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THE SET OF CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTIONS

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Let M be the set of all continuous real-valued functions defined on the interval $[0, 1]$ which do not have a finite derivative anywhere. It is shown that M forms a coanalytic, non-Borel, subset in the space of all real-valued continuous functions on $[0, 1]$ provided with the uniform norm.

Let C be the space of all real-valued continuous functions defined on the unit interval provided with the uniform norm. In the Scottish Book, Banach raised the question of the descriptive class of the subset D of C consisting of all functions which are differentiable at each point of $[0, 1]$. Banach pointed out that D forms a coanalytic subset of C and asked whether D is a Borel set. Later Mazurkiewicz showed that D is not a Borel set [3].

In this paper, we shall investigate the subset M of C consisting of all functions which do not have a finite derivative at any point of $[0, 1]$. It is well known that M is residual in C [2]. We shall prove the following theorem.

THEOREM A. *Let $M = \{f \in C: f$ does not have a finite derivative at any point of $[0, 1]\}$. The set M is a coanalytic subset of C which is not a Borel set.*

In order to see that $C - M$ is an analytic set, notice that a continuous function f has a finite derivative at some point x of $[0, 1]$ if and only if for each positive integer n, there is a positive integer m so that (*) if $0 < |h_1|$, $|h_2| < 1/m$ and $x + h_1$ and $x + h_2$ are both in $[0, 1]$, then

$$\left|\frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2}\right| \leqq \frac{1}{n}\,.$$

For each pair of positive integers (n, m), let $E(n, m) = \{(f, x) \in C \times [0, 1]: (*)$ holds$\}$. Then $C - M$ is the projection into C of $\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E(n, m)$. It may be checked that each set $E(n, m)$ is a closed subset of $C \times [0, 1]$. Thus, M is a coanalytic subset of C. The remainder of this paper is devoted to demonstrating that M is not a Borel set.

Let us make the following conventions. The set of positive integers will be denoted by N; by N^* shall be meant the set of all finite sequences of positive integers. We shall denote elements of

$J = N^N$ by Greek letters and the terms of such a sequence by its nearest Roman equivalent. Also, if $\sigma = \langle s_k \rangle_{k=1}^\infty \in J$ and $n \in N$, then $\sigma | n = \langle s_1, \cdots, s_n \rangle$.

For each element $s = \langle s_1, \cdots, s_k \rangle$ of N^*, let $I(s)$ be the left open, right closed interval with left end point

$$a(s) = 2^{-s_1} + 2^{-(s_1+s_2)} + \cdots + 2^{-(s_1+\cdots+s_k)}$$

and with right end point

$$b(s) = a(s) + 2^{-(s_1+\cdots s_k)} .$$

Notice, that $(0, 1] = \bigcup_{p=1}^\infty I(\langle p \rangle)$, $I(\langle s_1, \cdots, s_k \rangle) = \bigcup_{p=1}^\infty I(\langle s_1, \cdots, s_k, p \rangle)$ and if s and t are distinct elements of N^* having the same length, then $I(s)$ and $I(t)$ are disjoint. For each $\sigma \in J$, let $x(\sigma)$ be the point of intersection of $\bigcap_{k=1}^\infty I(\sigma | k)$. We have $x(\sigma) = \sum_{i=1}^\infty 2^{-(s_1+\cdots+s_i)}$. For each interval $(a, b]$, set

$$\varphi_{(a,b]}(x) = \begin{cases} x - a, & \text{if} \quad a < x \leqq (a+b)/2 , \\ b - x, & \text{if} \quad (a+b)/2 \leqq x \leqq b , \\ 0, & \text{otherwise} . \end{cases}$$

For each positive integer n, let $h_n = \Sigma \varphi_{I(s)}$, where the summation is taken over all elements of N^* which have length n. Also, let us set $h_0(x) = 1/2 - |x - 1/2|$, for $x \in [0, 1]$. For each n, h_n is a "sawtooth" function on $[0, 1]$. First we give three lemmas concerning these functions.

LEMMA 1. *For each n, h_n is nonnegative and $h_n(x) \leqq x/(2^{n+1} - 1)$, for each x in $[0, 1]$.*

Proof. It can be checked that the line through $(0, 0)$ and the highest point of the graph of h_n over the interval $I(\langle s_1, \cdots, s_n \rangle)$ has slope $1/1 + 2(1 + \sum_{i=1}^n 2^{s_i+\cdots+s_n}) \leqq 1/(2^{n+1} - 1)$. This means $h_n(x) \leqq x/(2^{n+1} - 1)$, for $x \in [0, 1]$.

We will also require the fact that the action of the functions h_p is being reproduced on each of the intervals $I(\langle q_1, \cdots, q_n \rangle)$. This is the content of the next lemma which may be proven by induction.

