Topology and its Applications 21 (1985) 87-94
North-Holland

A PARAMETRIZATION THEOREM

G. MÄGERL
Universität Erlangen-Nürnberg, Fed. Rep. Germany

R. Daniel MAULDIN*
North Texas State University, Denton, TX, U.S.A.

E. MICHAEL
University of Washington, Seattle, WA, U.S.A.

Received 13 April 1984
Revised 25 October 1984

The following result, and a closely related one, is proved: If $u: X \to Y$ is an open, perfect surjection, with X metrizable and with $\dim X = 0$ or $\dim Y = 0$, then there exists a perfect surjection $h: Y \times S \to X$ such that $u \circ h = \pi_Y$ (where S in the Cantor set and $\pi_Y: Y \times S \to Y$ is the projection). If moreover, $u^{-1}(y)$ is homeomorphic to S for all $y \in Y$, then h can be chosen to be a homeomorphism.

AMS (MOS) Subj. Class.: 54C10, 54C25, 54C65, 54F45, 54F99

Cantor set   perfect maps   open maps   homeomorphisms
continuous selections   zero-dimensional spaces

1. Introduction

The principal purpose of this paper is to prove the following two closely related results. All *maps* are continuous, and no separation properties are assumed unless indicated. We denote the Cantor set by S.

Theorem 1.1. *Suppose that* $u: X \to Y$ *is an open, perfect surjection, and that*: (*) X *is metrizable, and* $\dim X = 0$ *or* $\dim Y = 0$. *Then*:

(a) *There exists a perfect surjection* $h: Y \times S \to X$ *such that* $u \circ h = \pi_Y$.

(b) *If, moreover, each fiber* $u^{-1}(y)$ *is homeomorphic to* S, *then* h *can be chosen to be a homeomorphism.*

Theorem 1.2. *Theorem* 1.1 *remains true if assumption* (*) *is replaced by*: (**) $X \subset Y \times M$ *with* M *metrizable, with* u *the projection from* X *onto* Y, *and with* $\dim M = 0$ *or* $\dim Y = 0$.

* Supported in part by NSF Grant MCS 81-01581 and a Faculty Research Grant from North Texas State University.

In the above theorems, a map $u: X \to Y$ is called *perfect* if it is closed and every fiber $u^{-1}(y)$ is compact, $\pi_Y: Y \times S \to Y$ denotes the projection, and *dim* is the covering dimension.[1]

Let us briefly comment on the significance of the various hypotheses in Theorems 1.1 and 1.2. The assumption that u is open and perfect is clearly necessary, since $\pi_Y: Y \times S \to Y$ is open and perfect (because S is compact) and we want $u \circ h = \pi_Y$. The importance of our dimensional assumptions will be illustrated by Example 7.1. Finally, the special role played by the Cantor set S will be illustrated by Examples 7.2 and 7.3, which show that our results become false if S is replaced by a convergent sequence or by a closed interval.[2]

Theorem 1.1(b) may be regarded as a 0-dimensional analogue of some results of M.E. Hamstrom and E. Dyer in [3], and our Lemma 2.2(b) shows, in effect, that the map $u: X \to Y$ in Theorem 1.1(b) must be completely regular in the sense of [3]. Some other related topological results are obtained in [9] and [2], while related results on Borel and measurable functions can be found in [5] and [4] as well as in papers by Wesley, Cenzer and Mauldin, Bourgain, Ioffe and others which are cited in D.H. Wagner's survey article [10].

Section 2 establishes some notation and lemmas for function spaces, while Sections 3 and 4, which may be of independent interest, prove a general selection theorem for maps with 0-dimensional domain and a result on 0-dimensional hyperspaces. These results are then applied in Section 5 to prove a very special selection theorem from which our Theorems 1.1 and 1.2 then follow easily in Section 6. Section 7, finally, is devoted to examples.

The first author would like to thank S. Graf for several helpful discussions.

