

## The Volume Common to Two Congruent Circular Cones whose Axes Intersect Symmetrically

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The axes of two congruent right circular cones intersect at a point equidistant from the cones' vertices. We derive a formula for the volume of the set of points common to the interiors of both cones. Some of the work was done on the symbol manipulator MACSYMA.

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### 0. Introduction.

In this paper we find the volume of the intersection of two congruent circular cones whose axes cross at a common point  $P$ , with the vertices of the cones equidistant from  $P$ . The formula is expressible by elementary functions. The symbol manipulator MACSYMA was used in the derivations. The formula was needed for a practical application. It has not been found in the literature despite a thorough search.

### 1. The formula.

Let  $\alpha$  be the angle the generator of one cone makes with its axis, and  $\beta$  be the angle between that axis and the line joining the two vertices. Assume that  $\beta + \alpha$  does not exceed  $\pi/2$  (so that the required volume is finite) and that  $\alpha$  is no larger than  $\beta$  (so that the projection technique we use is applicable). Let  $d$  be the distance from either vertex to the intersection of the cones' axes. Then the geometry is as in Figure 1.

For the formula, define additionally

$$a := \tan\alpha, \quad \text{and} \quad b := \tan\beta;$$

then the volume of intersection of the cones is given by

$$V_{\text{cones}} = \frac{4}{3} \frac{d^3 a^2}{1-a^2 b^2} \left[ \frac{a(1+b^2)\sqrt{b^2-a^2}}{b^2(1+a^2)^2} + \frac{1}{\sqrt{1-a^2 b^2}} \tan^{-1} \left[ \frac{a(1+b^2)}{\sqrt{(b^2-a^2)(1-a^2 b^2)}} \right] \right]. \quad (1)$$

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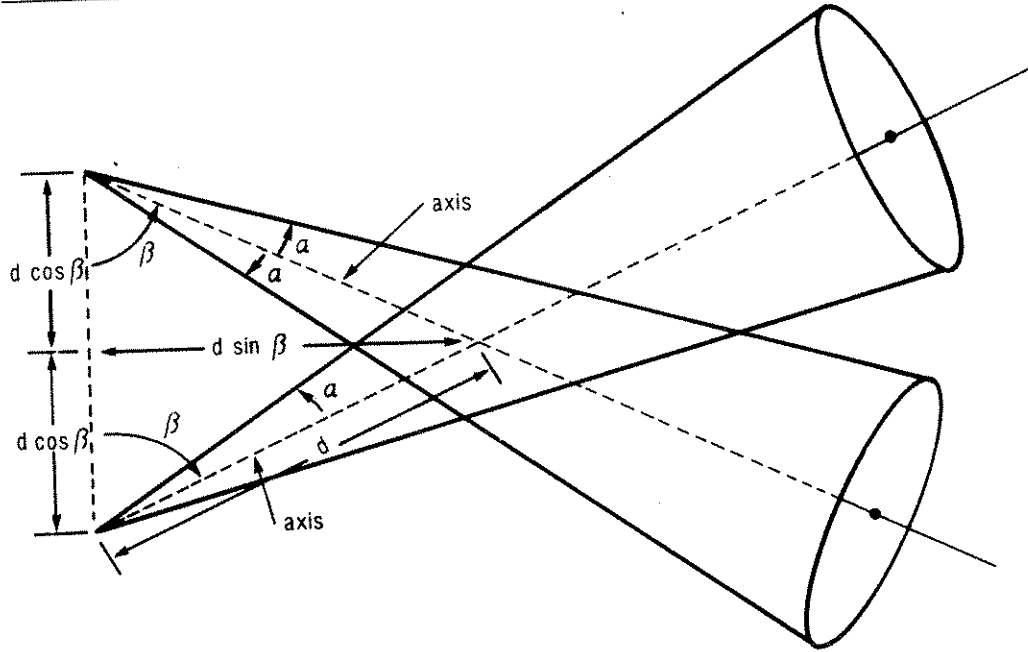


FIGURE 1

We note that  $1-a^2b^2$  is non-negative because  $\tan(\beta + \alpha)$  is positive, and that  $b^2-a^2$  is non-negative because  $\alpha$  is no larger than  $\beta$ .