LEMMA 2. *Let $\langle q_1, \cdots, q_n \rangle \in N^*$ and let*

$$g(x) = 2^{q_1+\cdots+q_n}x - \left(\frac{1}{2^{q_1}} + \cdots + \frac{1}{2^{q_1+\cdots+q_n}}\right) .$$

Then g maps $I(\langle q_1, \cdots, q_n \rangle)$ onto $(0, 1]$ and for each $p \geqq 0$, $h_p(g(x)) = (2^{q_1+\cdots+q_n})h_{n+p}(x)$, for $x \in I(\langle q_1, \cdots, q_n \rangle)$.

LEMMA 3. *Let* $\langle q_1, \cdots, q_{2^k}\rangle \in N^*$, *then*

$$h_{2^k+1}(x) \leqq \frac{1}{2^{2^k}}\left(x - \left(\frac{1}{2^{q_1}} + \cdots + \frac{1}{2^{q_1+\cdots+q_{2^k}}}\right)\right),$$

for each $x \in I(\langle q_1, \cdots, q_{2^k}\rangle)$.

Proof. By Lemma 2, for $x \in I(\langle q_1, \cdots, q_{2^k}\rangle)$

$$h_{2^k+1}(x) = \left(\frac{1}{2^{q_1+\cdots+q_{2^k}}}\right)h_{2^k}(g(x)),$$

where g is the appropriate function defined in Lemma 2. According to Lemma 1,

$$h_{2^k+1}(x) \leqq \left(\frac{1}{2^{q_1+\cdots+q_{2^k}}}\right)\frac{g(x)}{2^{2^k+1}-1}.$$

Substituting for $g(x)$ and noting that $2^{2^k} < 2^{2^k+1} - 1$:

$$h_{2^k+1}(x) \leqq \frac{1}{2^{2^k}}\left(x - \left(\frac{1}{2^{q_1}} + \cdots + \frac{1}{2^{q_1+\cdots+q_{2^k}}}\right)\right).$$

It can be shown that Σh_n does not have a finite derivative at any x in the $(0, 1]$, although we shall not use this fact. However, Theorem A will be demonstrated by continuously modifying a subsequence of $\{h_n\}_{n=1}^{\infty}$. We proceed as follows.

Let E be an analytic subset of the Cantor set K. Let H be a map from N^* into the clopen subsets of K so that

$$E = \bigcup_{\sigma \in J} \bigcap_{k=1}^{\infty} H(\sigma | k).$$

We may assume that $H(\sigma|k) \supseteqq H(\sigma|n)$ if $n > k$ and diam $(H(\sigma|k)) < 1/k$ [2].

For each $q = \langle q_1, q_2, \cdots, q_{2^i}\rangle \in N^{2^i}$, set

$$\lambda_q = 1 - \chi_{A(q)\cup H(\langle q_1,\cdots,q_{2^i-1}\rangle)}$$

where $A(q) = \cup\{H(s): s \in N^{2^i}$ and $|a(s) - b(q)| < 2^i/(2^{2^i+1} - 1 + 2^i)\}$. Of course, χ_B denotes the characteristic function of B on the Cantor set K.

For each $n \in N$, set

$$f_n(x, t) = \Sigma\lambda_s(t)\varphi_{I(s)}(\chi),$$

where summation is taken over all elements s of N^* of length 2^n.

Let $G(x, t) = \sum_{n=1}^{\infty} f_n(x, t)$ and $F(x, t) = t + \sqrt{x} + G(x, t)$, for $(x, t) \in [0, 1] \times K$. Finally, define the map Γ from K into C by setting $\Gamma(t) = F(\cdot, t)$, for each t in K. We next note three elementary properties of Γ.

First, notice that since $f_n(x, t) \leqq h_{2n}(x) < 2^{-n}$, for each n, the series $\Sigma f_n(x, t)$ converges uniformly over $[0, 1] \times K$. Since, for each t, the functions $f_n(\cdot, t)$ are continuous, the function $\Gamma(t)$ is an element of C. Since $F(0, t) = t$, Γ is one-to-one.

Second, notice that $\Gamma(t)$ does not have a finite derivative at 0. This is because $(\sqrt{x})'(0) = +\infty$ and $G(x, t) - G(0, t) \geqq 0$.

Third, notice that Γ is a Borel measurable map of K into C. This may be seen by as follows. Define $\Gamma_n \colon K \to C$ by

$$(\Gamma_n(t))(x) = t + \sqrt{x} + \sum_{p=1}^{n} f_p(x, t) .$$

Then $\{\Gamma_n\}_{n=1}^{\infty}$ converges uniformly to Γ. Also, note that if (X, M) is a measurable space, Y is a metric space and $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable maps from X into Y and this sequence converges uniformly to f, then f is a measurable map. This last fact may be used to verify that each function Γ_n is Borel measurable and then applied once again to show that Γ is Borel measurable.