2. Some notation and results on function spaces

Throughout this section, X will denote a metric space with metric d, and (as elsewhere in this paper) S denotes the Cantor set.

Let $C(S, X)$ be the space all of continuous $f: S \to X$, topologized by the sup metric σ, and let $H(S, X)$ be the set of all $f \in C(S, X)$ which are homeomorphisms into. Let $\mathcal{K}(X)$ be the space of non-empty, compact $A \subset X$ with the Hausdorff metric ρ,[3] and let $\mathcal{K}_S(X)$ be the set of all $A \in \mathcal{K}(X)$ which are homeomorphic to S. For $A \in \mathcal{K}(X)$, let $C_A(S, X)$ (resp. $H_A(S, X)$) be the set of all $f \in C(S, X)$ (resp. $f \in H(S, X)$) such that $f(S) = A$. (Observe that $C_A(S, X)$ is closed in $C(S, X)$, that $H_A(S, X)$ is closed in $H(S, X)$, and that $C_A(S, X)$ is non-empty (see Footnote 2).) Finally, 2^E will denote the collection of non-empty, closed subsets of a space E.

[1] Observe that, in Theorem 1.1, dim $X = 0$ actually implies dim $Y = 0$ because Y is the image of X under an open and closed map.

[2] The properties of S which are relevant to us, and which will be used in the proof of Lemma 2.2, are that it is a 0-dimensional compact metric space which is homeomorphic to each of its non-empty open-closed subsets, and that every non-empty compact metric space is a continuous image of it.

[3] $\rho(A, B) = \inf\{\varepsilon > 0: B \subset V_\varepsilon(A) \text{ and } A \subset V_\varepsilon(B)\}$, where V_ε denotes the ε-neighborhood.

Lemma 2.1. *If X is complete, then:*
(a) *$C(S, X)$ is complete with the sup metric σ.*
(b) *$H(S, X)$ is completely metrizable.*[4]

Proof. (a) Clear.
(b) Since $C(S, X)$ is complete with σ, we need only check that $H(S, X)$ is a G_δ in $C(S, X)$. Let

$$U_n = \{f \in C(S, X): f(s) \neq f(s') \text{ if } s, s' \in S \text{ and } |s - s'| \geq 1/n\}.$$

It is easy to check that U_n is open in $C(S, X)$ for all n, and that $H(S, X) = \bigcap_n U_n$. □

Lemma 2.2. *Let $\varepsilon > 0$, let $A, B \in \mathcal{K}(X)$ with $\rho(A, B) < \frac{1}{2}\varepsilon$, and let $f \in C_A(S, X)$. Then:*
(a) *There exists a $g \in C_B(S, X)$ such that $\sigma(f, g) < \varepsilon$.*
(b) *If $B \in \mathcal{K}_S(X)$, then this g can be chosen so that $g \in H_B(S, X)$.*

Proof. Choose a finite, disjoint open cover $\{S_1, \ldots, S_n\}$ of S such that $S_i \neq \emptyset$ and $\operatorname{diam} f(S_i) < \frac{1}{2}\varepsilon$ for $i = 1, \ldots, n$. Let

$$V_i = \{x \in B: d(x, f(S_i)) < \tfrac{1}{2}\varepsilon\}.$$

Clearly $\{V_1, \ldots, V_n\}$ is a cover of B by non-empty, relatively open subsets. For (a), we now pick surjective maps $g_i: S_i \to \bar{V}_i$ $(i = 1, \ldots, n)$; for (b), we first choose a *disjoint*, relatively open cover $\{U_1, \ldots, U_n\}$ of B such that $\emptyset \neq U_i \subset V_i$ for all i, and then pick homeomorphisms $g_i: S_i \to U_i$. (Here we use the properties of S listed in Footnote 2.) Now the map $g: S \to X$, defined by $g(s) = g_i(s)$ for $s \in S_i$, satisfies all our requirements. □

Corollary 2.3. (a) *The map $\theta: \mathcal{K}(X) \to 2^{C(S,X)}$, defined by $\theta(A) = C_A(S, X)$, is l.s.c.*[5]
(b) *The map $\theta': \mathcal{K}_S(X) \to 2^{H(S,X)}$, defined by $\theta'(A) = H_A(S, X)$, is l.s.c.*

Proof. This follows immediately from Lemma 2.2. □

3. A general selection theorem

In order to obtain Theorem 5.1 in Section 5 with no superfluous hypothesis on Y, we need the following generalization of [6, Theorem 2].

