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R. Daniel Mauldin

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ONE-TO-ONE SELECTIONS-MARRIAGE THEOREMS

By R. DANIEL MAULDIN*

Abstract. Let A be a Lebesgue measurable subset of $[0, 1] \times [0, 1]$ such that each vertical and each horizontal section of A has positive measure. Then there are Borel subsets E and F of $[0, 1]$ with measure one and a one-to-one Borel measurable map f of E onto F whose graph is a subset of A. Variations of this theorem are also considered.

A marriage theorem for finite sets is as follows [5]:

A necessary and sufficient condition that a relation $R \subseteq A \times B$ between finite sets A and B have a matching is that, for every positive integer k, every k-subset of A be related to a subset of B having at least k elements.

One can try a simple generalization to infinite sets. For example, let X be an infinite set and $R \subset X \times X$ such that for each subset A of X, $|A| \leq |R(A)|$ where $R(A) = \{y: \exists x[x \in A \wedge (x, y) \in R]\}$. It is easy to give examples of such relations R for which there is no marriage function or matching. However, there are some abstact marriage theorems.

Let κ be an infinite cardinal κ. The following abstract marriage theorem has been proven by Kaniewski and Rogers [2].

THEOREM A. *Let A be a subset of $X \times X$ so that for each x, $\kappa = |A_x| = |A^x|$, where $A_x = \{y:(x, y) \in A\}$ and $A^x = \{y:(y, x) \in A\}$. Then there is a one-to-one map of X onto X whose graph is a subset of A.*

This theorem is proved by transfinite induction. Perhaps the main benefit of this theorem is the possibilities it suggests. Kaniewski and Rogers prove for example that if X and Y are Polish spaces and E is a subset of $X \times Y$ which is a countable union of Borel rectangles and each X-section of E and each Y-section of E is uncountable, then there is a Borel isomorphism of X onto Y whose graph is a subset of E. In this paper we shall pursue a different line of thought.

Our first theorem concerns the descriptive character of the set of points of density of a Borel set. Let $I = [0, 1]$.

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THEOREM 1. *If A is a Borel subset of $I \times I$, then $D = \{(x, y): y$ is a density point of $A_x\}$ is a Borel set.*

Proof. Note that

$$D = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} T(n, m),$$

where

$$T(n, m) = \{(x, y): \forall h[(0 < h \leq 1/m) \rightarrow \lambda(A_x \cap [y - h, y + h]) \geq 2h(1 - 1/n)]\}.$$

It can be checked that

$$T(n, m) = \bigcap\{T(n, m, q): q \text{ is rational and } 0 < q \leq 1/m\},$$

where

$$T(n, m, q) = \{(x, y): \lambda(A_x \cap [y - q, y + q]) \geq 2q(1 - 1/n)\}.$$

In order to see that each set $T(n, m, q)$ is a Borel set, define a map f on $I \times I$ by setting

$$f(x, y) = \lambda(A_x \cap [y - q, y + q]).$$

For each x, $f(x, \cdot)$ is continuous and it is also well known that for each y, $f(\cdot, y)$ is a Borel measurable map. Thus, f itself is a Borel measurable function [3, p. 378] and since

$$T(n, m, q) = f^{-1}([2q(1 - 1/n), +\infty)),$$

$T(n, m, q)$ is a Borel set. Q.E.D.

THEOREM 2. *Let P and Q be closed subsets of I with $\lambda(P) > 0$. If A is a Borel subset of $P \times Q$ such that for each x in P, $\lambda(A_x) > 0$, then for each $\epsilon > 0$, there is a sequence $\{(H_i, y_i)\}_{i=1}^{\infty}$ such that*

1. *the sets H_i are closed, pairwise disjoint subsets of P*
2. $\operatorname{diam}(H_i) < \epsilon$ *and* $\lambda(H_i) > 0$, $i = 1, 2, 3, \ldots$

3. $y_i \neq y_j$ *if* $i \neq j$
4. *if* $x \in H_i$, *then* y_i *is a density point of* A_x
5. $\lambda(\cup H_i) = \lambda(P)$.