The unusual part of the proof of (1) invokes a projective transformation mapping the two cones to two circular cylinders - it seemed easier to integrate the Jacobian of this transformation over the region common to the two cylinders than to proceed otherwise. An independent check of (1) was made by integrating plane sections of the volume, each perpendicular to the plane containing the cones' axes but parallel to the vertex-connecting line. The resulting relatively complex formula, obtained partly with the help of the symbol manipulator MACSYMA, not only yielded the same volumes as (1) for several nontrivial values of the parameters, but also (eventually) was shown to reduce to (1) by judicious manipulation of MACSYMA itself.

Three other checks were made: the volume associated with the case when  $\alpha=\beta$  (so that the line connecting the two vertices lies on each cone) is computable independently as

$$V_{\alpha=\beta} = \frac{\pi d^3 \cos^2 \alpha (\sin 2\alpha)^2}{6(\cos 2\alpha)^{3/2}}$$

and coincides with that yielded by (1) in this case. Further, the volume of intersection of two circular cylinders of radius  $r$  (whose axes cross at an interior angle  $\pi-2\beta$ ), given by

$$V_{cyls} = 16 r^3 / (3 \sin 2\beta),$$

is attained in the following two limiting cases of (1): fix the radius  $r$  of the cones at the intersection of the two axes (measured perpendicular to either axis). For the first limit, fix  $\beta$  and let  $d$  increase without bound. For the second limit, instead of fixing  $\beta$  let  $2l$  be the distance between the cones' vertices, and let both  $d$  and  $l$  increase without bound but with  $d$  much greater than  $l$ . Then, for each case, the limit of  $V_{cyls} / V_{cones}$  is one.

## 2. Derivation of the formula.

We divide all three coordinates by  $d$  so that

$$V_{\text{cones}} = d^3 V,$$

where  $V$  corresponds to the intersection of the two cones in Figure 2.