Theorem 3.1. *Suppose $g: Z \to X$ is continuous, with $\dim Z = 0$ and X paracompact, and suppose $\theta: X \to 2^Y$ is l.s.c. with Y complete metric. Then $\theta \circ g: Z \to 2^Y$ has a continuous selection f.*[6]

[4] I.e., there is a complete metric on $H(S, X)$ which generates the same topology as the sup metric.
[5] A function $\theta: E \to 2^F$ is *l.s.c.* (= *lower semi-continuous*) if $\{x \in E: \theta(x) \cap V \neq \emptyset\}$ is open in E for every open V in F.
[6] Recall that $f: Z \to Y$ is a *selection* for $\psi: Z \to 2^Y$ if $f(z) \in \psi(z)$ for every $z \in Z$.

Proof. When $Z = X$ and g is the identity map, this theorem reduces to [6, Theorem 2]. The proof of that result in [6] begins by showing that, if X is normal and $\dim X = 0$, then every locally finite open cover of X has a disjoint open refinement. Analogously, and with essentially the same proof, one can show that, if $g: Z \to X$ is continuous with $\dim Z = 0$ and X normal, and if $\mathcal{V}$ is a locally finite open cover of X, then $g^{-1}(\mathcal{V})$ has a disjoint open refinement. Once that has been established, the proof of our theorem proceeds just like the proof of [6, Theorem 2]. □

4. A result on 0-dimensional spaces

The following result, which may be of independent interest and which was also proved by Eric van Douwen (private communication), will be used in the proof of Theorem 5.1.

Proposition 4.1. *If X is a metric space, and if* $\dim X = 0$, *then* $\dim \mathcal{K}(X) = 0$.[7]

Proof. Recall that, if X is metrizable and non-empty, then $\dim X = 0$ if and only if X has a base $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ with each $\mathcal{U}_n$ a disjoint open cover of X. Clearly we can choose these $\mathcal{U}_n$ so that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for all n.

Recall next that the Hausdorff metric on $\mathcal{K}(X)$ generates the *Vietoris topology*, a base for which consists of all collections of the form

$$\langle U_1, \ldots, U_k \rangle = \left\{ A \in \mathcal{K}(X) \colon A \subset \bigcup_{i=1}^{k} U_i,\ A \cap U_i \neq \emptyset \text{ for } i \leq k \right\},$$

with $U_1, \ldots, U_k$ open subsets of X and $k = 1, 2, \ldots$.

Now let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ be a base for X as in the first paragraph of this proof. For all n, let

$$\mathcal{V}_n = \{\langle U_1, \ldots, U_k \rangle \colon U_1, \ldots, U_k \in \mathcal{U}_n,\ k = 1, 2, \ldots\}.$$

and let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$. It is easy to check that each $\mathcal{V}_n$ is a disjoint open cover of $\mathcal{K}(X)$ and that $\mathcal{V}$ is a base for $\mathcal{K}(X)$, so $\dim \mathcal{K}(X) = 0$ by the first paragraph of this proof. □

5. A special selection theorem

The essence of Theorems 1.1 and 1.2 is contained in the following result.

Theorem 5.1. *Let $g : Y \to \mathcal{K}(X)$ be continuous, with X a metric space, and suppose* $\dim Y = 0$ *or* $\dim X = 0$. *Then*:

[7] For the small inductive dimension ind, this result is well known and easy to prove. It may also be known for dim, but we have been unable to find it in the literature.

(a) *There exists a continuous* $f: Y \to C(S, X)$ *such that* $[f(y)](S) = g(y)$ *for every* $y \in Y$.