We shall require some deeper properties of the function Γ.

LEMMA 4. *Suppose* $\sigma \in J$ *and* $\{t\} = \bigcap_{n=1}^{\infty} H(\sigma | n)$ *and* $x_0 = x(\sigma)$. *Then* $\Gamma(t)$ *has a left derivative at* x_0 *and* $G(\cdot, t)$ *has left derivative zero at* x_0.

Proof. It suffices to show that $G(\cdot, t)$ has left derivative zero at x_0.

Let $\varepsilon > 0$. Let n be a positive integer so that $2^{-n} < \varepsilon$. Let δ be a positive number so that $(x_0 - \delta, x_0] \subseteq I(\sigma | 2^n)$. Since $f_i(x_0, t) = 0$, for all i,

$$\left| \frac{G(x, t) - G(x_0, t)}{x - x_0} \right| \leqq \sum_{i=1}^{n} \left| \frac{f_i(x, t)}{x - x_0} \right| + \sum_{p=1}^{\infty} \left| \frac{f_{n+p}(x, t)}{x - x_0} \right| .$$

Let $x_0 - \delta < x < x_0$. If $1 \leqq i \leqq n$, then $f_k(x, t) = 0$. Suppose $p \geqq 1$. Set $\alpha = 2^{2^{n+p+1}} - 1$, $\beta = 2^{n+p}$ and $d = (\alpha/\alpha + \beta)x_0$. If $x \leqq d$, then

$$\left| \frac{f_{n+p}(x, t)}{x - x_0} \right| \leqq \frac{h_{2^{n+p}}(x)}{x_0 - x} .$$

Using Lemma 1 and the fact that $1/(x_0 - x) \leqq 1/(x_0 - d)$, we have

$$\left| \frac{f_{n+p}(x, t)}{x - x_0} \right| \leqq \frac{d}{\alpha} \cdot \frac{1}{x_0 - d} = 2^{-(n+p)} .$$

If $d < x < x_0$, then there is some $z = \langle z_1, \cdots, z_{2^{n+p}} \rangle$ so that

$$f_{n+p}(x, t) = \lambda_z(t)\varphi_{I(z)}(x) .$$

If $z = \sigma|2^{n+p}$ then $f_{n+p}(x, t) = 0$. Otherwise, $d < b(z) \leqq a(\sigma|2^{n+p}) < x_0$. Thus, $|a(\sigma|2^{n+p}) - b(z)| < x_0 - d = x_0(1 - \alpha/(\alpha + \beta)) \leqq \beta/(\alpha + \beta)$. This implies that t is in $A(z)$ and therefore $f_{n+p}(x, t) = 0$. These considerations lead to the conclusion that $G(\cdot, t)$ has left derivative zero at x_0. □

Let us make the following conventions. The set of all elements of J which are equal to one from some term on will be denoted by Q. Let $R(Q)$ denote the set of all x in $[0, 1]$ such that there is some element $\sigma \in Q$ for which $x = x(\sigma)$. Notice that Q and $R(Q)$ are countable sets and $\sigma \in J - Q$ if and only if $x(\sigma)$ is in the interior of $I(\sigma|k)$, for each k.

LEMMA 5. *Suppose* $\sigma \in J - Q$, $\{t\} = \cap H(\sigma|k)$, *and* $x_0 = x(\sigma)$. *Then* $\Gamma(t)$ *is differentiable at* x_0.

Proof. In view of Lemma 4, it suffices to show that $G(\cdot, t)$ has right derivative zero at x_0.

Let $\varepsilon > 0$. Let n be a positive integer so that $2^{-n} < \varepsilon$. Since $\sigma \in J - Q$, x_0 is in the interior of $I(\sigma|2^{n-1})$. Let δ be a positive number so that $[x_0, x_0 + \delta) \subseteqq I(\sigma|2^{n-1})$ and let x be between x_0 and $x_0 + \delta$. Since $f_k(x_0, t) = 0$, for all k, we have

$$\left|\frac{G(x, t) - G(x_0, t)}{x - x_0}\right| \leqq \sum_{i=1}^{n} \left|\frac{f_i(x, t)}{x - x_0}\right| + \sum_{p=1}^{\infty} \left|\frac{f_{n+p}(x, t)}{x - x_0}\right| .$$

It can be checked that if $1 \leqq i \leqq n$, then $f_i(x, t) = 0$.

