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ON THE BOREL CLASS OF THE DERIVED SET OPERATOR, II

BY

DOUGLAS CENZER and R. DANIEL MAULDIN (*)

RÉSUMÉ. — Soit X un espace non-énumerable topologique métrisable compact, 2^X l'espace topologique des compacts de X avec la topologie de Hausdorff et soit D la dérivation de Cantor. KURATOWSKI a démontré que D est borélienne et précisément de la deuxième classe, et a posé le problème de trouver la classe précise des dérivés successifs D^α. Nous démontrons que si n est fini, alors D^n est précisément de la classe $2n$ et si λ est un ordinal de seconde espèce et n fini, alors $D^{\lambda+n}$ est précisément d la classe $\lambda+2n+1$.

ABSTRACT. — KURATOWSKI showed that the derived set operator D, acting on the space of closed subsets of the Cantor space 2^N, is a Borel map of class exactly two and posed the problem of determining the precise classes of the higher order derivatives D^α. In part I of our work [*Bull. Soc. Math. France*, 110, 4, 1982, p. 357-380], we obtained upper and lower bounds for the Borel class of D^α and in particular showed that for limit ordinals λ, D^λ is exactly of class $\lambda+1$. The first author recently showed, using different methods (*cf.* [1]) that for finite n, D^n is exactly of Borel class $2n$. We now complete the solution of KURATOWSKI'S problem by showing that for any limit ordinal λ and any finite n, the operator $D^{\lambda+n}$ is of Borel class exactly $\lambda+2n+1$.

In this paper, we determine the exact Borel classes of the iterated derived set operators D^α, acting on the space $\mathscr{H}$ of closed subsets of the Cantor space 2^N with the usual Vietoris topology. This completes the solution of the problem of KURATOWSKI [3] which was begun in part I of our work [2].

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The results in the present paper depend strongly on those of its predecessor. We begin with some basic definitions and results from [2].

The derived set operator D maps $\mathscr{H}$ into $\mathscr{H}$ and is defined by :

$$D(F)=F'=\{x : x\in \mathrm{Cl}(F-\{x\})\}.$$

The α'th iterate D^α of the derived set operator map be defined for all ordinals α by letting $D^0(F)=F$, $D^{\alpha+1}(F)=D(D^\alpha(F))$ for all α and $D^\lambda(F)=\bigcap\{D^\alpha(F) : \alpha<\lambda\}$ for limit ordinals λ. The set F is said to be scattered if $D^{\alpha+1}(F)=\emptyset$ for some α; the derived set order $o(F)$ of F is the least such ordinal α.

The countable subset S of 2^N is defined to be $\{x : (\exists m)(\forall n>m), x(n)=0\}$. If 2^N is identified with the family $\mathscr{P}(N)$ of subsets of N, then S corresponds to the family of finite sets. Let $\bar{0}=(0, 0, 0, \ldots)$. The stitching operator Φ mapping $\mathscr{H}^N$ into $\mathscr{H}$ is defined as follows:

$$\Phi(F_0, F_1, F_2, \ldots)$$
$$=\{\bar{0}\}\cup\{(\overbrace{0, 0, \ldots, 0}^{n}, 1, x(0), x(1), \ldots) : x\in F_n\}.$$

Note that Φ preserves both finite intersections and unions, that is:

$$\Phi(F_0\cup G_0, F_1\cup G_1, \ldots)=\Phi(F_0, F_1, \ldots)\cup\Phi(G_0, G_1, \ldots)$$

and similarly for intersections. This also implies that Φ is monotone, that is, whenever $F_i\subset H_i$ for all i, then $\Phi(F_1, F_1, \ldots)\subset\Phi(H_0, H_1, \ldots)$. The two fundamental results on the stitching operator, Lemmas 3.7 and 3.8 of [2] concern the derived set order of the stitched set and the continuity of the stitching map. We actually need an extension of the former lemma to infinite ordinals; the proof goes through without difficulty.

LEMMA 1. — *For any sequence* $(F_0, F_1, \ldots)$ of sets from $\mathscr{H}\cap\mathscr{P}(S)$ and any ordinal α:

$$D^\alpha(\Phi(F_0, F_1, \ldots))=\begin{cases}\Phi(D^\alpha(F_0), D^\alpha(F_1), \ldots), \\ \quad\text{if } (\forall\beta<\alpha)\{n : D^\beta(F_n)\neq\emptyset\} \text{ is infinite,}\\ \Phi(D^\alpha(F_0), D^\alpha(F_1), \ldots)-\{\bar{0}\}, \text{ otherwise.}\end{cases}$$ □

LEMMA 2. — Let $(H_0, H_1, \ldots)$ be a sequence of continuous functions mapping a topological space X into the space $\mathscr{H}$ of closed subsets of 2^N

such that each $H_n(x) \subset S$. Then the function H, defined by $H(x) = \Phi(H_0(x), H_1(x), \ldots)$ is also continuous. □

Calculation of the exact Borel classes of the iterated derived set operators begins with Theorem 1.3 of [2].