(b) *If* $g: Y \to \mathcal{K}_S(X)$, *then* f *can be chosen so that* $f: Y \to H(S, X)$.

Proof. Without loss of generality, we may assume that X is complete (since it can always be replaced by its completion).

(a) Suppose first that $\dim Y = 0$. Let $\theta: \mathcal{K}(X) \to 2^{C(S,X)}$ be as in Corollary 2.3 (a). Now θ is l.s.c. by Corollary 2.3(a), $C(S, X)$ is complete by Lemma 2.1(a), and $\mathcal{K}(X)$ is metrizable and thus paracompact. Hence Theorem 3.1 implies that $\theta \circ g: Y \to 2^{C(S,X)}$ has a continuous selection f, and this f satisfies our requirements.

Now suppose $\dim X = 0$. Then $\dim \mathcal{K}(X) = 0$ by Proposition 4.1. We can therefore apply [6, Theorem 2][8] to obtain a continuous selection f^* for the map θ in Corollary 2.3(a). Letting $f = f^* \circ g$, we see that f satisfies our requirements.

(b) The proof is almost the same as for (a), except that the map θ of Corollary 2.3(a) is replaced by the map θ' of Corollary 2.3(b), and we invoke Lemma 2.1(b) instead of Lemma 2.1(a). □

6. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. (a) Since $u: X \to Y$ is open and closed, $u^{-1}: Y \to \mathcal{K}(X)$ is l.s.c. and u.s.c.[9] by [1, 1.7.17], and hence u^{-1} is continuous by [1, 2.7.20(d) and 4.5.22(a)]. Let $g = u^{-1}$, and choose $f: Y \to C(S, X)$ as in Theorem 5.1(a). Define $h: Y \times S \to X$ by $h(y, s) = [f(y)](s)$. Then h is also continuous, and clearly h is onto with $u \circ h = \pi_Y$. Hence $u \circ h$ is perfect (because S is compact), so h must also be perfect by [8, Corollary 1.4].

(b) If each $u^{-1}(y)$ is homeomorphic to S, we can choose the above f as in Theorem 5.1(b), which makes h one-to-one and thus (being perfect) a homeomorphism. □

Proof of Theorem 1.2. (a) As in the proof of Theorem 1.1, $u^{-1}: Y \to \mathcal{K}(X)$ is l.s.c. and u.s.c. Let $v: X \to M$ be the projection, and define $g: Y \to \mathcal{K}(M)$ by $g(y) = v(u^{-1}(y))$. Then g is also l.s.c. and u.s.c., and hence g is continuous. By Theorem 5.1(a) (with X replaced by M), there is a continuous $f: Y \to C(S, M)$ with $[f(y)](S) = g(y)$ for every $y \in Y$. Define $h: Y \times S \to X$ by $h(y, s) = (y, [f(y)](s))$. Then h is continuous and onto, and $u \circ h = \pi_Y$. Hence $u \circ h$ is perfect, so, since $h = (u \circ h, v \circ h)$, it follows from [8, Theorem 1.1] that h is also perfect.

(b) If each $u^{-1}(y)$ is homeomorphic to S, we can choose the above f as in Theorem 5.1(b), which makes h one-to-one and thus (being perfect) a homeomorphism. □

[8] Here we do not need the generalization of that result given in Theorem 3.1.

[9] Recall that a function $\psi: Y \to 2^X$ is *u.s.c.* (= *upper semi-continuous*) if $\{y \in Y : \psi(y) \subset U\}$ is open in Y for every open U in X.

Remark. We have derived Theorem 1.2 from Theorem 5.1. Conversely, it is not hard to derive Theorem 5.1 from Theorem 1.2. Thus Theorem 1.1 also follows from Theorem 1.2, which is not hard to verify directly. There seems to be no way, however, to derive Theorem 1.2 from Theorem 1.1.

