ON THE STIELTJES INTEGRABILITY OF DENSE SUBSETS OF $C[a, b]$

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In this paper the functions considered are real-valued functions and the method of integration considered is the Riemann-Stieltjes type defined by a refinement limit [1, p. 27].

If g is a function over the interval $[a, b]$, then the collection of all continuous g-integrable functions, $C_g[a, b]$, is $C[a, b]$ if and only if g is of bounded variation over the interval $[a, b]$. If g is not of bounded variation, then $C_g[a, b]$ is of the first category in $C[a, b]$. [2]. But even if g is not of bounded variation, $C_g[a, b]$ may be dense in $C[a,b]$, as would be the case if g were continuous over $[a, b]$, but not of bounded variation. In this paper, we show that $C_g[a, b]$ is dense in $C[a, b]$ if and only if g is continuous at a dense set of points in $[a, b]$. Of course, this theorem implies that g is continuous except at a set of the first category in $[a, b]$ if and only if $C_g[a, b]$ is dense in $C[a, b]$, [3, p. 33].

THEOREM. *If $C_g[a, b]$ is dense in C $[a, b]$, then g is continuous at a dense set of points in $[a, b]$.*

Proof. Suppose there is no dense subset of $[a, b]$ at which g is continuous. There exists a subinterval I of $[a, b]$ such that g is discontinuous at each point of I. It follows that there is a subinterval $[c, d]$ of I and a positive number ϵ such that the discontinuity of g at each point of $[c, d]$ is greater than ϵ. Suppose f is g-integrable over $[a, b]$; then g is f-integrable over $[a, b]$, and so g is f-integrable over $[c, d]$. If $\delta > 0$, there is a subdivision D of $[c, d]$ such that if E is a refinement of D, then $\sum_1 |f(q) - f(p)| < \delta$, where the sum is taken over all intervals $[p, q]$ of E for which $[p, q]$ contains numbers x and y such that $|g(x) - g(y)| > \epsilon$ [4, p. 86]. Since every subinterval of $[c, d]$ contains such numbers, it follows that f is of bounded variation over $[c, d]$ and that $V_a^b f = 0$. Therefore, if f is g-integrable over $[a, b]$, then f is constant over $[c, d]$. It follows that, if g is not continuous at a dense set of points in $[a, b]$, then $C_g[a, b]$ is not dense in $C[a, b]$.

THEOREM. *If g is continuous at a dense set of points in $[a, b]$, then $C_g[a, b]$ is dense in $C[a, b]$.*

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Proof. Suppose g is continuous at a dense set of points in $[a, b]$. If $[x, y]$ is a subinterval of $[a, b]$ then there exists an interval $[u, v]$ in the interior of $[x, y]$ over which g is bounded. The set of points between u and v at which g is continuous is an inner limiting (G_δ) set dense in $[u, v]$, so there exists a perfect subset K of $[u, v]$ such that g is continuous at each point of K. Let f be a continuous nondecreasing function over $[a, b]$ such that $f(x) = 0$ if x is in $[a, u]$, $f(x) = 1$ if x is in $[v, b]$, and f is constant over each component of $[a, b] - K$. Since f is constant over $[a, u]$ and $[v, b]$, g is f-integrable over these intervals. Since the set of points at which g is discontinuous is a set of μ-measure 0, where μ is the Borel measure determined by f and g is bounded on $[u, v]$, g is f-integrable over $[u, v]$. By integration by parts, f is g-integrable over $[a, b]$.

Thus, if x and y are numbers in $[a, b]$ there exists a nondecreasing function f in $C_g[a, b]$ such that $f(x) \neq f(y)$. Let L be the collection of all functions of bounded variation in $C_g[a, b]$. The collection L is a linear subspace of $C[a, b]$ containing all the constant functions over $[a, b]$. If f is in L, then g is f-integrable and $V_x^y|f| \leq V_x^y f$, for each subinterval $[x, y]$ of $[a, b]$; hence g is $|f|$-integrable over $[a, b]$ ([1], page 60), and $|f|$ is in L. By the Stone-Weierstrass approximation theorem, such a lattice of functions is dense in $C[a, b]$. Since $C_g[a, b]$ contains L, $C_g[a, b]$ also is dense in $C[a, b]$.

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