THE BOUNDEDNESS OF THE CANTOR-BENDIXSON ORDER OF SOME ANALYTIC SETS

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Let X and Y be complete separable metric spaces (Polish spaces). If E is a subset of $X \times Y$, and $x \in X$, then by the x-section of E, E_x, is meant $E \cap (\{x\} \times Y)$. By $P_S(E)$ is

$$P_S(E) = \{x: E_x \text{ is scattered}\} .$$

In this paper the following uniform boundedness principle for the Cantor-Bendixson order of analytic sets will be demonstrated.

THEOREM L. Let W be an analytic subset of $X \times Y$ and let M be an analytic subset of X such that $M \subset P_S(W)$. Then there is a countable ordinal α such that the αth Cantor-Bendixson derived set of E_x is empty, for each x in M.

Let us recall that if A is a subset of X, then the Cantor-Bendixson derivatives of A may be defined by transfinite induction as follows:

$$A^{(0)} = A$$

$A^{(\alpha)} = \bigcap_{\beta < \alpha} \{x: x \text{ is an acculumation point of } A^{(\beta)} \text{ and } x \in A\}$.

Recall that a subset, H, of a Polish space is scattered if, and only if H is a countable G_δ set, or equivalently, there is a countable ordinal γ such that the γth Cantor-Bendixson derived set, $H^{(\gamma)}$, of H is empty [5]. By the Cantor-Bendixson order of a subset H of a topological space is meant the first ordinal γ such that $H^{(\gamma)} = H^{(\gamma+1)}$. The Cantor-Bendixson order of every subset of a Polish space is necessarily less than ω_1 [5].

If $E \subset X \times Y$ and $M \subset X$, then E will be bounded on M provided there is an ordinal γ, $\gamma < \omega_1$, such that for each x in M, the Cantor-Bendixson order of E_x is $\leqq \gamma$; otherwise E will be said to be unbounded on M.

Let us note that in order to prove Theorem L it suffices to show that if E is an analytic subset of $X \times Y$ such that each x-section of E is scattered then E is bounded on the X projection of E, $\pi_X(E)$.

Theorem L has the following corollary:

COROLLARY 1. *Let X be an uncountable Polish space. Let $\mathscr{C}$ be any class of countable G_δ subsets of X which contains all the countable compact subsets of X except possibly countably many. Then no analytic set in X^2 can be universal for $\mathscr{C}$.*

Corollary 1 is proven by A. S. Kechris and D. A. Martin in [3]. Also, Corollary 1 has as corollaries:

COROLLARY 2. *There is no analytic set in the plane R^2 which is universal for the countable closed subsets of R.*

COROLLARY 3. *There is no Borel set in R^2, which is universal for the countable G_δ subsets of R.*

Corollaries 2 and 3 are proven by D. G. Larman and C. A. Rogers in [6]. A proof of Corollary 2 is given in the first section of this paper which involves techniques which differ from those used in [3], [6] and in the following parts of this paper.

The techniques of proof used in the second and third sections of this paper follow those used by Lusin in his deep book [7]. In fact, on page 186 of [7], Lusin states Theorem L for the space $R^n \times R$. However, Lusin does not give a proof.

In the second section a proof of another theorem of Lusin's is given [7, p. 247]:

THEOREM C. *Let A be an analytic subset of $X \times Y$ such that each X-section of A is countable. Then $A \subset B$, where B is a Borel subset of $X \times Y$ such that each X-section of B is also countable.*

The proof given here is in a slightly more general setting than Lusin's and the binary sieve of Lebesgue is used here instead of "the elementary sieve" as used by Lusin.

Theorem C is the major key to the proof given here of Theorem L.

Some definitions and notations are now given.
Let E be a subset of $X \times Y$ and M a subset of X.
If α is an ordinal, then

$$D_Y^{(\alpha)}(E) = \bigcup_{x \in X} (E_x)^{(\alpha)} .$$

A family $\mathscr{H}$ of subsets of $X \times Y$ is said to be simultaneously unbounded with respect to E on M provided that for each $\gamma < \omega_1$, there is some $x \in M$ such that

$$(H \cap D_Y^{(\gamma)}(E))_x \neq \varnothing ,$$

for all $H \subset \mathscr{H}$.