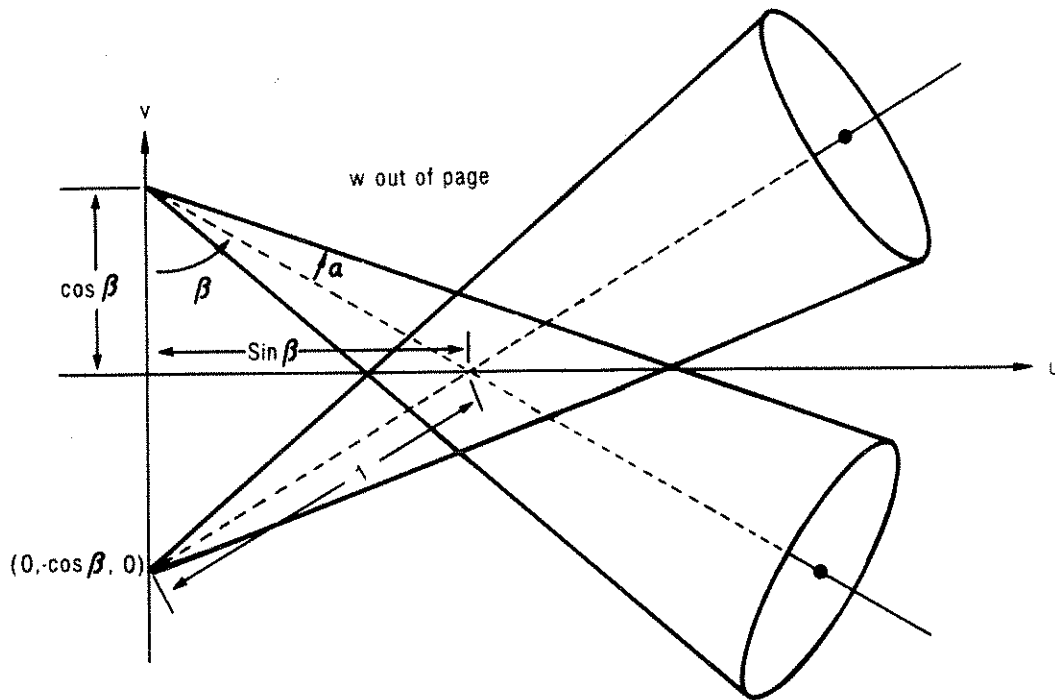


FIGURE 2

Consider, now, the transformation

$$T: (u, v, w) \rightarrow (x, y, z),$$

given by

$$\begin{aligned} x &= \sin\beta/u \\ T: \quad y &= v \tan\beta/u \\ z &= c w/u, \quad c > 0 \text{ a scaling constant to be determined later.} \end{aligned} \tag{2}$$

The motivation for  $T$  is to transform the two cones, for some  $c$ , to two circular cylinders.

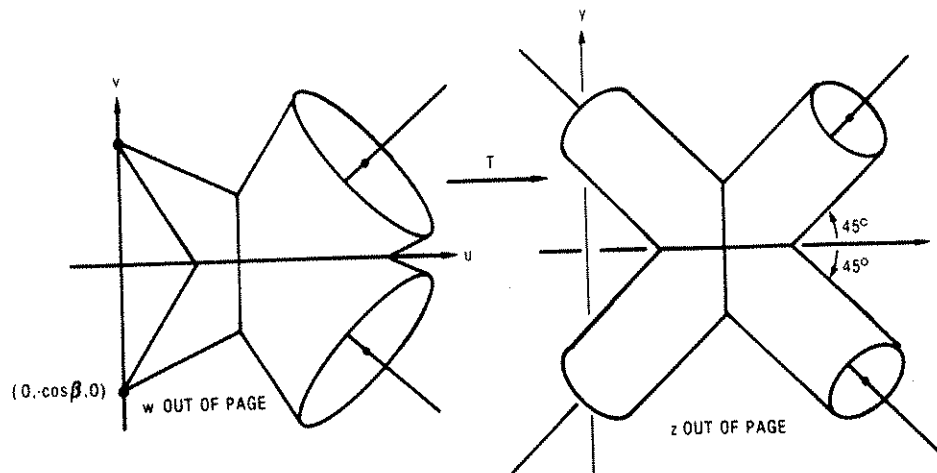


FIGURE 3

Under  $T$  (Figure 3):

- (a) The  $v-w$  plane ( $u=0$ ) goes to  $\infty$ .  
 (b) Points symmetric with respect to the  $u-w$  plane map to points symmetric with respect to the  $x-z$  plane, and points symmetric with respect to the  $u-v$  plane map to points symmetric with respect to the  $x-y$  plane.

For: if  $T(u, v, w) = (x, y, z)$ , then  $T(u, -v, w) = (x, -y, z)$  and  $T(u, v, -w) = (x, y, -z)$ .

- (c) The transformation of any line  $L$  through vertex  $(0, -\cos\beta, 0)$  is a line  $T(L)$  parallel to the  $x-y$  plane which has slope  $-1$  when projected onto that plane. A similar statement holds for lines through the other vertex  $(0, \cos\beta, 0)$ , except that their corresponding slope is  $+1$ . So each cone maps onto a cylinder.

For: any line through the first vertex may be written as

$$\begin{aligned} L: \quad u &= s \\ v &= ms - \cos\beta \\ w &= ns; \quad m, n \text{ constants, } s \text{ in } (-\infty, \infty); \end{aligned}$$

so that, via (2),

$$\begin{aligned} T(L): \quad x &= \sin\beta/s \\ y &= \tan\beta (ms - \cos\beta)/s \\ z &= c \, ns/s = \text{constant}; \end{aligned}$$

and hence, on  $T(L)$ ,

$$dx/ds = -\sin\beta/s^2, \quad dy/ds = \tan\beta [ms - (ms - \cos\beta)]/s^2 = \sin\beta/s^2;$$

so that  $dy/dx \equiv -1$  on  $T(L)$ .

