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ROTATIONS OF THE CIRCLE

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The ideas presented here having arisen from the consideration of the following problem of Erdős.

Let  $T$  be the unit circle and suppose  $S_1$  and  $S_2$  are subsets of  $T$  such that for each  $i, i=1,2$ , there is an infinite subset  $R_i$  of  $T$  so that the sets  $rS_i$ , where  $r \in R_i$  are pairwise disjoint. Is it true that the inner measure of  $S_1 \cup S_2$  is zero?

The answer to this question is yes and we shall present two solutions to this problem. Neither solution is difficult. But, each seems to lead to some interesting problems which will be formulated here.

Our first solution which is contained in the following theorem is based on the fact that the circle group is amenable.

**THEOREM 1.** Let  $G$  be a locally compact  $T_2$  group with left invariant Haar measure  $\lambda$  and such that  $G$  admits an invariant mean. Let  $S_1, \dots, S_k$  be subsets of  $G$  such that for each  $i, 1 \leq i \leq k$ , there is an infinite subset  $R_i$  of  $G$  so that the sets  $rS_i$ , where  $r \in R_i$ , are pairwise disjoint and  $\bar{R}_i$  is compact. Then the inner Haar measure of  $\bigcup_{i=1}^k S_i$  is zero.

**PROOF.** Since  $G$  admits an invariant mean, there is a nonnegative, finitely additive extension  $\mu$ , of  $\lambda$  to all subsets of  $G$  which is also left invariant.

It is enough to prove the theorem under the assumption that the sets  $S_i$  are pairwise disjoint.

Assume  $\bigcup S_i$  has positive inner measure. Let  $F$  be a compact set  $F \subset \bigcup S_i$ , with  $\lambda(F) > 0$ . Let  $D_i = F \cap S_i$ . Now, the sets  $rD_i$ , where  $r \in R_i$  are pairwise disjoint and  $\bigcup \{rD_i : r \in R_i\}$  is a subset of the compact set  $\bar{R}_i F$ . Therefore, for each  $i$ ,  $\mu(D_i) = 0$ .

This means  $\mu(\cup D_i) = 0 = \mu(F) = \lambda(F) > 0$ . This contradiction establishes the theorem.

Next we give an example to show that the conclusion of this theorem may be false if the group is not amenable.

Example. Let  $G$  be the orthogonal group on  $E^3$ . Notice that  $G$  acts transitively on  $S$ , the unit sphere of  $E^3$ . Let  $N$  be the north pole of  $S$  and let  $H$  be the stability subgroup of  $G$  at  $N$ .  $H = \{g \in G : g(N) = N\}$ . Then  $H$  is a closed subgroup of the compact group  $G$ . Let  $\theta$  be the one-to-one map of the left coset space  $G/H$  onto  $S : \theta(gH) = g(N)$ .

Now, according to Hausdorff

$$S = A \cup B \cup C \cup D,$$

where  $A, B, C$ , and  $D$  are disjoint,  $D$  is countable and  $(*) A \cong B \cup C, A \cong B, A \cong C$ .

Let  $E = \cup \{\theta^{-1}(d) : d \in D\}$ . Since  $\lambda(H) = 0$  and  $D$  is countable,  $\lambda(E) = 0$ . Let  $S_1 = \cup \{\theta^{-1}(a) : a \in A\}$   $S_2 = \cup \{\theta^{-1}(t) : t \in B \cup C\}$ . Then  $S_1 \cup S_2$  is a  $G_\delta$  set with  $\lambda(S_1 \cup S_2) > 0$ . Also, because of  $(*)$ , there are infinitely many pairwise disjoint translates of  $S_1$  and  $S_2$ .

Thus, the first method of proof leads to the following problem.

PROBLEM. Let  $G$  be a locally compact  $T_2$  group so that the conclusion of Theorem 1 holds. Is there a finitely additive left invariant extension of Haar measure to all subsets of  $G$ ?

Before giving the second method of proof, let us state the following lemma.

LEMMA. Let  $G$  be a locally compact group with left invariant Haar measure  $\lambda$ . Let  $F$  be a compact set such that  $0 < \lambda(F) < \infty$ . For each positive integer  $n$ , there is a neighborhood  $V$  of  $e$  so that if  $h_1, \dots, h_n$  are points of  $V$ , then  $\lambda(\cap \{h_i F : i \leq n\}) > 0$ .

The second method of proof is formulated for abelian groups.

THEOREM 2. Let  $G$  be a locally compact abelian group with Haar measure  $\lambda$  and let  $k$  be a positive integer. For each  $i, 1 \leq i \leq k$ , let  $S_i$  and  $R_i$  be subsets of  $G$  such that  $R_i$  is infinite,  $\bar{R}_i$  is compact, and the sets  $S_i + g, g \in R_i$  are pairwise disjoint. The  $\bigcup_{i=1}^k S_i$  has inner measure zero.

PROOF. Again, notice that we can and do assume that the sets  $S_i$  are disjoint. Let us assume that  $F$  is a compact set lying in  $\cup S_i$  and  $\lambda(F) > 0$ .