If $0 < \varepsilon_1 < \varepsilon_2$ and $\Gamma \subset X \times Y$, then the ε_1-band of Γ with respect to E is: $B_{\varepsilon_1}(\Gamma; E) = \{(x, z) \in \Gamma : \exists (x, y) \in E$ with the distance from y to z less than $\varepsilon_1\}$; the $(\varepsilon_1, \varepsilon_2)$-ring of Γ with respect to E is;

$R_{\varepsilon_1}^{\varepsilon_2}(\Gamma, E)\{(x, z) \in \Gamma: \exists (x, y) \in E$ such that the distance from y to z is between ε_1 and $\varepsilon_2\}$.

It is a pleasure to thank Robert R. Kallman for many valuable discussions concerning descriptive set theory.

Before proceeding to the arguments, the author would like to pose two of the problems which have occurred to him while working on this paper;

Is there an analytic subset of $I \times I$ which is universal for the family of all analytic subsets of I of Lebesgue measure zero? of the first category?

1. Applications of the Arsenin-Kunugui theorem.

In this section, the Arsenin-Kunugui theorem is applied to treat some problems discussed earlier.

THEOREM 1.1. *Let B be a Borel subset of $X \times Y$ such that for each x, B_x is a K_σ set (the union of countably many compact sets.) Then for each $\alpha < \omega_1$, $D_Y^{(\alpha)}(B)$ is a Borel set and each X-section of $D_Y^{(\alpha)}(B)$ is a K_σ set.*

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be a countable base for the topology of Y. For each n, let $W_n = \{x: [B \cap (X \times U_n)]_x$ consists of a single point$\}$. According to a theorem of Braun [1], W_n is a Borel subset of X. Then

$$D_Y^{(1)}(B) = B - \bigcup_{n=1}^{\infty} (W_n \times U_n),$$

and $D_Y^{(1)}(B)$ is a Borel set.

Clearly, each X-section of $D_Y^{(1)}(B)$ is the intersection of a closed set with the corresponding X-section of B. Thus, $D_Y^{(1)}(B)$ has the properties stated in the conclusion.

If $D_Y^{(\alpha)}(B)$ has the stated properties, then by the preceding argument $D_Y^{(\alpha+1)}(B)$ has the stated properties.

Now, suppose α is a countable limit ordinal. Let $\{\alpha_n\}_{n=1}^{\infty}$ be an increasing sequence of ordinals coverging to α. Clearly, $D_Y^{(\alpha)}(B) = \bigcap_{n=1}^{\infty} D_Y^{(\alpha_n)}(B)$. So, $D_Y^{(\alpha)}(B)$ is Borel. Let $x \in X$. Then $T = B_x$ is a K_σ set. If $(x, y) \in T - (D_Y^{(\alpha)}(B))_x$ then there is some n such that $U_n \cap T$ misses $(D_Y^{(\alpha)}(B))_x$. Thus, $T - (D_Y^{(\alpha)}(B))_x$ is open with respect to T and $(D_Y^{(\alpha)}(B))_x$ is the intersection of a closed set with the K_σ set T. Thus, $(D_Y^{(\alpha)}(B))_x$ is a K_σ set.

THEOREM 1.2. *Let E be a subset of $X \times Y$ which is universal*

for the family of countable, closed subsets of Y. *If* Y *is uncountable, then* E *is not analytic.*

Proof. Let us suppose that E is analytic and Y is uncountable. Let P be a compact perfect subset of Y. Let $A = (X \times P) \cap E$. Then A is an analytic subset of $X \times P$ which is universal for the countable closed subsets of P.

Let F be a closed subset of $(X \times P) \times I$ such that for each pair $(x, p) \in X \times P$, $F_{(x,p)}$ is uncountable if and only if $(x, p) \in A$ [4, p. 497].

Let φ be the map of X into, $2^{P \times I}$, the space of closed subsets of $P \times I$ defined by $\varphi(x) = F_x$, where F_x is regarded as a closed subset of $X \times (P \times I)$. $2^{P \times I}$ is considered to have the topology generated by the Hausdorff metric.

Now, it follows from the Arsenin-Kunugui theorem that φ is a Borel mapping of X into $2^{P \times I}$ [2, p. 379].

Let $M = [2^P \times 2^{P \times I} \times \varphi(X)] \cap G$, where $G = \{(D, F, B) \in 2^P \times 2^{P \times I} \times 2^{P \times I} : D = \pi_P(F)$, and $F \subset B$ and if V is open in P and $F \cap (V \times I) \neq \varnothing$, then $\exists x \varepsilon V$ such that F_x is uncountable$\}$.