- (d)  $T$  is a projective transformation, i.e., it has the form

$$\begin{aligned} x &= (a_{11}u + a_{12}v + a_{13}w + b_1)/D \\ y &= (a_{21}u + a_{22}v + a_{23}w + b_2)/D \\ z &= (a_{31}u + a_{32}v + a_{33}w + b_3)/D, \end{aligned}$$

where

$$D = a_{41}u + a_{42}v + a_{43}w + b_4.$$

Such maps take quadric surfaces onto quadric surfaces. Under the map  $T$ , each cone is mapped onto a cylinder of *elliptic* cross-section: To see this, take a  $u = \text{constant}$  section of one cone. From (2) we see that  $x = \text{constant}$ ; and that  $y = v \cdot \text{constant}$  and  $z = w \cdot \text{constant}$ . But  $(v, w)$  satisfies the equation of some ellipse; hence, so does  $(y, z)$ .

$V$  now may be expressed in the following form:

$$V = \iiint_V du \, dv \, dw = \int_{T(V)} \int \int \frac{\partial(u, v, w)}{\partial(x, y, z)} dx \, dy \, dz. \tag{3}$$

To compute this Jacobian determinant we invert (2):

$$\begin{aligned} u &= \sin\beta/x \\ v &= uy/\tan\beta = y \cos\beta/x \\ w &= uz/c = (\sin\beta/c) z/x. \end{aligned}$$

Then, the Jacobian is given by

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \det \begin{bmatrix} -\sin\beta/x^2 & 0 & 0 \\ \dots & \cos\beta/x & 0 \\ \dots & 0 & \sin\beta/(cx) \end{bmatrix} = -\sin^2\beta \cos\beta / (cx^4), \tag{4}$$

which is independent of  $y$  and of  $z$ .

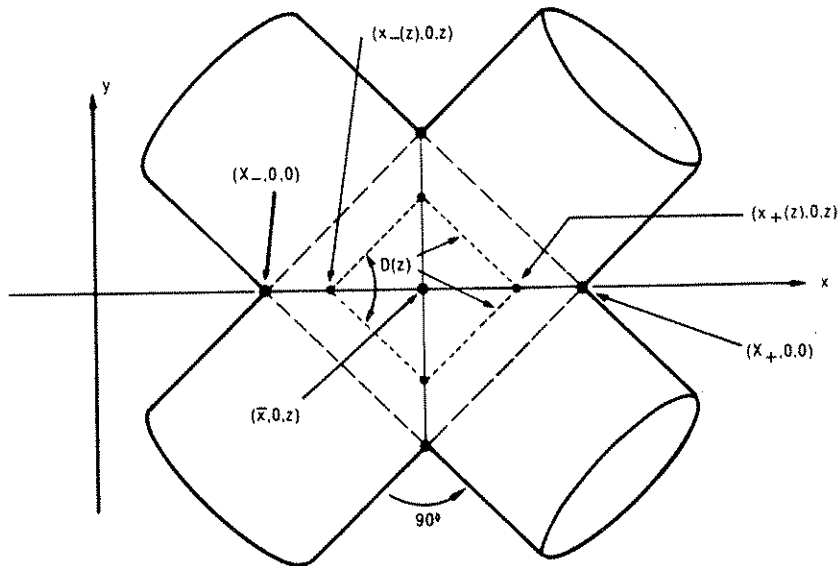


FIGURE 4

Now, the parameter  $c$  in (2) acts to scale the  $z$ -coordinate; thus, the cylinders have elliptical cross sections with semi-minor axes in the  $z$  direction if  $c$  is small and semi-major axes in the  $z$  direction for  $c$  large. As we shall see in section 3, there is one value of  $c$  for which the cylinders are circular; we presume henceforth that we have selected this  $c$ . Then we shall compute the required integral