Remark. Let h be as in the conclusion of Theorem 1.1(a), let (s_n) be dense in S, and let $f_n : Y \to X$ be defined by $f_n(y) = h(y, s_n)$. Then (f_n) is a sequence of continuous selections for u^{-1} such that $\{f_n(y): n = 1, 2, \ldots\}$ is dense in $u^{-1}(y)$ for every $y \in Y$. (The existence of such a sequence (f_n) also follows from [7, Theorem 5.1 and Example 5.4 (for $n = -1$)], where it is only assumed that $u: X \to Y$ is open with X and Y metrizable, with $\dim Y = 0$, and with each $u^{-1}(y)$ separable.)

7. Examples

Each of the following examples shows that Theorem 1.1(a) and (b) both become false if some of the hypotheses are omitted or modified. In Example 7.1, we do this by showing that $u^{-1}: Y \to \mathcal{K}(X)$ does not even have a continuous selection; in Examples 7.2 and 7.3, where $\dim Y = 0$ and where u^{-1} must therefore have a continuous selection by [6, Theorem 2], we have to use a different approach.

All spaces in our examples are compact metric, and thus all maps are perfect. As elsewhere in this paper, S denotes the Cantor set.

Our first example shows that the assumption that $\dim Y = 0$ or $\dim X = 0$ cannot be omitted from Theorem 1.1.

Example 7.1. *There exists an open, surjective map $u: X \to Y$, with X and Y compact metric and each fiber $u^{-1}(y)$ homeomorphic to S, such that u^{-1} does not have a continuous selection.*[10]

Proof. Let $I = [0, 1]$, and consider S as a subset of I in the usual way. Let X be the space obtained from $I \times S$ by identifying $(0, s)$ with $(1, 1 - s)$ for all $s \in S$. (In effect, X is a modified Möbius band.) Let Y be the circle obtained by identifying 0 and 1 in I, and let $u: X \to Y$ be the obvious map. If u^{-1} had a continuous selection $f: Y \to X$, then $f(Y)$ would be a circle in X which is mapped one-to-one onto Y by u. But clearly u maps every circle in X onto Y in a *two*-to-one fashion, and hence no such selection f exists. □

Our next example shows that S cannot be replaced in Theorem 1.1 by the convergent sequence $H = \{0\} \cup \{1/n: n \in N\}$.

[10] This implies that Theorem 1.1(a) is not satisfied (for if h is as in Theorem 1.1(a) and if $s \in S$, then $y \to h(y, s)$ is a continuous selection for u^{-1}). It follows, of course, that Theorem 1.1(b) is also not satisfied. Nevertheless, as kindly pointed out by I. Namioka, $Y \times S$ and X *are* homeomorphic in this example.

Example 7.2. *There exists an open, surjective map $u: X \to Y$, with X and Y both 0-dimensional compact metric spaces and with each fiber $u^{-1}(y)$ homeomorphic to H, such that there is no surjective map $h: Y \times H \to X$.*

Proof. Let $Y = H$. Let

$$B_n = \left\{\frac{1}{m}: m \in N, m < n\right\} \cup \left\{1 - \frac{1}{m}: m \in N, m \geq n\right\} \cup \{1\} \quad (n \in N),$$

$$X = (\{0\} \times H) \cup \bigcup_{n=1}^{\infty} \left(\left\{\frac{1}{n}\right\} \times B_n\right).$$

Define $u: X \to Y$ by $u(s, t) = s$.

Let us show that there is no surjective map $h: Y \times H \to X$. Recall that the *derived set* Z' of a space Z is the set of all non-isolated points of Z. Now it is easy to see that the second derived set $(Y \times H)''$ of $Y \times H$ is the singleton $\{(0, 0)\}$, whereas X'' consists of the two points $(0, 0)$ and $(0, 1)$. If there were a surjective map h: $Y \times H \to X$, then $h((Y \times H)'') \supset X''$ (because $Y \times H$ is compact and hence h is closed), and that is clearly impossible. □

Our last example shows that S cannot be replaced in Theorem 1.1 by the interval $I = [0, 1]$. As in Example 7.2, H denotes $\{0\} \cup \{1/n: n \in N\}$.