For the moment, let us assume that M is analytic. This implies that $\pi_1(M) = \{D \in 2^P : D$ is countable$\}$, is an analytic subset of 2^P. But, this set is known not to be analytic in 2^P, [5, p. 72].

Thus, the proof will be complete, once M is shown to be analytic. In order to see this it suffices to show that the set L of the next lemma is analytic.

LEMMA 1.3. *Let* $L = \{F \in 2^{P \times I} : $ *if* V *is an open subset of* P *and* $V \cap \pi_1(F) \neq \varnothing$, *then there is some* $x \in V$ *such that* F_x *is uncountable*$\}$. *Then* L *is an analytic subset of* $2^{P \times I}$.

Proof. Let $\{V_n\}_{n=1}^{\infty}$ be a countable base for the topology of P. Let $L_n = \{F \in 2^{P \times I} : $ either $V_n \cap \pi_1(F) = \varnothing$ or there is some $x \in V_n$ such that F_x is uncountable$\}$. Evidently, $L = \bigcap_{n=1}^{\infty} L_n$.

Let g be the map of $P \times 2^I$ into $2^{P \times I}$ defined by $g(x, T) = \{x\} \times T$. Let $H = \{T \in 2^I : T$ is uncountable$\}$. Then H is an analytic set [5, p. 72]. Clearly, g is continuous and therefore $W = g(P \times H)$ is an analytic subset $2^{I \times I}$. Let $S = \{(K, F) \in 2^{P \times I} \times 2^{P \times I} : K \in W$ and $K \subset F\}$. Then S is analytic in $2^{P \times I} \times 2^{P \times I}$ and

$$L_n = \pi_1(S) \cup \{F \in 2^{P \times I} : V_n \cap \pi_1(F) = \varnothing\}.$$

It follows from this expression that each L_n is analytic and therefore L is analytic.

Let us note that by following the arguments of this section, we have:

THEOREM 1.4. *Let Y be Polish and $\mathscr{C}$ a family of closed countable subsets of Y such that each closed countable subset of Y is a subset of some member of $\mathscr{C}$. If Y is uncountable and E is an universal set for $\mathscr{C}$ in $X \times Y$, then E is not analytic.*

Finally it was noted earlier that if B is Borel in $X \times P$ and each X-section of B is closed, then $\varphi: X \to 2^P$ defined by $\varphi(x) = B_x$ is a Borel map of X into 2^P. This leads to the following

EXAMPLE. There is an analytic subset A of $I \times I$ such that each vertical section of A is closed and yet $\{A_x: x \in I\}$ is not an analytic subset of 2^I of 2^I (regard each A_x as a subset of I).

Before constructing such a set, let us note the following

LEMMA. *If G is an analytic subset of 2^I, then $T = \{x: x \in H,$ for some $H \in G\}$ is an analytic subset of I.*

Proof. Let $M = \{(t, H): t \in H$ and $H \in 2^I\}$. Then M is a closed subset of $I \times 2^I$ and $T = \pi_1(M \cap (I \times G))$.

Construction. Let E be an analytic nonborel subset of I and let $A = \{(x, y): x = y$ or $(x \in E$ and $y = 0)\}$. Then A is an analytic subset of $I \times I$.

Suppose $J = \{A_x: x \in I\}$ is analytic in 2^I. Let $S = \{\{x\}: x \in I\}$. Then S is a closed subset of 2^I and $W = S \cap J$ is an analytic subset of 2^I. Therefore,

$$T = \{t: t \in H, \text{ for some } H \in W\}$$

is an analytic subset of I. But, $T = I - E$. This is a contradiction.

2. Theorem C. In this section Theorem C is proven. The proof follows to a great extent the outline given by Lusin in [7, p. 247].

Let A be an analytic subset of $X \times Y$ such that each X-section of A is countable. The proof is by contradiction. Thus, if B is a Borel set containing A, then some X-section of B is uncountable.

First, let us realize A as the projection of a G_δ set. Let $\{U_n\}_{n=0}^\infty$ be a decreasing sequence of open subsets of $(X \times Y) \times I$, where I is the unit interval such that

$$\pi_{XY}(G) = A \, ,$$

where

$$G = \bigcap_{n=0}^{\infty} U_n \,.$$

LEMMA 2.1. *There is an open parallelepiped* R *in* $X \times Y \times I$ *such that*

(i) *diameter* $(R) < 1$.