$$I := \int_{T(V)} \int \frac{dy \, dx}{x^4} dz \quad (5)$$

by taking sections parallel to the  $x$ - $y$  plane (Figure 4). Thus, consider the inner integral over the square diamond  $D(z)$  that comprises the intersection of a  $z$ =constant plane with  $T(V)$ . Two of its corners are at  $(x_{\pm}(z), 0, z)$ ; they lie on the ellipse that is the intersection of the cylinders with the  $x$ - $z$  plane and are equidistant from the center of the ellipse. This center has  $x$ -coordinate:

$$\bar{x} = \frac{X_+ + X_-}{2},$$

where  $X_{\pm}$  are the extreme values of  $x$  on that ellipse. We have (the integration being done on MACSYMA),

$$\int_{D(z)} \int \frac{dy \, dx}{x^4} = 2 \left[ \int_{x_-}^{\bar{x}} \frac{x - x_-}{x^4} dx + \int_{\bar{x}}^{x_+} \frac{x_+ - x}{x^4} dx \right] = \frac{1}{3} \left[ \frac{x_+^2 + x_-^2}{x_-^2 x_+^2} \right] - \frac{2}{3\bar{x}^2}.$$

Now, this is to be integrated with respect to  $z$  (Figure 5).

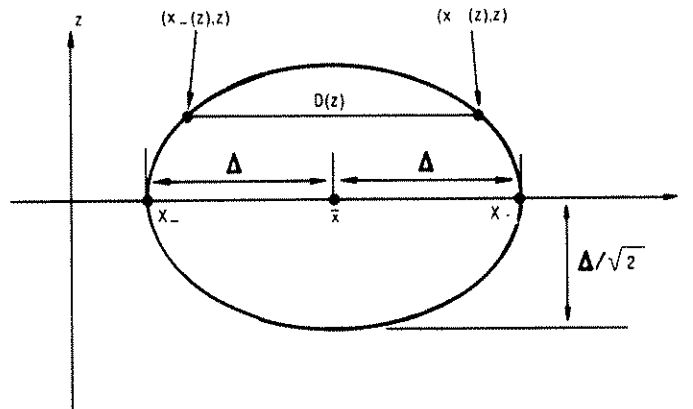


FIGURE 5

Here, again,  $X_{\pm}$  are the extreme values of  $x$  on the intersection of the two circular cylinders. Define

$$\Delta := \frac{X_+ - X_-}{2}. \quad (6)$$

The ellipse, being a 45 degree section of a circular cylinder (as we assume), has major axis

$\sqrt{2}$  times the diameter of the cylinder; so the radius is  $\Delta/\sqrt{2}$ ; the center of the ellipse is at  $\bar{x}$ ; so the equation of the ellipse is  $(x-\bar{x})^2 + 2z^2 = \Delta^2$ . In particular,

$$x_-^2 = \Delta^2 + 2\bar{x}x_- - \bar{x}^2 - 2z^2 \quad \text{and} \quad x_+^2 = \Delta^2 + 2\bar{x}x_+ - \bar{x}^2 - 2z^2;$$

so

$$x_-^2 + x_+^2 = 2\Delta^2 + 4\bar{x}^2 - 2\bar{x}^2 - 4z^2 = 2(\Delta^2 + \bar{x}^2 - 2z^2);$$

while

$$2x_-x_+ = (x_- + x_+)^2 - (x_-^2 + x_+^2) = 4\bar{x}^2 - (x_-^2 + x_+^2),$$

which, using the relation above it, yields

$$x_-x_+ = \bar{x}^2 - \Delta^2 + 2z^2.$$

Hence, using  $\bar{x}^2 - \Delta^2 = X_-X_+$ , we have, again using MACSYMA for integration,

$$\begin{aligned} I &= \int_{-\Delta/\sqrt{2}}^{\Delta/\sqrt{2}} dz \int_{D(z)} \frac{dydx}{x^4} = 2 \int_0^{\Delta/\sqrt{2}} \left[ \frac{1}{3} \left( \frac{x_-^2 + x_+^2}{x_-^2 x_+^2} \right) - \frac{2}{3\bar{x}^2} \right] dz \\ &= 2 \int_0^{\Delta/\sqrt{2}} \left[ \frac{1}{3} \frac{2(\bar{x}^2 + \Delta^2 - 2z^2)}{(\bar{x}^2 - \Delta^2 + 2z^2)^2} - \frac{2}{3\bar{x}^2} \right] dz; \end{aligned}$$

so,

$$I = \frac{2\sqrt{2}}{3X_-X_+} \left[ \frac{\Delta^3}{\bar{x}^2} + \frac{\Delta^2}{\sqrt{X_-X_+}} \tan^{-1} \left[ \frac{\Delta}{\sqrt{X_-X_+}} \right] \right]. \tag{7}$$

