frac{\gamma'(q, w)}{\alpha'(q, w)} A_j(q, w) + B_j(q, w)\right] \frac{\partial\beta(q, w)}{\partial w_j}. \tag{4.53}$$

So we still need $\gamma'(q, w)$ and $\alpha'(q, w)$. We use

$$(E_w[\varphi])' = E_w[\varphi \cdot (\log p - \alpha(q, w) \log t)], \tag{4.54}$$

to help in getting

$$\gamma'(q, w) = \frac{E_w[\log p \log w]}{E_w[\log t]} - \frac{E_w[\log w] \cdot E_w[\log p \log t]}{(E_w[\log t])^2}$$

$$- \alpha(q, w) \cdot \left\{ \frac{E_w[\log w \log t]}{E_w[\log t]} - \frac{E_w[\log w] \cdot E_w[(\log t)^2]}{(E_w[\log t])^2} \right\}, \tag{4.55}$$

while, as previously determined in Eq. 4.3),

$$\alpha'(q, w) = \frac{E_w[(\log p - \alpha(q, w) \log t)^2]}{E_w[\log t]} \tag{4.56}$$

Equation (4.45) is now immediate from Eq. (4.55), which gives $\gamma'(q, 1) = 0$, and the relation $B_j(q, 1) = -1$, $j = 1, ..., n$.

To complete the proof of the theorem, we have only to establish the lemma. In fixing α as we form the partial derivative with respect to w of a function $g(\alpha, w)$, we have to fix the value of the function $\alpha = \alpha(q, w)$. Owing to the w-dependence of $\alpha(q, w)$, the constancy of α implies a variation of q. Thus, regarding $g(\alpha, w)$ as a function of q and w via $\alpha = \alpha(q, w)$, we have

$$\left[\frac{\partial g(\alpha, w)}{\partial w_j}\right]_\alpha = \frac{\partial \bar{g}(q, w)}{\partial w_j} + \left[\frac{\partial q}{\partial w_j}\right]_\alpha \cdot \frac{\partial \bar{g}(q, w)}{\partial q}. \tag{4.57}$$

But from $\alpha = \alpha(q, w)$, and the one-to-one invertibility for $q = q(\alpha, w)$,

$$\left[\frac{\partial q}{\partial w_j}\right]_\alpha = -\frac{\partial \alpha(q, w)/\partial w_j}{\partial \alpha(q, w)/\partial q}, \tag{4.58}$$

so that

$$\left[\frac{\partial g(\alpha, w}{\partial w_j}\right]_\alpha = \frac{\partial \bar{g}(q, w)}{\partial w_j} - \frac{1}{\alpha'(q, w)} \frac{\partial \alpha(q, w)}{\partial w_j} \bar{g}'(q, w), \tag{4.59}$$

which is the content of Eqs. (4.46) and (4.47).

5. Gauge Invariance

The auxiliary functions now possess a new feature, namely certain covariance properties to transformation of the weights,

$$T(a, b)\colon w_i \to w_i' = w_i p_i^a t_i^b, \qquad i = 1, ..., n; \quad a, b \in \mathbb{R}. \tag{5.1}$$

We call these gauge transformations. They form an Abelian group under composition. Note that I_w is invariant.

THEOREM 5.1. *The translformation* (5.1) *of the weights is a symmetry of* $f(\alpha, w)$, *that is,*

$$f(\alpha, w) \to f(\alpha, w') = f(\alpha, w). \tag{5.2}$$

Proof. From Eq. (4.1), $T(a, b)$ induces the transformation of $\beta(q, w)$,

$$\beta(q, w) \to \beta(q, w') = -b + \beta(q + a, w), \tag{5.3}$$

so that

$$\alpha(q, w) \to \alpha(q, w') = \alpha(q + a, w), \tag{5.4}$$

while Eq. (4.5) gives

$$\gamma(q, w) \to \gamma(q, w') = b + a\alpha(q + a, w) + \gamma(q + a, w). \tag{5.5}$$

The result immediately follows from Eq. (4.4).