THEOREM 3. — For any finite n and any limit ordinal λ:

(*a*) D^n is of Borel class $2n$;

(*b*) $D^{\lambda+n}$ is of Borel class $\lambda+2n+1$. □

Proofs that the Borel classes cited in Theorem 3 are exact proceed as follows. First we note that $\{\emptyset\}$ is both a closed and an open subset of $\mathscr{H}$. Thus if D^n were of class $2n-1$, then $T_n = (D^n)^{-1}(\{\emptyset\})$ would have to be a Borel subset of $\mathscr{H}$ of both additive and multiplicative class $2n-1$; similarly, if $D^{\lambda+n}$ were of class $\lambda+2n$, then $T_{\lambda+n} = (D^{\lambda+n})^{-1}(\{\emptyset\})$ would be of both additive and multiplicative class $\lambda+2n$. To show that T_n is not of multiplicative class $2n-1$, we prove that T_n is actually universal for Borel sets of additive class $2n-1$; a similar result is given for $T_{\lambda+n}$. Both results will be proved by induction on n. We need two more propositions from [2]; the first is Proposition 4.1:

THEOREM 4. — For any F_σ subset B of N^N, there is a continuous function H mapping N^N into $\mathscr{H} \cap \mathscr{P}(S) \cap T_2$ such that, for all x, $x \in B$ if and only if $H(x) \in T_1$. □

We actually need the following improvement of Theorem 6.2 of [2].

THEOREM 5. — For any countable limit ordinal λ and any Borel subset B of N^N of additive class λ, there is a continuous function H mapping N^N into $\mathscr{H} \cap \mathscr{P}(S) \cap T_{\lambda+1}$ such that, for all x, $x \in B$ if and only if $H(x) \in T_\lambda$.

Proof. — Let B be a Borel subset of N^N of additive class λ. By Theorem 6.2 of [2], there is a continuous function G from N^N into $\mathscr{H} \cap \mathscr{P}(S) \cap T_{\lambda+2}$ such that, for all x, $x \in B$ if and only if $G(x) \in T_\lambda$; furthermore, $G(x)$ is also normal, as defined in 5.1 of [2]. Now let $C = C_\lambda$ be some canonical normal set with $o(C) = \lambda$ (*see* 5.10 of [2]). Define the function H by:

$$H(x) = G(x) \cap C_\lambda.$$

Recall from Lemma 5.2 of [2] that, for two normal sets F and G:

$$o(F \cap G) = \min(o(F), o(G)).$$

It follows that:

$$o(H(x)) = \min(o(G(x)), \lambda).$$

This implies that H maps into $T_{\lambda+1}$ and that, for any x, $x \in B$ if and only if $H(x) \in T_\lambda$. Recall from Lemma 5.12 of [2] that the intersection map is continuous for normal sets. Of course the constant map $F(x) = C_\lambda$ is continuous. It follows that H is also continuous. □

It should be pointed out that the proof of Theorem 6.2 in [2] required the introduction of a more complex stitching operator acting on the family of normal sets.

L. Pigtkiewicz has pointed out that in Proposition 5.8 of [2] $\theta(\hat{F})$ is actually normal if and only if $\gamma = \lim_{n \to \infty} (o(F_n) + 1)$; this does not affect the proof of Theorem 6.2.

The induction step in the proofs that T_n and $T_{\lambda+n}$ are universal depends on Lemmas 1 and 2 and the following well-known result (a version of which can be found in LUSIN's classic book [5]).

LEMMA 6. — *Let X be a topological space with a countable basis of clopen sets (such as 2^N and N^N). Then for any countable ordinal α and any Borel subset B of X of additive class α, B can be written as the disjoint countable union of Borel sets B_n, each of multiplicative class $< \alpha$.*

THEOREM 7. — (*a*) *For any natural number k and any Borel subset B of N^N of additive class $2k-1$, there is a continuous function H mapping N^N into $\mathscr{H} \cap \mathscr{P}(S) \cap T_{k+1}$ such that, for all x, $x \in B$ if and only if $H(x) \in T_k$.* (*b*) *For any countable limit ordinal λ, any natural number k and any Borel subset B of N^N of additive class $\lambda + 2k$, there is a continuous function H mapping N^N into:*

$$\mathscr{H} \cap \mathscr{P}(S) \cap T_{\lambda+k+1}$$

such that, for all x, $x \in B$ if and only if $H(x) \in T_{\lambda+k}$.