(ii) $\bar{R} \subset U_0$.

(iii) *if* B *is a Borel subset of* $X \times Y$ *containing* $\pi_{XY}(R \cap G)$, *then there are uncountably many x's such that B_x is uncountable.*

Proof. Let $\mathscr{B}$ be a countable base for the topology of U_0 consisting of open parallelepipeds, B, such that diam $(B) < 1$ and $\bar{B} \subset U_0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence consisting of all the members of $\mathscr{B}$ which have nonempty intersection with G.

Let us suppose that for each n, there are a Borel set B_n containing $\pi_{XY}(T_n \cap G)$ and a countable subset C_n of X such that if $x \notin C_n$, then B_{nx} is countable.

Let $B = (\bigcup_{n=1}^{\infty} B_n - \bigcup_{n=1}^{\infty} (C_n \times Y)) \cup (A \cap \bigcup_{n=1}^{\infty} (C_n \times Y))$.

Clearly, B is a Borel set containing A and every X-section of B is countable. This contradiction establishes Lemma 2.1.

Let L be a sieve which sifts $\pi_{XY}(R \cap G)(R \cap G)$ [5, 7]. Thus, L is a map from the rationals into $2^{X \times Y}$ such that $a \in A$ if and only if $I_L(a)$ is not well-ordered, where $I_L(a) = \{q \colon a \in L(q)\}$. Also, for each countable ordinal α, let $C_\alpha(L)$ be the αth constituent of $L \colon C_\alpha(L) = \{a \colon$ the order type of $I_L(a)$ is $\alpha\}$. Reinterpreting Lemma 2.1, we have

LEMMA 2.2. *For every* $\alpha < \omega_1$, *there is some* $x \in X$ *(in fact uncountably many) such that* $(\bigcup_{\alpha < \beta} C_\beta(L))_x$ *is uncountable.*

Proof. Let us assume the contrary. Then there is an $\alpha_0 < \omega_1$ such that for every x, $(\bigcup_{\alpha_0 < \beta} C(L))_x$ is countable.

But, then

$$B = X \times Y - \bigcup_{\beta \le \alpha_0} C_\beta(L)$$

is a Borel set containing $\pi_{XY}(R \cap G)$ and for every x, B_x is countable.

Let $\{F_n\}_{n=1}^{\infty}$ be a sequence consisting of all basic open rectangles, F, lying in $\pi_{XY}(R \cap U_1)$ such that there is a basic open subinterval, H, of I such that $\overline{F \times H} \subset R \cap U_1$, diam $(F \times H) < 1/2$ and $(F \times H) \cap G \neq \varnothing$.

For each n, let K_n map Q into the Borel subsets of $X \times Y$ by $K_n(q) = L(q) \cap F_n$. Thus, K_n is a sieve which sifts $\pi_{XY}(R \cap G) \cap F_n$. Moreover, the constituents are related as follows:

$$C_0(K_n) = C_0(L) \cup ((X \times Y) - F_n) \quad \text{and} \quad C_\alpha(K_n) = C_\alpha(L) \cap F_n \,,$$

for $\alpha > 0$.

LEMMA 2.3. *There are positive integers n and p such that $\bar{F}_n \cap \bar{F}_p = \varnothing$ and for every $\alpha < \omega_1$, there is some x such that $(\bigcup_{\alpha < \beta} C_\beta(K_n))_x$ and $(\bigcup_{\alpha < \beta} C_\beta(K_p))_x$ are both uncountable.*

Proof. For each $\alpha < \omega_1$, let $x(\alpha)$ be a point of X such that $W_\alpha = (\bigcup_{\alpha < \beta} C_\beta(L_0))_{x(\alpha)}$ is uncountable. Let $n(\alpha)$ and $p(\alpha)$ be positive integers such that $\overline{F_{n(\alpha)}} \cap \overline{F_{p(\alpha)}} = \varnothing$ and both $F_{n(\alpha)}$ and $F_{p(\alpha)}$ meet W_α in an uncountable set. There exist n and p such that $n = n(\alpha)$ and $p = p(\alpha)$ for uncountably many α's. Clearly, these integers have the required properties.

Let us reinterpret Lemma 2.3.