Remark 5.2. Since $\lambda_w = \lim_{q \to \infty} \alpha(q, w)$ and $\bar{\lambda}_w = \lim_{q \to -\infty} \alpha(q, w)$, both should be invariant by (5.4). This is the case, in fact, since λ_w and $\bar{\lambda}_w$ are specified in Eqs. (4.9) and (4.10), for $i \in I_w$, from the p_i and the t_i alone. From Eq. (4.8), we have also

$$e(w) \to e(w') = -b - a\lambda_w + e(w),$$

while from (4.12)

$$S(w) \to S(w') = S(w)$$
$$T(w) \to T(w') = bS(w) + a\lambda_w S(w) + T(w),$$

so that

$$g(w) = T(w)/S(w) \to b + a\lambda_w + T(w)/S(w) = b + a\lambda_w + g(w),$$

and we have the invariance of the limit,

$$\lim_{q \to \infty} f(q, w) = e(w) + g(w) \to e(w') + g(w') = e(w) + g(w).$$

Similarly, $\lim_{q \to -\infty} f(q, w) = \bar{e}(w) + \bar{g}(w)$ is also invariant. We note that if the sums in Eqs. (4.8) have only one term, then

$$e(w) = -\log w_{i_0}/\log t_{i_0},$$

where i_0 is the value of $i \in I_w$ for which $\lambda_w = \min(\log p_i/\log t_i)$. On the other hand, $S(w) = \log t_{i_0}$ and $T(w) = \log w_{i_0}$ in this case, and thus $\lim_{q \to \infty} f(q, w) = e(w) + g(w) = 0$. Similarly, $\lim_{q \to -\infty} f(q, w) = 0$ when the maximum value of $\log p_i/\log t_i$, $i \in I_w$, is taken for only one value of i. This generalizes the corresponding property holding for the $w = 1$ theory.

THEOREM 5.3. *Assume $n = 2$, and the weights are positive, and that $p_i \neq t_i^d$ for both $i = 1$ and 2, unless also there is a number h so that $t_i = w_i^h$, $i = 1, 2$. Then $f(\alpha, w)$ is independent of w, viz., $f(\alpha, w) = f(\alpha)$.*

Proof. We use the gauge invariance of $f(\alpha, w)$ and perform a transformation

$$(w_1, w_2) \to (w_1', w_2') = (p_1^a t_1^b w_1, p_2^a t_2^b w_2) = (1, 1).$$

The (a, b) which accomplish this satisfy

$$\log w_1^{-1} = a \cdot \log p_1 + b \cdot \log t_1$$
$$\log w_2^{-1} = a \cdot \log p_2 + b \cdot \log t_2$$

which always possess a solution unless the system is inconsistent. The latter is the case if and only if the determinant of the coefficients of (a, b) vanish while those of the augmented matrices fail to do so. These conditions are

$$\frac{\log p_1}{\log p_2} = \frac{\log t_1}{\log t_2} \neq \frac{\log w_1}{\log w_2}. \tag{5.6}$$

The equality in (5.6) implies $p_i = t_i^c$, $i = 1, 2$, with $c = d$. The inequality implies the nonexistence of a number h for which $t_i = w_i^h$, $i = 1, 2$. Hence, excepting possibly cases where the conditions (5.6) hold, $f(\alpha, w) = f(\alpha)$, for all w, when $n = 2$. ∎

PROBLEMS

1. Is it true that the sets K_q or, more generally, the sets $K^{q,w}$ are fractals in the sense of Taylor? Is the packing dimension of $K^{q,w} = f(q, w)$?

2. Can the hypotheses of Theorem 3.3 be relaxed? In particular, is it true that for all (pairwise disjoint) Moran constructions, the pointwise dimension of ρ at x is α if and only if $x \in K_\alpha$?

3. Regarding Theorem 4.8, is it true that $f(\alpha)$ is an absolute maximum of $f(\alpha, w)$, or at least a local maximum?

4. Does the multifractal decomposition in the presence of a nontrivial system of positive weights possess a ρ-measure completeness property like that expressed in Corollary 2.13? For example, does

$$\rho\left(\bigcup_{\alpha \in (\lambda, \bar{\lambda})} \bigcup_{b \in R} T(0, b) K_{q,w}^{\alpha,\gamma}\right) = \rho(K),$$

where

$$T(0, b) K_{q,w}^{\alpha,\gamma} = K_{q,w}^{\alpha,\gamma+b} ?$$

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Note added in proof. Experimental observations of non-concave $f(\alpha)$ curves recently has been reported by Paulis *et al.* [P]. A multifractal analysis was conducted of rat locomotor trajectories under the influence of cocaine. A shoulder-like non-concavity in the $f(\alpha)$ curve was observed in both the control where no cocaine was administered as well as when it was. In the highest dose case, a subgroup of the animals had $f(\alpha)$ curves with a very pronounced two humped structure similar to that shown in Fig. 4.4. For all doses including the control, the curves lay below the 45° line.

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