Proof. Let D be the density set of A. Since D is a measurable subset of $I \times I$ and for each x, $\lambda(D_x) = \lambda(A_x)$,

$$\lambda(\{y : \lambda(D^y) > 0\}) > 0.$$

Thus, there is some point y and closed subset H of P such that $\operatorname{diam}(H) < \epsilon$, $\lambda(H) > 0$ and if $x \in H$, then y is a density point of A_x.

Let $\mathcal{W}$ be the collection to which $\mathcal{H}$ belongs if and only if $\mathcal{H}$ is a collection of pairs (H_i, y_i) satisfying 1), 2), 3) and 4). Let $\mathcal{W}$ be partially ordered by inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{W}$. Choose $\mathcal{H}_n \in \mathcal{C}$ such that

$$\lim_{n \to \infty} \lambda(\mathcal{H}_n) = \sup\{\lambda(\mathcal{H}) : \mathcal{H} \in \mathcal{C}\}.$$

where

$$\lambda(\mathcal{H}) \equiv \lambda(\cup \{H : H \in \mathcal{H}\}).$$

Let $\mathcal{H}_0 = \cup\, \mathcal{H}_n$. Clearly, $\mathcal{H}_0 \in \mathcal{W}$ and $\mathcal{H}_0$ is an upper bound of $\mathcal{C}$. Let $\mathcal{M} = \{(H_i, y_i)\}_{i=1}^{\infty}$ be a maximal element of $\mathcal{W}$.

Suppose $\lambda(\cup \mathcal{M}) < \lambda(P)$. Let P^* be a closed set such that $P^* \subset P - \cup \{H : H \in \mathcal{M}\}$ and $\lambda(P^*) > 0$. Let $A^* = A \cap (P^* \times Q)$. Since

$$\lambda(\{y : \lambda(D^{*y}) > 0\}) > 0,$$

there is a point z and closed set $H \subset P^*$ such that $z \notin \{y_i : i = 1, 2, 3, \ldots\}$ $\operatorname{diam} H < \epsilon$, $\lambda(H) > 0$ and if $x \in H$, z is a density point of $A_x^* = A_x$. Thus, $\mathcal{M} \cup \{(H, z)\} \in \mathcal{W}$ and this contradicts the maximality of $\mathcal{M}$. Q.E.D.

THEOREM 3. *Let P and Q be closed subsets of I such that $\lambda(P) > 0$. Let A be a closed subset of $P \times Q$ such that for each x in P, $\lambda(A_x) > 0$. Then for each $\epsilon > 0$ and $\delta > 0$ there are closed subsets $P_1, \ldots, P_n$ of P and closed subsets $Q_1, \ldots, Q_n$ of Q such that*

(1) $P_i \cap P_j = \phi = Q_i \cap Q_j$, if $i \neq j$
(2) $\operatorname{diam}(P_i)$, $\operatorname{diam}(Q_i) < \epsilon$
(3) $x \in P_i \to \lambda(A_x \cap Q_i) > 0$
(4) $\lambda(\cup P_i) > \lambda(P) - \delta$.
(5) $\lambda(\cup Q_i) < \delta$.

Proof. Let $\epsilon > 0$ and $\delta > 0$. Let $\{(H_i, y_i)\}_{i=1}^{\infty}$ be a sequence satisfying the conclusion of Theorem 3 with respect to A. Choose n so that $\lambda(\cup\{H_i : 1 \leq i \leq n\}) > \lambda(P) - \delta$. Choose pairwise disjoint closed intervals (a_i, b_i), $i = 1, \ldots, n$ so that $\Sigma(b_i - a_i) < \delta$, and $a_i < y_i < b_i$ and $b_i - a_i < \epsilon$. Let $P_i = H_i$ and $Q_i = (a_i, b_i) \cap Q$. The sets P_i and Q_i satisfy the conclusion of the theorem. Q.E.D.