LEMMA 2.4. *If B_1 and B_2 are disjoint Borel sets containing $D_1 = F_n \cap \pi_{xy}(R \cap G)$ and $D_2 = F_p \cap \pi_{xy}(R \cap G)$ respectively, then there are uncountably many x's such that both B_{1x} and B_{2x} are uncountable.*

Proof. Let B_1 and B_2 be disjoint Borel sets containing D_1 and D_2 respectively. There is an ordinal $\alpha < \omega_1$ such that

$$E_\alpha^n = \bigcup_{\alpha < \beta} C_\beta(K_n) \subset B_1 \quad \text{and} \quad E_\alpha^p = \bigcup_{\alpha < \beta} C_\beta(K_p) \subset B_2 .$$

But, there are uncountably many x's such that $E_{\alpha x}^n$ and $E_{\alpha x}^p$ are uncountable.

LEMMA 2.5. *There is an open parallelepiped R_0 such that:*
(i) *diameter $(R_0) < 1/2$.*
(ii) $\overline{R_0} \subset R \cap U_1$.
(iii) $\pi_{xy}(R_0) \subset F_n$.
(iv) *if B_1 and B_2 are disjoint Borel sets containing*
$D_0 = \pi_{xy}(R_0 \cap G)$ *and* $D_2 = F_p \cap \pi_{xy}(R \cap G)$ *respectively, then there are uncountably many x's such that B_{1x} and B_{2x} are uncountable.*

Proof. Let $\{S_n\}_{n=1}^\infty$ be a sequence of all basic open parallelepipeds satisfying (i), (ii), and (iii) which also have nonempty intersection with G.

Suppose that for each n, there exist disjoint Borel sets M_n and N_n containing $\pi_{xy}(S_n \cap G)$ and D_2 respectively such that $C_n = \{x : M_{nx}$ and N_{nx} are uncountable$\}$ is countable.

Let $B_1 = \bigcup_{n=1}^\infty M_n$ and $B_2 = \bigcap_{n=1}^\infty N_n$. Then B_1 and B_2 are disjoint Borel sets containing D_1 and D_2 respectively. Let x be such that B_{1x} and B_{2x} are uncountable and yet $x \notin \bigcap_{n=1}^\infty C_n$. Then each M_{nx} is countable. This contradiction establishes Lemma 2.5.

By a similar argument one can show that there is an open parallelepiped R_1 such that

 (i)   diameter $(R_{01}) < 1/2$.

 (ii) $\overline{R_1} \subset R \cap U_{\cdot 1}$.

 (iii) $\pi_{xy}(R_1) \subset F_p$.

 (iv)   if B_1 and B_2 are disjoint Borel sets containing D_0 and $D_1 = \pi_{xy}(R_1 \cap G)$ respectively, then there are uncountably many x's such that B_{1x} and B_{2x} are uncountable.

Finally, it may be shown by induction:

THEOREM 2.6.   *For each finite sequence* $(e_1, \cdots, e_n)$ *of zeros and ones there is an open parallelepiped* $R_{e_1 \cdots e_n}$ *of* $(X \times Y) \times I$ *such that for each* n:

 (i)   *all* $R_{e_1 \cdots e_n e_{n+1}}$ *have diameter* $< 1/2^n$,

 (ii) $\overline{R_{e_1 \cdots e_n 0}} \cup \overline{R_{e_1 \cdots e_n 1}} \subset R_{e_1 \cdots e_n} \cap u_n$,

 (iii) $\overline{R_{e_1 \cdots e_n 0}} \cap \overline{R_{e_1 \cdots e_n 1}} = \varnothing$,

 (iv) $\pi_{xy}(R_{e_1 \cdots e_n 0}) \cap \pi_{xy}(R_{e_1 \cdots e_n 1}) = \varnothing$,

 (v)   *if* $B_{e_1 \cdots e_n e_{n+1}}$, *where* $(e_1, \cdots, e_{n+1}) \in \{0, 1\}^{n+1}$, *are disjoint Borel sets containing* $\pi_{xy}(R_{e_1 \cdots e_n e_{n+1}} \cap G)$, *then there are uncountably many* x's *such that all the sets* $(B_{e_1 \cdots e_{n+1}})_x$ *are uncountable.   For each finite sequence* $(e_1 \cdots e_n)$ *of* 0's *and* 1's, *let*

$$P(e_1 \cdots e_n) = (x(e_1 \cdots e_n), y(e_1 \cdots e_n), t(e_1 \cdots e_n))$$

be a point of $R_{e_1 \cdots e_n}$.