Suppose $p \geqq 1$. If $x \in I(\sigma|2^{n+p-1})$, then $f_{n+p}(x, t) = 0$. Suppose $b(\sigma|2^{n+p-1}) < x < x_0 + \delta$. There is some $q = \langle q_1, \cdots, q_{2^{n+p-1}}\rangle$ so that $x \in I(q)$. Using Lemma 3, we have

$$\left|\frac{f_{n+p}(x, t)}{x - x_0}\right| \leqq \frac{h_{2^{n+p}}(x)}{x - x_0} = \frac{h_{2^{n+p}}(x)}{x - a(q)} \cdot \frac{x - a(q)}{x - x_0} < 2^{-2(n+p-1)} < 2^{-(n+p)} .$$

It follows from these considerations that $G(\cdot, t)$ has right derivative zero at x_0.

LEMMA 6. *If* t *is in* $K - E$, *then* $\Gamma(t)$ *does not have a finite derivative at any point of* $[0, 1] - R(Q)$.

Proof. We have already noted that $\Gamma(t)$ does not have a finite derivative at 0. Thus, it suffices to show that $G(\cdot, t)$ does not have a finite derivative at any point of $(0, 1) - R(Q)$.

Let σ be an element of $J - Q$ and let $x_0 = x(\sigma)$.

Suppose there is a positive integer p_0 so that if $p \geqq p_0$, then $\lambda_{\sigma|2^p}(t) = 0$. If $p \geqq p_0$, then t is in $A(\sigma|2^p)$ or t is in $H(\sigma|2^{p-1})$. If t were in $H(\sigma|2^{p-1})$, for infinitely many p, then t would be in E. Thus, we may assume that if $p \geqq p_0$, then t is in $A(\sigma|2^p)$. For each $p \geqq p_0$, there is a point $q^p = \langle q_1^p, \cdots, q_{2^p}^p \rangle$ in N^{2^p} so that t is in $H(q^p)$ and $|a(q^p) - b(\sigma|2^p)| < 2^p/(2^{2^p+1} - 1 + 2^p)$. This implies that the sequence $\{a(q^p)\}_{p=1}^{\infty}$ converges to x_0. Since x_0 is in the interior of $I(\sigma|1)$, this implies that there is a positive integer n_1 so that if $p > n_1$, then $q_1^p = s_1$. This means that t is in $H(\sigma|1)$. Similar considerations show that for each i, t is in $H(\sigma|i)$. This implies that t is in E. This contradicts the assumption that t is not in E. Thus, there are infinitely many p such that $\lambda_{\sigma|2^p}(t) = 1$.

Let $\delta > 0$. Choose p so that $\lambda_{\sigma|2^p}(t) = 1$ and $(a, b] = I(\sigma|2^p)$ is a subset of $(x_0 - \delta/2, x_0 + \delta/2)$. Let $m = (a + b)/2$. Since $m \neq x_0$ and

$$\begin{aligned}
&\left|\frac{G(b, t) - G(a, t)}{b - a} - \frac{G(b, t) - G(m, t)}{b - m}\right| \\
&= \left|\frac{G(b, t) - G(a, t)}{b - a} - \frac{G(m, t) - G(a, t)}{m - a}\right| \\
&= \left|\frac{f_p(b, t) - f_p(a, t)}{b - a} - \frac{f_p(b, t) - f_p(m, t)}{b - m}\right| \\
&= \left|\frac{f_p(b, t) - f_p(a, t)}{b - a} - \frac{f_p(m, t) - f_p(a, t)}{m - a}\right| = 1\,,
\end{aligned}$$

it follows that $G(\cdot, t)$ does not have a finite derivative at x_0 [4, pp. 114-116].

Let us collect the preceding lemmas together.

THEOREM B. *There is a countable subset Y of* $[0, 1]$ *such that for each analytic subset E of K there is a one-to-one Borel measurable map Γ of K into C and a countable subset S of E so that* (1) *if t is in $E - S$, then $\Gamma(t)$ has a finite derivative at some point of* $[0, 1] - Y$ *and* (2) *if t is in $K - E$, then $\Gamma(t)$ does not have a finite derivative at any point of* $[0, 1] - Y$.

A proof of Theorem A can now be given. Let $Y = \{y_n\}_{n=1}$ be a countable subset of $[0, 1]$ so that Theorem B holds. Let $D(Y) = \{f \in C$: f has a finite derivative at some point of $[0, 1] - Y\}$. It can be shown that $D(Y)$ is an analytic subset of C (in fact, if Y is any coanalytic subset of $[0, 1]$, then $D(Y)$ is an analytic subset of C). Now, if $D(Y)$ were a Borel subset of C, then by applying Theorem B, every analytic subset of K would be a Borel subset of K. This contradiction establishes that $D(Y)$ is not a Borel subset of C. If

M were a Borel subset of C, then $D(Y)$ would be a Borel set, since

$$D(Y) = (C - M) - \bigcup_{n=1}^{\infty} D_n ,$$

where $D_n = \{f \in C\colon f$ has a finite derivative at $y_n\}$, and each set D_n is an $F_{\sigma\delta}$ subset of C. This contradiction establishes Theorem A.

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