It remains to find the value of the parameter  $c$  in (2) which makes the two cylinders circular.

### 3. Scaling for circular cylinders.

First we maximize the  $z$  component of  $T(u,0,w)$  on the ellipse  $E$  that is the section in the  $u-w$  plane of either cone. For this we assume  $E$  has unknown height  $h$ ; so we next determine  $h$ . The appropriate value of  $c$  is then found. And, finally, we combine everything to determine the required volume  $V_{cones}$  (1).

So, let  $E$  be the ellipse comprising the section in the  $u-w$  plane of either cone (Figure 6).

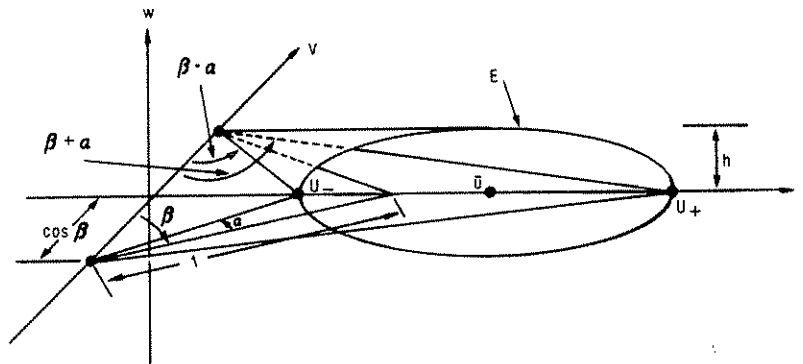


FIGURE 6

$E$  has horizontal axis

$$\overline{U_- U_+}, \text{ where } U_{\pm} := \cos\beta \tan(\beta \pm \alpha);$$

its half-length and center, respectively, are

$$\delta := (U_+ - U_-)/2 \text{ and } \bar{u} := (U_+ + U_-)/2.$$

We take  $h$  to be the height of  $E$  (i.e., half its vertical axis), to be computed explicitly later. Then the equation for  $E$  is

$$(u - \bar{u})^2/\delta^2 + w^2/h^2 = 1.$$

On  $T(E)$  we have (from (2))  $z = cw/u$ .  $z$  is a maximum, then, when  $dz/du = 0$ ; i.e., when  $0 = c(u \, dw/du - w)/u^2$ ; or when

$$dw/du = w/u. \quad (8)$$

On  $E$ , however,

$$\frac{u - \bar{u}}{\delta^2} + \frac{w}{h^2} \frac{dw}{du} = 0. \quad (9)$$

Combining (8) and (9), we want  $(u, w)$  on  $E$  such that

$$\frac{w^2}{h^2} = -\frac{u(u - \bar{u})}{\delta^2} = \frac{(\bar{u} - u)(u - \bar{u}) - \bar{u}(u - \bar{u})}{\delta^2} = -\frac{(u - \bar{u})^2}{\delta^2} - \bar{u} \frac{(u - \bar{u})}{\delta^2}.$$