It can be checked that for each infinite sequence $e = \{e_k\}_{k=1}^{\infty}$ from 2^ω the sequence $\{P(e \mid n)\}_{n=1}^{\infty}$ converges to some point $P(e) = (x(e), y(e), t(e))$ of G.   It follows from Theorem 2.6, that if $e \neq e'$ then $y(e) \neq y(e')$.

It is now shown that if $e \neq e'$, then $x(e) = x(e')$:

It follows from the properties listed in Theorem 2.6 that for each n, there is some x_n of X and points y_n and y'_n of Y such that $(x_n, y_n) \in \pi_{xy}(R_{e \mid n})$ and $(x_n, y'_n) \in \pi_{xy}(R_{e' \mid n})$.

Therefore $x_n \longrightarrow x(e)$ and $x_n \longrightarrow x(e')$.   Thus, $x(e) = x(e')$.

This last result implies that $A_{x(e)}$ is uncountable.   This contradiction establishes Theorem C.

3.   **A Proof of Theorem L.**   In this section, Theorem L, as stated in the introduction, is proven.   As was noted in the introduction, it suffices to consider the following situation: E is an analytic subset of $X \times Y$ such that each section of E is scattered and for every α, $\alpha < \omega_1$, $D_Y^{(\alpha)}(E) \neq \varnothing$.

The aim of this section is to prove Theorem L by showing that these assumptions concerning E lead to some X-section of E con-

taining a dense-in-itself set. From this contradiction, it follows that Theorem L holds.

To begin with, we have from Theorem C that $E = \bigcup_{n=1}^{\infty} \Gamma_n$, where for each n, Γ_n is an analytic set which is uniform with respect to X. (Recall that Γ_n is uniform means each $(\Gamma_n)_x$ has cardinality less than 2.)

All rings and bands considered in this section are with respect to E.

Let n be such that $\Gamma = \Gamma_n$ has the property that for every countable ordinal α, $\Gamma \cap D_Y^{(\alpha)}(E) \neq \varnothing$. The existence of such a Γ_n follows from a simple cardinality argument.

LEMMA 3.1. *For each $\varepsilon > 0$, there is a positive number ε', $\varepsilon' < \varepsilon$ such that Γ and $\mathscr{R}_{\varepsilon'}^{\varepsilon}(\Gamma)$ are simultaneously unbounded.*

Proof. Let $\varepsilon > 0$. For each α, $\alpha < \omega_1$, let $(x_\alpha, y_\alpha) \in \Gamma \cap D_y^{(\alpha+1)}(E)$ and let $\varepsilon(\alpha)$ be a positive rational such that $(\mathscr{R}_{\varepsilon(\alpha)}^{\varepsilon}(\Gamma))_{x(\alpha)} \cap D_Y^{(\alpha)}(E) \neq \varnothing$. There must be some ε' which is $\varepsilon(\alpha)$ for uncountably many α's. Clearly, Γ and $\mathscr{R}_{\varepsilon'}^{\varepsilon}(\Gamma)$ are simultaneously unbounded.

Now, let ε' be such that Γ and $\mathscr{R}_{\varepsilon'}^{1}(\Gamma)$ are simultaneously unbounded.

According to Theorem C,

$$\mathscr{R}_{\varepsilon'}^{1}(\Gamma) = \bigcup_{n=1}^{\infty} T_n \, ,$$

where each T_n is an analytic set which is uniform with respect to X.

LEMMA 3.2. *There exists some k such that Γ and $T = T_k$ are simultaneously unbounded.*

This lemma also follows by a simple cardinality argument.

The set of all irrationals will be denoted by J. If $k_1 \cdots k_p$ is a finite sequence of positive integers, $\mathscr{N}_{k_1 \cdots k_p}$ denotes the set of irrationals with $k_1 \cdots k_p$ as the initial sequence in its continued fraction expansion.

Let φ_0 be a continuous map of J onto Γ; φ_1 a continuous map of J onto T, and ψ a continuous map of J onto $\pi_X(E)$.

Let $\delta = \min(\varepsilon'/2, 1/2)$. Then the bands $B_\delta(\varphi_0(J))$ and $B_\delta(\varphi_1(J))$ are disjoint.