Example 7.3. *There exists an open, surjective map $u: X \to Y$, with X and Y compact metric*, $\dim Y = 0$, *and each fiber $u^{-1}(y)$ homeomorphic to I, with the following two properties*:

(a) *$Y \times I$ is not homeomorphic to X.*

(b) *There is no surjective map $h: Y \times I \to X$ with $u \circ h = \pi_Y$.*[11]

Proof. Let $Y = H$. Let

$$X_n = \left\{\left(t, \sin\frac{1}{t}\right): \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) \leq t \leq \frac{1}{n}\right\} \quad (n \in N),$$

$$X_\infty = \{0\} \times [-1, 1].$$

Let $X = (\bigcup_{n=1}^{\infty} X_n) \cup X_\infty$, and define $u: X \to Y$ by $u(X_n) = \{1/n\}$ and $u(X_\infty) = \{0\}$.

(a) To verify (a), we need only observe that every point in $Y \times I$ has a base of neighborhoods U such that $U \cap E$ is connected for every component E of $Y \times I$, while X lacks the corresponding property at each point of X_∞.

(b) To verify (b), consider the following property of a collection $\mathcal{D}$ of subsets of a space E:

(†) For every $x \in E$ and neighborhood V of x in E, there exists a (not necessarily open) neighborhood $U \subset V$ of x in E and an integer n such that $U \cap D$ is the union of $\leq n$ connected sets for every $D \in \mathcal{D}$.

[11] However, there does exist a surjective map $h: Y \times I \to X$.

It is easy to check that, if $h: E \to F$ is a perfect map, and if $\mathcal{D}$ satisfies (†) in E, then $h(\mathcal{D})$ satisfies (†) in F.

Suppose there were a surjective map h: $Y \times I \to X$ with $u \circ h = \pi_Y$. Let $\mathcal{D} = \{\{1/n\} \times I: n \in N\}$. Then $\mathcal{D}$ clearly satisfies (†) in $Y \times I$, while $h(\mathcal{D}) = \{X_n: n \in N\}$ does not satisfy (†) in X. By the previous paragraph, that is impossible. □

Added in Proof. In Lemma 2.2 the hypothesis $\rho(A, B) < \frac{1}{2}\varepsilon$ can be weakened to $\rho(A, B) < \varepsilon$. For (a), see Math. Ann. 162 (1965) 87–88, Lemma 2.1.

References

[1] R. Engelking, General Topology, Monografie Matematyczne, Tom. 60 (PWN-Polish Scientific Publishers, Warszawa, 1977).

[2] I. Ekeland and M. Valadier, Representation of set-valued mappings, J. Math. Anal. Appl. 35 (1971) 621–629.

[3] M.E. Hamstrom and E. Dyer, Completely regular mappings, Fund. Math. 45 (1957) 103–118.

[4] J.M. Ingram, Geometric problems in measure theory and parametrizations, Doctoral Dissertation, North Texas State University, Denton, TX, 1981.

[5] R.D. Mauldin and H. Sarbadhikari, Continuous one-to-one parametrizations, Bull. Sci. Math. (2) 105 (1981) 435–444.

[6] E. Michael, Selected selection theorems, Amer. Math. Monthly 63 (1956) 233–238.

[7] E. Michael, Dense families of continuous selections, Fund. Math. 47 (1959) 173–178.

[8] E. Michael, A theorem on perfect maps, Proc. Amer. Math. Soc. 28 (1971) 633–634.

[9] C.P. Pixley, Continuously choosing a retraction of a separable metric space onto each of its arcs, Illinois J. Math. 20 (1976) 22–29.

[10] D.H. Wagner, Survey of measurable selection theorems: An update, in: D. Kölzow, ed., Proc. Conf. Measure Theory, Oberwolfach 1979, Lecture Notes in Math. 794 (Springer, Berlin, 1980) 176–219.