But on  $E$ ,  $w^2/h^2 = 1 - (u - \bar{u})^2/\delta^2$  also. So

$$1 = \frac{\bar{u}}{\delta} \frac{(\bar{u} - u)}{\delta},$$

or

$$\frac{\bar{u} - u}{\delta} = \frac{\delta}{\bar{u}} \text{ implying } u = \bar{u} - \delta^2/\bar{u};$$

also  $w^2/h^2 = 1 - [(u - \bar{u})/\delta]^2 = 1 - \delta^2/\bar{u}^2$  implying  $w = h\sqrt{1 - \delta^2/\bar{u}^2}$ . Thus

$$\frac{w}{u} = \frac{h\sqrt{1 - \delta^2/\bar{u}^2}}{\bar{u} - \delta^2/\bar{u}} = h \frac{\sqrt{\bar{u}^2 - \delta^2}}{\bar{u}^2 - \delta^2} = \frac{h}{\sqrt{\bar{u}^2 - \delta^2}}.$$

But (as was true for  $\bar{x}$  and  $\Delta$ )  $\bar{u}^2 - \delta^2 = U_+ U_-$ . So

$$w/u = h/\sqrt{U_+ U_-}. \quad (10)$$

Now, on  $T(E)$  (from (2)),  $z_{\max} = cw/u$ . But, as below (6),  $z_{\max} = \Delta/\sqrt{2}$ . So, we conclude that  $T$  maps the cones onto *circular* cylinders if

$$c = \frac{\Delta u}{\sqrt{2} w}. \quad (11)$$

To use this with (10) we need the height,  $h$ , of this ellipse  $E$  that comprises the section of either cone with the  $u-w$  plane (Figure 6, above). For this we need, first, an equation for the surface of one of the cones. The geometry is as in Figure 7.



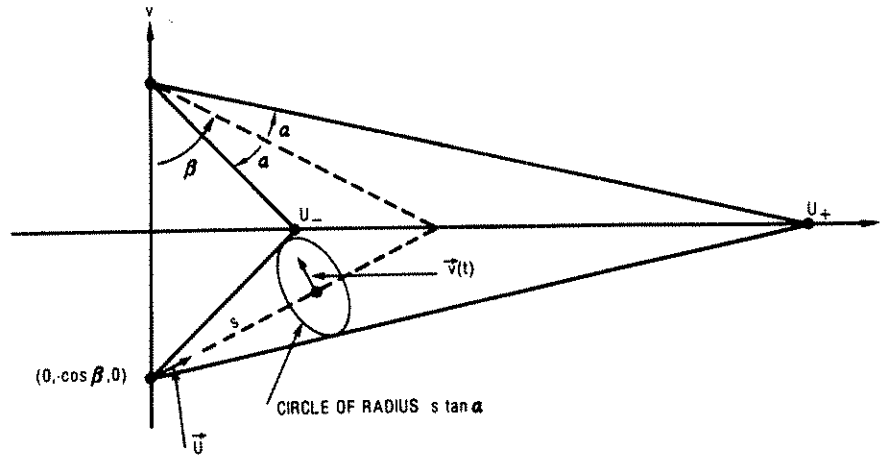


FIGURE 7

The extreme values of  $u$  on the cones' intersection is, as before,  $U_{\pm} = \cos\beta \tan(\beta \pm \alpha)$ . Let  $\vec{U}$  be the unit vector along the lower cone's axis, and  $\vec{V}(t)$  be a unit vector perpendicular to  $\vec{U}$  and rotating around it:

$$\vec{U} := (\sin\beta, \cos\beta, 0) \quad \text{and} \quad \vec{V}(t) := (-\cos\beta \cos t, \sin\beta \cos t, \sin t).$$

Then points  $\vec{C}(s, t)$  lie on the lower cone if (and only if)

$$\vec{C}(s, t) := (0, -\cos\beta, 0) + s \vec{U} + s \tan\alpha \vec{V}(t); \tag{12}$$

here  $s \geq 0$  and, say,  $0 \leq t < 2\pi$ . Now on the ellipse  $E$  the  $v$ -coordinate on this cone is 0: i.e.,  $E$  is the image of the set of  $(s, t)$  such that

$$s = 1/(1 + \tan\alpha \tan\beta \cos t).$$

On  $E$ , then, from (12),

$$w = s \tan\alpha \sin t = \tan\alpha \sin t / (1 + \tan\alpha \tan\beta \cos t). \tag{13}$$