LEMMA 3.3. *There is a positive rational δ', $\delta' < \delta$, such that the sets $\varphi_0(J)$, $\varphi_1(J)$, $\mathscr{R}_{\delta'}^{\delta}(\varphi_0(J))$ and $\mathscr{R}_{\delta'}^{\delta}(\varphi_1(J))$ are simultaneously unbounded on $\psi(J)$.*

Proof. For each α, $\alpha < \omega_1$, let $x(\alpha)$ be a point of X such that $(\varphi_i(J))_{x(\alpha)} \cap D_Y^{(\alpha+1)}(E) \neq \phi$, $i = 0, 1$. For each countable ordinal α, let $\delta(\alpha)$ be a rational, $0 < \delta(\alpha) < \delta$ such that the intersection of $D_Y^{(\alpha)}(E)$ with each of $(\varphi_0(J))_x$, $(\varphi_1(J))_x$, $(\mathscr{R}_{\delta(\alpha)}^\delta(\varphi_0(J)))_x$ and $(\mathscr{R}_{\delta(\alpha)}^\delta(\varphi_1(J)))_x$ is nonempty. There must be some δ' which is $\delta(\alpha)$ for uncountably many α's. Clearly, δ' satisfies the requirements of the lemma.

Since the two rings of the preceding lemma, may be expressed as the union of countably many analytic sets which are uniform with respect to X, we have:

LEMMA 3.4. *There exist continuous functions* φ_{11} *and* φ_{12} *of* J *into* E *such that* $\varphi_{11}(J) \subset \mathscr{R}_{\delta'}^\delta(\varphi_0(J))$, $\varphi_{12} \subset \mathscr{R}_{\delta'}^\delta(\varphi_1(J))$ *and* $\varphi_0(J)$, $\varphi_1(J)$, $\varphi_{11}(J)$, *and* $\varphi_{12}(J)$ *are simultaneously unbounded on* X.

Since $\pi_x(E) = \bigcup_{i=1}^\infty \psi(\mathscr{N}_i)$, there must be some t_1 such that $\varphi_0(J)$, $\varphi_1(J)$, $\varphi_{11}(J)$, and $\varphi_{12}(J)$ are simultaneously unbounded on $\psi(\mathscr{N}_{t_1})$. There must be an integer s_1^0 such that $\varphi_0(\mathscr{N}_{s_1^0})$, $\varphi_1(J)$, $\varphi_{11}(J)$ and $\varphi_{12}(J)$ are simultaneously unbounded on $\psi(\mathscr{N}_{t_1})$. Continuing in this manner, we have:

LEMMA 3.5. *There exist integers* t_1, s_1^0, s_1^1, s_1^{11}, s_2^{12} *such that* $\varphi_0(\mathscr{N}_{s_1^0})$, $\varphi_1(\mathscr{N}_{s_1^1})$, $\varphi_{11}(\mathscr{N}_{s_1^{11}})$, *and* $\varphi_{12}(\mathscr{N}_{s_1^{12}})$ *are simultaneously unbounded on* $\psi(\mathscr{N}_{t_1})$. *Also, there is a positive number* ε_1 *such that the* ε_1-*bands of the first four sets are disjoint.*

This completes the first stage of an induction process. The second stage is completed by the following procedure. First, let $\delta = \min(\varepsilon_1, 1/4)$. Second, obtain a δ', $0 < \delta' < \delta$ such that the eight sets: $\varphi_0(\mathscr{N}_{s_1^0})$, $\varphi_1(\mathscr{N}_{s_1^1})$, $\varphi_{11}(\mathscr{N}_{s_1^{11}})$, $\varphi_{12}(\mathscr{N}_{s_1^{12}})$, $\mathscr{R}_{\delta'}^\delta(\varphi_0(\mathscr{N}_{s_1^0}))$, $\mathscr{R}_{\delta'}^\delta(\varphi_1(\mathscr{N}_{s_1^1}))$, $\mathscr{R}_{\delta'}^\delta(\varphi_{11}(\mathscr{N}_{s_1^{11}}))$ and $\mathscr{R}_{\delta'}^\delta(\varphi_{12}(\mathscr{N}_{s_1^{12}}))$ are simultaneously unbounded on $\psi(\mathscr{N}_{t_1})$. Third, obtain continuous functions φ_{21}, φ_{22}, φ_{23}, φ_{24} of J into the respective rings such that the first four sets of the preceding sentence and $\varphi_{2i}(J)$, $i = 1, 2, 3, 4$, are simultaneously unbounded on $\psi(\mathscr{N}_{t_1 t_2})$. Fourth, fix t_2 so that these eight graphs are unbounded on $\Psi(\mathscr{N}_{t_1 t_2})$. Finally obtain integers: s_1^0, s_2^1, s_2^{11}, s_2^{12} and integer pairs: (s_1^{21}, s_2^{21}), (s_1^{22}, s_2^{22}), (s_1^{23}, s_2^{23}), (s_1^{24}, s_2^{24}) such that the sets: $\varphi_0(\mathscr{N}_{s_1^0 s_2^0})$ $\varphi_1(\mathscr{N}_{s_1^1 s_2^1})$ and $\varphi_{ij}(\mathscr{N}_{s_{12}^{ij}})$, $i = 1, 2$, $j \leq 2^i$ are simultaneously unbounded on $\psi(\mathscr{N}_{t_1 t_2})$.