By Seq we mean the space of all finite sequences of positive integers which is a tree rooted at $\langle 0 \rangle$, the empty sequence, when provided with the lexicographical order.

THEOREM 4. *Let R be a Lebesgue measurable subset of $I \times I$ such that $\lambda(\{x : \lambda(R_x) > 0\}) = 1$. Then for each $\epsilon > 0$, there exist a closed subset D of I and a one-to-one continuous function f from D into I whose graph is a subset of R such that $\lambda(D) \geq 1 - \epsilon$ and $\lambda(f(D)) = 0$.*

Proof. Let $\epsilon > 0$. Let $\{F_n\}_{n=1}^{\infty}$ be an increasing sequence of closed sets such that $\cup F_n \subset R$ and $\lambda(R - \cup F_n) = 0$. For each n, let $K_n = \{x : \lambda(F_{nx}) > 0\}$. Since $\lambda(\cup K_n) = 1$, there is some n so that $\lambda(K_n) > 1 - \epsilon$. Let P be a closed subset of K_n so that $\lambda(P) > 1 - \epsilon$.

By iterating Theorem 3, we find that there is a subset T of Seq and maps P, Q from T into the space of closed subsets of I such that

(1) $P(\langle 0 \rangle) = P$ and $Q(\langle 0 \rangle) = I$,

(2) T is a tree rooted at $\langle 0 \rangle$ and each vertex of T has only finitely many edges emerging from it,

(3) if $\langle i_1, \ldots, i_k \rangle$ is a vertex of T, then there is a positive integer n such that $\langle i_1, \ldots, i_k, i \rangle$, $i = 1, 2, \ldots, n$ form the set of all vertices of T with one edge at $\langle i_1, \ldots, i_k \rangle$ and

- (a) $P(\langle i_1, \ldots, i_k, i \rangle)$, $Q(\langle i_1, \ldots, i_k, i \rangle)$ are closed subsets of I with diameters $< 1/2^k$
- (b) the sets $P(\langle i_1, \ldots, i_k, i \rangle)$; $i = 1, \ldots, n$ are pairwise disjoint subsets of $P(\langle i_1, \ldots, i_k \rangle)$,
- (c) the sets $Q(\langle i_1, \ldots, i_k, i \rangle)$; $i = 1, \ldots, n$ are pairwise disjoint subsets of $Q(\langle i_1, \ldots, i_k \rangle)$,
- (d) if $x \in P(\langle i, \ldots, i_k, i \rangle)$, then $\lambda(R_x \cap Q(\langle i_1, \ldots, i_k, i \rangle)) > 0$.
- (e) $\cup P(\langle i_1, \ldots, i_k, i \rangle) \subset P\langle i_1, \ldots, i_k \rangle$,

(4) for each k, $\lambda(\cup\{P(\langle i_1, \ldots, i_k \rangle) : \langle i_1, \ldots, i_k \rangle \in T\}) > 1 - \epsilon$.

(5) for each k, $\lambda(\cup\{Q(\langle i_1, \ldots, i_k \rangle) : \langle i_1, \ldots, i_k \rangle \in T\}) < 2^{-k}$.

Now, for each n, let

$$H(n) = \bigcup \{P(\langle i_1, \ldots, i_n \rangle) \times Q(\langle i_1, \ldots, i_n \rangle) : \langle i_1, \ldots, i_n \rangle \in T\}.$$

Let $G = \bigcap H(n)$ and let $D = \pi_1(G)$.

It is easy to check that G is the graph of a one-to-one continuous function f of D into I such that $\lambda(D) \geq 1 - \epsilon$ and $\lambda(f(D)) = 0$. Q.E.D.