At the maximum of  $w$  on  $E$ ,  $dw/dt = 0$ , i.e.,

$$(1 + \tan\alpha \tan\beta \cos t) \tan\alpha \cos t - \tan\alpha \sin t (-\tan\alpha \tan\beta \sin t) = 0,$$

or

$$\cos t + \tan\alpha \tan\beta = 0. \tag{14}$$

For this value of  $t$ , using (13) and then (14),

$$h = w_{\max} = \tan\alpha \sin t / (1 + \tan\alpha \tan\beta \cos t) = \tan\alpha \sin t / (1 - \cos^2 t) = \tan\alpha / \sin t,$$

or

$$h = \tan\alpha / \sqrt{1 - \tan^2\alpha \tan^2\beta}.$$

From (10), then,

$$w/u = h / \sqrt{U_+ U_-} = \frac{\tan\alpha}{\sqrt{(1 - \tan^2\alpha \tan^2\beta) \tan(\beta + \alpha) \tan(\beta - \alpha) \cos\beta}},$$

or

$$\frac{w}{u} = \frac{\tan\alpha}{\sqrt{\tan^2\beta - \tan^2\alpha}} \frac{1}{\cos\beta};$$

so that the parameter  $c$  yielding circular cylinders is given from (11) by

$$c = \frac{\Delta}{\sqrt{2}} \frac{u}{w} = \frac{\Delta}{\sqrt{2}} \frac{\sqrt{\tan^2\beta - \tan^2\alpha} \cos\beta}{\tan\alpha}. \quad (15)$$

We now combine all we have done to find, from  $V_{cones} = d^3V$ , from (4), and from (15) and (7), that

$$V_{cones} = \frac{4}{3} d^3 \sin^2\beta \frac{\tan\alpha}{\sqrt{\tan^2\beta - \tan^2\alpha}} \frac{\Delta}{X_+ X_-} \left[ \frac{\Delta}{\bar{x}^2} + \frac{1}{\sqrt{X_+ X_-}} \tan^{-1} \left[ \frac{\Delta}{\sqrt{X_+ X_-}} \right] \right].$$

Now, from (2),  $X_{\mp} = \sin\beta/U_{\pm} = \tan\beta/\tan(\beta \pm \alpha)$ , so that with  $b := \tan\beta$  and  $a := \tan\alpha$ , (hence, e.g., then  $1/\cos^2\beta = 1+b^2$ ) we may verify the following relations:

$$\Delta = \frac{X_+ - X_-}{2} = \frac{b}{2} \left[ \frac{1+ab}{b-a} - \frac{1-ab}{b+a} \right]$$

$$= \frac{ab}{\cos^2\beta (b^2 - a^2)},$$

$$\bar{x} = \frac{X_+ + X_-}{2}$$

$$= \frac{b^2}{\cos^2\alpha (b^2 - a^2)},$$

$$X_+ X_- = \frac{b^2(1-a^2b^2)}{b^2 - a^2},$$

$$\frac{\Delta}{\sqrt{X_+ X_-}} = \frac{a}{\cos^2\beta \sqrt{(b^2 - a^2)(1 - a^2b^2)}},$$

$$\frac{\Delta}{\bar{x}^2} = \frac{a (b^2 - a^2) \cos^4\alpha}{\cos^2\beta b^3},$$

and

$$\frac{\Delta}{X_+ X_-} = \frac{a}{b \cos^2\beta (1 - a^2b^2)}.$$

We conclude that

$$V_{cones} = \frac{4}{3} d^3 \sin^2\beta \frac{a}{\sqrt{b^2 - a^2}} \frac{a}{b \cos^2\beta (1 - a^2b^2)} \times \left[ \frac{a (b^2 - a^2) \cos^4\alpha}{\cos^2\beta b^3} + \frac{\sqrt{b^2 - a^2}}{b \sqrt{1 - a^2b^2}} \tan^{-1} \left[ \frac{a}{\cos^2\beta \sqrt{(b^2 - a^2)(1 - a^2b^2)}} \right] \right],$$

which reduces to (1).