Let  $n > k$ . According to the lemma, there is some neighborhood  $V$  of  $e$  so that if  $h_p \in V, p = 1, \dots, n^k$ , then  $\lambda(M) > 0$ , where

$$M = F \cap (\cap \{F - h_p : p \leq n^k\})$$

For each  $i, 1 \leq i \leq k$ , obtain  $n+1$  distinct points  $g_{i0}, \dots, g_{in}$  of  $R_i$  so that for each  $k$ -tuple,  $p = (p_1, \dots, p_k)$  of the first  $n$  positive integers,

$$h_p = d_{1p_1} + \dots + d_{kp_k}$$

is in  $V$ , where  $d_{it} = g_{i0} - g_{it}$ .

According to the lemma, there is some  $x$  in  $F \cap (\cap \{F - h_p : p = (p_1, \dots, p_k) \in \{1, \dots, n\}^k\})$ .

For each  $i$ , let  $M_i = \{p : x + h_p \in S_i\}$ . The sets  $M_i$  are pairwise disjoint and each  $k$ -tuple of the first  $n$ -integers is in some  $M_i$ .

A contradiction will be reached by examining the cardinalities of the sets  $M_i$ . Notice that  $M_i$  has the following property. If  $(p_1, \dots, p_i, \dots, p_k) \in M_i$  and  $(r_1, \dots, r_i, \dots, r_k)$  is such that  $r_j = p_j$ , if  $j \neq i$  and  $r_i \neq p_i$ , then  $(r_1, \dots, r_k) \notin M_i$ . The reason for this is that if they were both in  $M_i$ , then  $(x + d_{1p_1} + \dots + d_{ip_i} + d_{kp_k}) + g_{ip_i} = x + d_{1p_1} + \dots + g_{i0} + \dots + d_{kp_k} = (x + d_{1r_1} + \dots + d_{ir_i} + \dots + d_{kr_p}) + g_{ir_k}$ . Thus, the sets  $S_i + g_{ip_i}$  and  $S_i + g_{ir_i}$  would not be disjoint.

Now, because of this property of the sets  $M_i$ , we know that,  $\text{card}(M_i) \leq n^{k-1}$ . Therefore,

$$n^k = \text{card}(\cup M_i) \leq kn^{k-1}.$$

This contradiction establishes the theorem. Q.E.D.

Both of these proofs raise the question of estimating the size of a subset  $S$  of  $T$  which can be partitioned into  $k$  sets each of which has  $n$  pairwise disjoint rotations.

As Mycielski pointed out to us, one can use the extension of Haar measure to all subsets of  $T$  and argue along the lines of Theorem 1 to obtain the following estimate.

Theorem 3. If  $S \subset T$  and  $S \subset \cup \{\Lambda_i : i \leq k\}$  where each set  $\Lambda_i$  has  $n$  pairwise disjoint rotations then the inner measure of  $S$  is  $\leq k/n$ .

The proof of Theorem 2 was based on some simple combinational properties of finite sets. The problem we pose is that of estimating the size of a measurable subset  $M$  of  $T$  from the fact that  $M$  possesses two sets of rotations which avoid the contradiction obtained in Theorem 2. We formulate this as follows.

STATEMENT 1. Let  $0 < \alpha$ . There is a positive integer  $n_0(\alpha)$  so that if  $M$  is a measurable subset of  $T$  with  $\lambda(M) > \alpha$  and  $R_1$  and  $R_2$  are subsets of  $T$  with  $|R_1|, |R_2| > n_0(\alpha)$ , then there are points  $g_{1i}$  of  $R_1$ ,  $g_{2i}$  of  $R_2$ ,  $i = 0, 1, \dots, 4$  such that

$$M \cap (\cap \{M - h_p : p = (p_1, \dots, p_3) \in \{1, \dots, 3\}^2\}) \neq \phi,$$

where

$$h_p = (g_{10} - g_{1p_1}) + (g_{20} - g_{2p_2}).$$

In fact, statement 1 leads to the consideration of the following statement.

STATEMENT 2. Suppose  $\alpha > 0$ . There is a positive integer  $t_0(\alpha)$  and a  $\beta > 0$  so that if  $M \subset T$ ,  $\lambda(M) > \alpha$  and  $R \subset T$ ,  $|R| > t_0(\alpha)$ , then there are points  $g_0, g_1, g_2, g_3$  of  $R$  so that

$$\lambda\left(\bigcap_{i=0}^3 M + (g_0 - g_i)\right) > \beta.$$

Clearly, statement 2 implies statement 1.

We have been unable to determine whether statement 2 is true. However, we have been led to the following statement.

STATEMENT 3. For each  $c > 0$ , there is an integer  $\ell_0(c)$  and an integer  $N_0(c)$  so that if  $\ell > \ell_0(c)$  and  $N > N_0(c)$  and

$$1 \leq a_1 < a_2 < \dots < a_t \leq N \text{ where } t > cN,$$

then for each  $\ell$  integers

$$b_1 < \dots < b_\ell < N,$$

there is an arithmetic progression of three terms among the  $a$ 's of difference some  $b_j - b_i$ .

At this time, we do not know which if any, of the preceding three statements is true.