THEOREM 5. *Let A be a Lebesgue measurable subset of $I \times I$ such that $\lambda(\{x : \lambda(A_x) > 0\}) = 1$. Then there is a Borel set D and a one-to-one Borel measurable map f of D into I such that $\lambda(D) = 1$, $\lambda(f(D)) = 0$ and the graph of f is a subset of A.*

Proof. Let $\mathcal{W}$ be the family of all $\mathcal{H}$ such that $\mathcal{H}$ is a collection of pairs (E, g) such that E is a closed subset of I, g is a one-to-one function from E into I, $\lambda(E) > 0$, the graph of g is a subset of A, $\lambda(g(E)) = 0$, and if $(F, h) \in \mathcal{H}$ and $(F, h) \neq (E, g)$, then $F \cap E = \phi$ and $h(F) \cap g(E) = \phi$. Consider $\mathcal{W}$ to be partially ordered by inclusion. Clearly, if $\mathcal{C}$ is a chain in $\mathcal{W}$, then $\bigcup \mathcal{C} \in \mathcal{W}$. It follows that if $\mathfrak{M}$ is a maximal element in $\mathcal{W}$, then $\lambda(D) = 1$, where $D = \bigcup \{E : \exists g[(E, g) \in \mathfrak{M}]\}$ and the function f defined by $f|E = g$, where $(E, g) \in \mathfrak{M}$ satisfy the conclusion of the theorem. Q.E.D.

THEOREM 6. *Let A be a Lebesgue measurable subset of $I \times I$ such that $\lambda(F_1) = \lambda(F_2) = 1$, where $F_1 = \{x : \lambda(A_x) > 0\}$ and $F_2 = \{y : \lambda(A^y) > 0\}$. Then there are Borel subsets E and F of I with measure one and a Borel isomorphism of E onto F whose graph is a subset of A.*

Proof. Let D_1 be a Borel subset of F_1 and g a one-to-one Borel measurable map of D_1 onto a Borel subset E_1 of I such that $\lambda(D_1) = 1$, $\lambda(E_1) = 0$. Let D_2 be a Borel subset of F_2 and h a one-to-one Borel measurable map of D_2 onto E_2 so that $(\mathrm{Gr}(h))^{-1} \subset A$, $\lambda(D_2) = 1$, and $\lambda(E_2) = 0$. Let $K_1 = h(D_2 - E_1)$ and define f on $K_1 \cup D_1$ by $f|K_1 = h^{-1}$ and $f|D_1 - K_1 = g|D_1 - K_1$. Clearly, f is a Borel isomorphism of $K_1 \cup D_1$ into I whose graph is a subset of A and the domain and range of f have measure one. Q.E.D.

These results suggest other possibilities.

Question 1. Let B be a Borel subset of $I \times I$ such that for each x, $\lambda(B_x) > 0$ and $\lambda(B^x) > 0$. Is there a Borel (or universally measurable) isomorphism of I onto I whose graph is a subset of B?

We note that such a set B does contain the graph of a Borel map from

I into I and much more [1, 4]. Of course, category analogues of the measure theorem are also suggested. For example:

Question 2. Let A be a Borel subset of $I \times I$ such that for each x, B_x and B^x are not meager. Is there a Borel (or universally measurable) isomorphism of I onto I whose graph is a subset of B?

Again, there are some results along these lines [4, 6].

Finally, one could consider purely descriptive set theoretic properties.

Question 3. Let A be a Borel subset of $I \times I$ such that for each x, A_x and A^x are uncountable. Does the conclusion of Theorem 6 hold?

Concerning this last question we give the following example.

Example. There is a Borel subset B of $I \times I$ such that for each x in I, B^x and B_x are uncountable and B does not contain the graph of a Borel measurable map of I into I.

Construction. Let C be the standard Cantor middle third set. Let D be a G_δ subset of $(I - C) \times I$ such that for each x in $I - C$, D_x is uncountable and such that D does not contain the graph of a Borel measurable map from $I - C$ into I [4]. Let $B = (C \times I) \cup D$.

Added in proof. The author has determined that Question 3 has a positive answer.

NORTH TEXAS STATE UNIVERSITY

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