Continuing this process by induction, we have:

THEOREM 3.6. *For each* $n \in N$ *and* $p \leq 2^n$, *there is a continuous map* φ_{np} *of* J *into* A, *a point* $\sigma^{np} = (s_1^{np}, s_2^{np}, \cdots)$ *of* J *and points* $\sigma^0 = (s_1^0, s_2^0, \cdots)$, $\sigma^1 = (s_1^1, s_2^1, \cdots)$, *and* $\tau = (t_1, t_2, t_3, \cdots)$ *of* J *such that*

(i) *the sets $\varphi_{np}(J)$ are disjoint and uniform with respect to* X,

(ii) *if $n \in N$, then the sets $\varphi_{kp}(\mathcal{N}_{\sigma^{kp}|n})$, $k \leq n$, $p \leq 2^k$, and $\varphi_0(\mathcal{N}_{\sigma^0|n})$ and $\varphi_1(\mathcal{N}_{\sigma^1|n})$ are simultaneously unbounded on $\psi(\mathcal{N}_{\tau|n})$,*

(iii) *if $1 \leq k < l$ and $p \leq 2^k$, then there is some $j \leq 2^l$ such that $\varphi_{lt}(J)$ lies in the 2^{-l}-band of $\varphi_{kp}(\mathcal{N}_{\sigma^{kp}|n})$.*

We now proceed with the

Proof of Theorem L. Let $x = \psi(\tau)$ and $(x_{kp}, y_{kp}) = \varphi_{kp}(\sigma^{kp})$, $k \in N$, $p \leq 2^k$. Thus, $(x_{kp}, y_{kp}) \in E$, for $k \in N$ and $p \leq 2^k$.

It is now shown that $x_{kp} = x$. For each $i > k$, $\varphi_{kp}(\mathcal{N}_{\sigma^{kp}|i})$ is unbounded on $\psi(\mathcal{N}_{\tau|i})$. Let $z_i \in \mathcal{N}_{\sigma^{kp}|i}$ with $\varphi_{kp}(z_i) = (x_i, y_i)$ and $x_i \in \psi(\mathcal{N}_{\tau|i})$. As $i \to \infty$, $z_i \to \sigma^{kp}$ and therefore $x_i \to x_{kp}$. But $x_i \to x$. So, $x = x_{kp}$.

Thus, E_x contains $D = \{(x_{kp}, y_{kp}): k \in N \text{ and } p \leq 2^k\}$.

It is now shown that D is dense-in-itself. To see this consider a point (x, y_{kp}) of D and a positive number ε. Pick $n > k$ so that $2^{-n} < \varepsilon$. By property (iii) of Theorem 3.6, there is some $j \leq 2^n$ such that $\varphi_{nj}(J)$ lies in the $1/2n$-band of $\varphi_{kp}(\mathcal{N}_{\sigma^{kp}|n})$. Thus, $\varphi_{nj}(\sigma^{nj})$ and $\varphi_{kp}(\sigma^{kp})$ both lie in this band. Since the points are different and have the same X-coordinate, y_{nt} lies in the ε-neighborhood of y_{kp} and $y_{nt} \neq y_{kp}$.

But, by the hypothesis of Theorem L, E_x is scattered. This contradiction implies that the conclusion of Theorem L holds.

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