Proof. — The proofs of parts (*a*) and (*b*) proceed from, respectively, Theorems 4 and 5 in a similar manner. We will give the proof of (*b*), which is of course by induction on k. Theorem 5 covers the case $k=0$. Suppose therefore that the result is true for k and let B be a Borel subset of N^N of additive class $\lambda+2k+2$. Since $N^N \setminus B$ is of multiplicative class $\lambda+2k+2$, there is a decreasing sequence $\{C_n : n \in N\}$ of sets of additive class $\lambda+2k+1$ such that $N^N \setminus B = \bigcap_n C_n$. Now by Lemma 6, there exists for each n a disjoint sequence $\{C_{n,m} : m \in N\}$ of sets of multiplicative class $\lambda+2k$ such that $C_n=\bigcup_m C_{n,m}$. It is now easy to see that, for all x:

(i) $x\in B \leftrightarrow \{(n,m) : x\in C_{n,m}\}$ is finite.

Let $(n_0, m_1), (n_1, m_1), \ldots$ be some one-to-one enumeration of $N\times N$ and let $A_i=N^N\setminus C_{n_i,m_i}$. By the induction hypothesis, there exists a sequence $\{H_i : i\in N\}$ of continuous functions from N^N into:

$$\mathscr{H}\cap\mathscr{P}(S)\cap T_{\lambda+k+1}$$

such that, for all x:

(ii) $x\in A_i \leftrightarrow H_i(x)\in T_{\lambda+k}$.

The desired reduction H of B to $T_{\lambda+k}$ is now defined by:

(iii) $H(x)=\Phi(H_0(x), H_1(x), \ldots)$.

H is continuous by Lemma 2. We must now calculate the possible derived set order of $H(x)$. First of all, from the induction hypothesis $D^{\lambda+k+1}(H_i(x))=\emptyset$; it follows from Lemma 1 that $D^{\lambda+k+2}(H(x))=\Phi(\emptyset, \emptyset, \ldots)\setminus\{\bar{0}\}=\emptyset$. Thus $H(x)\in T_{\lambda+k+2}$ for any x. Next suppose that $x\in B$. Then by (i) and the definition of the A_i, $\{i : x\notin A_i\}$ is finite. It follows from (ii) that:

$$\{i : D^{\lambda+k}(H_i(x))\neq\emptyset\}.$$

is finite. Then by Lemma 1, $D^{\lambda+k+1}(H(x))=\emptyset$ as desired. Finally, suppose that $x\notin B$. Then again using (i) and (ii), it follows that:

$$\{i : D^{\lambda+k}(H_i(x))\neq\emptyset\}$$

is infinite. Applying Lemma 1 and the fact that each $D^{\lambda+k+1}(H_i(x))=\emptyset$, we obtain:

$$D^{\lambda+k+1}(H(x))=\Phi(\emptyset, \emptyset, \ldots)=\{\bar{0}\},$$

so that $H(x)\notin T_{\lambda+k+1}$. □

THEOREM 8. — *(a) For any natural number k, T_k is a Borel subset of $\mathscr{H}$ of additive class $2k-1$ but not of multiplicative class $2k-1$. (b) For any countable limit ordinal λ and any finite k, $T_{\lambda+k}$ is a Borel subset of $\mathscr{H}$ of additive class $\lambda+2k$ but not of multiplicative class $\lambda+2k$.*

Proof. — The positive direction is proved by induction, as follows. $T_1=\{F : F \text{ is finite}\}$ is an F_σ set by Lemma 1.1 of [2]. For any limit ordinal λ, $T_\lambda=\bigcup_{\alpha<\lambda} T_\alpha$ and will therefore be of additive class λ

if the result is assumed for $\alpha<\lambda$. Finally, $T_{\alpha+1}=D^{-1}(T_\alpha)$; since D is a mapping of Borel class 2, the result can always be extended from α to $\alpha+1$. The other direction has similar proofs for parts (a) and (b); we give the proof of (b). Let B be an arbitrary subset of N^N which is of additive class $\lambda+2k$ but not of multiplicative class $\lambda+2k$ (*see* [4], p. 425). By Theorem 7, there is a continuous function H such that $B=H^{-1}(T_{\lambda+k})$. Now if $T_{\lambda+k}$ were of multiplicative class $\lambda+2k$, it would follow that B must also be of multiplicative class $\lambda+2k$, contradicting our choice of B.

We can now give the complete solution of the problem of Kuratowski.

THEOREM 9. — (a) *For any natural number k, the iterated derived set operator D^k is of Borel class exactly $2k$.* (b) *For any countable limit ordinal λ and any natural number k, $D^{\lambda+k}$ is of Borel class exactly $\lambda+2k+1$.*

Proof. — One direction is given by Theorem 3. The other direction has similar proofs for parts (a) and (b); we give the proof of (b). Recall that $\{\emptyset\}$ is a closed subset of $\mathscr{K}$. Thus if $D^{\lambda+k}$ were of Borel class $\lambda+2k$, then:

$$T_{\lambda+k}=(D^{\lambda+k})^{-1}(\{\emptyset\})$$

would have to be a Borel set of multiplicative class $\lambda+2k$, which would contradict Theorem 8.

The finite cases of Theorems 7, 8 and 9 were previously obtained by the first author in [1] using different methods.

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